Motivic Zeta Functions for Quasi-Ordinary Hypersurface Singularities

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The International Conference
on Singularity Theory and Applications
Hefei, China
25th-31st July 2011

Milnor fibration

$f : \mathbb{A}_{\mathbb{C}}^{d+1} \to \mathbb{A}_{\mathbb{C}}^{1}$ non-constant algebraic or analytic morphism

$x \in f^{-1}\{0\}$ singular point.

For $0 < \delta << \epsilon < 1$

- $f : f^{-1}(\mathbf{D}_{\delta}^{*}) \cap \mathbb{B}(x, \epsilon) \to \mathbf{D}_{\delta}^{*}$ locally trivial C^{∞}-fibration
- $F_{f,x} := f^{-1}\{t\} \cap \mathbb{B}(x, \epsilon)$ **Milnor fibre** of f at the point x.

Topology of $F_{f,x}$

- Betti numbers $b_i(F_{f,x}) := \dim_{\mathbb{C}} H^i(F_{f,x}, \mathbb{C})$
- Euler characteristic $\chi(F_{f,x}) := \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(F_{f,x}, \mathbb{C})$

If x is an **isolated** singular point of $f^{-1}\{0\}$,

$$b_i(F_{f,x}) = \dim_{\mathbb{C}} H^i(F_{f,x}, \mathbb{C}) = \begin{cases} 1 & \text{if} \quad i = 0, \\ \mu(f,x) & \text{if} \quad i = d, \\ 0 & \text{if} \quad i \neq 0, d \end{cases}$$

where $\mu(f,x) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x_1,\ldots,x_{d+1}]]}{(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_{d+1}})}$ is the **Milnor number**.

Geometric Monodromy The diffeomorphism of $F_{f,x}$ corresponding to going once around the boundary of $\mathbf{D}_\delta$.

Algebraic Monodromy Operator of the cohomology ring induced by the geometric monodromy

$$M^\bullet_{f,x} : H^\bullet(F_{f,x}, \mathbb{C}) \rightarrow H^\bullet(F_{f,x}, \mathbb{C}).$$

Theorem (Monodromy Theorem)

The endomorphism $M_{f,x}$ is quasi-unipotent: $\exists\, A, B \in \mathbb{N}$ such that

$$(M^A_{f,x} - I)^B = 0.$$

$\Longrightarrow$ *The eigenvalues of $M_{f,x}$ are roots of unit.*

Theorem (Steenbrink, Saito, Navarro-Aznar)

$H^i(F_{f,x},\mathbb{Q})$ enjoys a mixed Hodge structure compatible with $M_{f,x}$:

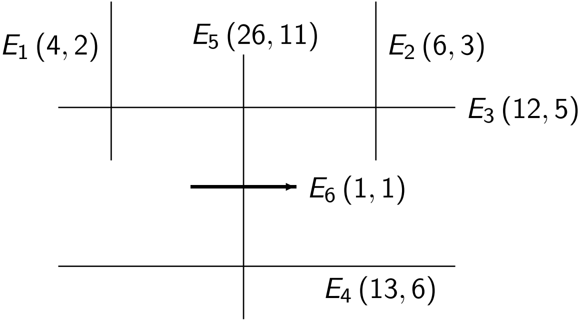
$$[H^i(F_{f,x},\mathbb{Q})] := \sum_m [Gr_m^W H^i(F_{f,x},\mathbb{Q})] \in K_0(HS^{mon}).$$

Hodge-Steenbrink spectrum of f at a singular point

$$hsp(f,x) := \sum_{\alpha\in\mathbb{Q}} n_\alpha(f,x) t^\alpha,$$

where $n_\alpha(f,x) := \sum_i (-1)^i \dim_{\mathbb{C}} Gr_F^{\lfloor d+1-\alpha \rfloor} \tilde{H}^{d+i}(F_{f,x},\mathbb{C})_{e^{-2\pi\alpha}}$

Example: $f = (y^2 - x^3)^2 - x^5 y$.



$F_{f,0}$ is a genus 8 surface with a hole (due to the intersection with the strict transform E_6 of $f^{-1}\{0\}$).

$$\mu(f,0) = b_1(F_{f,0}) = 16 \qquad \text{and} \qquad \chi(F_{f,0}) = -15$$

A'Campo Formula for the Monodromy Zeta Function:

$$\xi_{M_{f,x}}(s) = \frac{(1-s^4)(1-s^6)(1-s^{13})}{(1-s^{12})(1-s^{26})}$$

Formulas by Schrauwen, Steenbrink & Stevens (resolution) + Saito & Nemethi (Puiseux pairs)

$$hsp(f,x) = t^{5/12} + t^{11/12} + t^{13/12} + t^{19/12} + \sum_{i=0}^{11} t^{\frac{15+2i}{26}}$$

Idea (Denef-Loeser):

Substitute the Milnor fibre $F_{f,x}$ and its monodromy operator $M_{f,x}$ by the **motivic Milnor fibre** $S_{f,x} \in K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$.

$K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ is the **Grothendieck ring** of complex algebraic varieties endowed with a good $\hat{\mu}$-action, where

$$\mu_n := \mathrm{Spec}\mathbb{C}[x]/(x^n - 1) \quad \text{and} \quad \hat{\mu} := \varprojlim \mu_n.$$

Notation: $\mathbb{L} := [\mathbb{A}^1_{\mathbb{C}}]$ and $\mathcal{M}^{\hat{\mu}} := K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$.

Theorem (Deligne)

If X is a complex algebraic variety, $H^i_c(X,\mathbb{Q})$ has a mixed Hodge structure. Furthermore, if X had a good $\hat{\mu}$-action, then the mixed Hodge structure is endowed with a quasi-unipotent homomorfism.

Hodge Caracteristic

$$\chi_h^{mon} : K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}}) \to K_0(HS^{mon})$$

$$[X,\hat{\mu}] \mapsto \sum_i (-1)^i [H^i_c(X,\mathbb{Q}), M]$$

and there is a notion of spectrum $Sp([X,\hat{\mu}).$

$\mathcal{L}(\mathbb{A}_{\mathbb{C}}^{d+1})_0$ **arcs of** $\mathbb{A}_{\mathbb{C}}^{d+1}$ **centered at** 0

$\varphi \equiv (\varphi_1(t), ..., \varphi_{d+1}(t)) \in (\mathbb{C}[[t]])^{d+1}$ such that $\varphi_i(0) = 0$ for all $1 \leq i \leq d+1$

$\mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^{d+1})_0$ **n-jets of** $\mathbb{A}_{\mathbb{C}}^{d+1}$ **centered at** 0**:**

$\varphi \equiv (\varphi_1(t), ..., \varphi_{d+1}(t)) \in (\mathbb{C}[t]/(t^{n+1}))^{d+1}$ such that $\varphi_i(0) = 0$ for all $1 \leq i \leq d+1$

Truncation map $\pi_n : \mathcal{L}(\mathbb{A}_{\mathbb{C}}^{d+1})_0 \to \mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^{d+1})_0$

$f : \mathbb{A}^{d+1}_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ non-constant morphism with $f(0) = 0$ and $\varphi \in \mathcal{L}(\mathbb{A}^{d+1}_{\mathbb{C}})_0$

$$\Longrightarrow f \circ \varphi = a_s t^s + a_{s+1} t^{s+1} + \cdots, a_s \neq 0$$

- $\mathrm{ord}_t f \circ \varphi := s \qquad \mathrm{ac}(f \circ \varphi) := a_s$
- $\mathcal{X}_n := \{\varphi \in \mathcal{L}_n(\mathbb{A}^{d+1}_{\mathbb{C}})_0 \mid \mathrm{ord}_t f \circ \varphi = n\}$
- $\mathcal{X}_{n,1} := \{\varphi \in \mathcal{X}_n \mid \mathrm{ac}(f \circ \varphi) = 1\}$

Lemma

The set $\mathcal{X}_n$ (resp. $\mathcal{X}_{n,1}$) is a **constructible** *subset of $\mathcal{L}_n(\mathbb{A}^{d+1}_{\mathbb{C}})_0$ and $\mathbb{C}^*$ (resp. μ_n) acts on it.*

Denote $\mathcal{Z}_{n,(1)}$ the preimage of $\mathcal{X}_{n,(1)}$ in $\mathcal{L}(\mathbb{A}_{\mathbb{C}}^{d+1})_0$.

The **motivic measure** is given by

$$\mu_{mot}(\mathcal{Z}_{n,(1)}) := [\mathcal{X}_{n,(1)}] \cdot \mathbb{L}^{-n(d+1)} \in K_0^{(\hat{\mu})}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$$

Motivic Zeta Function (Denef and Loeser)

$$Z^{naive}(f, T) := \sum \mu_{mot}(\mathcal{Z}_n) T^n,$$

$$Z(f, T) := \sum \mu_{mot}(\mathcal{Z}_{n,1}) T^n.$$

Rationality result:

Theorem (Denef-Loeser)

These zeta functions are rational functions w.r.t. T. With help of embedded resolution of the singularity $(f^{-1}\{0\}, 0)$, there are expressions like

$$Z(f, T) = \sum_{I \subset J} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^{\circ}, \tilde{\mu}] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

Two connections with monodromy:

Monodromy Conjecture (Igusa):

If $s_o = \frac{\nu}{N}$ is a pole of the topological zeta function $\chi_{top}(Z^{naive}(f, \mathbb{L}^s))$ then $e^{-2\pi i s_o}$ is an eigenvalue of the monodromy of f at some point of $f^{-1}\{0\}$.

The **Motivic Milnor Fibre** is defined as

$$\mathcal{S}_{f,x} := -\lim_{T\to\infty} Z(f,T) \in K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}].$$

Theorem (Denef-Loeser)

$$\chi_h(F_{f,x}) = \chi_h(S_{f,x}) \in K_0(HS^{mon})$$

$\Longrightarrow S_{f,x}$ recovers cohomological invariants of $F_{f,x}$; e.g.

$$hsp(f,x) = Sp((-1)^d(S_{f,x} - 1))$$

Example: $f = (y^2 - x^3)^2 - x^5 y$.

Formulas by Loeser and Veys

$$\chi_{top}(Z^{naive}(f, \mathbb{L}^s)) = \frac{10}{5+12s} - \frac{1}{15}\left(14\frac{11}{11+26s} + \frac{1}{1+s}\right)$$

Formula by Guibert

$$S_{f,0} = [\{(x,y) \in \mathbb{C}^2 \mid (y^2 - x^3)^2 = 1\}]$$
$$+[\{(x,y) \in \mathbb{C}^2 \mid y^2 - x^{13} = 1\}] - [\mu_2]\mathbb{L} - \mathbb{L} + 1$$

A germ $(S,0) \subset (\mathbb{A}^{d+1}_{\mathbb{C}},0)$ of complex hypersurface is **quasi-ordinary** if there exists a proper and finite morphism onto $(\mathbb{A}^{d}_{\mathbb{C}},0)$ whose discriminant lies on $x_1 \cdots x_d = 0$.

Theorem (Abhyankar-Jung)

Let $f \in \mathbb{C}\{x_1,...,x_d\}[Y]$ be an quasi-ordinary polynomial with $n = \deg f$. Then, there exits a root $\xi \in \mathbb{C}\{x_1^{1/n},...,x_d^{1/n}\}$ of f.

Characteristic Exponents:

$$\xi^{(s)} - \xi^{(t)} = X^{\lambda_{st}} H_{st} \text{ with } \lambda_{st} \in \frac{1}{n}\mathbb{Z}^d.$$

where $X^{\lambda_{st}} := x_1^{\lambda_{st,1}} \cdots x_d^{\lambda_{st,d}}$ and $H_{st}(0) \neq 0$.

Theorem (Lipman-Gau)

If f is a irreducible quasi-ordinary polynomial, the finite set of characteristic exponents is totally ordered

$$\lambda_1 < \lambda_2 < \cdots < \lambda_g.$$

Furthermore, the (normalized) characteristic exponents determine and are determined by the embedded topological type.

Examples of quasi-ordinary hypersurface singularities

- Plane algebraic curve singularities
- Whitney Umbrella: $z^2 - x^2 y \quad (\lambda_1 = (1, 1/2))$
- $(z^2 - xy^3)^4 - x^4 y^{13} \quad (\lambda_1 = (1/2, 3/2), \lambda_2 = (1/2, 7/4))$
- Also generalize some aspects of toric singularities: The normalization is an affine normal toric variety (González Pérez).

Theorem (González-Pérez & —)

The naive and motivic zeta functions $Z(f, T)$, the motivic Milnor fibre $S_{f,0}$ and the spectrum $hsp(f, 0)$ of an irreducible quasi-ordinary hypersurface singularity are determined by the embedded topological type.

Explicit formulas for all these invariants in terms of the characteristic exponents.

Corolary (Budur, González-Pérez & —)

The log canonical threshold of an irreducible quasi-ordinary hypersurface singularity is also determined by the embedded topological type.

Previous work:

- For plane curve sinsularities ($d = 1$) by Guibert.
- Artal, Cassou-Nogués, Luengo & Melle studied the "poles" of the naive zeta function of quasi-ordinary singularities with help of the arc space and **Newton transformations**.

The computation of the motivic zeta function requires another approach.

Our Tools:

- Describe the contact of the arcs with the hypersurface with help of the **toric embedded resolution**.
- Use **semi-roots of** f to subdivide the space of arcs.
- Measure the subsets of arcs with help of **change of variables formula** of Kontsevich, Denef & Loeser.

Embedded resolution of irreducible quasi-ordinary hypersurfaces (González-Pérez)

$$W \xrightarrow{\pi_2} U \xrightarrow{\pi_1} \mathbb{A}_{\mathbb{C}}^{d+1}$$

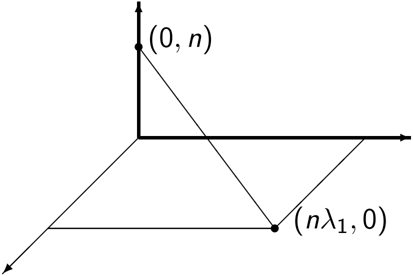
- π_1 **canonical partial resolution**, sequence of g toric morphisms determined by the characteristic exponents.
- U singular space (toric singularities)
- π_2 standard **toric embedded resolution** of U.

Canonical partial resolution

The **Newton polygon** of f is determined by λ_1 and $n = deg_y f$

$$f = (Y^{n_1} - x^{n_1\lambda_1})^{e_1} + \cdots \quad \text{where } x^{n_1\lambda_1} := x_1^{n_1\lambda_{1,1}} \cdots x_d^{n_1\lambda_{1,d}}$$

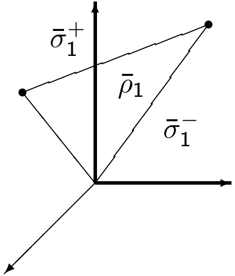
Denote $M_0 = \mathbb{Z}^d$, $\quad M_0' = M_0 \times \mathbb{Z}$
$M_1 = M_0 + \lambda_1\mathbb{Z}$, $\quad n_1 = [M_1 : M_0]$ and $n = n_1 e_1$.



Denote $\rho = \mathbb{R}^d_{\geq 0}$ $\qquad \rho' = \rho \times \mathbb{R}_{\geq 0}$
N_0 and N_0' dual lattices to M_0 and M_0'.

The Newton polygon induces a **dual subdivision** $\Sigma_1 = \{\bar{\sigma}_1^+, \bar{\sigma}_1^-, \bar{\rho}_1\}$ of the cone ρ'. It is rational with respect to N_0'.

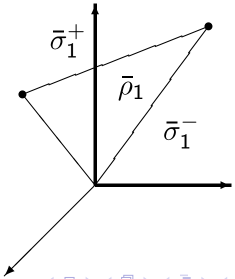
It gives a **toric morphism** $\Pi_1 : Z_1 \to Z_0 = \mathbb{C}^{d+1}$



The affine toric subvariety $Z_{\bar{\rho}_1, N_0'}$ of Z_1 associated to $\bar{\rho}_1$ has a product structure: $Z_{\bar{\rho}_1, N_0'} \equiv Z_{\rho, N_1} \times \mathbb{C}^*$

The restriction of the projection $Z_{\bar{\rho}_1, N_0'} \equiv Z_{\rho, N_1} \times \mathbb{C}^* \to Z_{\rho, N_1}$ to the strict transform $(S^{(1)}, o_1)$ of $(S, o) = (f^{-1}\{0\}, o)$ is an unramified finite cover over the torus T_{N_1} of Z_{ρ, N_1}.

$(S^{(1)}, o_1)$ is a **toric quasi-ordinary** hypersurface singularity with $g - 1$ characteristic exponents $\lambda_2 - \lambda_1, \ldots, \lambda_g - \lambda_1$



Previous construction gives a first canonical morphism determined by λ_1 and $n = deg_Y f$.

Iterating we get the **canonical partial resolution**

$$\pi_1 : U \longrightarrow \mathbb{C}^{d+1}$$

The strict transform $(S^{(g)}, o)$ of (S, o) is isomorphic to the normal affine variety Z_{ρ, N_g}; i.e. it is the normalization.

Semiroots

We associate to f a set of **semi-roots**

$$Y = f_0,\ f_1,\ \ldots,\ f_g = f \in \mathbb{C}\{\mathbf{x}\}[Y],$$

parametrized by **truncations** of a root $\xi(x_1^{1/n}, \ldots, x_d^{1/n})$ of f.

Example:

$$f = (Y^2 - x_1 x_2^3)^4 - x_1^4 x_2^{13} \qquad (\lambda_1 = (1/2, 3/2), \lambda_2 = (1/2, 7/4))$$

$$f_0 = Y \quad f_1 = Y^2 - x_1 x_2^3 \quad f_2 = f$$

Classifying arcs

Given an arc φ we measure its **contact order** with the coordinate hyperplanes and the semi-roots.

$$\mathbf{ord}(\mathbf{x},\mathbf{f})(\varphi) = (\mathrm{ord}_t(x_1 \circ \varphi), \ldots, \mathrm{ord}_t(x_d \circ \varphi), \mathrm{ord}_t(Y \circ \varphi),$$

$$\mathrm{ord}_t(f_1 \circ \varphi), \mathrm{ord}_t(f_2 \circ \varphi), \ldots, \mathrm{ord}_t(f_g \circ \varphi)) \in \mathbb{Z}_{\geq 0}^{d+g+1}.$$

Rewrite

$$Z(f,T) = \sum_{\mathbf{k} \in \mathbb{Z}_{>0}^{d+g+1}} \mu(H_{\mathbf{k},1}) T^{k_{d+g+1}}$$

with

$$H_{\mathbf{k},1} := \{\varphi \mid \mathbf{ord}(\mathbf{x},\mathbf{f})(\varphi) = \mathbf{k}, ac(f \circ \varphi) = 1\} \text{ with } \mathbf{k} \in \mathbb{Z}_{\geq 0}^{d+g+1}$$

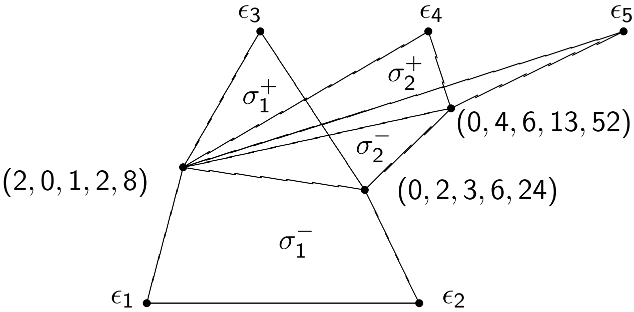
The indices **k** such that $H_{\mathbf{k},1} \neq \emptyset$ are the integer points of the interior of a $d+1$ **dimensional fan** $\Theta \subset \mathbb{R}_{\geq 0}^{d+g+1}$.

$$\Theta = \bigsqcup_{j=1}^{g+1} \Theta_j \qquad \Theta_j = \{\sigma_j^+, \sigma_j^-, \rho_j\} \qquad \Theta_{g+1} = \sigma_{g+1}$$

$$\dim \sigma_j^\bullet = d+1, \qquad \dim \rho_j = d$$

Θ is determined by the characteristic exponents.

$$Z(f,T)_0 = \sum_{j=1}^{g+1} \sum_{\theta \in \Theta_j} \sum_{k \in \mathring{\theta} \cap \mathbf{Z}^{d+g+1}} \mu(H_{k,1}) T^{\eta(k)}.$$



Projectivization of the fan $\Theta \subset \mathbf{Z}^5_{\geq 0}$ of $f = (z^2 - xy^3)^4 + x^4y^{13}$.

With help of the toric resolution π_2 and the **Change of Variable Theorem** of Kontsevitch and Denef & Loeser compute the **motivic measure** of $H_{\mathbf{k},1}$

$$\mu_{mot}(H_{\mathbf{k},1}) = c_1(\sigma) \cdot \mathbb{L}^{-\xi_\sigma(\mathbf{k})}$$

where σ is the unique cone of Θ such that **k** belongs to its interior.

- $c_1(\sigma) \in K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$.
- $\xi_\sigma(\mathbf{k})$ is a linear function on **k**.
- Both $c_1(\sigma)$ and ξ_σ depend only on $\sigma \subset \Theta$ and can be expressed in terms of the characteristic exponents.

Fixed $\mathbf{k}$ there are uniques j and $\theta \in \Theta_j$ with $\mathbf{k} \in \mathrm{int}(\theta) \cap \mathbb{Z}^{d+g+1}$.

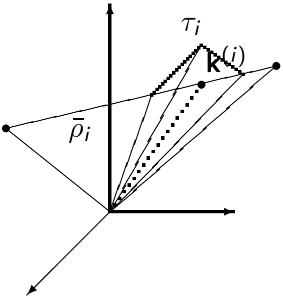
$\mathbf{k}$ determines a unique element $\mathbf{k}^{(i)} \in \bar{\rho}_i \cap N'_{i-1}$ for all $i < j$.

Take a regular subdivision Σ'_i of Σ_i with a cone τ_i as in the figure

$$\dim \tau_i \cap \bar{\rho}_i = d$$

$$H^{(i)}_{\mathbf{k},1} := \{\varphi \mid (\pi_{\tau_i} \circ \cdots \circ \pi_{\tau_1})(\varphi) \in H_{\mathbf{k},1}\}$$

By Change of variables
$\mu_{mot}(H_{\mathbf{k},1}) = \int_{H^{(i)}_{\mathbf{k},1}} \mathbb{L}^{-ord_t Jac(\pi_{\tau_i} \circ \cdots \circ \pi_{\tau_1})} d\mu_{mot} =$
$c_1(\theta)(\mathbb{L}-1)^{\dim \theta - 1} \mathbb{L}^{-\xi_j(\mathbf{k})}$



Finally

$$Z(f,T) = \sum_{\mathbf{k}\in\mathbb{Z}_{>0}^{d+g+1}} \mu(H_{\mathbf{k},1})T^{k_{d+g+1}} = \sum_{\sigma\in\Theta} c_1(\sigma)\cdot S_\sigma$$

where S_σ are **rational functions of degree** 0 **w.r.t.** T

S_σ are calculated with help of the generating functions Φ_σ of the cone $\sigma \in \Theta$ and the linear form $\xi_\sigma(\mathbf{k})$.

Example:

$$f = (z^2 - xy^3)^4 - x^4y^{13}$$

$$\lambda_1 = (1/2, 3/2),\ \lambda_2 = (1/2, 7/4)$$

The motivic Milnor fibre is

$$S_{f,0} = c_1(\rho_1)(1-\mathbf{L}) + [\mu_8]\mathbf{L} + c_1(\rho_2)(1-\mathbf{L}) - [\mu_4]\mathbf{L}(1-\mathbf{L}) + (1-\mathbf{L})^2.$$

with

$$c_1(\rho_1) = [\{(x,y) \in (\mathbb{C}^*)^2 | (y^2 - x)^4 = 1\}]$$

and

$$c_1(\rho_2) = [\{(x,y) \in (\mathbb{C}^*)^2 | y^4 - x = 1\}].$$

The Hodge-Steenbrink spectrum is

$$hsp(f,0) = \frac{1-t}{1-t^{1/8}}t - t = t^{\frac{9}{8}} + \cdots + t^{\frac{15}{8}}$$

Log canonical threshold

$\mu : Y \to U \subset \mathbb{C}^{d+1}$ a log resolution of f

E_i for $i \in J$ be the irreducible components of $\mu^{-1}(0)$.
a_i the order of vanishing of $f \circ \mu$ along E_i,
k_i the order of vanishing of $\det(\mathrm{Jac})_\mu$ along E_i

The **log canonical threshold of f at the origin** is defined as

$$\mathrm{lct}_0(f) := \min \left\{ \frac{k_i + 1}{a_i} \mid i \in J \right\}.$$

f is **log canonical at** 0 iff $\mathrm{lct}_0(f) = 1$.

Theorem (Budur, González-Pérez & —)

Let $f \in \mathbb{C}\{x_1, \dots, x_d\}[y]$ be a quasi-ordinary irreducible polynomial.

- *The number $lct_0(f)$ is determined by the embedded topological type of the germ defined by $f = 0$ at the origin.*
- *f is log canonical if and only if $g = 1$ and either $\lambda_{1,i} \in \{1, \frac{1}{2}\}$ or $\lambda_{1,1} = \frac{1}{n_1}$.*
- $lct_0(f) = \begin{cases} \min\{1, A_1\} & \text{if} \quad \lambda_{1,1} \neq \frac{1}{n_1}, \quad \text{or if} \quad g = 1, \\ \min\{A_2, A_3\} & \text{if} \quad \lambda_{1,1} = \frac{1}{n_1}, \quad g > 1. \end{cases}$
  where

$$A_1 = \frac{1+\lambda_{1,1}}{e_0\lambda_{1,1}},\ A_2 = \frac{n_1(1+\lambda_{2,1})}{e_1(n_1(1+\lambda_{2,1})-1)} \text{ and } A_3 = \frac{1+\lambda_{2,\ell}}{e_1\lambda_{2,\ell}}.$$

Computation of $\mathrm{lct}_0(f)$

Theorem (Halle and Nicaise / Veys and Zuñiga-Galindo)

The biggest pole of $Z_{mot,f}(\mathbb{L}^{-s})_0$ is equal to $-\mathrm{lct}_0(f)$.

Proposition (González-Pérez & —)

If $f \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ is an irreducible quasi-ordinary polynomial then the poles of $Z_{mot,f}(\mathbb{L}^{-s})_0$ are contained in the set $\{-\frac{\xi_\sigma(k)}{k_{d+g+1}} \mid k$ is generating vector of $\sigma, \sigma \in \Theta\}$.

References:

P.D. González-Pérez & —, *Motivic Milnor fiber of a Quasi-Ordinary Hypersurface*, `arXiv:1105.2480`

N. Budur, P.D. González-Pérez & —, *Log Canonical Thresholds of Quasi-Ordinary Hypersurface Singularities*, `arXiv:1105.2794`

Thank you for your attention!!