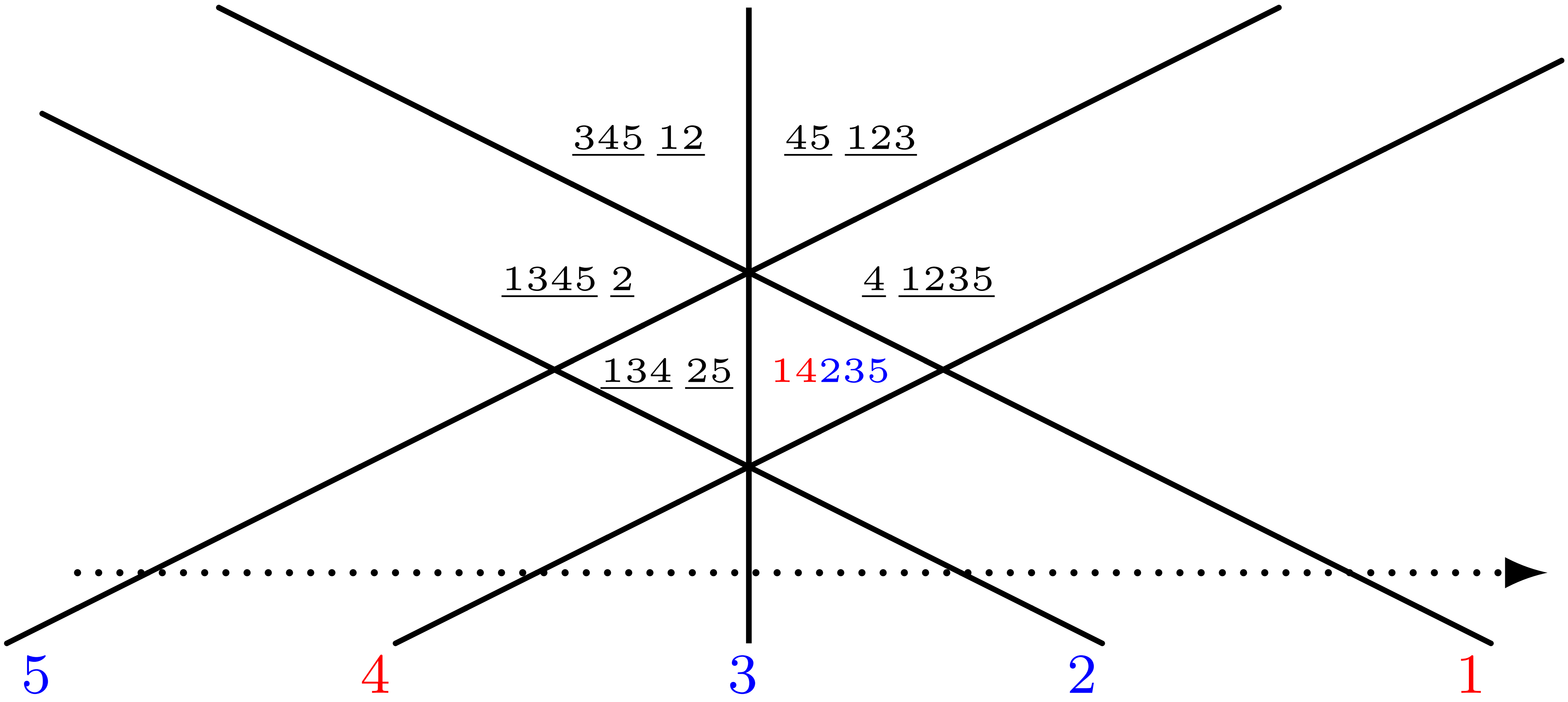
Minimal Stratification for Line Arrangements

Masahiko Yoshinaga (Kyoto)

SINGULARITY THEORY CONFERENCE,
中国科学技術大学, 合肥, 2011年7月27日



OUR OBJECTS

$\mathcal{A} = \{H_1, \ldots, H_n\}, H_i \subset \mathbb{R}^2$ lines.

$M = M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{i=1}^{n} H_i \otimes \mathbb{C}$, complement.

GOAL

Proposing a new relation between real structure and topology of $M(\mathcal{A})$, which is also related to minimality of $M(\mathcal{A})$.

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Contents

§1 Backgrounds:

- Real structure v.s. topology of $M(\mathcal{A})$.
- Minimality of $M(\mathcal{A})$.

§2 Minimal Stratification.

§3 Fundamental group $\pi_1(M(\mathcal{A}))$.

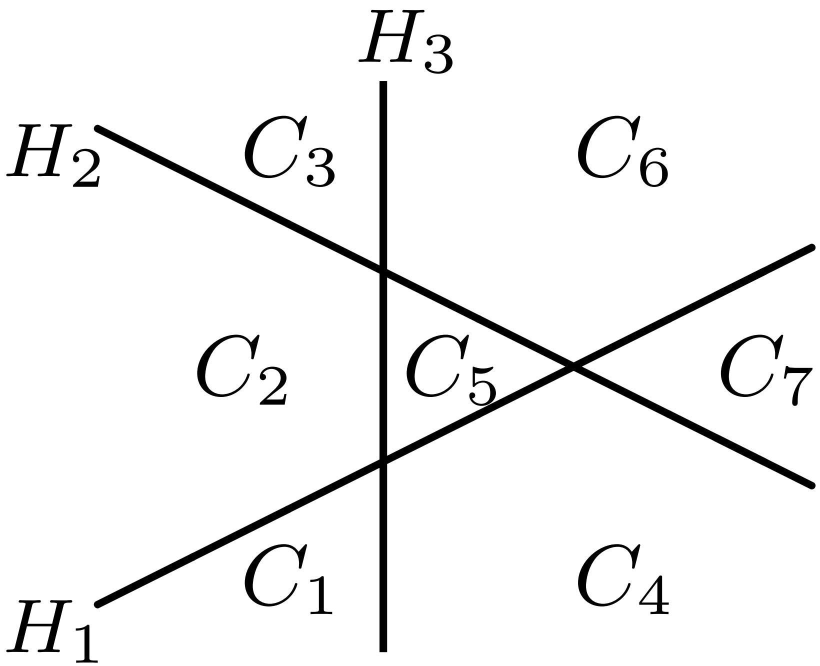
1 Backgrounds: 1. Real structure

2 Backgrounds: 1. Real structure

Def. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ arrangement in $\mathbb{R}^\ell$.

$$\mathrm{ch}(\mathcal{A}) := \{C : \text{connected compo's of } \mathbb{R}^n \setminus \bigcup H_i\}$$

$$\mathrm{bch}(\mathcal{A}) := \{C : \text{bounded chambers}\}.$$



$$\mathcal{A} = \{H_1, H_2, H_3\}$$

$$\mathrm{ch}(\mathcal{A}) = \{C_1, \ldots, C_7\}$$

$$\mathrm{bch}(\mathcal{A}) = \{C_5\}$$

2 Backgrounds: 1. Real structure

Known facts on “real structrue v.s. $M(\mathcal{A})$”:

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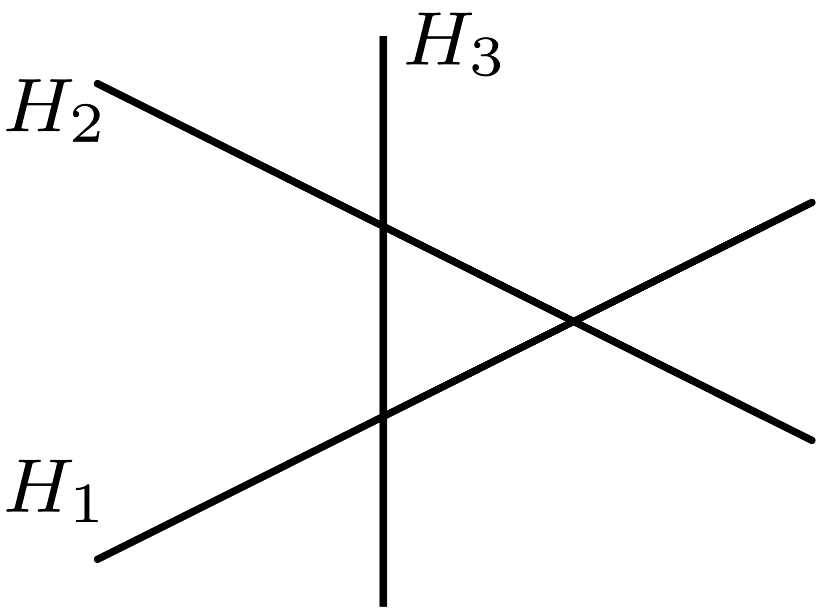
Thm. (Zaslawsky) $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup H_i \otimes \mathbb{C}$,

$$|\operatorname{ch}(\mathcal{A})| = \sum b_i(M(\mathcal{A}))$$

$$|\operatorname{bch}(\mathcal{A})| = (-1)^\ell \chi(M(\mathcal{A}))$$



$$\mathcal{A} = \{H_1, H_2, H_3\}$$

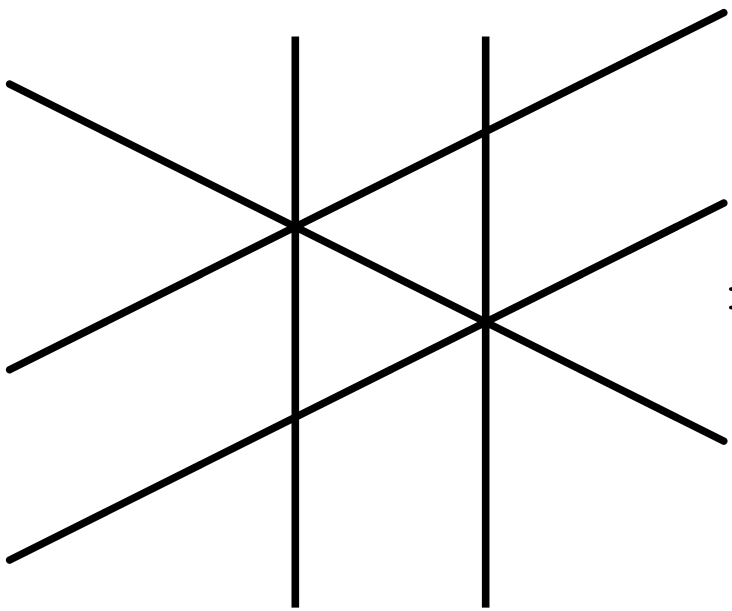
$$b_0 + b_1 + b_2 = 7$$

$$b_0 - b_1 + b_2 = 1$$

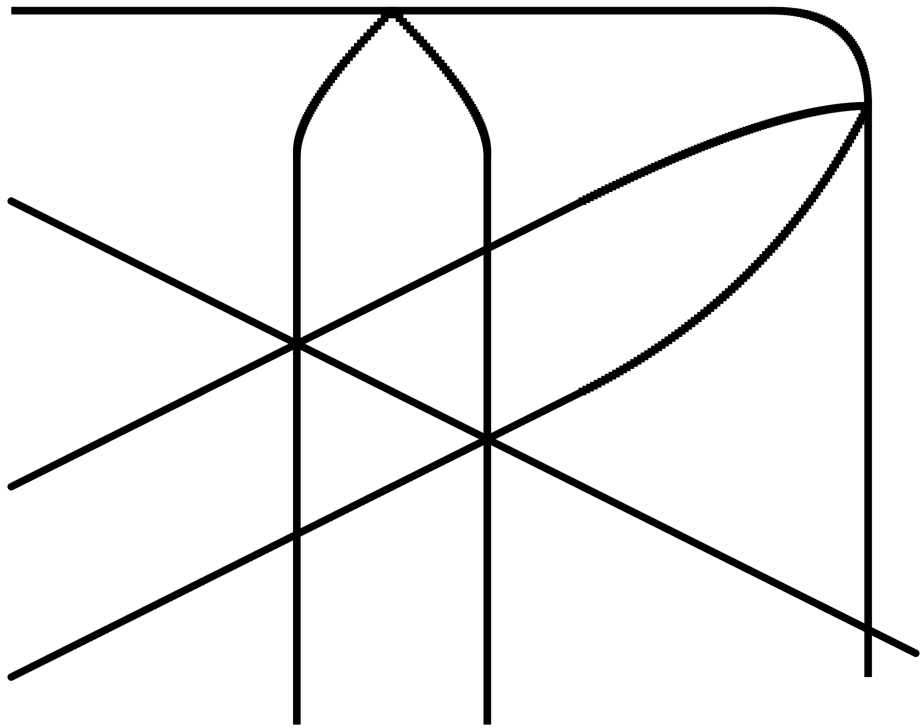
2 Backgrounds: 1. Real structure

Known facts on “real structrue v.s. $M(\mathcal{A})$”:

<u>Thm.</u> (Deligne) $\mathcal{A} = \{H_1, \dots, H_n\}$ induces a simplicial decomposition of $\mathbb{RP}^\ell$, $M(\mathcal{A})$ is $K(\pi, 1)$.



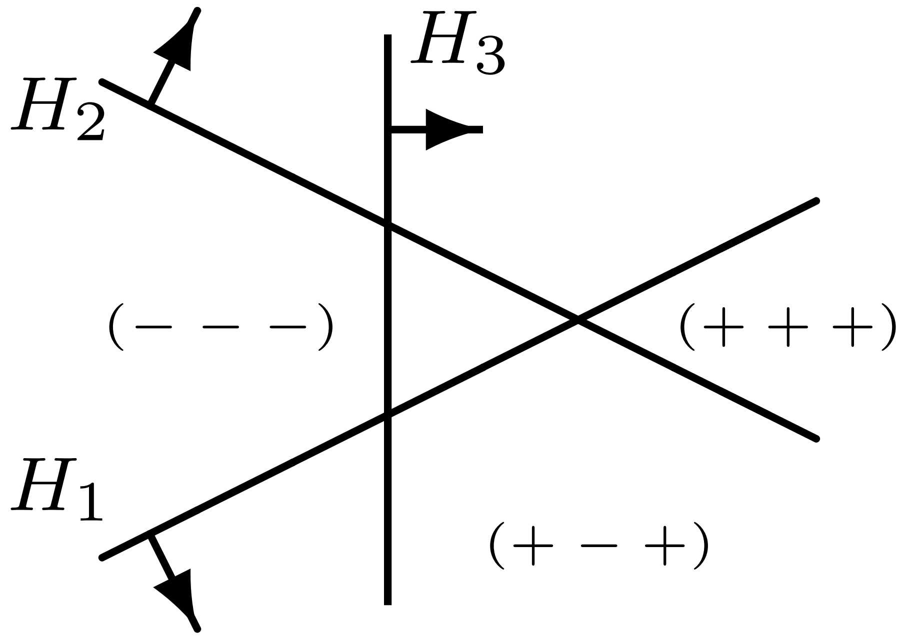
$$\Longrightarrow$$



2 Backgrounds: 1. Real structure

Known facts on “real structrue v.s. $M(\mathcal{A})$”:

<u>Thm.</u> (Salvetti) Homotopy type of $M(\mathcal{A})$ is recovered from oriented matroid of $\mathcal{A}$.



Oriented Matroid of $\mathcal{A}$:

$(+ - -), (- - -), (- + -), (+ - +),$
$(- - +), (- + +), (+ + +),$

$(0 - -), (-0-), (+ - 0), (- - 0),$
$(- + 0), (0 - +), (-0+), (+0+),$
$(0 + +),$
$(0 - 0), (-00), (00+).$

2 Backgrounds: 1. Real structure

Known facts on “real structrue v.s. $M(\mathcal{A})$”:

Thm. (Aomoto, et al.) Let $\mathcal{L}$ be a generic complex rank one local system on $M(\mathcal{A})$. Then

$$H^k(M(\mathcal{A}), \mathcal{L}) = \begin{cases} 0 & , \quad k \neq \ell \\ \bigoplus_{C \in \mathrm{bch}(\mathcal{A})} \mathbb{C} \cdot [C] & , \quad k = \ell. \end{cases}$$

2 Backgrounds: 2. Minimality of $M(\mathcal{A})$

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Thm. (Dimca-Papadima, Randell)
$M(\mathcal{A})$ is homotopic to a finite *minimal CW complex* X (i.e. $\sharp(k\text{-cells}) = b_k(M(\mathcal{A}))$.)

Rem. In general

$$\sharp(k\text{-cells of } X) \geq b_k(X).$$

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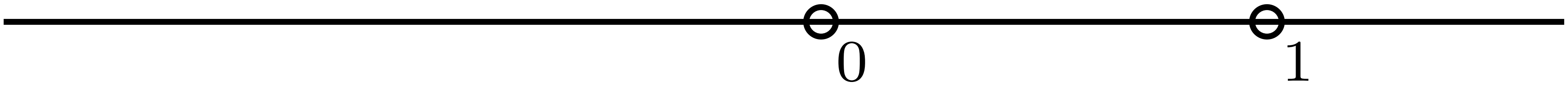
It is proved by Morse theoretic arguments.

2 Backgrounds: 2. Minimality of $M(\mathcal{A})$

Thm. $M(\mathcal{A})$ is homotopic to a minimal CW complex X.

Proof for $\ell = 1, \mathcal{A} = \{0, 1\}$.



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$$\varphi(z) = \left| \frac{(z+1)^2}{\sqrt{z(z-1)}} \right|$$

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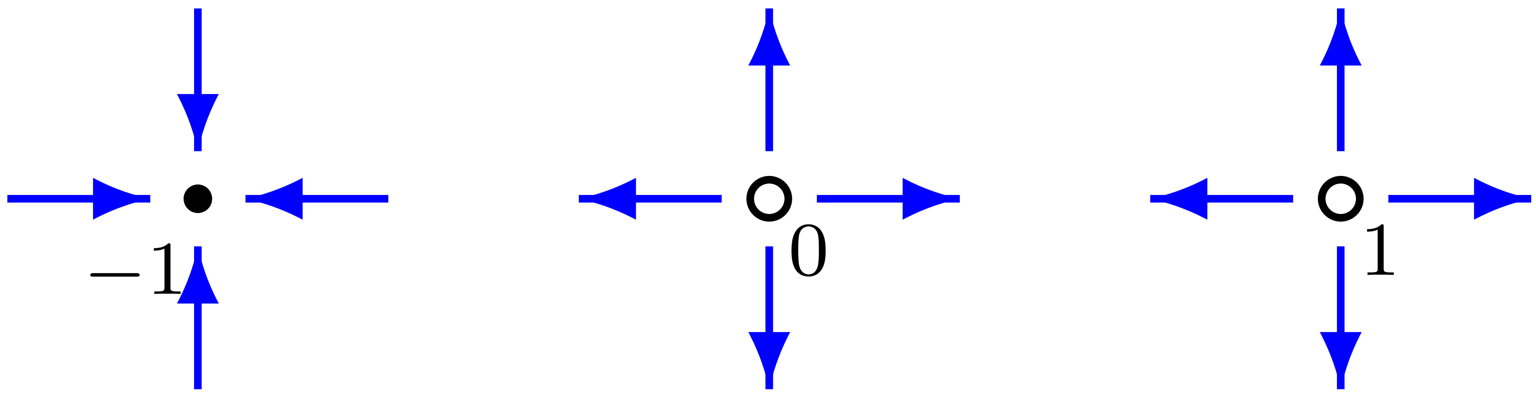
$$-\operatorname{grad}\varphi$$



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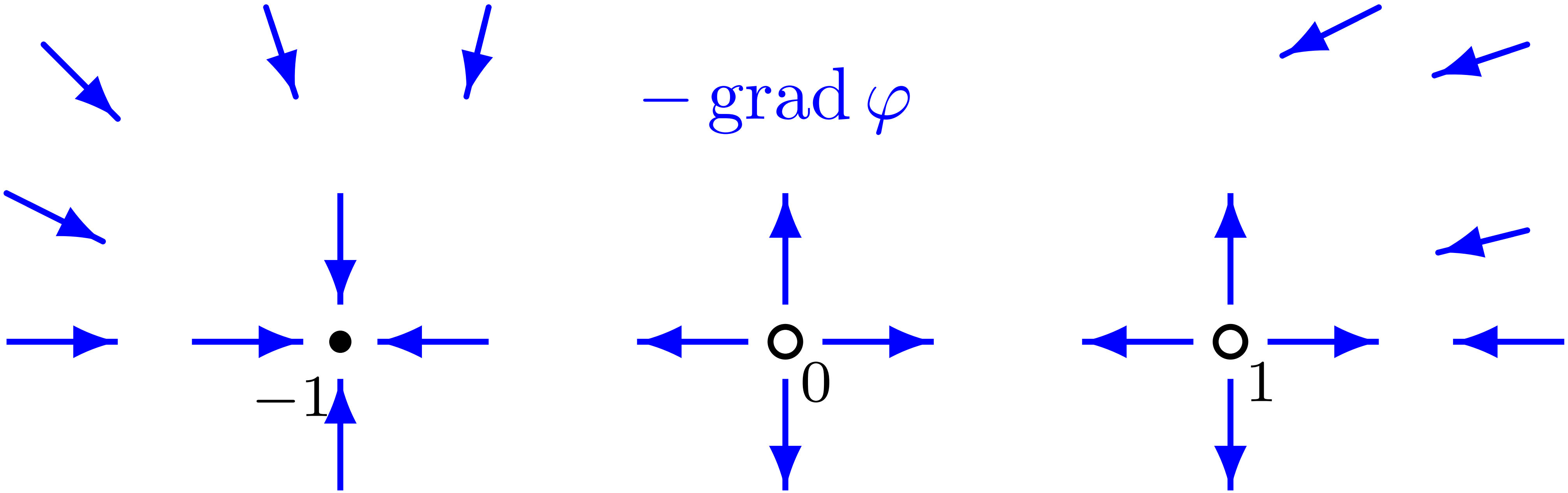

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<u>Thm.</u> $M(\mathcal{A})$ is homotopic to a minimal CW complex X.

<u>*Proof for $\ell = 1, \mathcal{A} = \{0, 1\}$.*</u>

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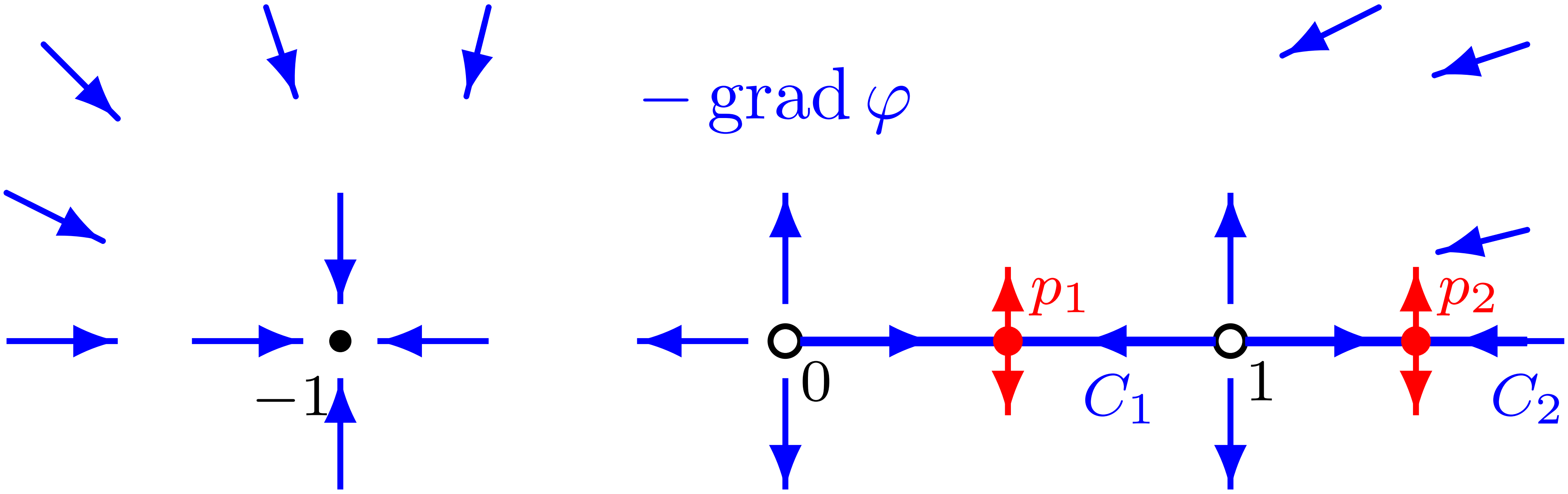

$$\mathrm{Cr}(\varphi) = \{p_1, p_2\}$$

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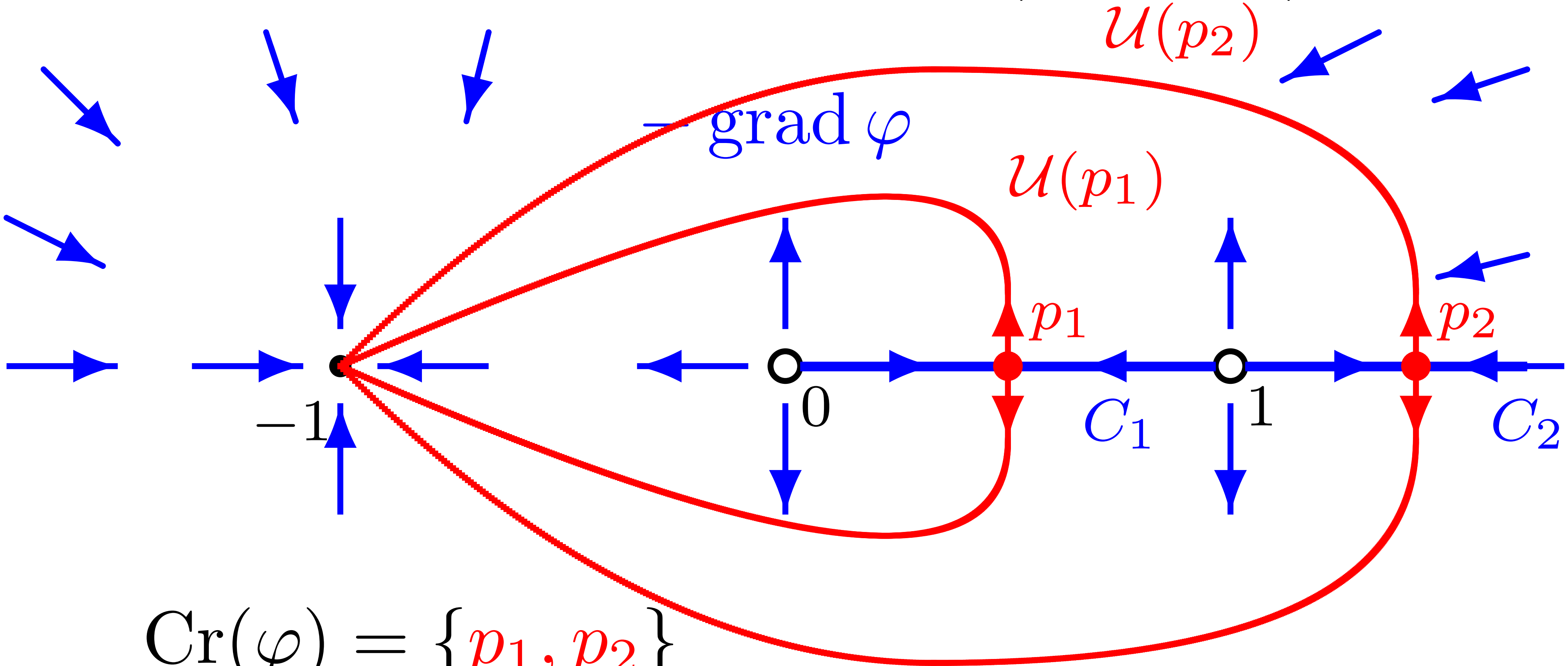


$$\mathrm{Cr}(\varphi) = \{p_1, p_2\}$$

2 Backgrounds: 2. Minimality of $M(\mathcal{A})$

Hence

$$M(\mathcal{A}) \approx \{-1\} \cup \mathcal{U}(p_1) \cup \mathcal{U}(p_2)$$

gives a minimal complex. (q.e.d.)

This is an "existence" result.
$\Longrightarrow$ The attaching maps are not clear.

2 Backgrounds: 2. Minimality of $M(\mathcal{A})$

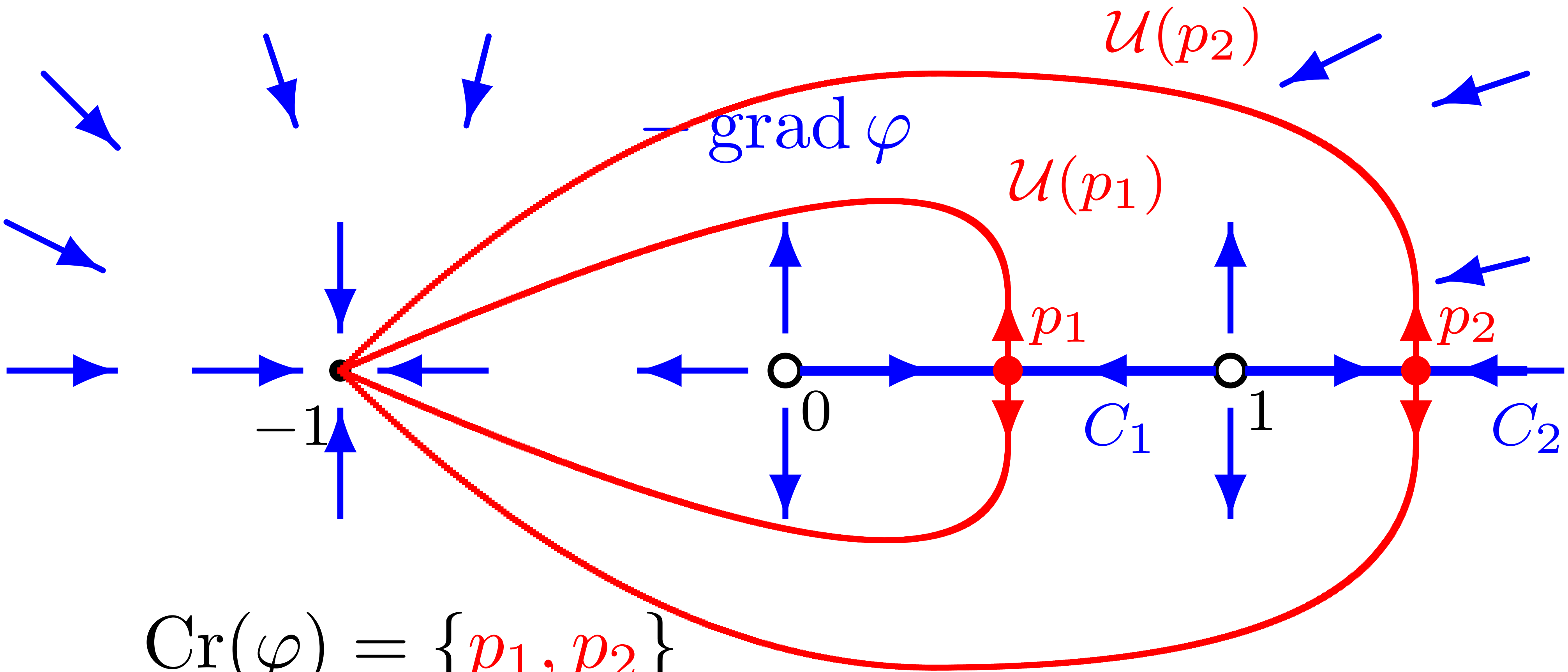
There are two approaches to describe attaching maps:

(i) Analyzing Lefschetz hyperplane section theorem (Y.).

(ii) Using discrete Morse theory on Salvetti complex. (Salvetti-Settepanella, et al.)

Today: Describe "dual" stratification.

2 Backgrounds: 2. Minimality of $M(\mathcal{A})$



$\mathrm{Cr}(\varphi) = \{p_1, p_2\}$

Observation: C_1 and C_2 are stable cells.

2 Backgrounds: 2. Minimality of $M(\mathcal{A})$

$U := M(\mathcal{A}) \setminus S_1 \cup S_2$

Key observation: $M(\mathcal{A}) = U \sqcup S_1 \sqcup S_2$,

$$S_1 = (0,1) = \left\{ z \in \mathbb{C} \mid \frac{z-1}{z} \in \mathbb{R}_{<0} \right\},$$

$$S_2 = (1,\infty) = \left\{ z \in \mathbb{C} \mid \frac{-1}{z-1} \in \mathbb{R}_{<0} \right\},$$

S_1, S_2 and U are contractible.

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S_1, S_2 and U are contractible.

We will generalize this semi-algebraic stratification to $\ell = 2$.

3 Minimal Stratification

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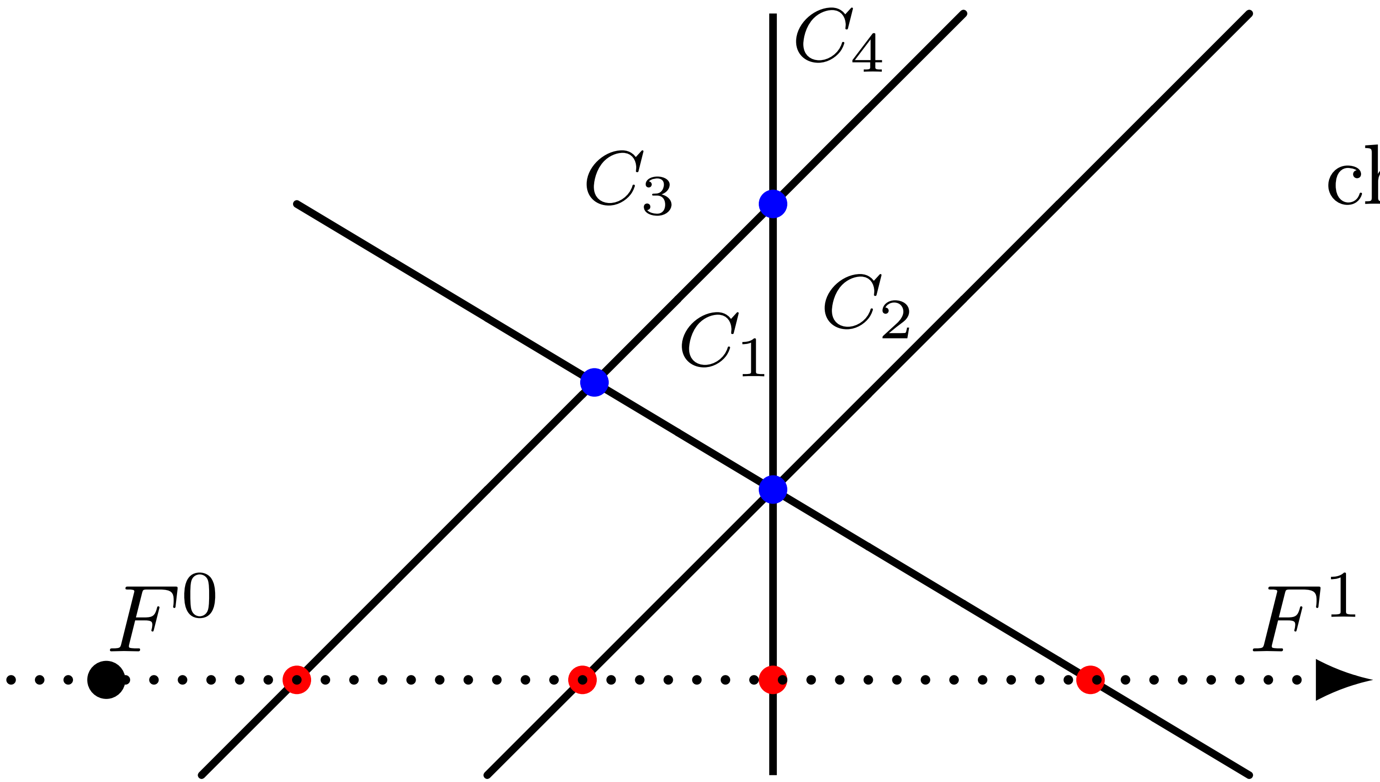
Setting: ∘ $\mathcal{A} = \{H_1, \dots, H_n\}, H_i \subset \mathbb{R}^2$.

∘ Fix a generic flag $F^0 \subset F^1$ (with orientation)

s.t. F^1 does not separate intersections of $\mathcal{A}$, and

F^0 does not separate $\mathcal{A} \cap F^1$.



Def.

$$\mathrm{ch}_2(\mathcal{A}) = \{C \mid C \cap F^1 = \emptyset\} = \{C_1, C_2, C_3, C_4\}$$

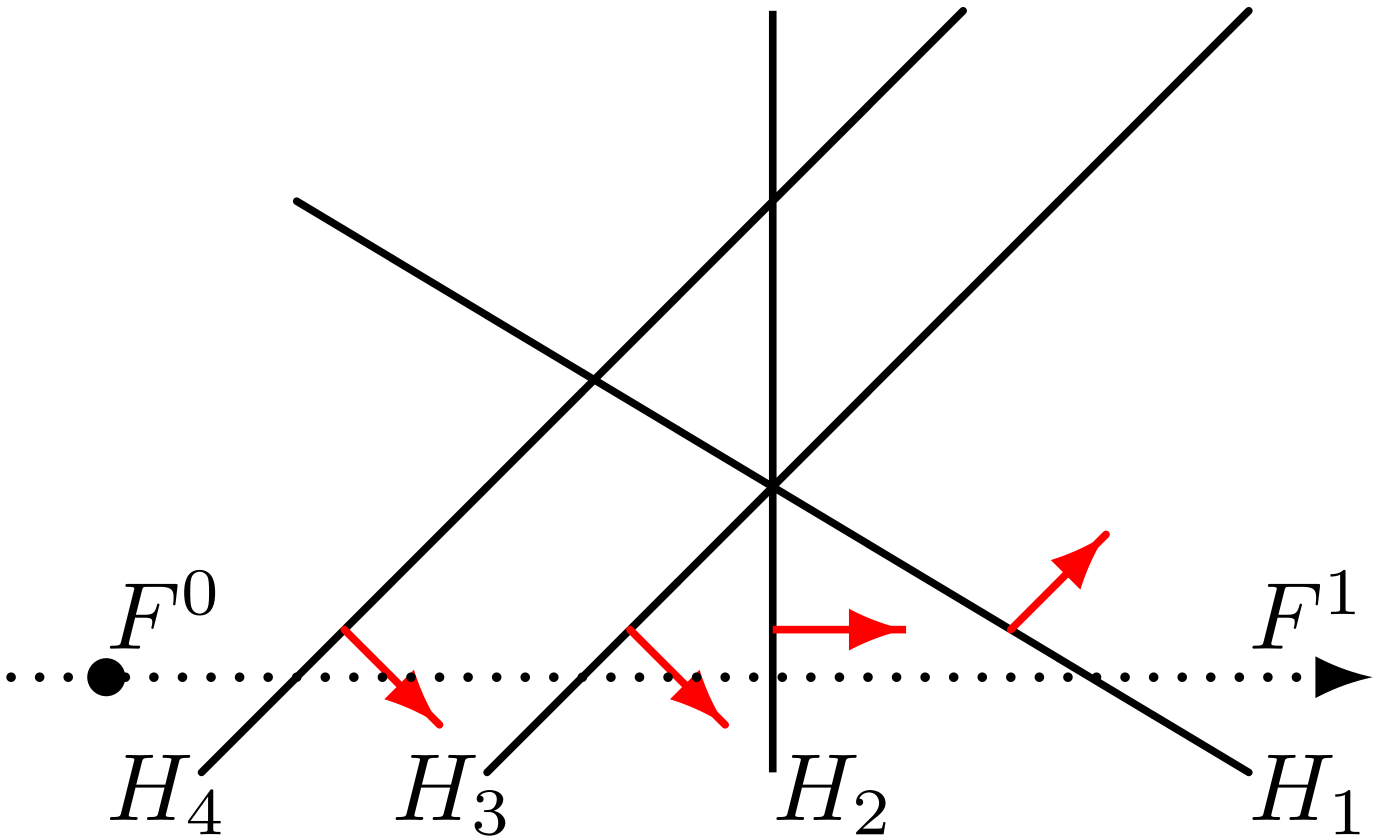
Prop. $|\mathrm{ch}_2| = b_2$.

3 Minimal Stratification

Numbering: ∘ $F^0 < H_n \cap F^1 < \cdots < H_1 \cap F^1$.

∘ $\alpha_i(F^0) < 0$, $(i = 1, \ldots, n)$

(Put $\alpha_0 \equiv -1$ for simplicity.)

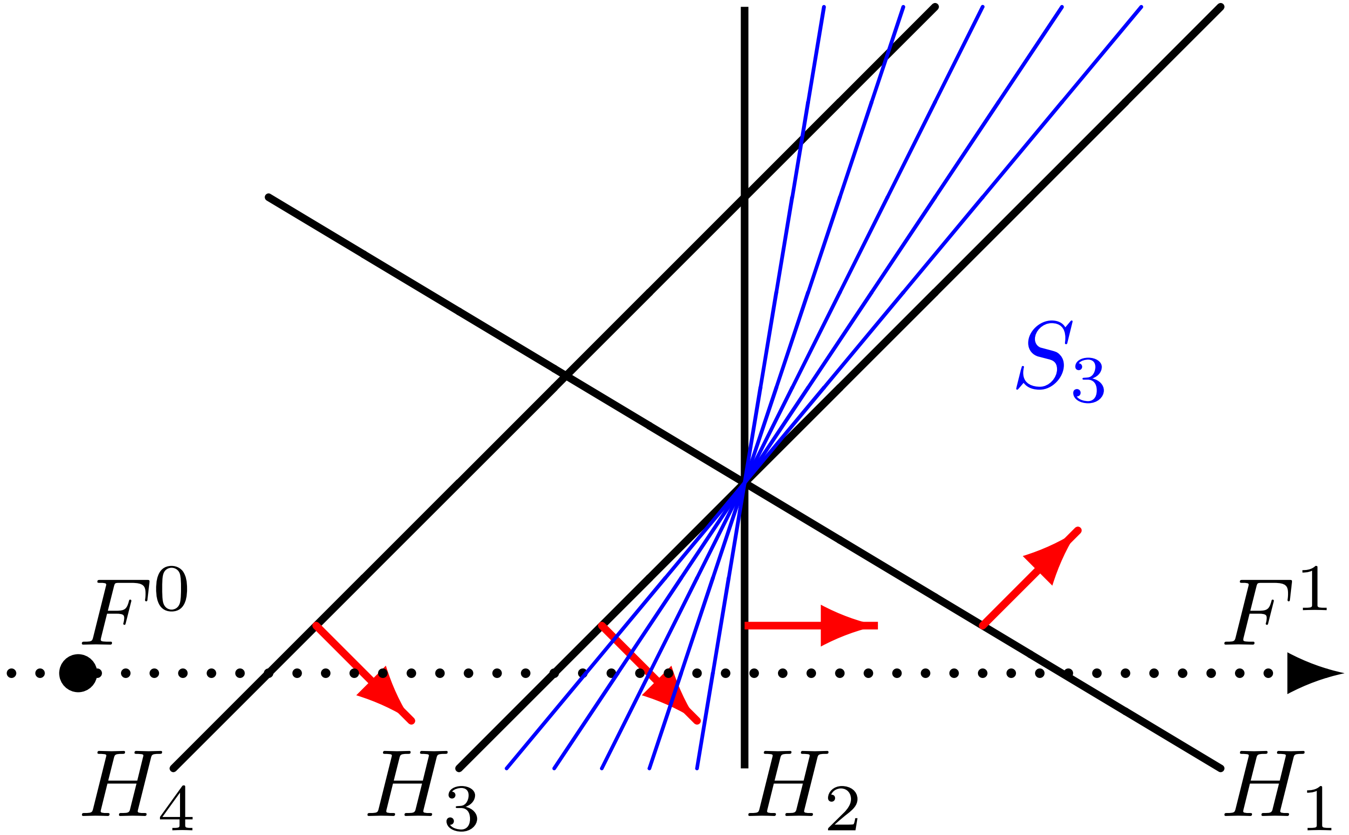


$$H_i = \{\alpha_i = 0\}$$
$$F^0 \subset \{\alpha_i < 0\}$$

3 Minimal Stratification

Key Definition:

$$S_i = \left\{ z \in M(\mathcal{A}) \,\middle|\, \frac{\alpha_{i-1}(z)}{\alpha_i(z)} \in \mathbb{R}_{<0} \right\}$$



Rem.

If $a < b$ in $\mathbb{R}$,

$$\{z \in \mathbb{C} \mid \frac{z-b}{z-a} \in \mathbb{R}_{<0}\}$$
$$= (a, b)$$

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Thm.

(0) S_i is an orientable 3-manifold.

(1) $S_i \pitchfork S_j$, and $S_i \cap S_j$ is union of $C \in \mathrm{ch}_2(\mathcal{A})$ satisfying $\frac{\alpha_i(C)}{\alpha_{i-1}(C)} < 0$ and $\frac{\alpha_j(C)}{\alpha_{j-1}(C)} < 0$.

(2) $S_i^\circ = S_i \setminus \bigsqcup_{C \in \mathrm{ch}_2} C$ is contractible 3-mfd.

(3) $U = M(\mathcal{A}) \setminus \bigsqcup_{i=1}^n S_i$ is contractible 4-mfd.

Rem. $M(\mathcal{A}) = U \sqcup \bigsqcup_{i=1}^n S_i \sqcup \bigsqcup_{C \in \mathrm{ch}_2} C$

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Consider $S = \{(z_1, z_2) \in \mathbb{C}^2 \mid \frac{z_1}{z_2} \in \mathbb{R}_{<0}\}$

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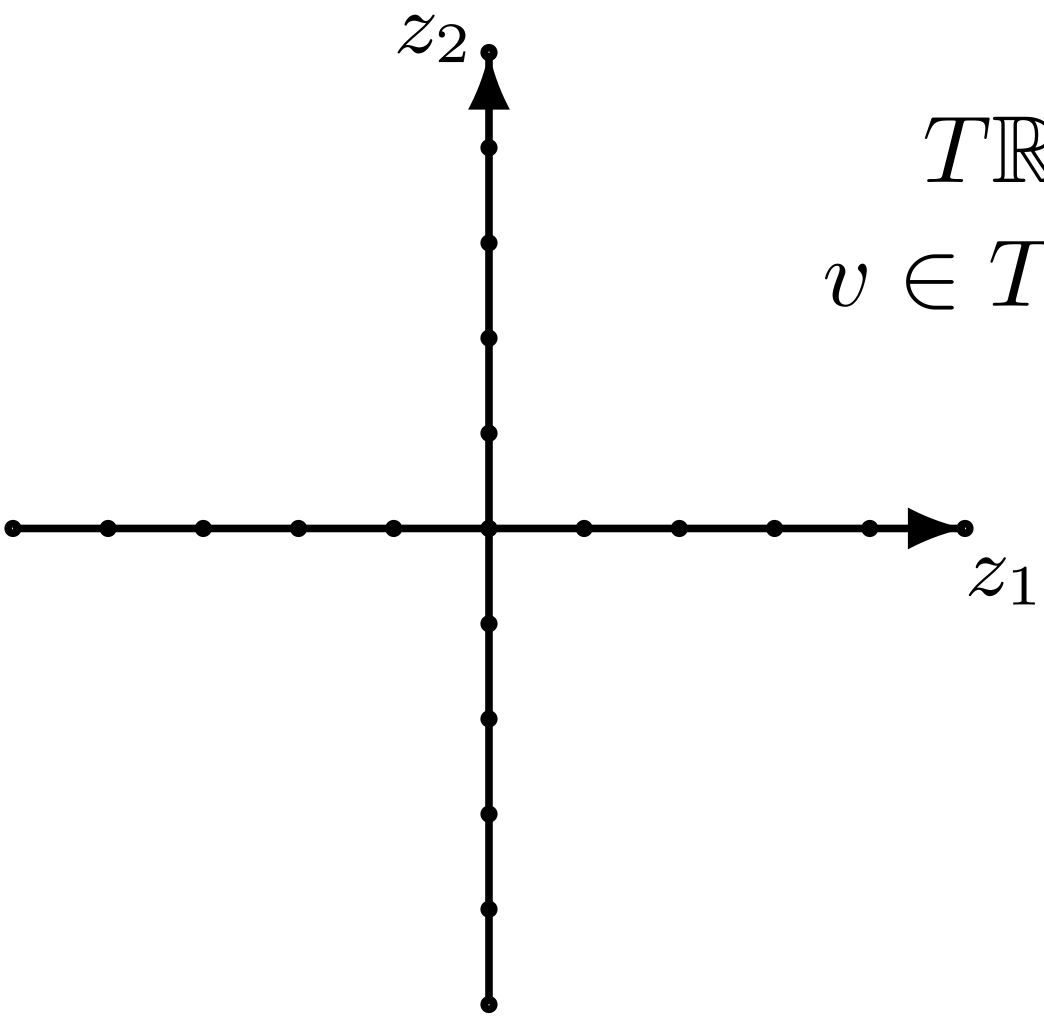

$$\begin{array}{ccc} T\mathbb{R}^2 & \cong & \mathbb{C}^2 \\ v \in T_x\mathbb{R}^2 & \mapsto & x + \sqrt{-1}v \end{array}$$

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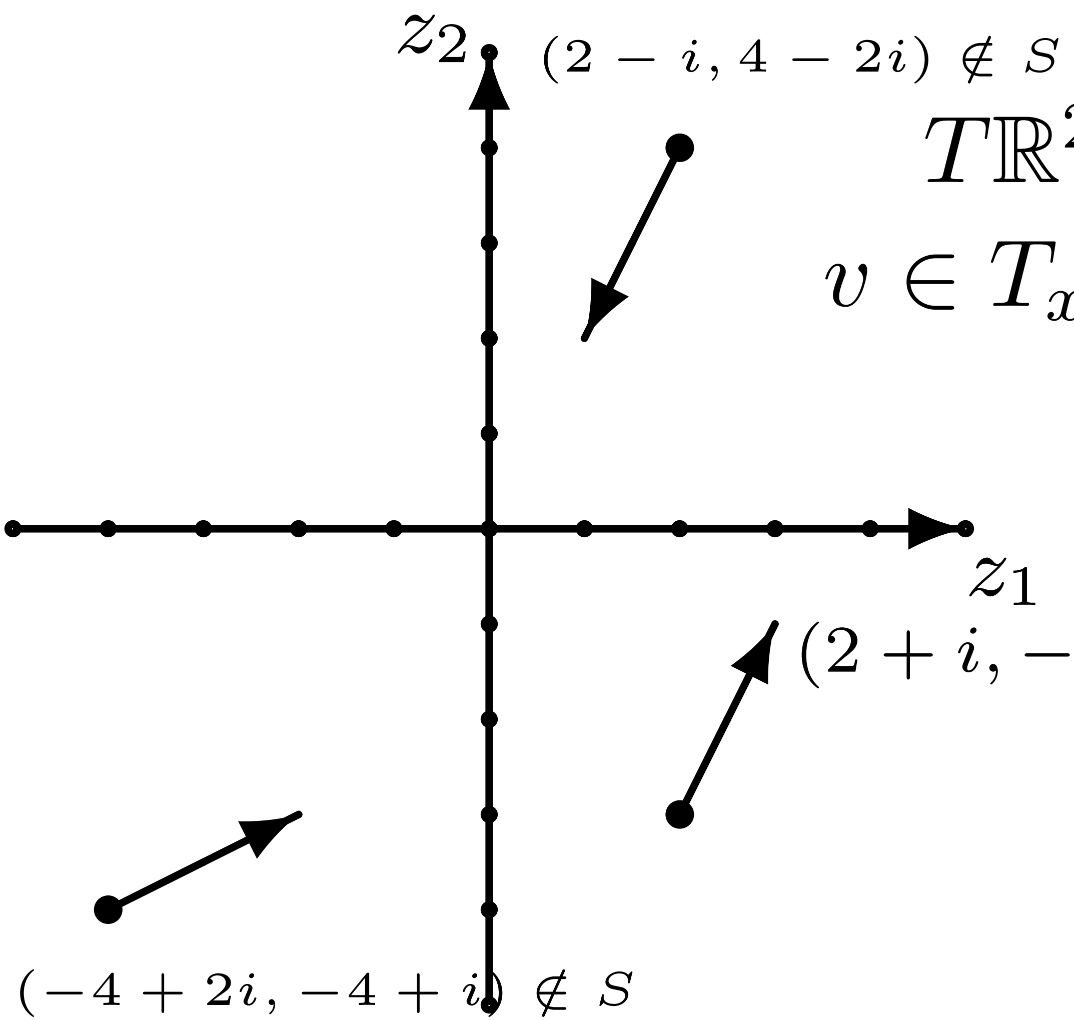

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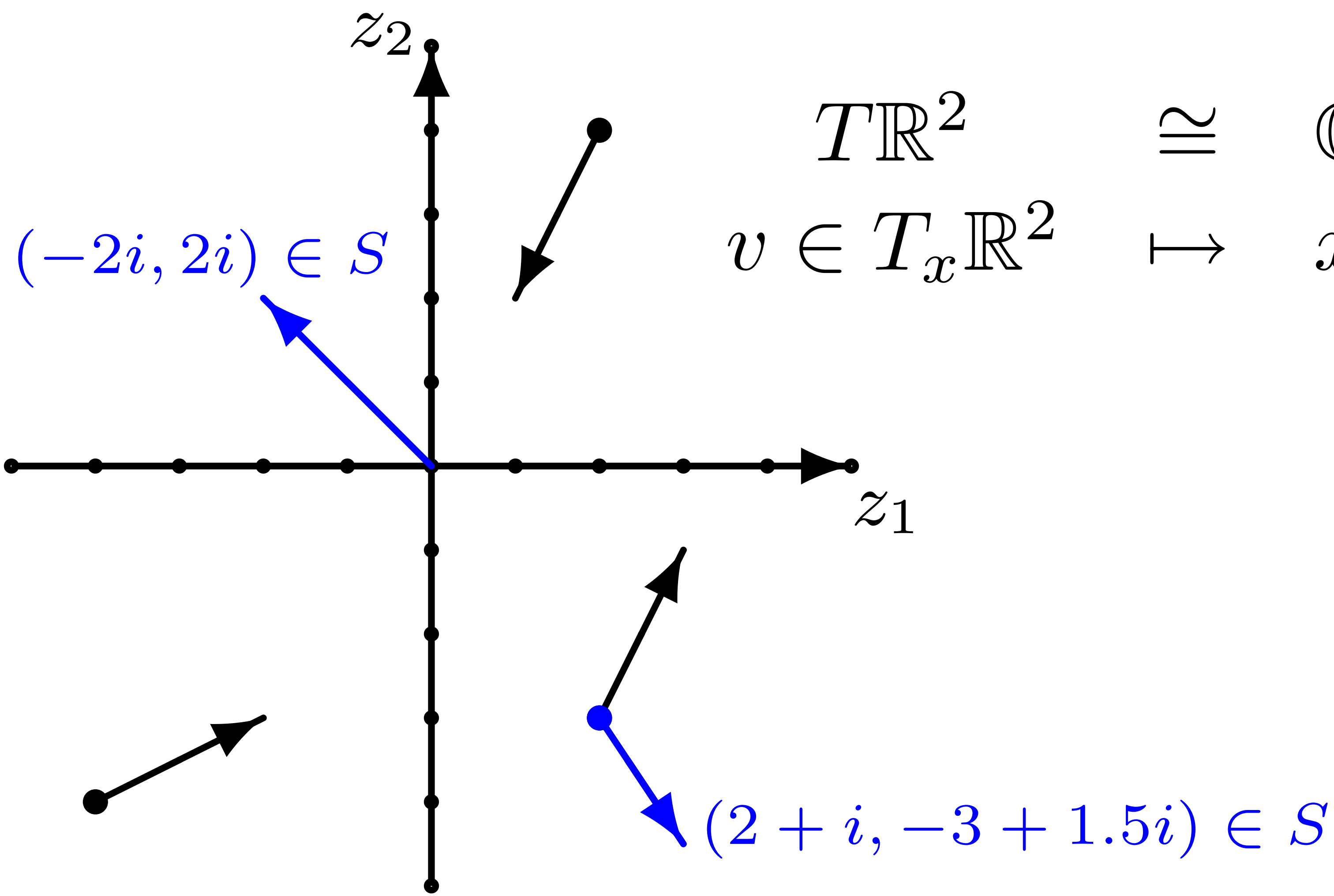

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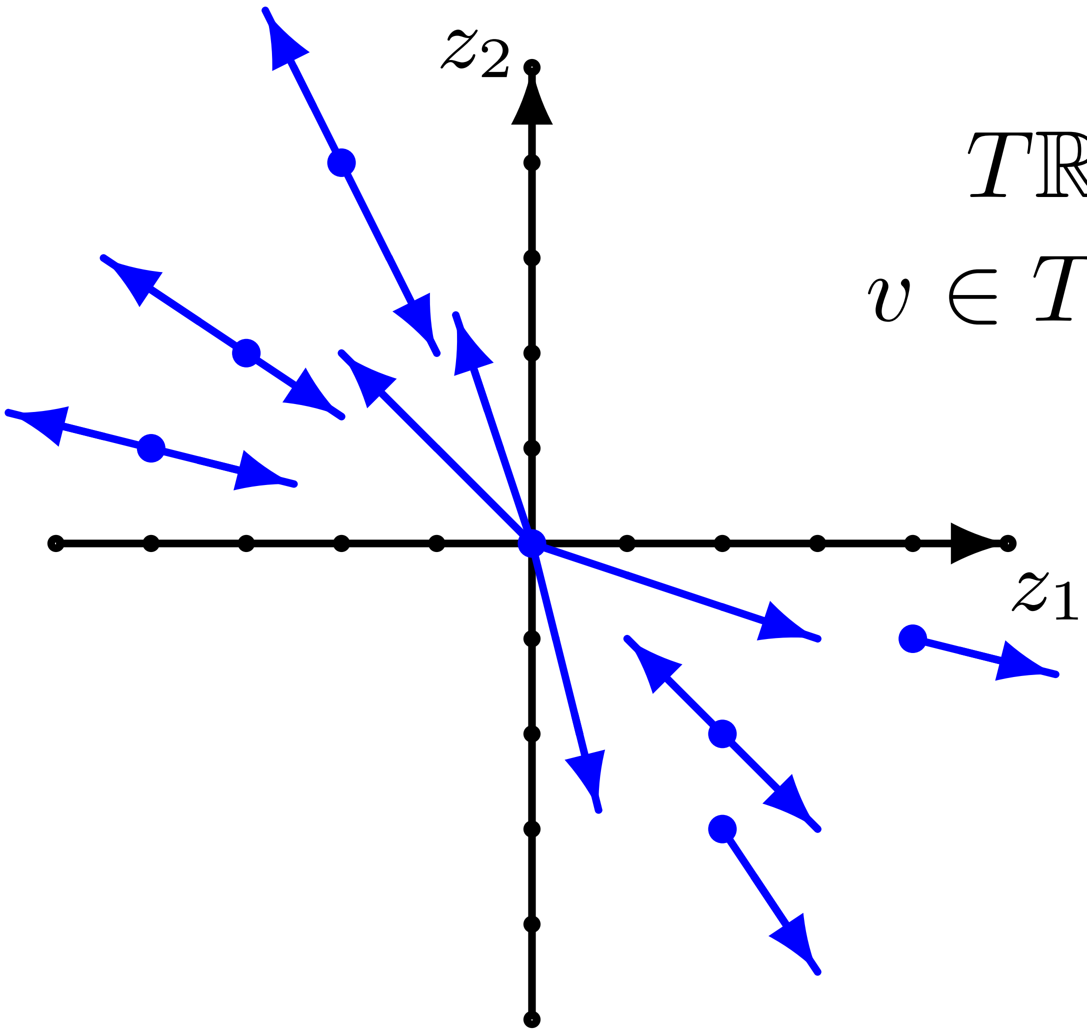

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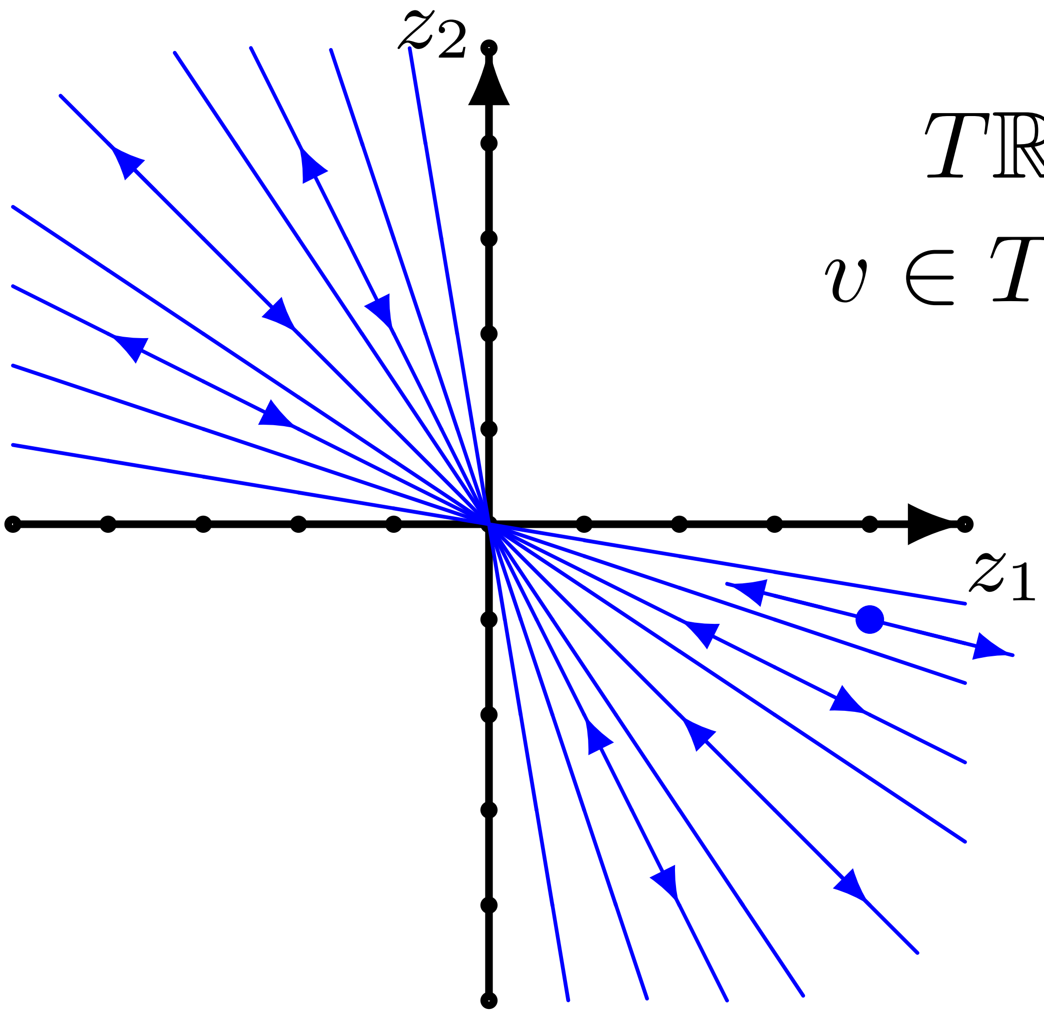

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Key Definition: $S_i = \left\{ z \in M(\mathcal{A}) \,\middle|\, \frac{\alpha_{i-1}(z)}{\alpha_i(z)} \in \mathbb{R}_{<0} \right\}$

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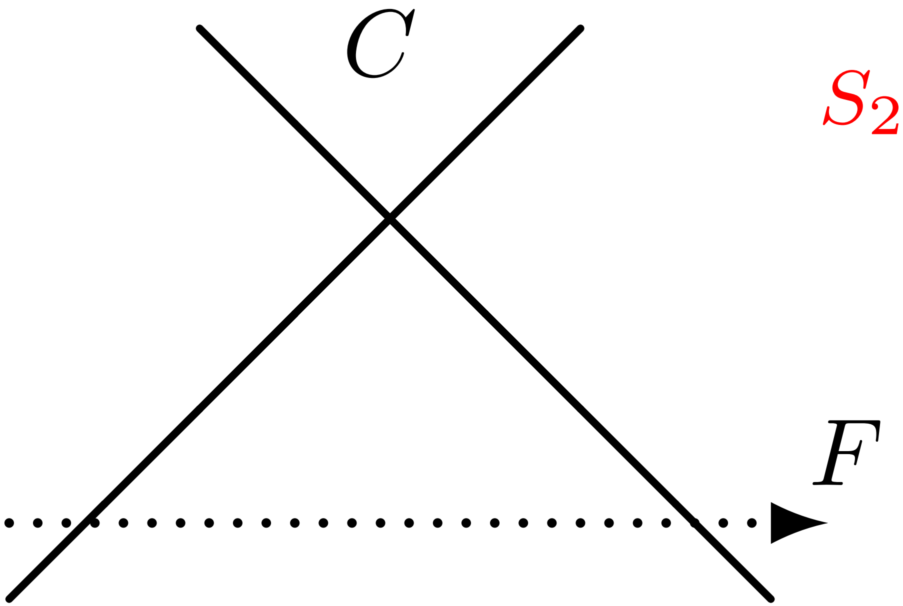
3 Minimal Stratification

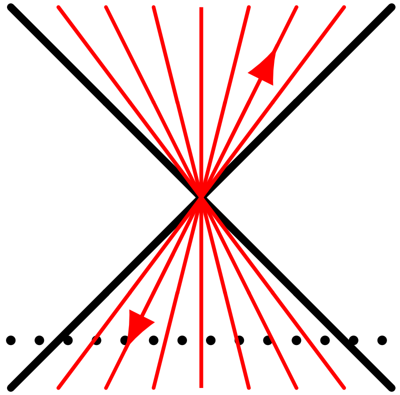
Example.

We shall outline the proof with:

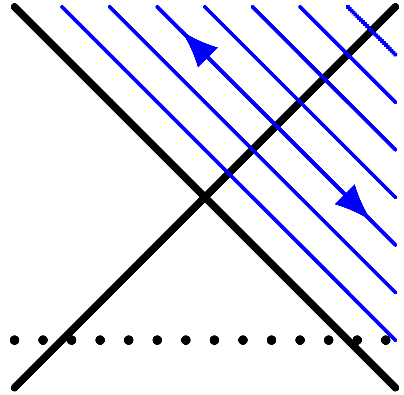


$H_2 = \{\alpha_2 = 0\}$ $\quad$ $H_1 = \{\alpha_1 = 0\}$

$(\mathrm{ch}_2(\mathcal{A}) = \{C\})$

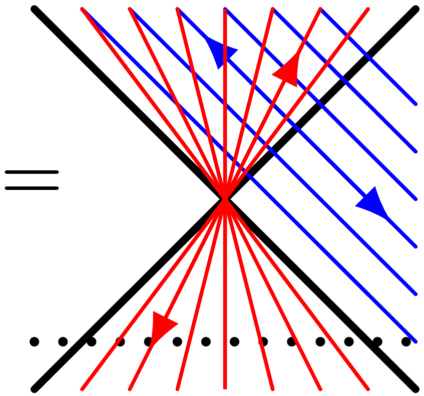


$S_2 = \{\frac{\alpha_2}{\alpha_1} \in \mathbb{R}_{<0}\}$



$S_1 = \{\alpha_1 \in \mathbb{R}_{>0}\}$

$S_1 \pitchfork S_2 =$

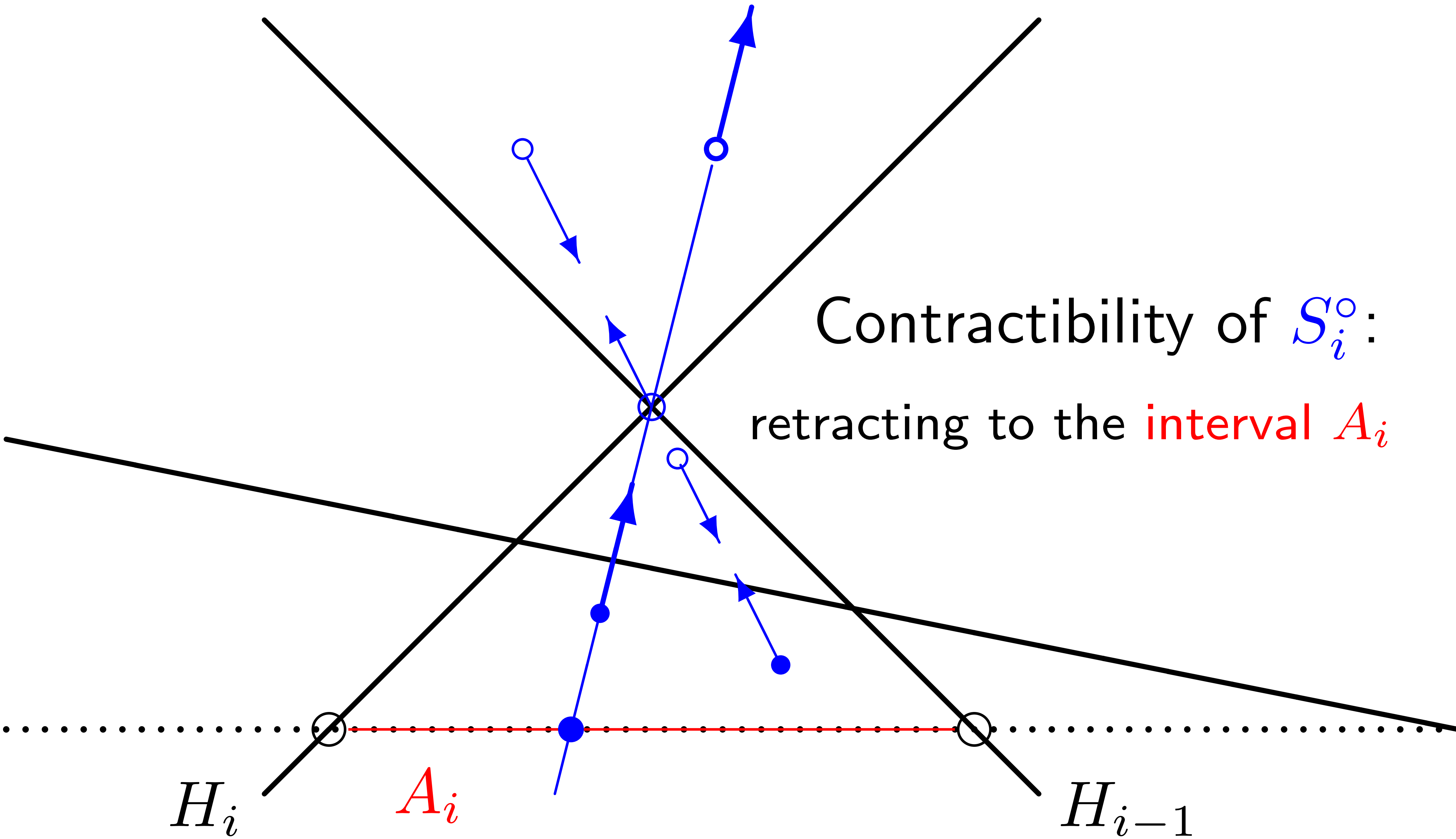


$= C$

3 Minimal Stratification

Contractibility of S_i°:

retracting to the interval A_i



3 Minimal Stratification

We skip the contractibility of U. (Complicated.)

4 Fundamental group $\pi_1(M(\mathcal{A}))$

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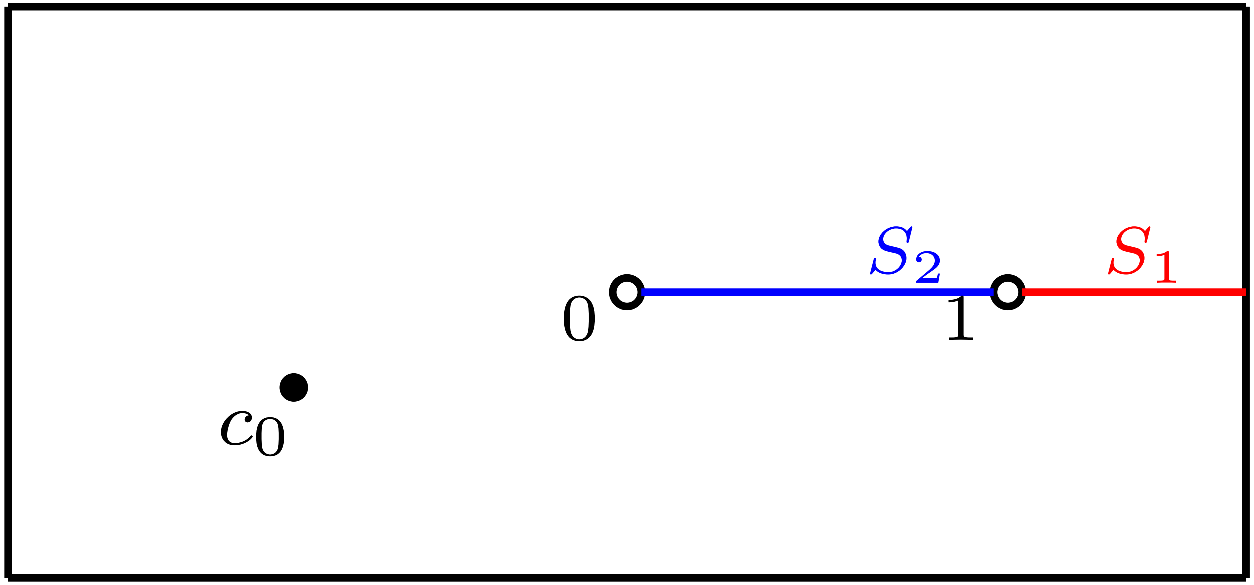


$\mathcal{A} = \{0, 1\}$

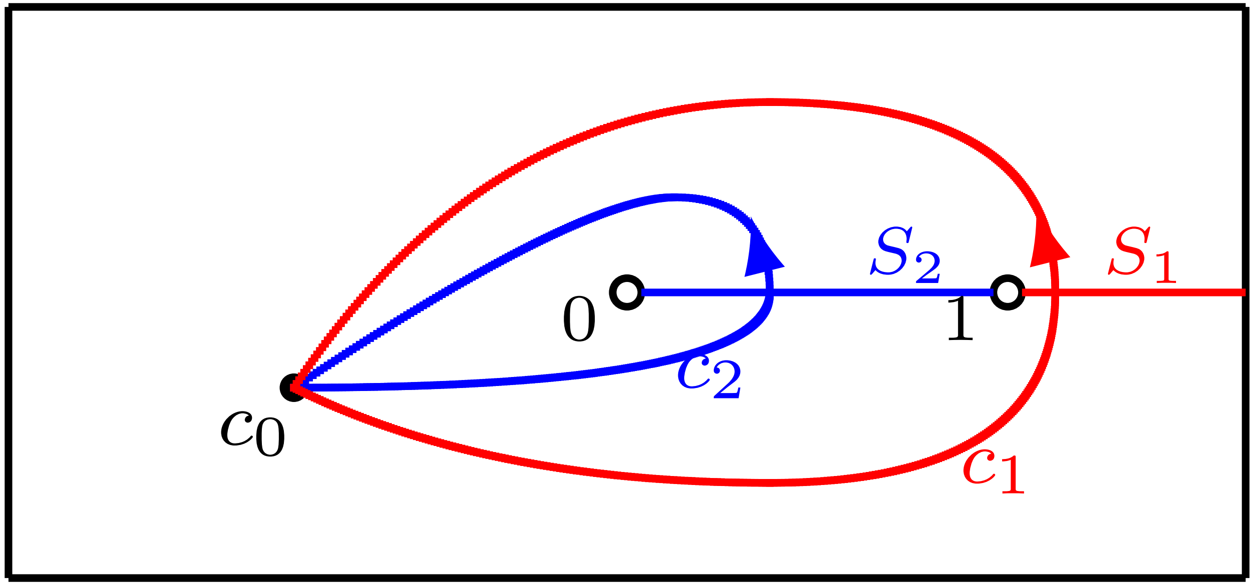
$M(\mathcal{A}) = \mathbb{C} \setminus \{0, 1\}$

$S_2 = (0, 1)$

$S_1 = (1, \infty)$

How to recover π_1 from stratification?

4 Fundamental group $\pi_1(M(\mathcal{A}))$



$\mathcal{A} = \{0, 1\}$

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Since strata are contractible,
S_1 and S_2 determine transversal generators
c_1 and c_2 uniquely (up to homotopy).

How to recover π_1 from stratification?

2 Minimal Stratification

Example. Σ_2: surface of genus $g = 2$.

Computing $\pi_1(\Sigma_2)$ from stratification.

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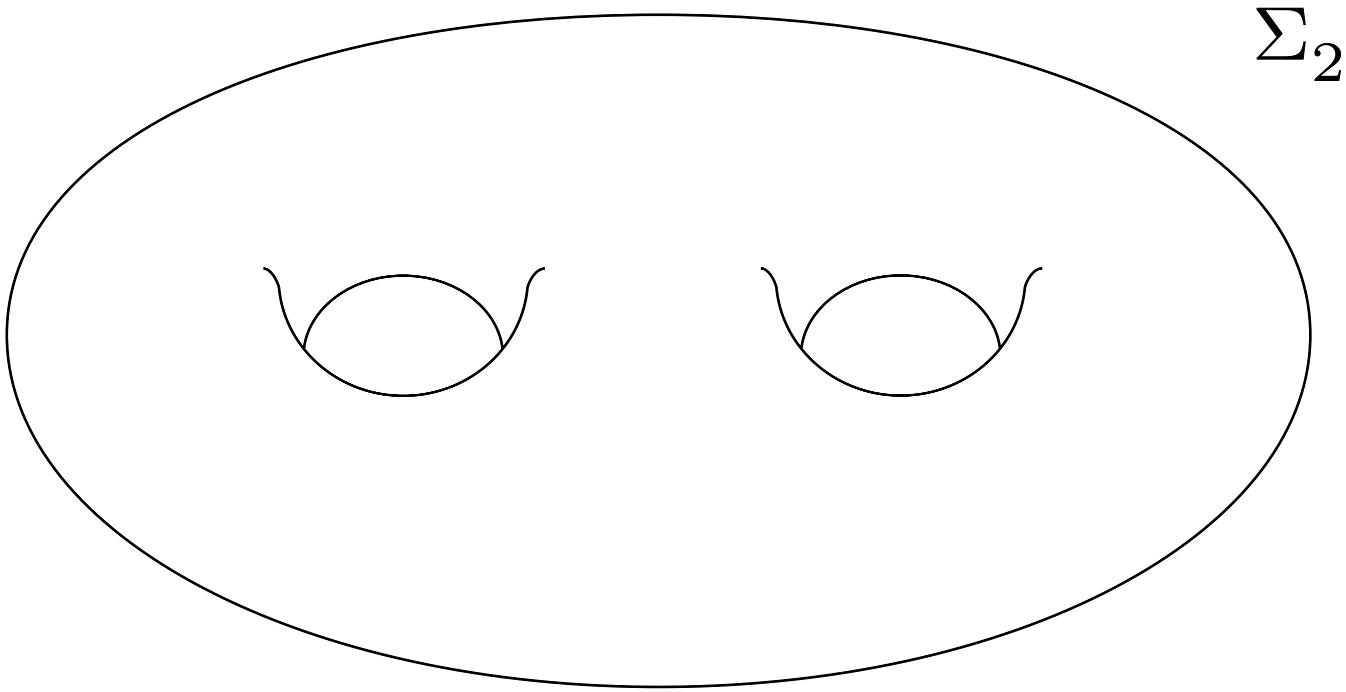

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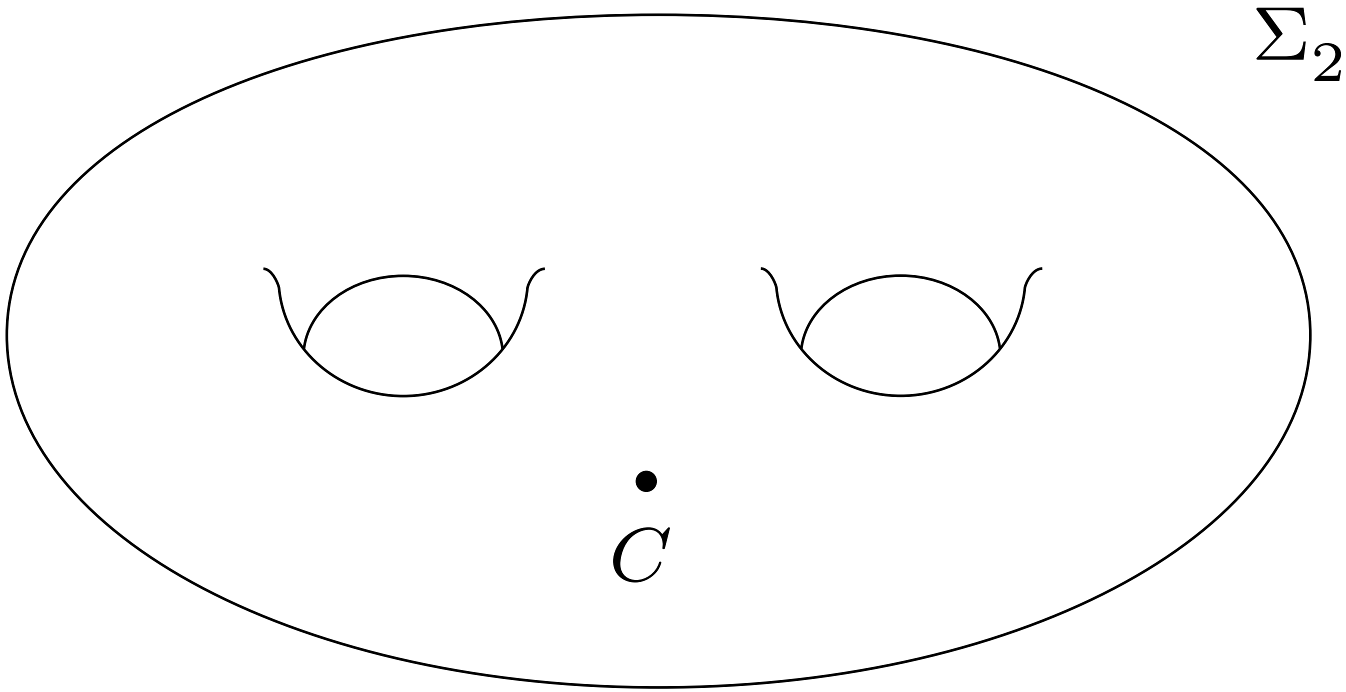

C: 0-dim stratum.

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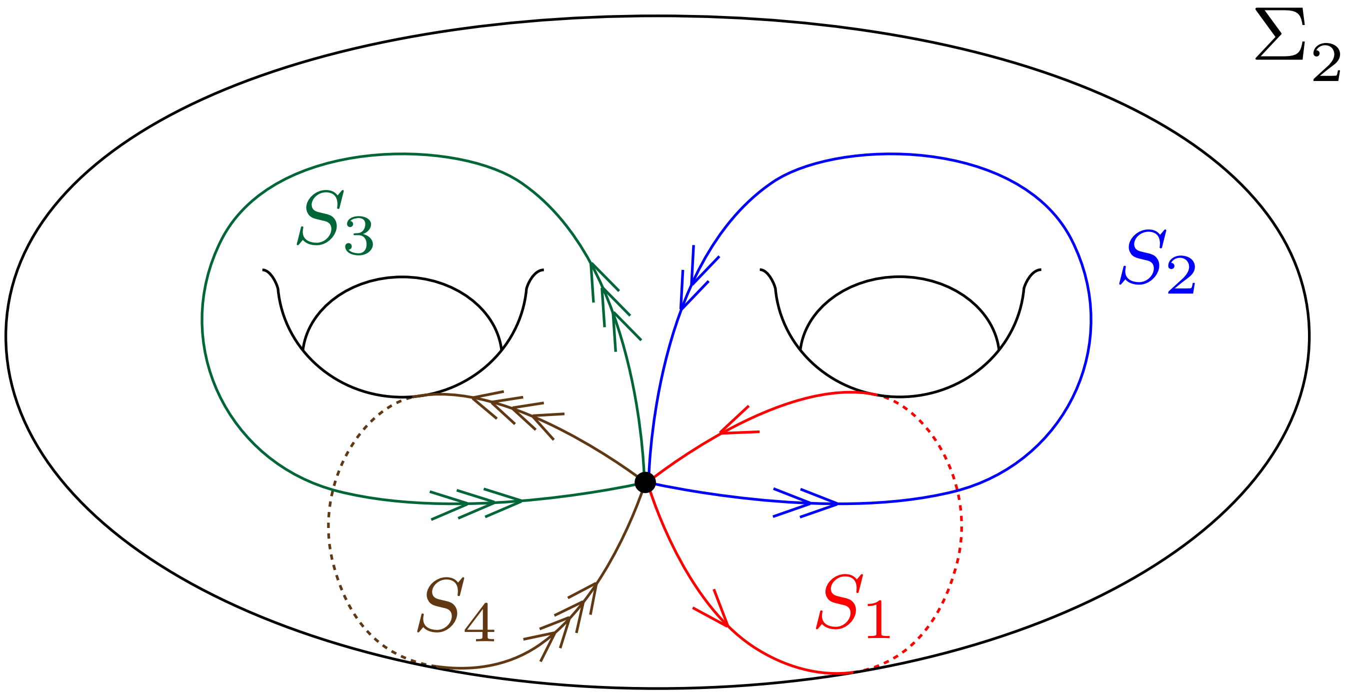

S_i: 1-dim strata.

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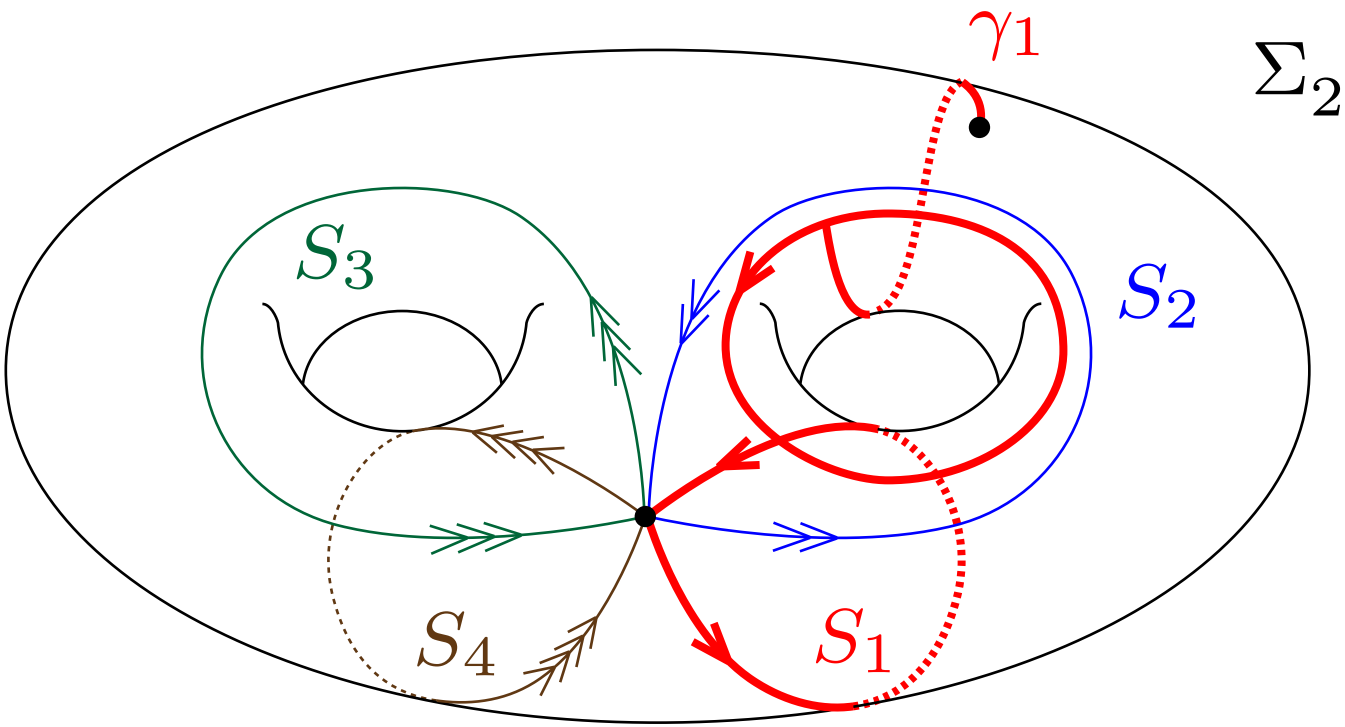

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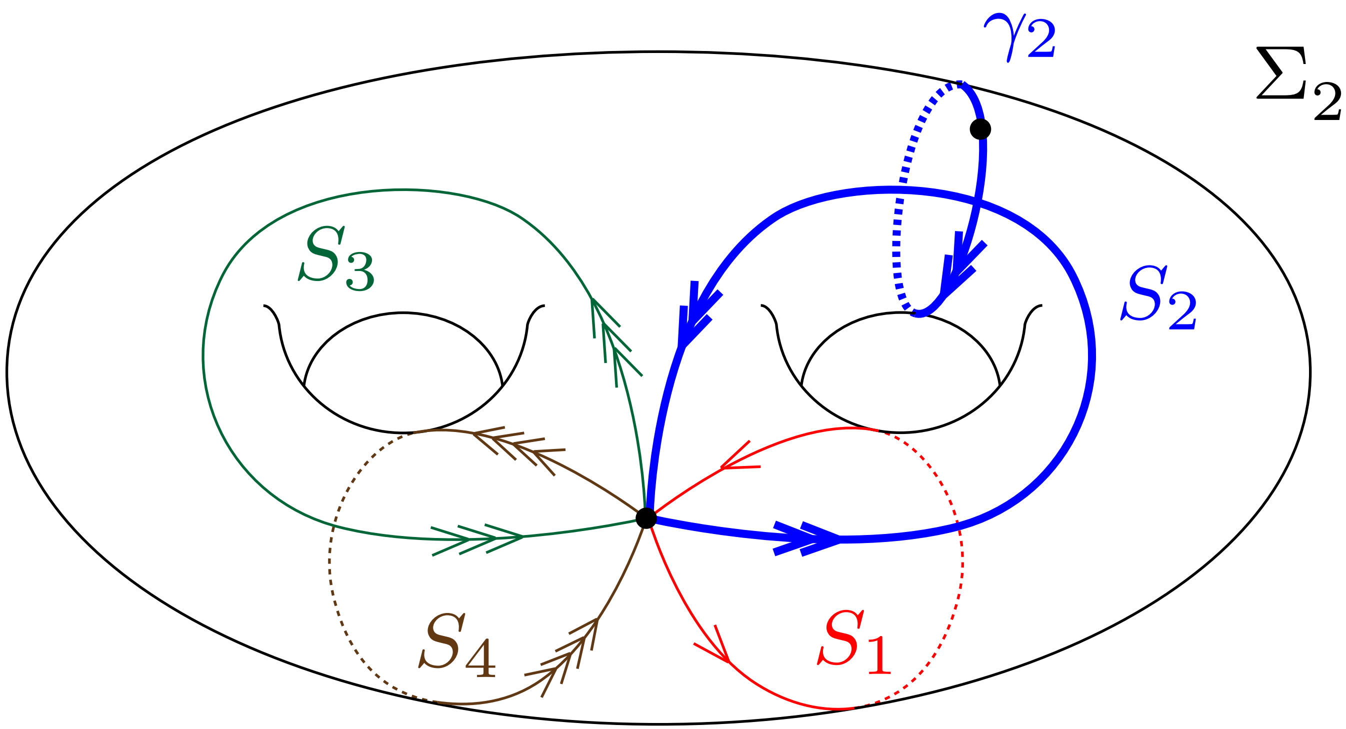
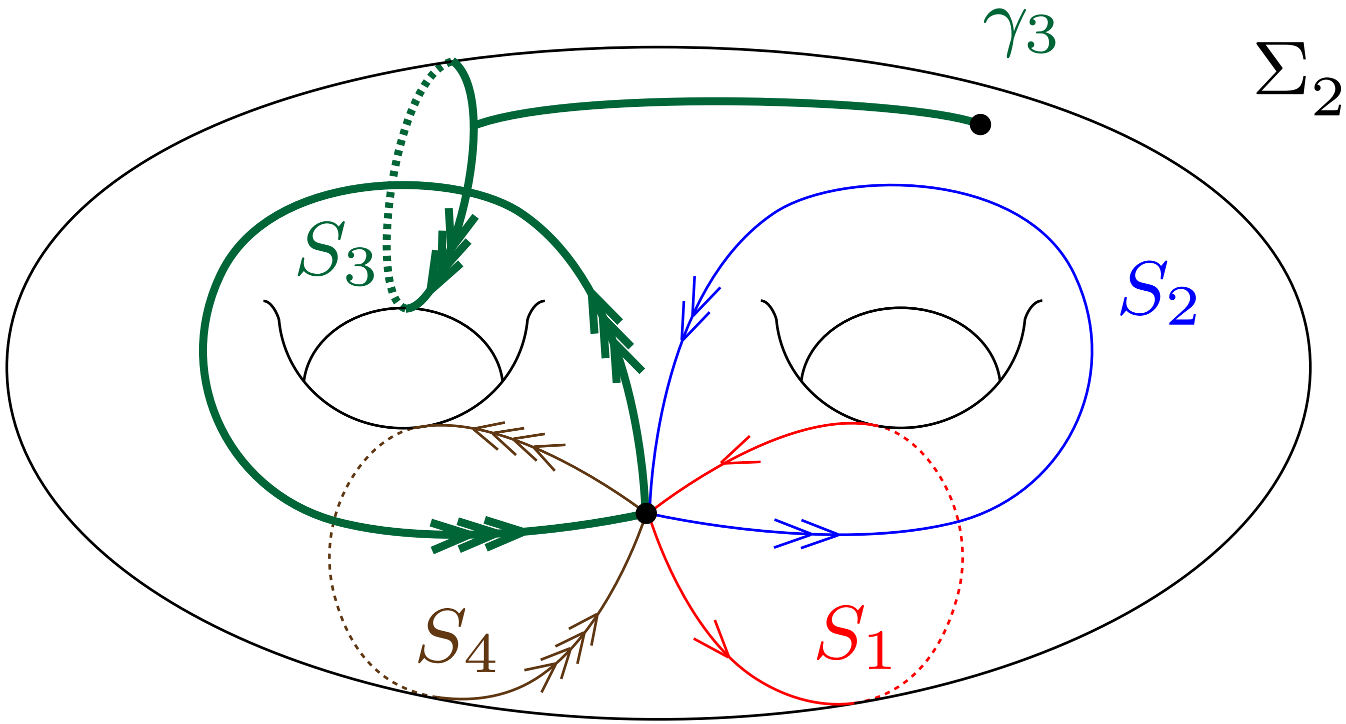


2 Minimal Stratification

Example. Σ_2: surface of genus $g = 2$.

Computing $\pi_1(\Sigma_2)$ from stratification.



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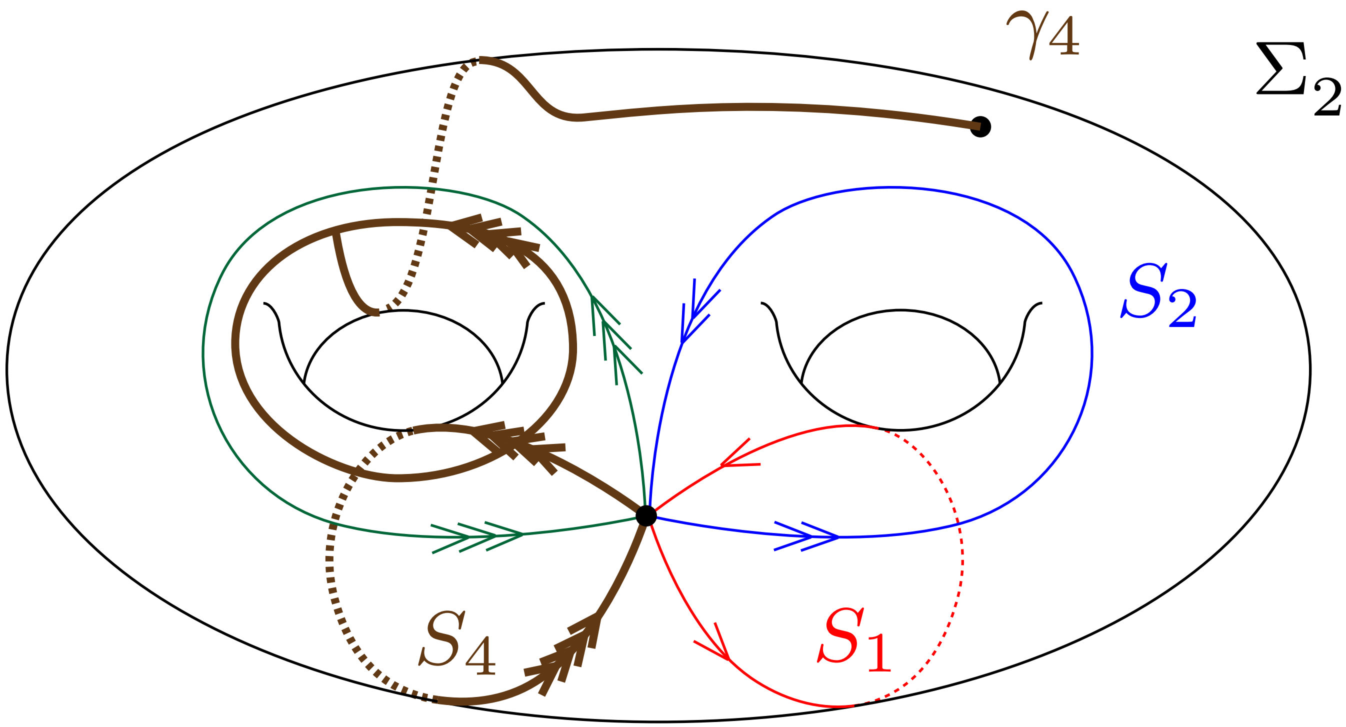


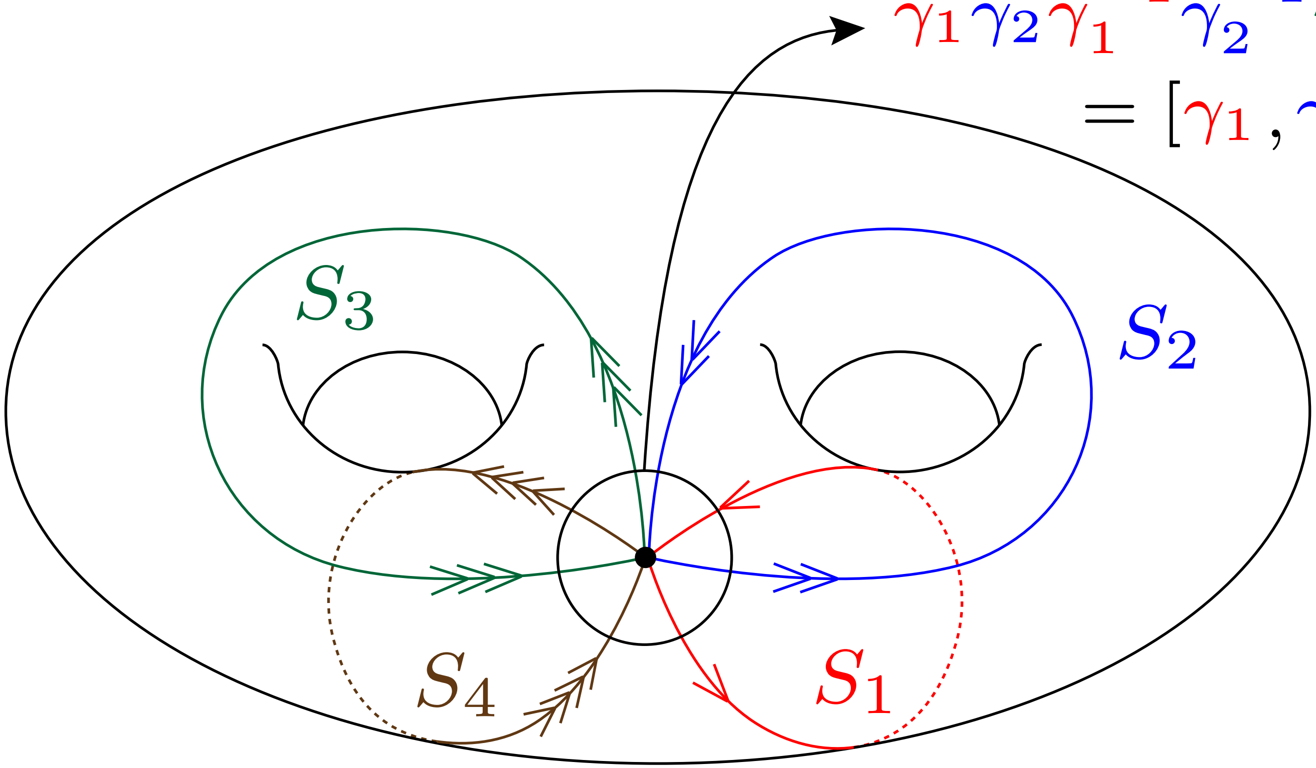
Where is the relations?

2 Minimal Stratification

Example. Σ_2: surface of genus $g = 2$.

Computing $\pi_1(\Sigma_2)$ from stratification.

$$\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\gamma_3\gamma_4\gamma_3^{-1}\gamma_4^{-1}$$
$$= [\gamma_1, \gamma_2][\gamma_3, \gamma_4] = 1$$

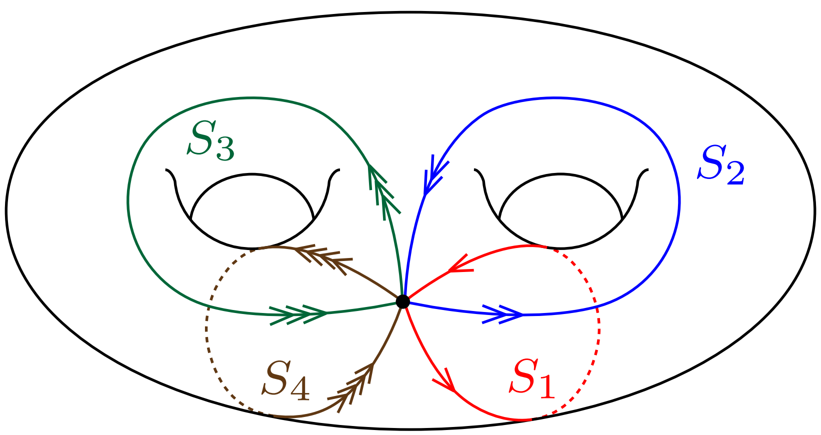


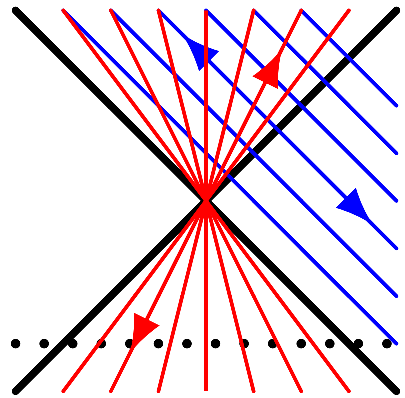
Look at $\mathrm{codim} = 2$ stratum.

2 Minimal Stratification

To compute $\pi_1(M(\mathcal{A}))$, we will follow the Strategy:

1. Stratify $M = \bigsqcup_\lambda S_\lambda$ by contractible strata.
2. Generator $\longleftrightarrow \mathrm{codim} = 1$ strata.
3. Relation $\longleftrightarrow \mathrm{codim} = 2$ strata.

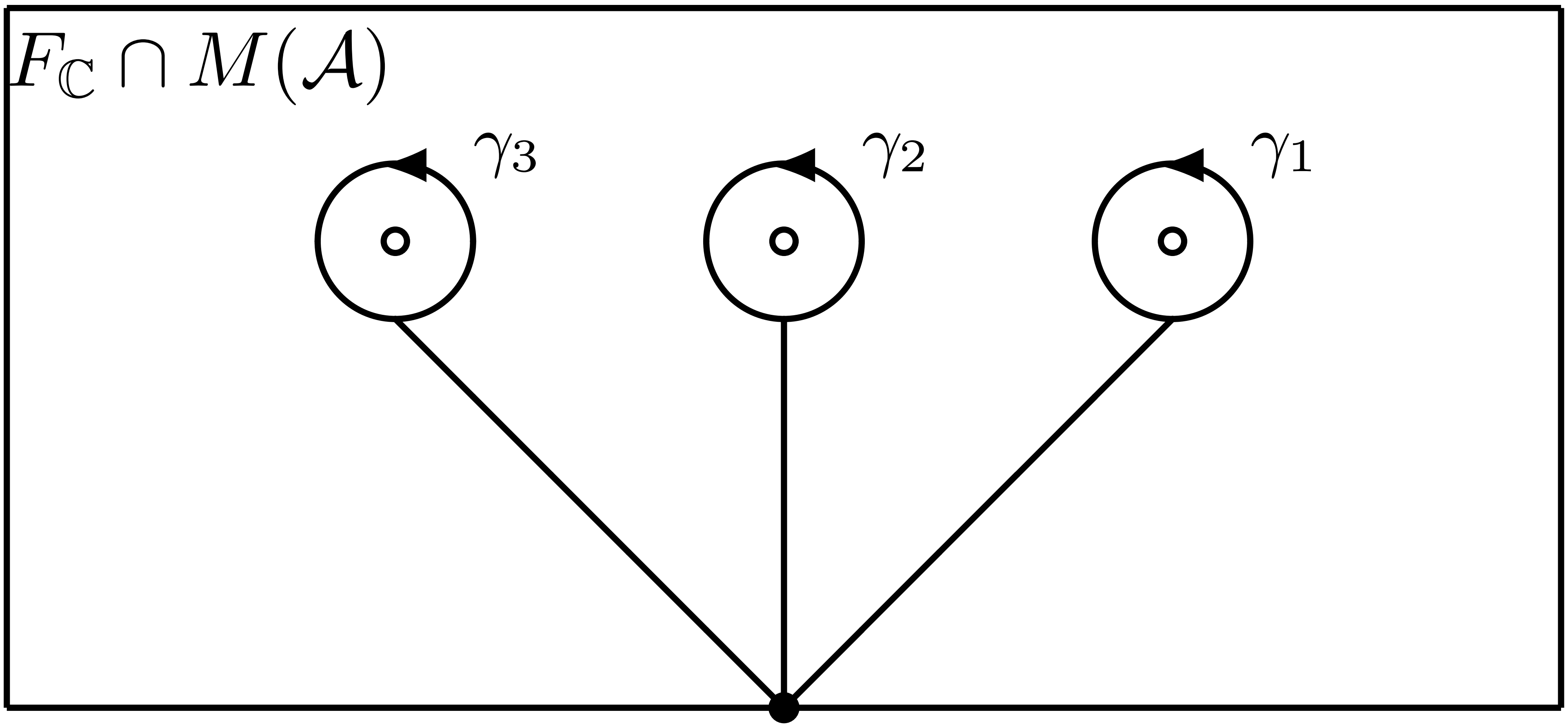




$M(\mathcal{A}) = U \sqcup {\color{blue}S_1^\circ} \cup {\color{red}S_2^\circ} \cup C \triangleright$

4 Fundamental group $\pi_1(M(\mathcal{A}))$

Generators: Meridians (with base $-\sqrt{-1}$).



4 Fundamental group $\pi_1(M(\mathcal{A}))$

Relations:

(1) Attach to $C \in \mathrm{ch}_2(\mathcal{A})$ a permutation $(i_1, i_2, \ldots, i_n)$ of $(1, \ldots, n)$ as

$$\underbrace{i_1 < \cdots < i_k}_{\text{Passing through right side of } C},\ \underbrace{i_{k+1} < \cdots < i_n}_{\text{Passing through left side of } C}$$

(2) Associate to C a relation

$$R(C) : \gamma_1\gamma_2\ldots\gamma_n = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_n}.$$

Thm. $\pi_1(M) \cong \langle \gamma_1, \ldots, \gamma_n \mid R(C), C \in \mathrm{ch}_2(\mathcal{A}) \rangle$

4 Fundamental group $\pi_1(M(\mathcal{A}))$

Thm.

$$\pi_1(M) \cong \langle \gamma_1, \ldots, \gamma_n \mid$$

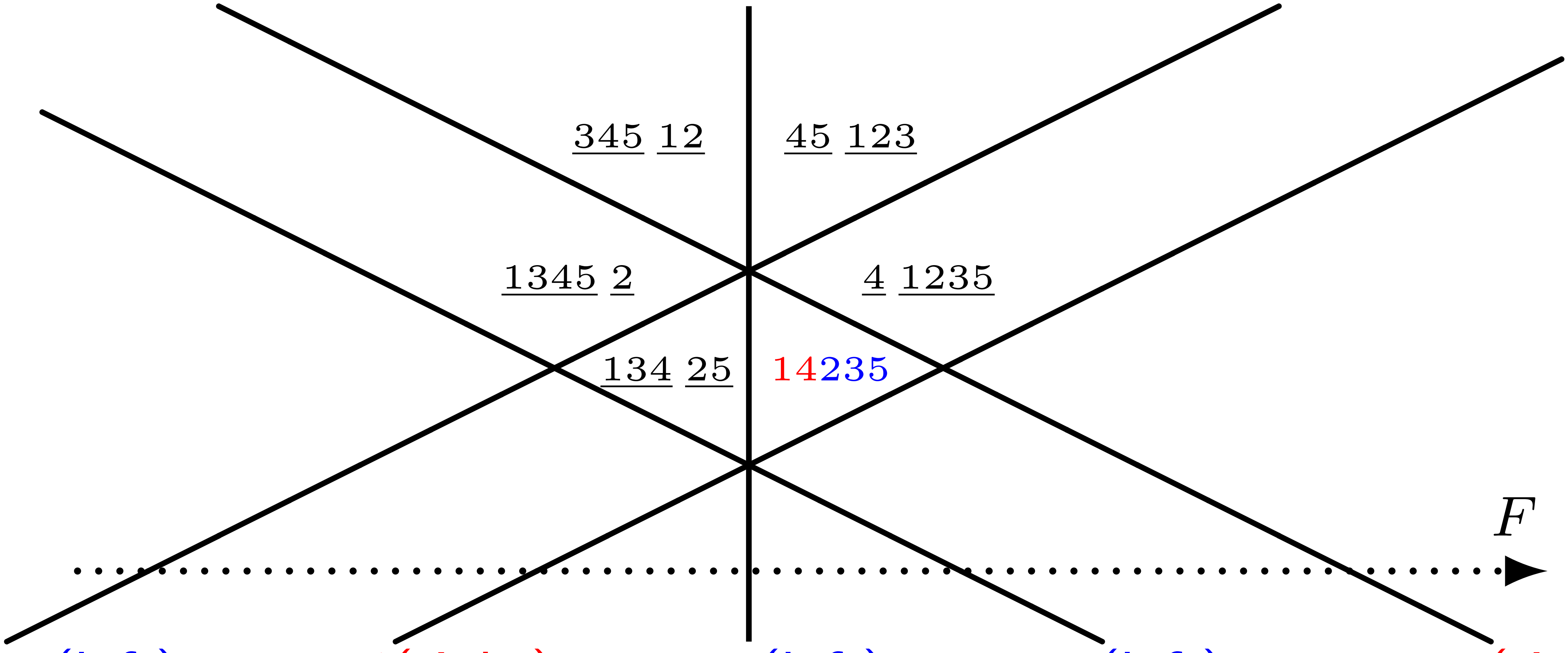
$$R(C) : \gamma_1 \cdots \gamma_n = \gamma_{i_1(C)} \cdots \gamma_{i_n(C)},$$

$$C \in \mathrm{ch}_2(\mathcal{A}) \rangle.$$

4 Fundamental group $\pi_1(M(\mathcal{A}))$

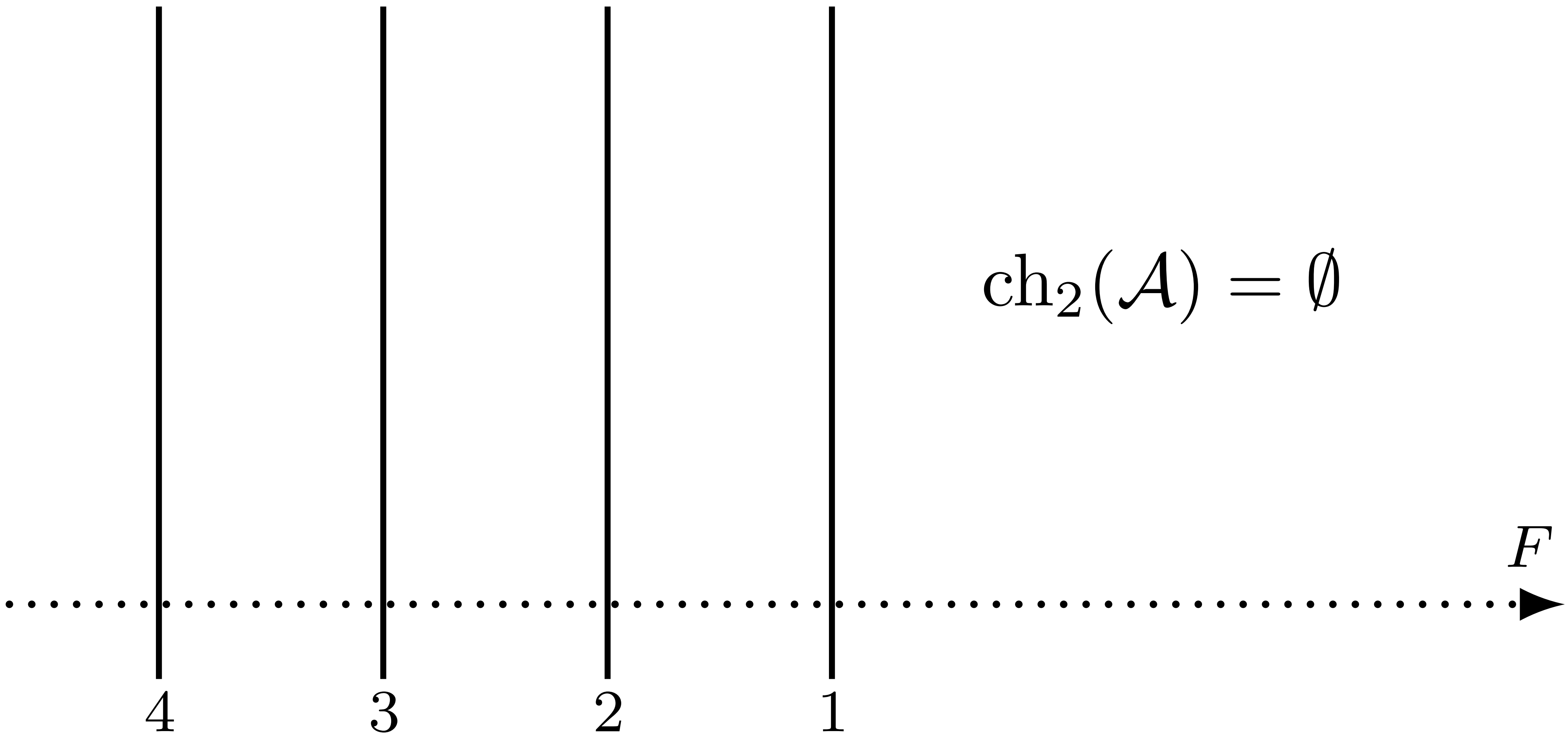
Example

$$\pi_1(M) \cong \left\langle \gamma_1, \ldots, \gamma_5 \;\middle|\; \begin{array}{l} 12345 \\ = 14235 = 13425 = 13452 \\ = 34512 = 45123 = 41235 \end{array} \right\rangle$$



4 Fundamental group $\pi_1(M(\mathcal{A}))$

Example

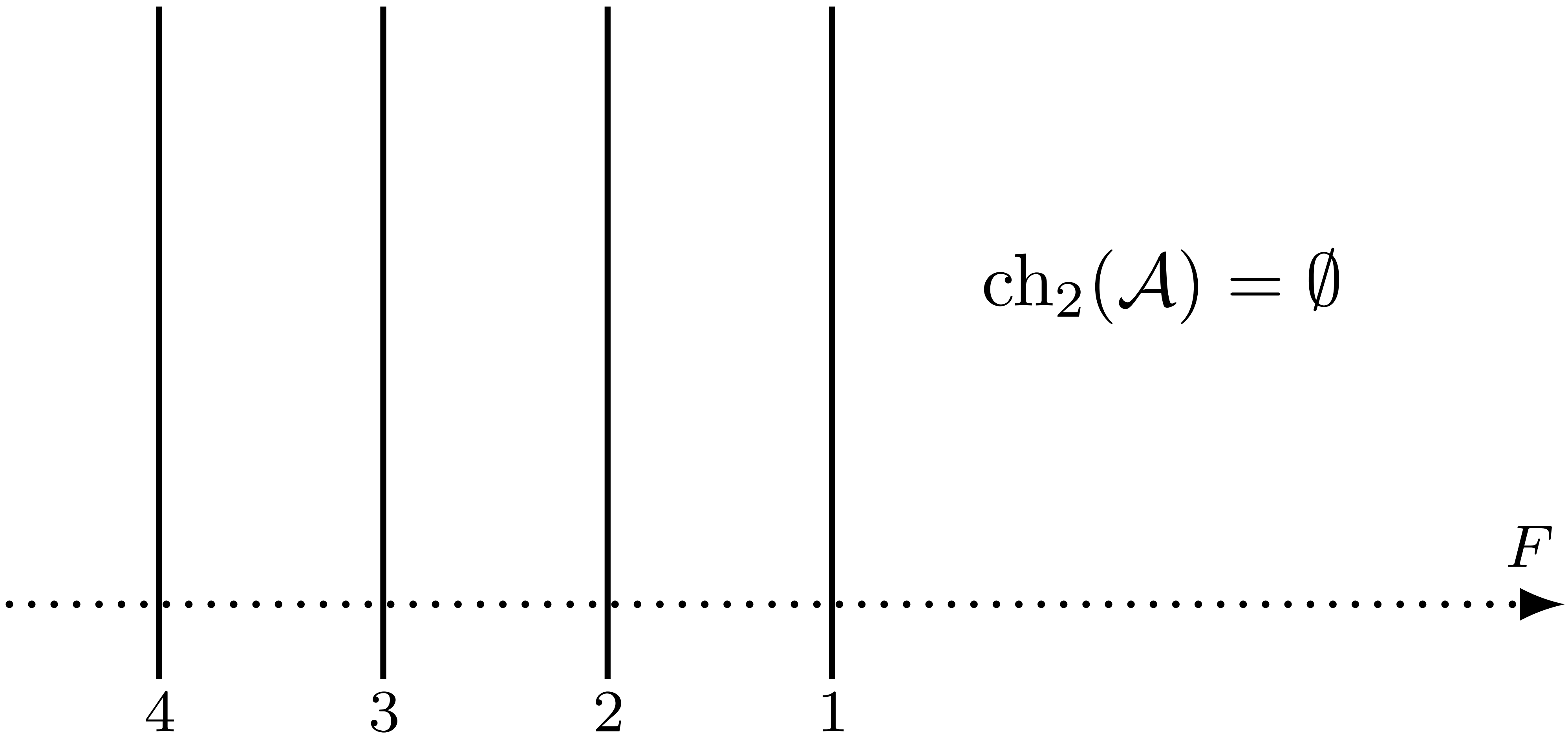


$$\mathrm{ch}_2(\mathcal{A}) = \emptyset$$

4 Fundamental group $\pi_1(M(\mathcal{A}))$

Example

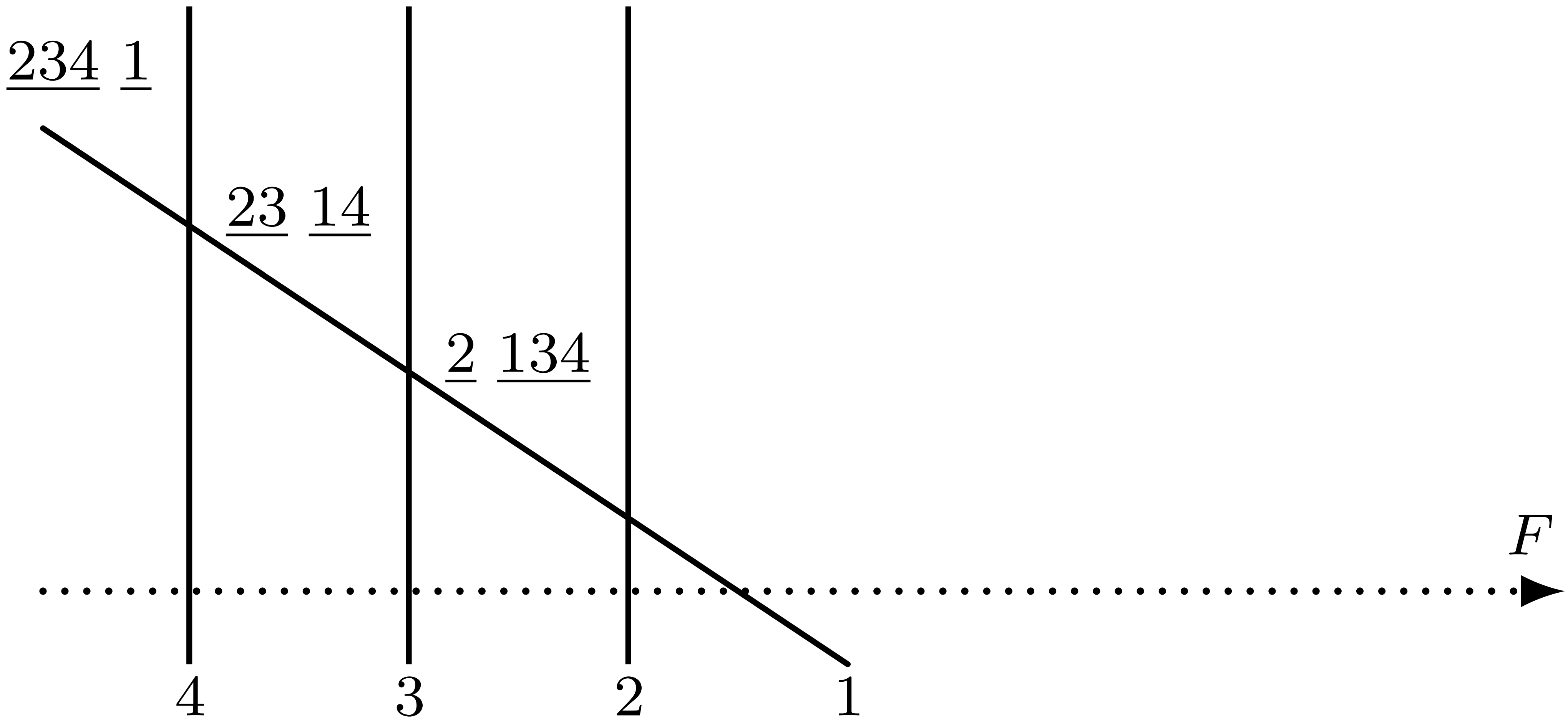
$$\pi_1(M) \cong \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid \text{no relations } \rangle$$



4 Fundamental group $\pi_1(M(\mathcal{A}))$

Example

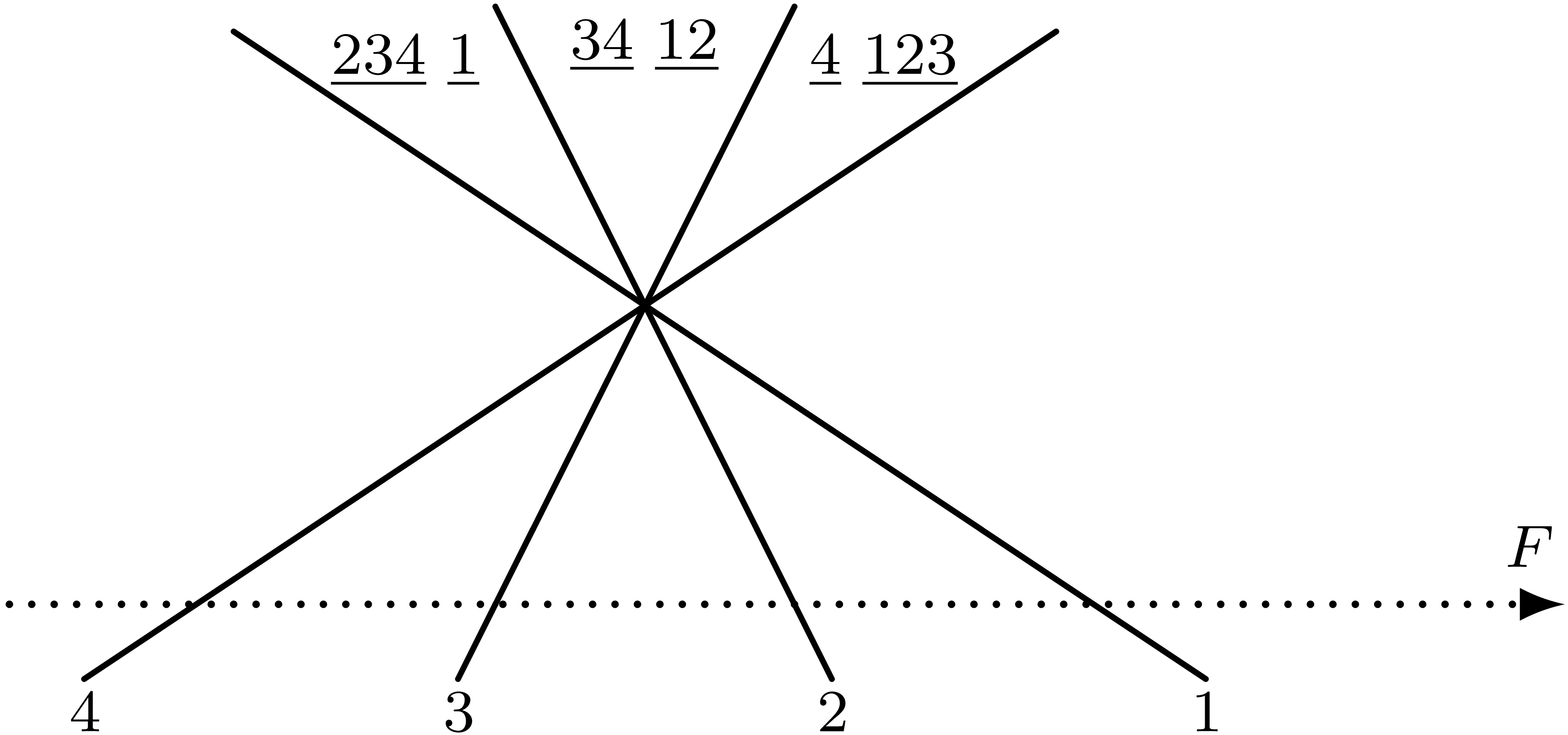
$$\pi_1(M) \cong \left\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \;\middle|\; \begin{array}{l} 1234 = 2134 \\ = 2314 = 2341 \end{array} \right\rangle$$



4 Fundamental group $\pi_1(M(\mathcal{A}))$

Example

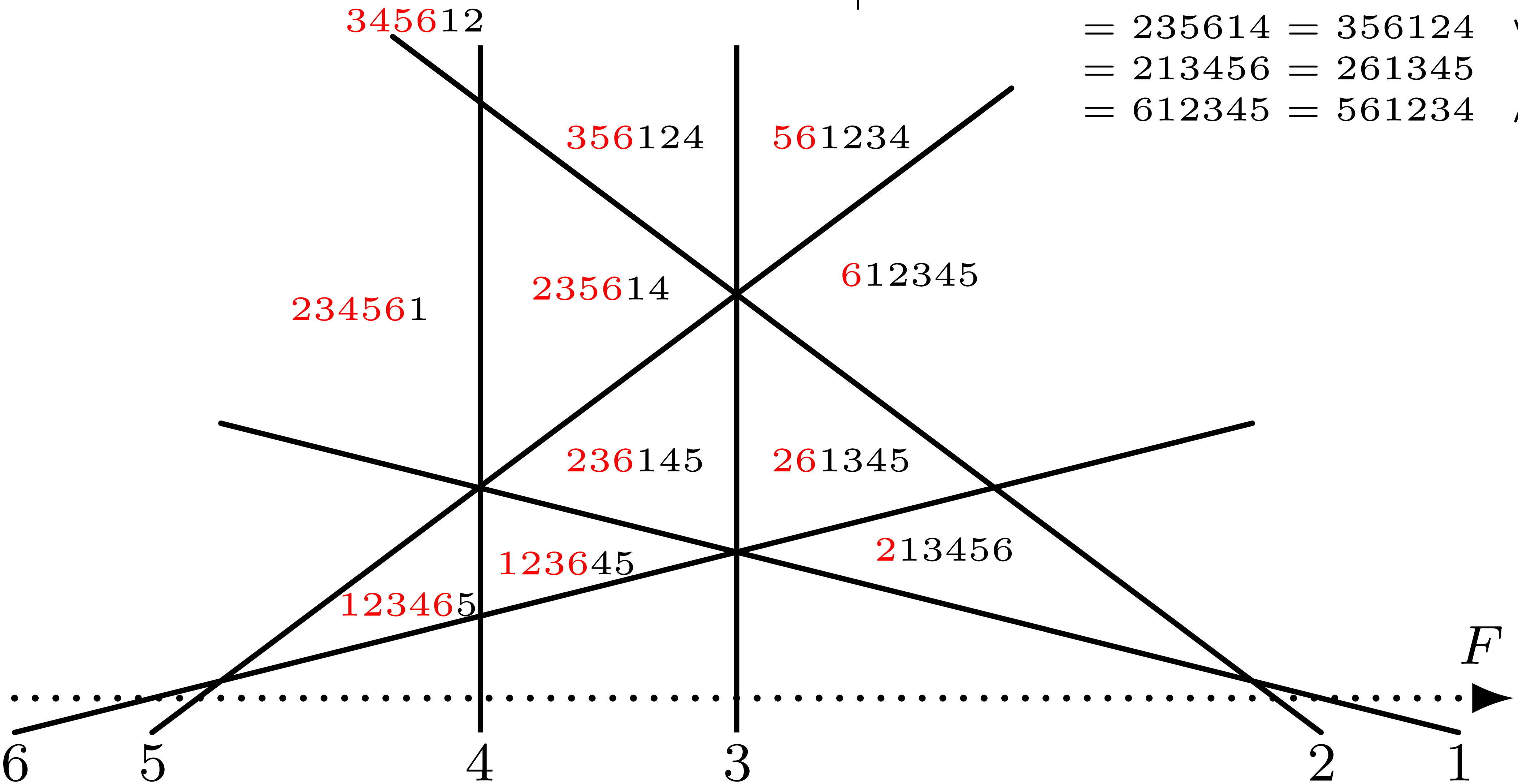
$$\pi_1(M) \cong \left\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \;\middle|\; \begin{array}{l} 1234 = 2341 \\ = 3412 = 4123 \end{array} \right\rangle$$



4 Fundamental group $\pi_1(M(\mathcal{A}))$

Example

$$\pi_1(M) \cong \left\langle \gamma_1, \ldots, \gamma_6 \;\middle|\; \begin{array}{l} 123456 = 123465 \\ = 234561 = 345612 \\ = 123645 = 236145 \\ = 235614 = 356124 \\ = 213456 = 261345 \\ = 612345 = 561234 \end{array} \right\rangle$$



4 Fundamental group $\pi_1(M(\mathcal{A}))$

Thm.

$$\pi_1(M(\mathcal{A})) = \langle \gamma_1, \ldots, \gamma_n \mid \gamma_1 \cdots \gamma_n = \gamma_{i_1(C)} \cdots \gamma_{i_n(C)},\ C \in \mathrm{ch}_2 \rangle$$

Cor. $\pi_1(M(\mathcal{A}))$ has a positive homogeneous presentation.

4 Fundamental group $\pi_1(M(\mathcal{A}))$

Potential applications:

- Qualitative study on π_1 (word problem etc.)
- $\pi_2(M(\mathcal{A}))$. (Let $\varphi : S^2 \longrightarrow M(\mathcal{A})$ be differentiable and generic. Then the isotopy class

$$\varphi^{-1}(\bigcup S_i) \subset S^2$$

determines $[\varphi] \in \pi_2(M(\mathcal{A}))$.)

Reference:

M. Yoshinaga, Minimal stratifications for line arrangements and positive homogeneous presentations for fundamental groups. arXiv:1105.1857

Thank you.