Topological complexity of hyperplane complement

Sergey Yuzvinsky

University of Oregon

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Let X be a topological space. Let PX be the space of all free paths on X (with the compact - open topology) and $\pi : PX \to X \times X$ be the standard fibration assigning to each path $p : [0,1] \to X$ the pair of its ends $(p(0), p(1))$.

Definition

The topological complexity of (motion planning on) X is the smallest number n such that $X \times X$ is partitioned into Euclidean neighborhood retracts (*local domains*) X_i ($i = 1, 2, \ldots, n$) and on each X_i there exists a section (*local rule*) $s_i : X_i \to PX$ *of* π *(i.e.,* $\pi \circ s_i = id_{X_i}$*). Each choice of a partition* $X \times X = \bigcup_i X_i$ *and sections* (s_i) *is called a motion planning algorithm for* X*.*

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This definition was given by Michael Farber about 10 years ago. We abbreviate the topological complexity of X as $\mathrm{TC}(X)$.

TC is a specialization of the Schwartz genus which was defined and studied for an arbitrary fibration by Albert Schwarz in 60s; the Schwartz genus is in turn a generalization of the Lusternik - Schnirelman category. Other specializations of the Schwartz genus have been used by Smale and Vassiliev for different fibrations.

(1) TC(X) is an invariant of the homotopy type of X.

(2) TC(X) = 1 if and only if X is contractible.

(3) TC(X) $< 2 \dim X + 2$.
If X is r-connected then

$$\mathrm{TC(X)} < \frac{2 \dim \mathrm{X} + 1}{\mathrm{r} + 1} + 1.$$

(4) $\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y) - 1$.

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(5) Cohomological lower bound.

Let k be a field and $A = H^*(X; k)$ (considered as a graded algebra). Define the graded algebra structure on $A \otimes A$ via

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2.$$

where a_i, b_i are homogeneous elements from A.

Then the multiplication in A is the graded algebra homomorphism $\mu : A \otimes A \to A$ whose kernel J is called *the ideal of zero divisors*. *The zero divisor cup length* $\ell(J)$ is the length of the longest non-vanishing product in J. Then

$$\mathrm{TC(X)} > \ell(\mathrm{J}).$$

1. If $X = S^1$ then $\mathrm{TC}(X) = 2$.
Indeed by (3) $\mathrm{TC}(X) \leq 3$; by (2) (or (5)) it is greater than 1. But it is easy to design a motion algorithm with the partition into two sets: $X_1 = \{(A, B) | A = -B\}$, $X_2 = X \times X \setminus X_1$ and move B to A along a fixed orientation for $(A, B) \in X_1$ and along the shortest arc otherwise.

2. If $X = S^n$, n is odd then again $\mathrm{TC}(X) = 2$.
The proof is similar using a tangent non-vanishing vector field.

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3. For even n we have $\mathrm{TC}(S^n) = 3$.
To find a lower bound let u be a generator of $H^n(S^n, \mathbb{Z})$. Then $v = u \otimes 1 - 1 \otimes u \in J$ and $v^2 = 2u \otimes u \neq 0$. Thus $\mathrm{TC}(S^n) \geq 3$. For the opposite inequality one can use a vector field with one 0 at A_0 and the partition $X_1 = \{(A_0, -A_0)\}$, $X_2 = \{(A, -A) | A \neq A_0\}$ and $X_3 = S^n \times S^n \setminus (X_1 \cup X_2)$.

Let X be a simply connected finite polyhedron of dimension $2n$; k is a field of zero characteristic. Suppose that there exists $u \in H^2(X;k)$ such that $u^n \neq 0$. Then $\mathrm{TC}(X) = 2n+1$.

Indeed the inequality $\leq$ follows from (3) with $r=1$. The opposite inequality follows from (5):

$$(u \otimes 1 - 1 \otimes u)^{2n} = \pm \binom{2n}{n} u^n \otimes u^n \neq 0.$$

As a corollary, if X is a simply connected symplectic manifold

$$\mathrm{TC(X)} = 2\dim_{\mathbb{C}}(\mathrm{X}) + 1.$$

In particular

$$\mathrm{TC}(\mathbb{C}\mathrm{P}^{\mathrm{n}}) = 2\mathrm{n} + 1.$$

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Real Projective Spaces

Here we give an example showing that in general TC is a very non-trivial invariant.

Theorem

Let $X = \mathbb{R}P^n$*.*
(i) If $n = 1, 3,$ *or* 7 *then* $TC(X) = n + 1$*;*
(ii) For every other value of n *the number* $TC(X)$ *coincides with the smallest number* k *such that* X *admits an immersion into* $\mathbb{R}^{k-1}$*.*

The theorem says that the problem of calculating $\mathrm{TC}(\mathbb{R}P^n)$ is equivalent to the classical immersion problem for the real projective spaces. The latter is a topological problem with a long history and a lot of research although the general answer is not found. The known results allow one to calculate $\mathrm{TC}(\mathbb{R}P^n)$ for $n \leq 23$. More precisely $\mathrm{TC}(\mathbb{R}P^{23}) = 39$.

Definition

A (complex linear) hyperplane arrangement is a set $\mathcal{A}$ of n linear hyperplanes in $\mathbb{C}^r$. The complement of $\mathcal{A}$ is the topological space $M = \mathbb{C}^r \setminus \bigcup_{H \in \mathcal{A}} H$.

Among the arrangement complements there are, for instance, $K[\pi, 1]$ spaces for all pure Artin groups and all pure Artin type groups for all finite complex reflection groups.

Example. Consider $\binom{r}{2}$ hyperplanes given by the equations $x_i = x_j$ for all $1 \leq i < j \leq r$. This arrangement is called Braid arrangement because $\pi_1(M)$ is the pure Braid group on r strings. Moreover M is the $K[\pi, 1]$ for that group.

An arrangement is *essential* if the intersection of all hyperplanes is 0. In the rest of the talk, we will be assuming that $\mathcal{A}$ is an **essential** arrangement.

The intersections of all subsets of hyperplanes form a *geometric intersection lattice* $L = L(\mathcal{A})$ if ordered opposite to inclusions.

An equivalent combinatorial structure is a *simple matroid*. The latter structure consists of all independent subsets of hyperplanes (see below). The *rank* $\operatorname{rk}\mathcal{A}$ of an arrangement $\mathcal{A}$ is the maximal length of its independent subsets. For an (essential) arrangement $\operatorname{rk}\mathcal{A} = r$. An independent subset of size r is called a *base*.

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For each $X \in L$ we put $\mathcal{A}_X = \{H \in \mathcal{A} | X \subset H\}$. The *rank of X* $\operatorname{rk} X = \operatorname{rk} \mathcal{A}_X$ that is equal to $\operatorname{codim} X$. Notice that $\operatorname{rk} 0 = \operatorname{rk} \mathcal{A} = r$.

If a property holds fro all arrangements $\mathcal{A}_X$ including $X = 0$ we say it holds locally for $\mathcal{A}$.

The cohomology algebra $H^*(M)$ over $\mathbb{C}$ (or $\mathbb{Z}$) is known. Ffix for each hyperplane $H \in \mathcal{A}$ a linear form α_H such that $H = \ker \alpha_H$ and denote by ω_H the differential 1-form $\frac{d\alpha_H}{\alpha_H}$. We can identify $\mathcal{A}$ with the set of α_H. When it is needed we will fix a linear order on hyperplanes and abbreviate α_{H_i} as α_i.

Theorem

De Rham map reduces to the graded algebra isomorphism $\rho : A \to H^*(M; \mathbb{C})$ where A is the subalgebra of the algebra Ω^* of differential forms on M generated by $\{\omega_H | H \in \mathcal{A}\}$.

An immediate corollary is that M is a formal space.

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An immediate corollary is that M is a formal space.

Let E be the exterior algebra over $\mathbb{C}$ generated by $\{e_H | H \in \mathcal{A}\}$. Then A is a quotient of E by an ideal. We do not need to know the ideal. It suffices to say that for every $S \subset \{1, \dots, n\}$ the product e_S of e_i, $i \in S$ is in the ideal if and only if the respective set of hyperplanes is dependent.

Thus the monomials with independent supports in *A* generate the whole algebra. However they are not in general linearly independent. As a monomial basis for *A* one can use a Gröbner basis for any ordering of $\mathcal{A}$.

TCM has been calculated in the past for the two classes of arrangements - general position ones and the sets of reflection hyperplanes of classical series of Coxeter groups. In this talk we give the answer for an arbitrary arrangement.

Definition

Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^r$ and $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ a partition. The partition is called *basic* if $|\mathcal{B}| = r$ (i.e., $\mathcal{B}$ is a base) and there exists a total ordering of $\mathcal{A}$ such that both blocks give monomials of some Gröbner basis.

Of course not all arrangements have basic partitions. A simple necessary (but not sufficient) condition is $n \leq 2r$, even (more subtle) $n \leq 2r - 1$.

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Now we show that the property to have a basic partition can be formulated without any reference to a basis.

Lemma

An arrangement $\mathcal{A}$ is basic if and only if there is no $X \in L$ such that $|\mathcal{A}_X| \geq 2\,\mathrm{rk}\,X$.

Definition

A subarrangement (subset) of an arrangement $\mathcal{A}$ is *basic* if it has the same rank as $\mathcal{A}$ and admits a basic partition. The maximal cardinality of basic subsets of $\mathcal{A}$ is the *basic number* $b(\mathcal{A})$.

Lemma above implies $b(\mathcal{A}) \leq 2r - 1$.

In order to find a lower bound for $\mathrm{TC}(M(\mathcal{A}))$ we use the property (5), i.e., the zero-divisor-cup-length of the Orlik-Solomon algebra A.

For each i consider the element

$$\overline{e_i} = 1 \otimes e_i - e_i \otimes 1 \in (A \otimes A)_1 .$$

Clearly $\overline{e_i} \in J$ (the zero-divisor ideal) and we want to study the product

$$\pi_I = \prod \overline{e_i}$$

where the product is taken over some set I of indexes.

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Theorem

Let I be a subset of an arrangement $\mathcal{A}$ of full rank . Then $\pi_I \neq 0$ if and only if I is basic.

The 'only if' statement is easier. It can be calculated that $\pi_I = 0$ for all $\mathcal{A}$ of rank r and the cardinality of I equal $2r$. Thus if I is not basic it has a subset with such values of parameters whence π_I has a zero subproduct.

The 'if' statement is harder. The proof based on the fact that the non-zero monomial term $m_{\mathcal{B}} \otimes m_{\mathcal{C}}$ corresponding to the basic partition of I in the decomposition of π_I can not get canceled.

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Corollary

$TC(M) \geq b(\mathcal{A}) + 1.$

In order to obtain an upper bound for $\mathrm{TC}(M)$ we construct an explicit motion planning algorithm for M.

We say that a pair (P, Q) of points from M lies in a local domain D_i ($i = 0, 1, \ldots, n$) if the interval $[P, Q]$ of the real (affine) line PQ intersects with precisely i hyperplanes from $\mathcal{A}$.

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Let $\widetilde{PQ}$ be the complex line through PQ. The local rule (section) s is defined for a pair (P, Q) as the path that goes from P to Q with constant speed along PQ untill it reaches either Q or the interval of small length centered at some P_i. Then it continues on the semicircle centered at P_i lying $\widetilde{PQ}$.

The half plane for the choice of the semicircle is given by the vector $\sqrt{-1}\overrightarrow{PQ}$.

The main nn-trivial property of the algorithm is the following.

Theorem

By gluing together the domains D_i for $i \geq b(\mathcal{A})$ and gluing the respective s_i one still gets a motion planning algorithm with $b(\mathcal{A}) + 1$ domains.

Theorem

$TC(M) = b(\mathcal{A}) + 1.$

In particular this Theorem proves the conjecture that TS(M) is larger by 1 than the zero-divisor-cup-length. This in turn shows that TC(M) is a combinatorial invariant (while the homotopy type of M is not).

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Remarks

1. In matroid theory, there has been a popular topic about covering a matroid by independent sets. In particular, a theorem of Jack Edmonds has the following corollary. A matroid can be covered by two independent sets if and only if $|U| \leq 2\,\mathrm{rk}\,U$ for every subset U where $\mathrm{rk}\,U$ is the rank of the subset.

Our definition of a basic set is similar but an effect of our inequality can be a lot stronger.

Example

Let k be a positive integer and the arrangement is: $x_i, y_i, x_i \pm y_i$ where $i = 1, 2, \ldots, k$. Clearly $r = 2k$ and the arrangement satisfies the weaker inequality. Indeed it can be partitioned into two bases with $2k$ elements in each. On the other hand, one of the maximal basic sets can be constructed by deleting all the differences $x_i - y_i$ whence the basic number is $3k$.

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Here are more distant and vague relations of basic number to Bernstein-Sato polynomials (b-functions). These are polynomials of one indeterminate defined for given polynomial of several variables (e.g., for the defining polynomials of arrangements). The construction relates to holomorphic continuations and D-modules.

If $f(x)$ is a polynomial in several variables then there is a non-zero polynomial $b(s)$ and a differential operator $P(s)$ with polynomial coefficients such that
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Moderate type

There are several famous conjectures about roots of b-functions. The simplest conjecture specified to arrangements says that the number $-\frac{r}{n}$ is the largest root of the b-function for an arrangement.
The first significant progress about the conjecture (for arrangements) was a proof of it for so called *moderate type* arrangements. This notion requires the function $q(X) = \frac{\operatorname{rk} X}{|\mathcal{A}_X|}$ on L to decrease, i.e., $q(X) \geq q(Y)$ for every $Y > X$.

Notice that the definition of basic arrangement in this language is: $q(X) > \frac{1}{2}$ for every X. Thus an arrangement of moderate type is basic if $q(0) = \frac{r}{n} > \frac{1}{2}$, i.e., $n < 2r$.

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THANK YOU FOR YOUR ATTENTION!