Monodromy representations of conformal field theory

Toshitake Kohno

The University of Tokyo

July 2011

Plan

- The space of conformal blocks

Plan

- The space of conformal blocks
- Quantum representations of mapping class groups

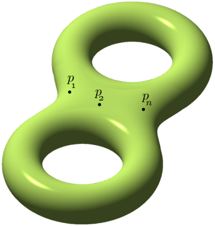
Plan

- The space of conformal blocks
- Quantum representations of mapping class groups
- Monodromy representations of braid groups and Gauss-Manin connections

Plan

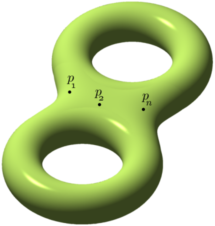
- The space of conformal blocks
- Quantum representations of mapping class groups
- Monodromy representations of braid groups and Gauss-Manin connections
- Images of quantum representations of mapping class groups

Conformal Field Theory



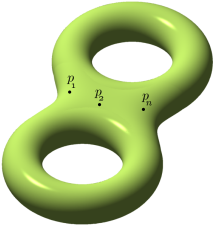
$(\Sigma, p_1, \cdots, p_n)$: Riemann surface with marked points
$\lambda_1, \cdots, \lambda_n$: level K highest weights

Conformal Field Theory



$(\Sigma, p_1, \cdots, p_n)$: Riemann surface with marked points
$\lambda_1, \cdots, \lambda_n$: level K highest weights
$\mathcal{H}_\Sigma(p, \lambda)$: **space of conformal blocks**
vector space spanned by holomorphic parts of the WZW partition function.

Conformal Field Theory



$(\Sigma, p_1, \cdots, p_n)$: Riemann surface with marked points
$\lambda_1, \cdots, \lambda_n$: level K highest weights
$\mathcal{H}_\Sigma(p, \lambda)$: **space of conformal blocks**
vector space spanned by holomorphic parts of the WZW partition function.
Geometry : vector bundle over the moduli space of Riemann surfaces with n marked points with projectively flat connection.

Representations of $sl_2(\mathbf{C})$

$\mathfrak{g} = sl_2(\mathbf{C})$ has a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

λ : non-negative integer
V_λ : irreducible highest weight representation of $sl_2(\mathbf{C})$ with highest weight vector v such that

$$Hv = \lambda v, \ Ev = 0$$

Representations of an affine Lie algebra

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$: affine Lie algebra with commutation relation

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \mathrm{Res}_{\xi=0}\, df\, g\ \langle X, Y \rangle c$$

K a positive integer (level)
$\widehat{\mathfrak{g}} = \mathcal{N}_+ \oplus \mathcal{N}_0 \oplus \mathcal{N}_-$
c acts as $K \cdot \mathrm{id}$.

Representations of an affine Lie algebra

$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$: affine Lie algebra with commutation relation

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \mathrm{Res}_{\xi=0}\, df\ g\ \langle X, Y \rangle c$$

K a positive integer (level)
$\widehat{\mathfrak{g}} = \mathcal{N}_+ \oplus \mathcal{N}_0 \oplus \mathcal{N}_-$
c acts as $K \cdot \mathrm{id}$.

λ : an integer with $0 \leq \lambda \leq K$
$\mathcal{H}_\lambda$: irreducible quotient of $\mathcal{M}_\lambda$ called the integrable highest weight modules.

Geometric background

G : the Lie group $SL(2, \mathbf{C})$
$LG = \mathrm{Map}(S^1, G)$: loop group
$\mathcal{L} \longrightarrow LG$: complex line bundle with $c_1(\mathcal{L}) = K$

G : the Lie group $SL(2, \mathbf{C})$
$LG = \mathrm{Map}(S^1, G)$: loop group
$\mathcal{L} \longrightarrow LG$: complex line bundle with $c_1(\mathcal{L}) = K$

The affine Lie algebra $\widehat{\mathfrak{g}}$ acts on the space of sections $\Gamma(\mathcal{L})$.
The integrable highest weight modules $\mathcal{H}_\lambda$, $0 \leq \lambda \leq K$, appears as sub representations.
As the infinitesimal version of the action of the central extension of $\mathrm{Diff}(S^1)$ the Virasoro Lie algebra acts on $\mathcal{H}_\lambda$.

The space of conformal blocks - definition -

Suppose $0 \leq \lambda_1, \cdots, \lambda_n \leq K$.
$p_1, \cdots, p_n \in \Sigma$
Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.
$\mathcal{H}_j$: irreducible representations of $\widehat{\mathfrak{g}}$ with highest weight λ_j at level K.

Suppose $0 \leq \lambda_1, \cdots, \lambda_n \leq K$.
$p_1, \cdots, p_n \in \Sigma$
Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.
$\mathcal{H}_j$: irreducible representations of $\widehat{\mathfrak{g}}$ with highest weight λ_j at level K.

$\mathcal{M}_p$ denotes the set of meromorphic functions on Σ with poles at most at $p_1, \cdots, p_n$.

The space of conformal blocks - definition -

Suppose $0 \leq \lambda_1, \cdots, \lambda_n \leq K$.
$p_1, \cdots, p_n \in \Sigma$
Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.
$\mathcal{H}_j$: irreducible representations of $\widehat{\mathfrak{g}}$ with highest weight λ_j at level K.

$\mathcal{M}_p$ denotes the set of meromorphic functions on Σ with poles at most at $p_1, \cdots, p_n$.

The space of conformal blocks is defined as

$$\mathcal{H}_\Sigma(p, \lambda) = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} / (\mathfrak{g} \otimes \mathcal{M}_p)$$

where $\mathfrak{g} \otimes \mathcal{M}_p$ acts diagonally via Laurent expansions at $p_1, \cdots, p_n$.

Conformal block bundle

Σ_g : Riemann surface of genus g
$p_1, \cdots, p_n$: marked points on Σ_g
Fix the highest weights $\lambda_1, \cdots, \lambda_n$.

Conformal block bundle

Σ_g : Riemann surface of genus g
$p_1, \cdots, p_n$: marked points on Σ_g
Fix the highest weights $\lambda_1, \cdots, \lambda_n$.

The union

$$\bigcup_{p_1, \cdots, p_n} \mathcal{H}_{\Sigma_g}(p, \lambda)$$

for any complex structures on Σ_g forms a vector bundle on $\mathcal{M}_{g,n}$, the moduli space of Riemann surfaces of genus g with n marked points.

Conformal block bundle

Σ_g : Riemann surface of genus g
$p_1, \cdots, p_n$: marked points on Σ_g
Fix the highest weights $\lambda_1, \cdots, \lambda_n$.

The union

$$\bigcup_{p_1, \cdots, p_n} \mathcal{H}_{\Sigma_g}(p, \lambda)$$

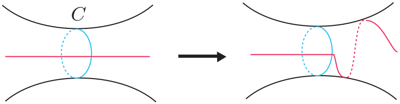
for any complex structures on Σ_g forms a vector bundle on $\mathcal{M}_{g,n}$, the moduli space of Riemann surfaces of genus g with n marked points.

This vector bundle is called the conformal block bundle and is equipped with a natural projectively flat connection. The holonomy representation of the mapping class group is called the quantum representation.

Mapping class groups

$\Gamma_{g,n}$: **mapping class group** of the Riemann surface of genus g with n marked points (orientation preserving diffeomorphisms of Σ upto isotopy)

$\Gamma_{g,n}$ is generated by Dehn twists.

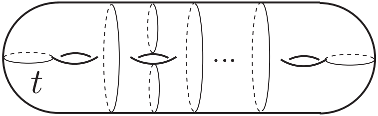


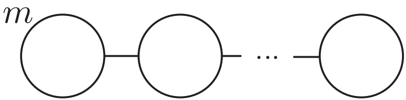
Dehn twist along the curve C

$\Gamma_{g,n}$ acts on $\mathcal{H}_\Sigma$: **quantum representation** ρ_K.

Action of Dehn twists

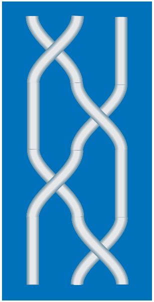
A basis of the space of conformal blocks is given by trivalent graphs labelled by highest weights dual to pants decomposition of the surface.

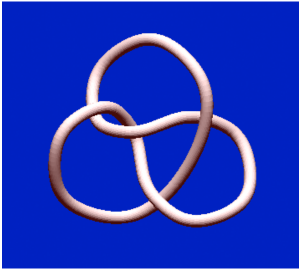




The Dehn twist along t acts as $e^{2\pi i \Delta_m}$
(Δ_m : conformal weight)

A braid and its closure (figure 8 knot)





genus 0 case : The flat connection is the **KZ connection**, which is interpreted as **Gauss-Manin connection** via hypergeometric integrals.

The case $g = 0$

$p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$ with $p_{n+1} = \infty$
Assign highest weights $\lambda_1, \cdots, \lambda_{n+1} \in \mathbf{Z}$ to $p_1, \cdots, p_{n+1}$.

The case $g = 0$

$p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$ with $p_{n+1} = \infty$
Assign highest weights $\lambda_1, \cdots, \lambda_{n+1} \in \mathbf{Z}$ to $p_1, \cdots, p_{n+1}$.

We have a flat vector bundle over the configuration space

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j, \ i \neq j\}.$$

with KZ connection.

$p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$ with $p_{n+1} = \infty$
Assign highest weights $\lambda_1, \cdots, \lambda_{n+1} \in \mathbf{Z}$ to $p_1, \cdots, p_{n+1}$.

We have a flat vector bundle over the configuration space

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j, \ i \neq j\}.$$

with KZ connection.

The monodromy representation is the quantum representation of the braid groups.

KZ connections

$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.
$\Omega = \sum_\mu I_\mu \otimes I_\mu$
$r_i : \mathfrak{g} \to End(V_i)$, $1 \leq i \leq n$ representations.

KZ connections

$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.
$\Omega = \sum_\mu I_\mu \otimes I_\mu$
$r_i : \mathfrak{g} \to End(V_i)$, $1 \le i \le n$ representations.

Ω_{ij} : the action of Ω on the i-th and j-th components of $V_1 \otimes \cdots \otimes V_n$.

$$\omega = \frac{1}{\kappa} \sum_{i,j} \Omega_{ij} d\log(z_i - z_j), \quad \kappa \in \mathbf{C} \setminus \{0\}$$

ω defines a flat connection for a trivial vector bundle over the configuration space X_n with fiber $V_1 \otimes \cdots \otimes V_n$ since we have

$$\omega \wedge \omega = 0$$

Monodromy representations of braid groups

As the holonomy we have representations

$$\theta_\kappa : P_n \to GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

$$\theta_\kappa : B_n \to GL(V^{n\otimes}).$$

Monodromy representations of braid groups

As the holonomy we have representations

$$\theta_\kappa : P_n \to GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

$$\theta_\kappa : B_n \to GL(V^{n\otimes}).$$

We shall express the horizontal sections of the KZ connection : $d\varphi = \omega\varphi$ in terms of homology with coefficients in local system homology on the fiber of the projection map

$$\pi : X_{m+n} \longrightarrow X_n.$$

$X_{n,m}$: fiber of π, $\quad Y_{n,m} = X_{n,m}/S_m$

Gauss-Manin connection

$\mathcal{L}$: rank 1 local system over $Y_{n,m}$
$m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_n - \lambda_{n+1})$
$\mathcal{H}_{n,m}$: local system over X_n with fiber $H_m(Y_{n,m}, \mathcal{L})$

Theorem

There is an injective bundle map from the conformal block bundle

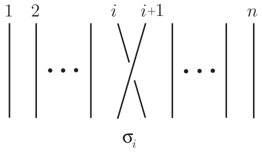
$$\bigcup \mathcal{H}_{\mathbf{C}P^1}(p, \lambda) \longrightarrow \mathcal{H}_{n,m}$$

via hypergeometric integrals. The KZ connection is interpreted as Gauss-Manin connection.

Asymptotic faithfulness

Any two elements of the mapping class group are distinguished by the quantum representation for sufficiently large K (J. Andersen).

$B_n[k]$: normal subgroup of the braid group B_n generated by σ_i^k, $1 \leq i \leq n-1$.



Theorem (L. Funar and T. Kohno)

For any infinite set $\{k\}$, *we have* $\bigcap_k B_n[2k] = \{1\}$.

A positive answer to Squier's conjecture.

Images of quantum representations

The quantum representations are projectively unitary.

$$\rho_K : \Gamma_g \longrightarrow PU(\mathcal{H}_{\Sigma_g})$$

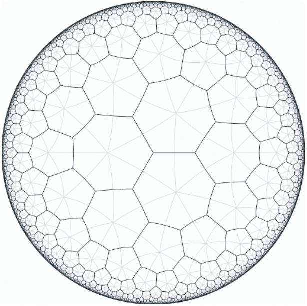
The k-th Johnson subgroup acts trivially on the k-th lower central series of the fundamental group $\pi_1(\Sigma_g)$.

The image of the quantum representation is "big" in the following sense.

Theorem (L. Funar and T. Kohno)

Suppose $g \geq 4$ and K sufficiently large. Then the image of any Johnson subgroup by ρ_K contains a non-abelian free group.

Images of braid groups B_3 in the mapping class group by the quantum representation ρ_K are related to Schwarz triangle groups.



tessellation of the Poincaré disc by the triangle group

Lattices and cyclotomic integers

Gilmer and Masbaum show in the case $\kappa = K + 2$ is odd prime, the image of the quantum representation ρ_K is contained in

$$PU(\mathcal{O}_\kappa)$$

where $\mathcal{O}_\kappa$ is the ring of cyclotomic integers:

$$\mathcal{O}_\kappa \subset \mathbf{Q}(e^{2\pi i/\kappa})$$

Suppose g, K sufficiently large.

Theorem (L. Funar and T. Kohno)

$\rho_K(\Gamma_g)$ is of infinite index in $PU(\mathcal{O}_\kappa)$.

Reference: L. Funar and T. Kohno, On images of quantum representations of mapping class groups, arXiv:0907.0568