tom Dieck-Kosniowski-Stong localization theorem and a differential operator

—A joint work with Qiangbo Tan

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Outline

- Background—equivariant cobordism classification
- tom Dieck-Kosniowski-Stong localization theorem
- A differential operator
- Main result—an equivalent description of tom Dieck-Kosniowski-Stong localization theorem
- Application

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Cobordism classification

In 1954, Thom invented the cobordism theory

All closed manifolds are classified up to cobordism.

Two key points:

- cobordism relation $\longleftrightarrow$ a complete invariant (characteristic numbers)
- Determination of

$$\mathfrak{N}_* = \sum_{n\geq 0} \mathfrak{N}_n \text{ (unoriented case)}$$

and

$$\Omega_* = \sum_{n\geq 0} \Omega_n \text{ (orientable case)}$$

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Equivariant cobordism classification

In 1960s, Conner and Floyd:

Equivariant cobordism theory.

A fundamental problem of equivariant cobordism theory

To classify all closed G-manifolds up to equivariant cobordism

G: a compact Lie group.

Unlike non-equivariant case, the ring structures of $\mathfrak{N}_*^G$ and Ω_*^G are still open.

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Equivariant cobordism classification

Throughout this talk, assume $G = (\mathbb{Z}_2)^k$.

Theorem (tom Dieck in 1970s)

A closed $(\mathbb{Z}_2)^k$-mfd $\sim 0 \Longleftrightarrow$ all equiv. Stiefel-Whitney numbers are zero.

RK. 1) Each equiv. Stiefel-Whitney number is a polynomial in $H^*(B(\mathbb{Z}_2)^k; \mathbb{Z}_2)$.

2) It is still difficult to determine $\mathfrak{N}_*^G$.

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Equivariant cobordism classification

By Conner-Floyd's idea, consider the natural homomorphism

$$\Phi : \mathfrak{N}_*^{(\mathbb{Z}_2)^k} \longrightarrow \mathfrak{N}_*(BO)$$

by mapping

$$\{M\} \longmapsto \sum_{F \subset M^G} \{\nu \longrightarrow F\}$$

where $\mathfrak{N}_*^{(\mathbb{Z}_2)^k}$: equivariant Thom cobordism ring of all closed $(\mathbb{Z}_2)^k$-mfds

$\mathfrak{N}_*(BO)$: the cobordism ring of all $(\mathbb{Z}_2)^k$-vector bundles over closed mfds.

Theorem (Stong)

$\Phi : \mathfrak{N}_*^{(\mathbb{Z}_2)^k} \longrightarrow \mathfrak{N}_*(BO)$ is a monomorphism

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Theorem (Stong)

$\Phi : \mathfrak{N}_*^{(\mathbb{Z}_2)^k} \longrightarrow \mathfrak{N}_*(BO)$ is a monomorphism

A natural question

Question

Given a family of vector G-bundles over closed mfds $\sqcup_{F^k}\nu^{n-k} \longrightarrow F^k$, under what condition do these bundles can become the normal bundle of some $G \curvearrowright M^n$?

Localization theorem

- **Borel construction**

Given $G \curvearrowright X$ （where X: a top. space）
$EG \longrightarrow BG$: universal principal G-bundle.
We obtain $G \curvearrowright EG \times X$ by
$(g,(x,y)) \longmapsto (xg^{-1}, gy)$.

$$X_G := EG \times X/G(i.e., EG \times_G X)$$

is called Borel construction。

Localization theorem

- **Equivariant cohomology**
  Equivariant cohomology of $G \curvearrowright X$ is defined as
  $$H_G^*(X) := H^*(EG \times_G X)$$
- **Remark.** $X \longrightarrow EG \times_G X \longrightarrow BG$ implies that $H_G^*(X)$is a $H^*(BG)$-module

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Localization theorem

$$G := (\mathbb{Z}_2)^k$$

Localization Theorem (Borel)

Let $G \curvearrowright X$, where X is a paracompact G-space. Then the localized restriction homomorphism

$$S^{-1}H_G^*(X; \mathbb{Z}_2) \longrightarrow S^{-1}H_G^*(X^G; \mathbb{Z}_2)$$

is an isomorphism where $S = H^*(BG; \mathbb{Z}_2) - \{0\}$.

Remark

Localization theorem implies that the **global information** of $G \curvearrowright X$ can be described by the **local information**.

Localization theorem—explicit formula

—**Case**: $G = \mathbb{Z}_2 \curvearrowright M^n$, a smooth closed mfd
$\nu \longrightarrow M^{\mathbb{Z}_2} = \sqcup_{F^k \subset M^{\mathbb{Z}_2}} \nu^{n-k} \longrightarrow F^k$: normal bundle

Kosniowski-Stong formula

Let $\mathbb{Z}_2 \curvearrowright M^n$. Then for any symmetric function f of deg $= n$

$$f(x_1, ..., x_n)[M^n] = \sum_{F^k \subset M^{\mathbb{Z}_2}} \frac{f(1+y_1, ..., 1+y_{n-k}, z_1, ..., z_k)}{\prod_{i=1}^{n-k}(1+y_i)}[F^k]$$

where $w(\tau(M)) = \prod_{i=1}^{n}(1+x_i)$: total Stiefel-Whitney class
$w(\nu^{n-k}) = \prod_{i=1}^{n-k}(1+y_i)$
$w(\tau(F^k)) = \prod_{i=1}^{k}(1+z_i)$

Localization theorem—explicit formula

—**General case**: $G=(\mathbb{Z}_2)^k \curvearrowright M^n$ a smooth closed mfd

$\nu \longrightarrow M^G = \sqcup_{F^k\subset M^G}\nu^{n-k} \longrightarrow F^k$: normal bundle

Write $w^G(\tau(M))=\prod_{i=1}^n(1+x_i)$: total equivariant Stiefel-Whitney class

Kosniowski–Stong formula

Let $(\mathbb{Z}_2)^k \curvearrowright M^n$. Then

$$f(x_1,...,x_n)[M^n]=\sum_{F^k\subset M^G}\frac{f|_F}{\chi^G(\nu^{n-k})}[F^k]\in H^*(BG;\mathbb{Z}_2)$$

where $f|_F$ is the restriction of f to F^k
$\chi^G(\nu^{n-k})$ is the equivariant Euler class of $\nu^{n-k}\longrightarrow F^k$

Recall

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tom Dieck, Kosniowski, Stong gave a partial answer.

Recall

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tom Dieck-Kosniowski-Stong localization theorem

Assume $(\mathbb{Z}_2)^k \curvearrowright M^n$ fixes only l isolated points $p_1, ..., p_l$
$\tau_1, ..., \tau_l$: n-dim real $(\mathbb{Z}_2)^k$-representations

tom Dieck-Kosniowski-Stong localization theorem

$\tau_1, ..., \tau_l$ become the tangent representations at $p_1, ..., p_l$ of $(\mathbb{Z}_2)^k \curvearrowright M^n \Leftrightarrow$ for any symmetric $f(x_1, ..., x_n)$

$$\sum_{i=1}^{l} \frac{f(\tau_i)}{\chi^G(\tau_i)} \in H^*(B(\mathbb{Z}_2)^k; \mathbb{Z}_2) = \mathbb{Z}_2[t_1, ..., t_k]$$

where $\chi^G(\tau_i)$ is a product of n nonzero elements in $H^1(B(\mathbb{Z}_2)^k; \mathbb{Z}_2) = \text{Span}\{t_1, ..., t_k\}$
$f(\tau_i)$ implies that n variables $x_1, ..., x_n$ are replaced by those degree-one factors in $\chi^G(\tau_i)$

To equivariant cobordism classification

Case

The fixed point set consists of isolated pts.

Let $\mathcal{Z}_*((\mathbb{Z}_2)^k) \subset \mathfrak{N}_*^{(\mathbb{Z}_2)^k}$ consist of smooth closed $(\mathbb{Z}_2)^k$-mfds fixing isolated pts, which forms a graded commutative algebra over $\mathbb{Z}_2$.

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To equivariant cobordism classification

The restriction to $\mathcal{Z}_*((\mathbb{Z}_2)^k)$ of Stong's monomorphism

$$\Phi : \mathfrak{N}_*^{(\mathbb{Z}_2)^k} \longrightarrow \mathfrak{N}_*(BO)$$

induces

$$\Phi : \mathcal{Z}_*((\mathbb{Z}_2)^k) \longrightarrow R_*((\mathbb{Z}_2)^k)$$

by

$$\{M\} \longmapsto \sum_{p \in M^{(\mathbb{Z}_2)^k}} [\tau_p M]$$

where $R_*((\mathbb{Z}_2)^k)$ is an algebra over $\mathbb{Z}_2$ introduced by Conner and Floyd, generated by all irreducible $(\mathbb{Z}_2)^k$-representations.

$R_*((\mathbb{Z}_2)^k)$: called the **Conner-Floyd algebra**

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To equivariant cobordism classification

A characterization result

$$\mathcal{Z}_*((\mathbb{Z}_2)^k) \cong \mathrm{Im}\Phi \subset \mathcal{R}_*((\mathbb{Z}_2)^k)$$

In particular, $\tau_1 + \cdots + \tau_l \in \mathrm{Im}\Phi \Longleftrightarrow$ for any symmetric polynomial function f,

$$\sum_{i=1}^{l} \frac{f(\tau_i)}{\chi^{(\mathbb{Z}_2)^k}(\tau_i)} \in H^*(B(\mathbb{Z}_2)^k; \mathbb{Z}_2)$$

Conner and Floyd showed that when $k = 1$ $\mathcal{Z}_*(\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Conner-Floyd-Kosniowski-Stong

When $k = 2$, $\mathcal{Z}_*((\mathbb{Z}_2)^2) \cong \mathbb{Z}_2[u]$ where u denotes the class of $\mathbb{R}P^2$ with the standard linear $(\mathbb{Z}_2)^2$-action.

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A differential operator

A reformulation of Conner-Floyd algebra

- Each nontrivial irreducible real $(\mathbb{Z}_2)^k$-representation is 1-dim, and has the form

$$\lambda_\rho : (\mathbb{Z}_2)^k \times \mathbb{R} \longrightarrow \mathbb{R},\ (g, x) \longmapsto (-1)^\rho x$$

where $\rho \in \text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$ is nonzero. Thus

$$\{\text{all nontrivial irr. real } (\mathbb{Z}_2)^k\text{-rep.}\} \longleftrightarrow \text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)\backslash\{0\}$$

- Conner-Floyd algebra $\mathcal{R}_*((\mathbb{Z}_2)^k)$ is isomorphic to the algebra $\mathbb{Z}_2[\text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)]$.

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Dual algebra

$\mathrm{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^k)$: all homomorphisms from $\mathbb{Z}_2 \longrightarrow (\mathbb{Z}_2)^k$

$$\Downarrow$$

$\mathbb{Z}_2[\mathrm{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^k)]$: polynomial algebra generated by nonzero elements of $\mathrm{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^k)$.

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A differential operator

- Define a **differential operator** d on $\mathbb{Z}_2[\mathrm{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^k)]$ as follows: for each monomial $s_1 \cdots s_i$

$$d_i(s_1 \cdots s_i) = \begin{cases} \sum_{j=1}^{i} s_1 \cdots s_{j-1} \widehat{s_j} s_{j+1} \cdots s_i & \text{if } i > 1 \\ 1 & \text{if } i = 1. \end{cases}$$

且$d_0(1) = 0$

Basic fact

$H_i(\mathbb{Z}_2[\mathrm{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^k)]; \mathbb{Z}_2) = 0$

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Next purpose

Purpose

To discuss an equivalent description of tom Dieck-Kosniowski-Stong localization theorem

An equivalent description

$\text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$ and $\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^k)$ are dual to each other by the following pair

$$\langle \cdot, \cdot \rangle : \text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^k) \times \text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2) \longrightarrow \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_2)$$

where $\langle \xi, \rho \rangle = \rho \circ \xi$.

Notations

- a degree-k homogeneous polynomial $g = \sum_i a_{i,1} \cdots a_{i,k} \in \mathbb{Z}_2[\text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)]$ is said to be **faithful** if $a_{i,1}, \cdots, a_{i,k}$ form a basis of $\text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$.
- By the pair, g determines a unique degree-k homogeneous polynomial $g^* = \sum_i s_{i,1} \cdots s_{i,k}$ in $\mathbb{Z}_2[\text{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^k)]$, called the **dual polynomial** of g.

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An equivalent description

Recall that $\tau_1 + \cdots + \tau_l \in \mathrm{Im}\Phi_n \subset \mathcal{R}_n((\mathbb{Z}_2)^k) \Longleftrightarrow$ for any symmetric polynomial function f,

$$\sum_{i=1}^{l} \frac{f(\tau_i)}{\chi^{(\mathbb{Z}_2)^k}(\tau_i)} \in H^*(B(\mathbb{Z}_2)^k; \mathbb{Z}_2)$$

and $\mathcal{R}_*((\mathbb{Z}_2)^k) \cong \mathbb{Z}_2[\mathrm{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)]$

Theorem (Lü-Tan) Another characterization of $g \in \mathrm{Im}\Phi_n$ in the case $k = n$

Let $g = \sum_{i=1}^{l} a_{i,1} \cdots a_{i,n} \in \mathbb{Z}_2[\mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)]$ be faithful. Then $g \in \mathrm{Im}\Phi_n$ (i.e., $a_{1,1} \cdots a_{1,n}, \ldots, a_{l,1} \cdots a_{l,n}$ become the fixed data of some $(\mathbb{Z}_2)^n \curvearrowright M^n$) $\Longleftrightarrow d(g^*) = 0$

An algebra corollary

Algebraic corollary

Let $g = \sum_{i=1}^{l} a_{i,1}\cdots a_{i,n} \in \mathbb{Z}_2[\mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)]$ be faithful. Then $d(g^*) = 0$ in $\mathbb{Z}_2[\mathrm{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^n]$ $\Longleftrightarrow$ for any symmetric function $f(x_1, ..., x_n)$

$$\sum_{i=1}^{l} \frac{f(a_{i,1}, \cdots, a_{i,n})}{a_{i,1}\cdots a_{i,n}} \in \mathbb{Z}_2[\mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)]$$

Proof of Theorem

- key point is to use GKM theory, established by Goresky, Kottwitz and MacPherson.
- A faithful G-polynomial $g \in \mathbb{Z}_2[\mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)]$ belongs to $\mathrm{Im}\Phi_n$ if and only if it is the coloring polynomial of a colored graph (Γ, α).
- $d(g^*) = 0$ if and only if g is the coloring polynomial of a colored graph.

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- $d(g^*) = 0$ if and only if g is the coloring polynomial of a colored graph.

Colored graph and coloring polynomial

Let Γ be a finite regular graph of valence n without loops. If there is a map α from the set E_Γ of all edges of Γ to all nontrivial elements of $\mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)$ with the following properties:

1) for each vertex p of Γ, $\prod_{x \in E_p} \alpha(x)$ is faithful in $\mathbb{Z}_2[\mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)]$, where E_p denotes the set of all edges adjacent to p;

2) for each edge e of Γ, $\alpha(E_p) \equiv \alpha(E_q) \mod \alpha(e)$ in $\mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)$ where p and q are two endpoints of e;

then the pair (Γ, α) is called a **colored graph** of Γ, and $g_{(\Gamma,\alpha)} = \sum_{p \in V_\Gamma} \prod_{x \in E_p} \alpha(x)$ in $\mathbb{Z}_2[\mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)]$ is called the **coloring polynomial** of (Γ, α).

Colored graph and coloring polynomial

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1) for each vertex p of Γ, $\prod_{x \in E_p} \alpha(x)$ is faithful in $\mathbb{Z}_2[\mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)]$, where E_p denotes the set of all edges adjacent to p;

2) for each edge e of Γ, $\alpha(E_p) \equiv \alpha(E_q) \mod \alpha(e)$ in $\mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)$ where p and q are two endpoints of e;

then the pair (Γ, α) is called a **colored graph** of Γ, and $g_{(\Gamma,\alpha)} = \sum_{p \in V_\Gamma} \prod_{x \in E_p} \alpha(x)$ in $\mathbb{Z}_2[\mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2)]$ is called the **coloring polynomial** of (Γ, α).

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Application

In paper 【Z. Lü, 2-torus manifolds, cobordism and small covers, Pacific J. Math. 241 (2009), 285–308】,

$\mathfrak{M}_n$ consisting of equivariant cobordism classes of smooth closed n-mfds with effective $(\mathbb{Z}_2)^n$-actions was introduced and studied.

Structure of $\mathfrak{M}_3$

$\dim \mathfrak{M}_3 = 13$

In addition, the following conjecture was also posed

Conjecture $(*)$

Each class of $\mathfrak{M}_n$ contains a small cover as its representative. It was showed that when $n \leq 3$, this is true.

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A remark

Small cover

- Small covers were introduced by M. Davis and T. Januszkiewicz.
- An n-dimensional **small cover** is an n-dim smooth closed mfd M^n with a locally standard $(\mathbb{Z}_2)^n$-action such that its orbit space is an n-dim simple convex polytope P^n.
- A small cover is a topological version of a real toric variety.
- A canonical example is $\mathbb{R}P^n$ with a $(\mathbb{Z}_2)^n$-action defined by

$$((g_1, ..., g_n), [x_0, x_1, ..., x_n]) \longmapsto [x_0, (-1)^{g_1} x_1, ..., (-1)^{g_n} x_n]$$

whose orbit space is an n-simplex.

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The restriction to $\mathfrak{M}_n$ of Stong homomorphism gives a monomorphism $\Phi_n : \mathfrak{M}_n \longrightarrow \mathcal{R}_n((\mathbb{Z}_2)^n)$ by

$$\Phi_n(\{M^n\}) = \sum_{p \in M^G} \{\tau_p M\}$$

Using the equivalence description of tom Dieck-Kosniowski-Stong localization theorem, we obtain

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Let $\mathfrak{M}_* = \sum_{n\geq 1} \mathfrak{M}_n$. Then $\mathfrak{M}_*$ forms a noncommutative ring.

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$\mathfrak{M}_*$ is generated by equivariant cobordism classes of all small covers over $\Delta^{n_1} \times \cdots \times \Delta^{n_\ell}$, where Δ^{n_i} is an n_i-simplex.

We also determine $\mathfrak{M}_4$

$\dim \mathfrak{M}_4 = 510$ and $\mathfrak{M}_4$ is generated by small covers over $\Delta^2 \times \Delta^2$.

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$$g=\sum_{i=1}^{l} a_{i,1}\cdots a_{i,n}\in \mathbb{Z}_2[\mathrm{Hom}((\mathbb{Z}_2)^n,\mathbb{Z}_2)]\text{: faithful.}$$

A summary–The following are equivalent

- $g \in \mathrm{Im}\Phi_n$
- g is the coloring polynomial of a colored graph (Γ, α)
- for any symmetric polynomial function $f(x_1, ..., x_n)$

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