Motivic Milnor fibers and Newton polyhedra

(joint work with Y. Matsui and A. Esterov)

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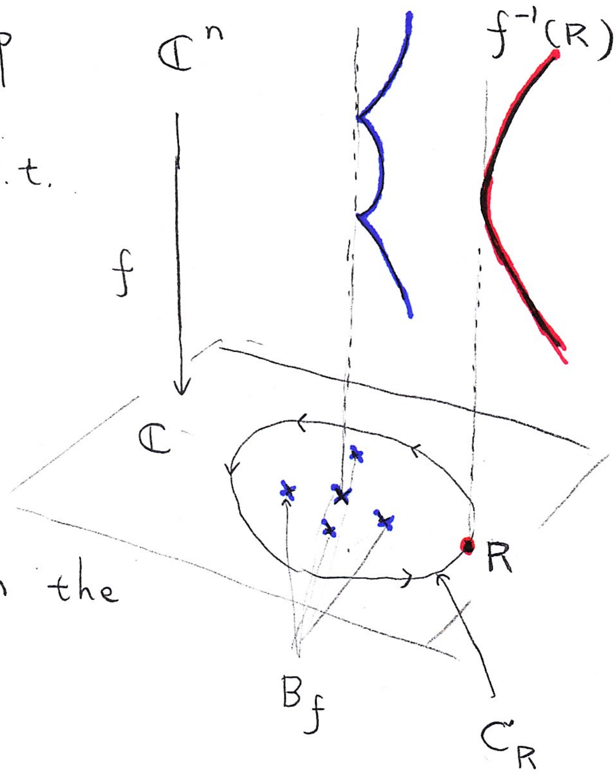
§ 1. Monodromies at ∞

$f : \mathbb{C}^n \longrightarrow \mathbb{C}$: polynomial map

$\Longrightarrow$ $^\exists B_f \subset \mathbb{C}$: finite set s.t.

$$\mathbb{C}^n \setminus f^{-1}(B_f) \xrightarrow[f]{} \mathbb{C} \setminus B_f$$

is a locally trivial fibration.



$\Longrightarrow$ For $R >> 0$ we obtain the monodromies at ∞ :

$$\Phi_j^\infty : H^j(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R); \mathbb{C})$$

$(j = 0, 1, 2, \cdots)$.

Rem $^\exists$ monodromy representations

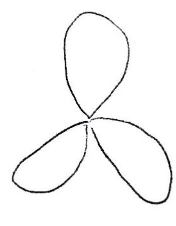
$$\Phi : \pi_1(\mathbb{C} \setminus B_f) \longrightarrow \mathrm{Aut}\left(H^j(f^{-1}(R); \mathbb{C})\right)$$

(cf. Dimca – Némethi, Duke math 2001).

Thm (Broughton, Invent. math, 1988)

If f is tame at ∞ ($\Longleftrightarrow$ $\partial f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is proper over a n.h.d of $0 \in \mathbb{C}^n$), then

$$\underset{(R>>0)}{f^{-1}(R)} \underset{\text{homot}}{\sim} S^{n-1} \vee \cdots\cdots \vee S^{n-1}.$$



$\Longrightarrow$ $H^j(f^{-1}(R); \mathbb{C}) = 0 \quad (j \neq 0, n-1)$.

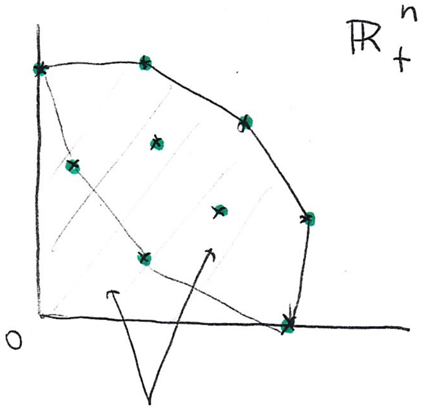
Known results

◎ Eigenvalues of Φ_{n-1}^{∞} (Libgober – Sperber, Garcia-Lopez – Némethi, G-Z-Luengo-M-H, M-T, Siersma – Tibar etc.)

◎ Jordan blocks of Φ_{n-1}^{∞} (Dimca – Saito, G-L-N etc.)

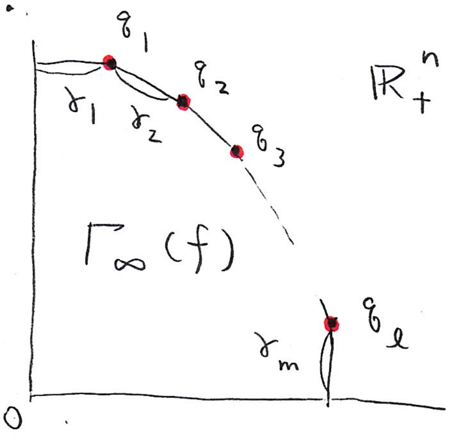
§ 2. Jordan normal form of Φ_{n-1}^{∞}

Assumptions f is convenient and non-deg at ∞ ($\Rightarrow$ tame at ∞).



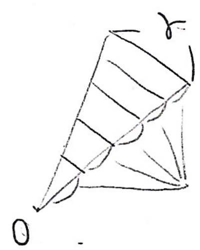
$\Gamma_\infty(f) :=$ the convex hull of $\{0\} \cup NP(f)$.

q_i : 0-dim faces of $\Gamma_\infty(f)$ s.t. $q_i \in \mathrm{Int}(\mathbb{R}_+^n)$.



γ_i : 1-dim faces of $\Gamma_\infty(f)$ s.t. $\mathrm{rel.int}(\gamma_i) \subset \mathrm{Int}(\mathbb{R}_+^n)$.

Def $\gamma \prec \Gamma_\infty(f)$ is adapted to $\lambda \in \mathbb{C}$
$\underset{def}{\Longleftrightarrow}$ $\lambda^{d(\gamma)} = 1$.

(


$d(\gamma)$ steps !)

For $\lambda \in \mathbb{C}$ and γ_i $(1 \leq i \leq m)$ we set

$$n(\gamma_i)_\lambda \underset{\text{def}}{:=} \mathrm{length}_{\mathbb{Z}}(\gamma_i) - \#\Big\{ \underbrace{\nu}_{\text{0-dim faces}} < \gamma_i \text{ adapted to } \lambda \Big\}.$$

Thm (M–T) For $\lambda \neq 1$ we have

(i) $$\#\left\{ \underbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}_{n} \in \Phi^{\infty}_{n-1} \right\} = \#\left\{ \gamma_i : \text{adapted to } \lambda \right\}.$$

maximal possible size!

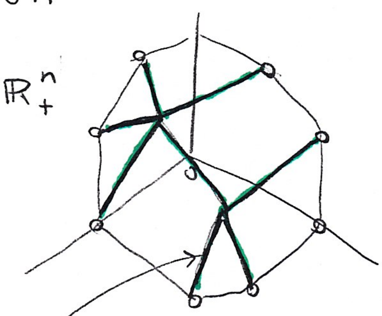
(ii) $$\#\left\{ \underbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}_{n-1} \in \Phi^{\infty}_{n-1} \right\} = \sum_{\gamma_i : \text{ adapted to } \lambda} n(\gamma_i)_\lambda .$$

Thm (M–T) For the eigenvalue 1 we have

(i) $$\#\left\{ \underbrace{\begin{pmatrix} 1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}}_{n-1} \in \Phi^{\infty}_{n-1} \right\} = \#\left\{ \begin{array}{l} \text{the lattice points on} \\ \text{1-skeleton of} \\ \partial \Gamma_\infty(f) \cap \mathrm{Int}(\mathbb{R}^n_+) \end{array} \right\}.$$

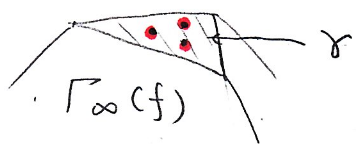
maximal possible size!

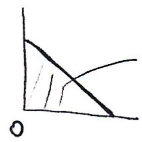
$\mathbb{R}^n_+$



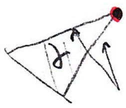
(ii) $$\#\left\{ \underbrace{\begin{pmatrix} 1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}}_{n-2} \in \Phi^{\infty}_{n-1} \right\}$$

$$= \sum_{\substack{\mathrm{rel.int.}(\gamma) \subset \mathrm{Int}(\mathbb{R}^n_+) \\ \dim \gamma = 2}} (2)\ \#\left\{ \mathrm{rel.int}(\gamma) \cap \mathbb{Z}^n \right\}.$$



Remark (i) If f is quasi-homog ( $\Gamma_\infty(f)$)

$\Longrightarrow$ $\bar{\Phi}^{\infty}_{n-1}$ is semisimple.

(ii) $\exists$ Algorithm to compute the Jordan normal form of $\bar{\Phi}^{\infty}_{n-1}$. $\exists$ Closed formulas if $\forall\, \gamma < \Gamma_\infty(f)$ s.t $0 \notin \gamma$ are prime ( $\dim \gamma$ edges).

(iii) $\exists$ Analogue of the Steenbrink conj for the Hodge spectrum at ∞ of $\bar{\Phi}^{\infty}_{n-1} \circlearrowright H^{n-1}(f^{-1}(R)\,;\,\mathbb{C})$.

§3. Motivic Milnor fibers at ∞

For $d \in \mathbb{Z}_{>0}$ set $\mu_d \underset{\text{def}}{:=} \{ t \in \mathbb{C} \mid t^d = 1 \}$

$\simeq \mathbb{Z}/\mathbb{Z}d$ and

$$\hat{\mu} \underset{\text{def}}{:=} \varprojlim_{d} \mu_d \quad \left(\begin{array}{ccc} \mu_{dd'} & \twoheadrightarrow & \mu_d \\ \Psi & & \Psi \\ t & \longmapsto & t^{d'} \end{array} \right).$$

Def (Denef – Loeser, J.A.G 1998, 2001)

(i) $\mathrm{Var}_{\mathbb{C}} := \{ \text{varieties} / \mathbb{C} \}$,

$$K_0(\mathrm{Var}_{\mathbb{C}}) := \left\{ \sum_{\text{finite}} \underset{\mathbb{Z}}{\underset{\cap}{a_i}} \, [\underset{\mathrm{Var}_{\mathbb{C}}}{\underset{\cap}{Z_i}}] \right\} \Bigg/ \begin{array}{l} X \simeq Y \\ \quad \Downarrow \\ [X] = [Y], \\ [X] = [Y] + [X \setminus Y] \\ \quad \uparrow \\ Y \subset X : \text{Zariski closed} \end{array}$$

(ii) X has a good action of $\hat{\mu} \underset{\text{def}}{\Longleftrightarrow}$

$$\begin{tikzcd} \hat{\mu} \arrow[r] \arrow[dr] & \mathrm{Aut}(X) \\ & \exists\, \mu_d \arrow[u, dashed] \end{tikzcd}$$

$+$ some condition.

(iii) $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}) := \left\{ \sum\limits_{\text{finite}} \underset{\mathbb{Z}}{\overset{\cap}{a_i}} \, [Z_i] \right\} \Big/ \sim .$

Z_i: varieties with good action of $\hat{\mu}$.

$\Downarrow$

$\mathbb{L} \underset{\text{def}}{:=} [\,\mathbb{C}$ with the trivial action of $\hat{\mu}\,]$

Lefschetz motive

(iv) $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}} := K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}].$

Sketch of proof

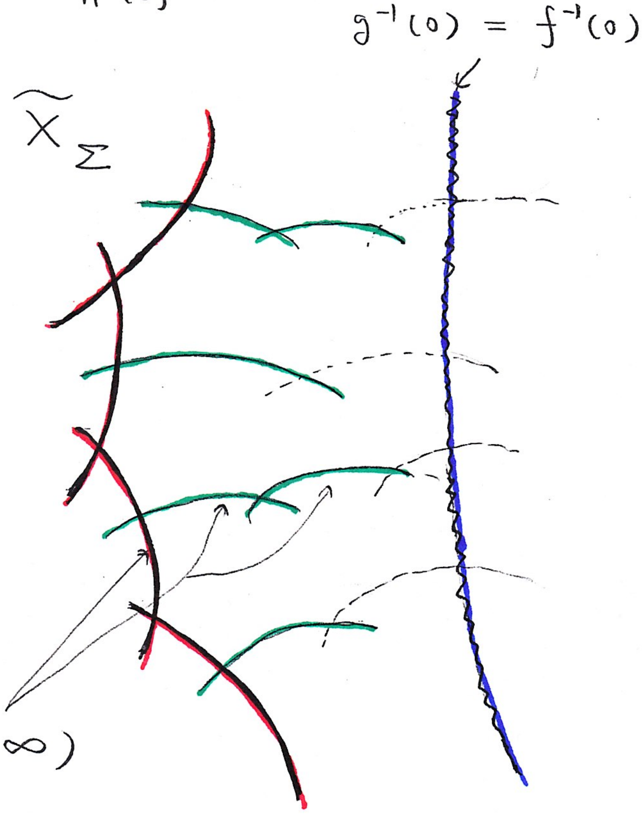
$\Phi_{n-1}^{\infty,c} \circlearrowright H_c^{n-1}(f^{-1}(R);\mathbb{C})$ compact support !

$$\Longrightarrow \quad \mathcal{F}^{\bullet} \underset{\text{def}}{:=} \psi_h(j_! Rf_! \mathbb{C}_{\mathbb{C}^n}) \in D^b(\{\infty\}).$$

$$\left(\begin{array}{l} j : \mathbb{C} \hookrightarrow \mathbb{P}^1 \ni \infty = \{h = 0\} \ \text{and} \\ \psi_h : D^b(\mathbb{P}^1) \longrightarrow D^b(\underbrace{\{\infty\}}_{h^{-1}(0)}) \end{array} \right)$$

ψ_h: Deligne's nearby cycle

$$\begin{tikzcd}
\mathbb{C}^n \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, "f"'] & & \widetilde{X}_\Sigma \arrow[dl] \arrow[dd, bend left, "\exists g"] \\
& X_\Sigma \arrow[d, dashed, "\times"] & \\
\mathbb{C} \arrow[r, hook, "j"'] & \mathbb{P}^1 \ni \infty & \mathbb{P}^1
\end{tikzcd}$$



$\exists$ Hodge realization map

Hodge str with action

$$\begin{array}{ccc} \chi_h : \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} & \longrightarrow & K_0(HS^{mon}) \\ \Psi & & \Psi \\ [Z] & \longmapsto & \sum_{j \in \mathbb{Z}} (-1)^j \left[H_c^j(Z;\mathbb{Q}) \right]. \end{array}$$

Then by using the isomorphism

$\mathcal{F}^{\bullet} \simeq \mathbb{R} g_! \, \psi_{h \circ g}(\iota_! \, \mathbb{C}_{\mathbb{C}^n})$ we obtain.

<u>Prop</u> (M-T) $\exists\, \mathcal{S}_f^{\infty} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ (<u>the motivic Milnor fiber at ∞ of f</u>) s.t.

monodromy filt

$$\chi_h(\mathcal{S}_f^{\infty}) = \sum_{j \in \mathbb{Z}} (-1)^j \left[Gr^{W}(H^j \mathcal{F}^{\bullet}) \right]$$

in $K_0(HS^{mon})$.

$\Phi_{n-1}^{\infty} \circlearrowright H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R >> 0)$

nearby cycle sheaf
$\psi_h(j_! \, \mathbb{R}f_! \, \mathbb{C}_{\mathbb{C}^n})$

We used also Sabbah's very deep theory

motivic Milnor fiber at ∞
$\mathcal{S}_f^{\infty} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$

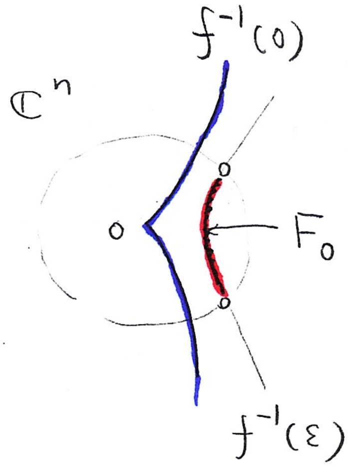
(M-T, Raibaut)

nearby cycle of a mixed Hodge module of Saito

Denef-Loeser (JAG 1998)
Guibert-Loeser-Merle (Duke math 2006)

§ 4. Symmetry between local and global

$f \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ s.t. $f(0) = 0$.



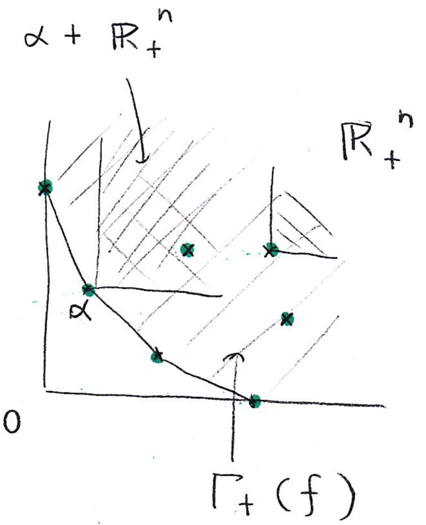
$F_0 \underset{\text{def}}{:=}$ the Milnor fiber of f at $0 \in \mathbb{C}^n$.

$\Longrightarrow$ $\exists$ Milnor monodromies

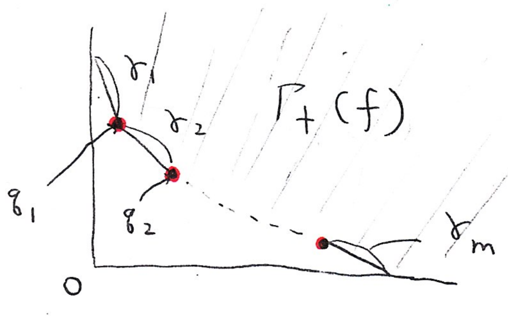
$$\Phi_j \circlearrowright H^j(F_0 ; \mathbb{C}) \quad (j = 0, 1, \cdots).$$

Assumptions (i) $f^{-1}(0)$ has an isolated sing point at $0 \in \mathbb{C}^n$. (ii) f is convenient and non-deg at $0 \in \mathbb{C}^n$.

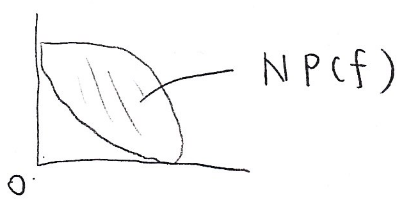
$\Gamma_+(f) :=$ the convex hull of $\bigcup_{\alpha \in \operatorname{supp} f} (\alpha + \mathbb{R}^n_+)$.



If we set



as before, then we find that the results for Φ_{n-1} are completely parallel to the one $\Phi_{n-1}^{(\infty)}$ at ∞ !!



(cf. Danilov and Tanabe etc.)

§ 5 Monodromies over C. I. in $\mathbb{C}^n$

For $0 \leq k \leq n$ let

$$W = \{ f_1 = \cdots\cdots = f_{k-1} = 0 \}$$

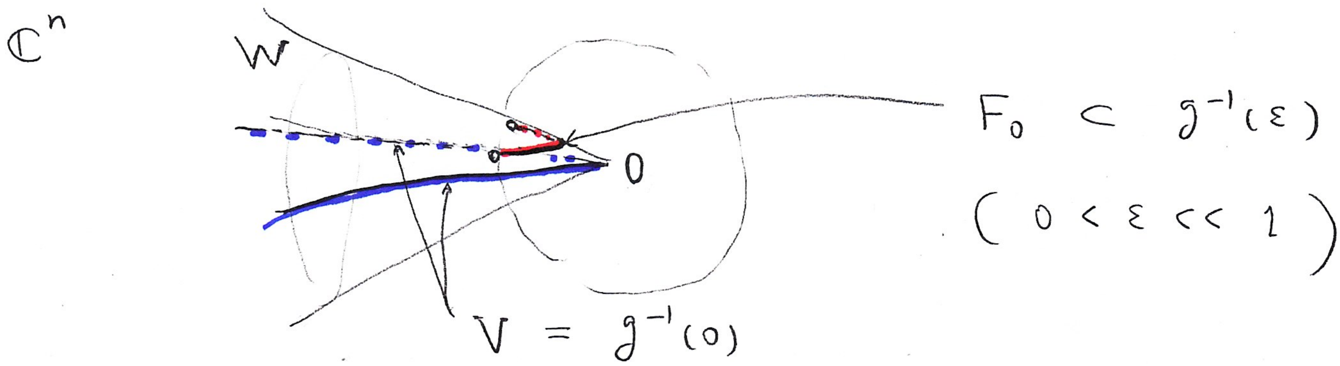
$$\supset V = \{ f_1 = \cdots\cdots = f_{k-1} = f_k = 0 \} \ni 0$$

be C. I. subvarieties in $\mathbb{C}^n$. Set

$$g \underset{\text{def}}{:=} f_k \Big|_W : W \longrightarrow \mathbb{C}.$$

and

$$F_0 := \text{the Milnor fiber of } g : W \to \mathbb{C}$$
$$\text{at } 0 \in V = g^{-1}(0).$$



<u>Assumptions</u> (i) W and V have isolated sing points at 0 ($\underset{\text{Hamm}}{\Longrightarrow}$ $H^j(F_0; \mathbb{C}) = 0$ ($j \neq 0, n-k$))

(ii) $f_1, \cdots, f_k$ are convenient and W and V are non-deg at $0 \in \mathbb{C}^n$ (see Oka's book).

The eigenvalues of $\Phi_{n-k} \circlearrowright H^{n-k}(F_0; \mathbb{C})$ can be determined by Oka's theorem.

Jordan normal form of Φ_{n-k} (cf. Ebeling–Steenbrink ⑨ and Tanabe etc.)

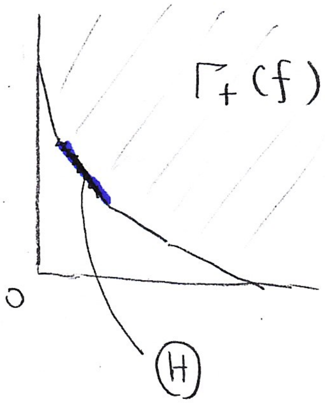
Def Set $f := (f_1, f_2, \cdots, f_k)$ and

$$\Gamma_+(f) := \Gamma_+(f_1) + \cdots + \Gamma_+(f_k).$$

For each face $\Theta < \Gamma_+(f)$

there exist $\gamma_j^{\Theta} < \Gamma_+(f_j)$ s.t.

$$\Theta = \gamma_1^{\Theta} + \cdots + \gamma_k^{\Theta}.$$



Def (i) (Strumfels) For each compact face $\Theta < \Gamma_+(f)$ and $J \subset \{1, 2, \cdots, k-1\}$ we set

$$\delta(\Theta, J) \underset{\text{def}}{:=} \dim\Big(\gamma_k^{\Theta} + \sum_{j \in J} \gamma_j^{\Theta}\Big) - \# J.$$

(ii) When $\min_J \delta(\Theta, J) = 0$ we denote the (unique) maximal element of the set $\{J \mid \delta(\Theta, J) = 0\} \neq \emptyset$ by J_0

For $\lambda \neq 1$ and each cpt face $\Theta < \Gamma_+(f)$ we set

$$E(\Theta)_\lambda := \begin{cases} MV(\gamma_{j_1}^{\Theta}, \cdots, \gamma_{j_m}^{\Theta}) & \left(\begin{array}{l}\min_J \delta(\Theta, J) = 0 \text{ and} \\ \lambda^{d_{\Theta, J_0}} = 1\end{array}\right) \\ \quad J_0 = \{j_1, \cdots, j_m\} & \\ 0 & (\text{otherwise}). \end{cases}$$

mixed volume!

Theorem (E–T) For $\lambda \neq 1$ we have

(i) $\exists$ Algorithm to compute the Jordan normal form of Φ_{n-k} for the eigenvalue λ-part.

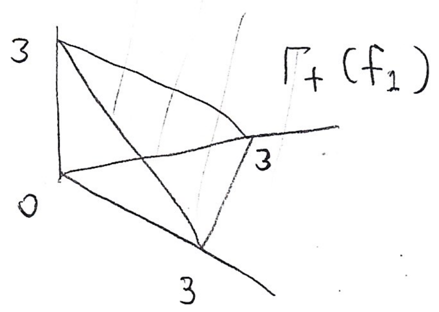
(ii)
$$\#\left\{ n-k+1 \left(\underbrace{\begin{array}{|ccc|} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{array}}_{n-k+1} \right) \in \Phi_{n-k} \right\}$$

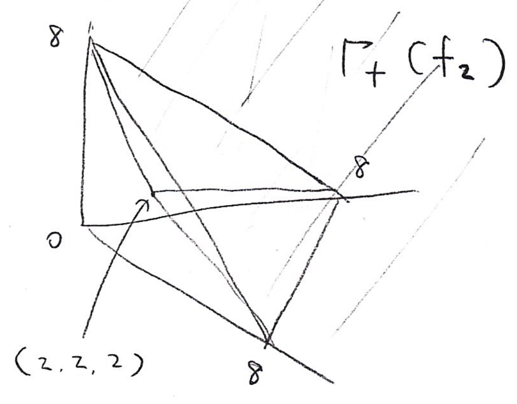
$\leftarrow$ maximal possible size!

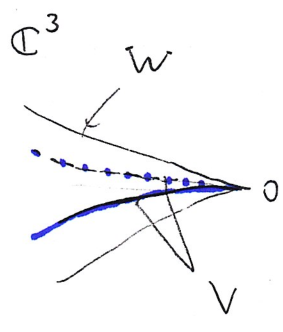
$$= \sum_{\substack{\text{rel.int } \textcircled{H} \subset \operatorname{Int}(\mathbb{R}^n_+) \\ \dim \textcircled{H} \geqq k-1}} (-1)^{\dim \textcircled{H} - (k-1)} E(\textcircled{H})_\lambda .$$

(iii) $\exists$ A closed formula for $\#\left\{ n-k \left(\underbrace{\begin{array}{|ccc|} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{array}}_{n-k} \right) \in \Phi_{n-k} \right\}$.

Example ($n = 3$, $k = 2$)







$$g = f_2 \Big|_W : W = \{ f_1 = 0 \} \longrightarrow \mathbb{C} .$$

Then by Oka's theorem

$$\#\{ \text{eigenvalues } -1 \text{ in } \Phi_{n-k} \} = 18 .$$

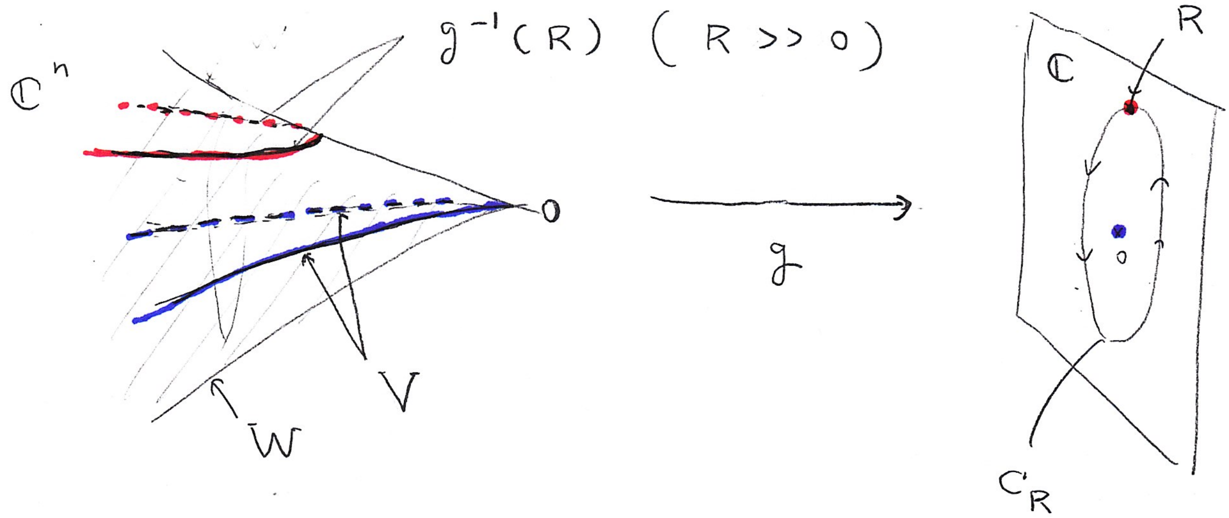
By our Thm (ii) we have $\#\left\{ \begin{array}{|cc|} -1 & 1 \\ 0 & -1 \end{array} \in \Phi_{n-k} \right\} = 8 .$

Hence the eigenvalue (-1)-part of $\Phi_{\underset{\substack{\shortparallel \\ 2}}{(n-k)}} \circlearrowright H^{n-k}(F_0 ; \mathbb{C})$ is conjugate to the matrix :

$$\underbrace{\left.\begin{pmatrix} \begin{smallmatrix}-1&1\\0&-1\end{smallmatrix} & & & & & \bigcirc \\ & \begin{smallmatrix}-1&1\\0&-1\end{smallmatrix} & & & & \\ & & \ddots & & & \\ & & & \begin{smallmatrix}-1&1\\0&-1\end{smallmatrix} & & \\ & & & & -1 & \\ \bigcirc & & & & & -1 \end{pmatrix}\right\} 18}_{18}$$

monodromy at ∞ over C. I.

We have also similar results for the monodromy at ∞ of $g : W \longrightarrow \mathbb{C}$.



Summary

	local	global (at ∞)
on $\mathbb{C}^n$	○	○
on C. I. in $\mathbb{C}^n$	△	△

Eigenvalue 1 ?