KZ connections and generalized Brauer algebras.

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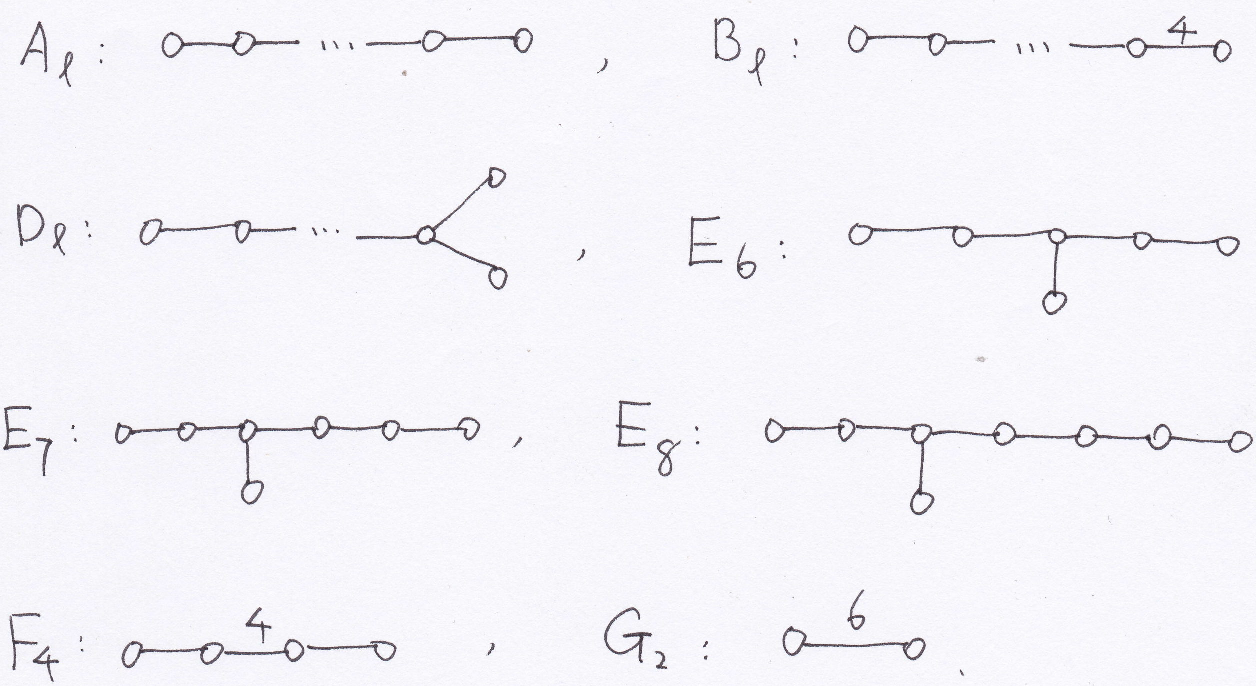
1. Dynkin Diagram, Coxeter Matrix.



Coxeter Matrix :

$\Gamma = (m_{ij})_{n \times n}$:

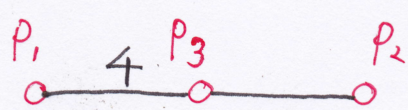
- $m_{ij} = m_{ji}$;
- $m_{ii} = 1$;
- $m_{ij} \in \{2, 3, 4, \dots, \infty\}$, $i \neq j$.

$\Downarrow$

diagram Γ :

ex : $\Gamma = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{pmatrix}$,

diagram :



2 Coxeter Groups, Hecke algebras.

$\Gamma = (m_{ij})_{n\times n}$.

- $W_\Gamma = \langle s_1, \dots, s_n \mid \underbrace{s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_i \cdots}_{m_{ij}} \text{ for } i \neq j,\ s_i^2 = 1 \rangle$.

- $A_\Gamma = \langle \delta_1, \dots, \delta_n \mid \underbrace{\delta_i \delta_j \cdots}_{m_{ij}} = \underbrace{\delta_j \delta_i \cdots}_{m_{ij}} \text{ for } i \neq j \rangle$.

- $H_\Gamma(q) = \langle s_1, \dots, s_n \mid s_i s_j \cdots = s_j s_i \cdots \text{ for } i \neq j,\ (s_i - 1)(s_i + q) = 0 \rangle$.

Rem: 1. When Γ is finite type, $\dim H_\Gamma(q) = \# W_\Gamma$.

2. For generic q, when Γ is finite type.
$H_\Gamma(q) \cong \mathbb{C} W_\Gamma$. be semi simple, so there is a bijection: (In this sense, W_Γ is "deformable".).

$$\{\text{Irre Reps of } H_\Gamma(q)\} \xleftrightarrow{1\sim1} \{\text{Irre Reps of } W_\Gamma\}$$

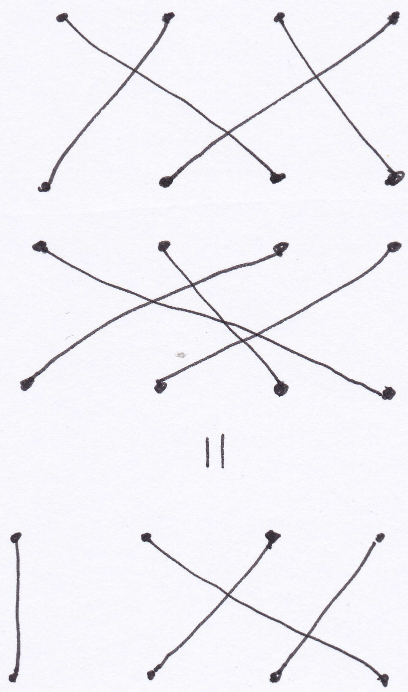
this can be described by Cherednik's KZ equation.

3. $H_\Gamma(q) = \mathbb{C} A_\Gamma / \sim$.

3. S_n. Brauer Algebras $B_n(\tau)$. BMW algebras $B_n(\tau,\ell)$.

Diagram Representation of permutations:

$$\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix} \longrightarrow$$



$$\times$$

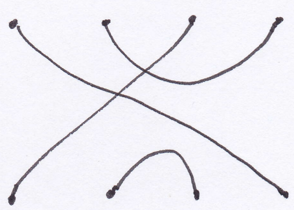
$$\begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix}$$

$$\|$$

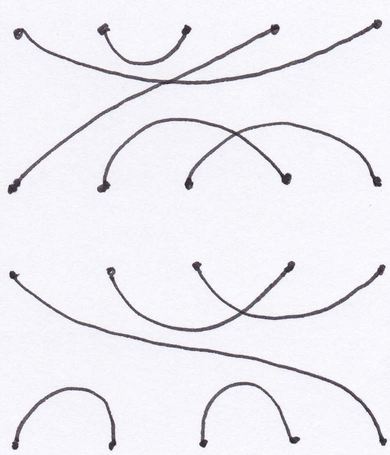
$$\begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}$$

$B_n(\tau)$ is spanned by more diagrams like:

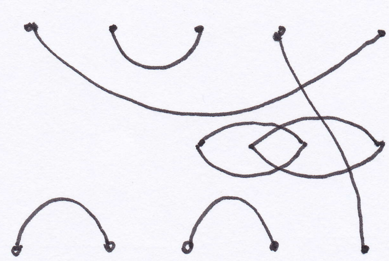
$\dim = (2n-1)!!$



, example of multiplication:



$=$



$= \tau^2$

cannonical presentation of $B_n(\tau)$, $B_n(\tau, \ell)$.

Recall presentation of S_n:

s_1 s_2 s_{n-2} s_{n-1}

$s_i =$ 

:

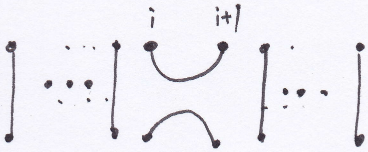
- $s_i s_j = s_j s_i$ $\quad |i-j|>1$
- $s_i s_j s_i = s_j s_i s_j$ $\quad |i-j|=1$.
- $s_i^2 = 1$. $\quad \forall i$

of $B_n(\tau)$:

s_1 s_2 s_{n-2} s_{n-1}

e_1 e_2 e_{n-2} e_{n-1}

where:

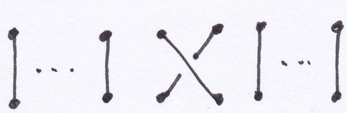
$e_i =$ 

- $s_i s_j = s_j s_i$; $s_i e_j = e_j s_i$; $e_i e_j = e_j e_i$; $\quad |i-j|>1$.
- $s_i s_j s_i = s_j s_i s_j$; $s_i s_j e_i = e_j s_i s_j$; $e_i s_j e_i = e_i$; $\quad |i-j|=1$.
- $s_i^2 = 1$; $s_i e_i = e_i s_i = e_i$; $e_i^2 = \tau e_i$ $\quad \forall i$

of $B_n(\tau, \ell)$.

X_1 X_2 ... X_{n-2} X_{n-1}

E_1 E_2 E_{n-2} E_{n-1}.

where: $X_i =$ 

$E_i =$ 

- $X_i X_j = X_j X_i$; $X_i E_j = E_j X_i$; $E_i E_j = E_j E_i$; $\quad |i-j|>1$.
- $X_i X_j X_i = X_j X_i X_j$; $X_i X_j E_i = E_j X_i X_j$; $E_i X_j E_i = \ell E_i$; $\quad |i-j|=1$.
- $\ell(X_i^2 + mX_i - 1) = mE_i$; Kauffman skein; $X_i E_i = \ell^{-1} E_i$; $E_i^2 = \tau E_i$. $\quad \forall i$

$m := \frac{\ell - \ell^{-1}}{1-\tau}$.

Rem: 1. $B_n(\tau)$ is semisimple iff $\tau \notin \mathbb{Z}$ or

2. For generic τ, ℓ. $B_n(\tau, \ell) \cong B_n(\tau)$. So we have a bije:

$$\left\{ \text{Irre Reps of } B_n(\tau) \right\} \xrightarrow{1\sim 1} \left\{ \text{Irre Reps of } B_n(\tau, \ell) \right\}.$$

3. $B_n(\tau, \ell) = \mathbb{C} B_n / \sim$.

4. Generalization to simply laced cases (Cohen, Wales ··· 2005.)

$\Gamma = (m_{ij})_{n\times n}$: • $m_{ij} \in \{2, 3\}$ if $i \neq j$.

$B_\Gamma(\tau)$		$B_\Gamma(\tau, \ell)$	
$s_1 \ s_2 \ \cdots \ s_n$ $e_1 \ e_2 \ \cdots \ e_n$		$X_1 \ X_2 \ \cdots \ X_n$ $E_1 \ E_2 \ \cdots \ E_n$	
$s_i s_j = s_j s_i$; $s_i e_j = e_j s_i$; $e_i e_j = e_j e_i$;	$m_{ij} = 2$	$X_i X_j = X_j X_i$; $X_i E_j = E_j X_i$; $E_i E_j = E_j E_i$;	$m_{ij} = 2$.
$s_i s_j s_i = s_j s_i s_j$; $s_i s_j e_i = e_j s_i s_j$; $e_i s_j e_i = e_i$;	$m_{ij} = 3$.	$X_i X_j X_i = X_j X_i X_j$; $X_i X_j E_i = E_j X_i X_j$; $E_i X_j E_i = \ell E_i$;	$m_{ij} = 3$.
$s_i^2 = 1$; $s_i e_i = e_i s_i = e_i$; $e_i^2 = \tau e_i$.	$\forall i$.	$\ell(X_i^2 + mX_i - 1) = mE_i$; $X_i E_i = \ell^{-1} E_i$; $E_i^2 = \tau E_i$;	$\forall i$

Rem: The following are proved (Cohen, Wales ···)
For finite type Γ ;

- $B_\Gamma(\tau)$, $B_\Gamma(\tau, \ell)$ are semisimple for generic τ, ℓ ;
- $\dim B_\Gamma(\tau, \ell) = \dim B_\Gamma(\tau) < \infty$;
- $B_\Gamma(\tau)$, $B_\Gamma(\tau, \ell)$ has cellular structure.

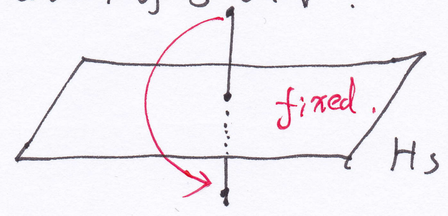
problem: For general Γ.

?	$m_{ij} = 4, 5 \cdots$	?	$m_{ij} = 4, 5 \cdots$

5. Complementary spaces. Artin Groups.

Reflection $s \in GL(V)$, $V \cong \mathbb{R}^n$: $s \sim \mathrm{diag}(-1, 1, \dots, 1)$; $H_s :=$ 1-eigenspace of s.

action of s on V :



Reflection Group finite $W \subseteq GL(V)$: generated by reflections :

We have a ~~rough~~ 1-1 correspondence:

$$\left\{ \begin{array}{c} \text{Irreducible} \\ \text{reflection} \\ \text{Groups} \end{array} \right\} \underset{\text{Tit's Geometric Representation}}{\overset{\text{Weyl chamber}}{\rightleftarrows}} \left\{ \begin{array}{c} \text{finite type} \\ \text{coxeter groups} \end{array} \right\}$$

Suppose Γ is finite type. $W_\Gamma \curvearrowright V \cong \mathbb{R}^n$ as a reflection group.

Let R be the set of reflections in W_Γ.

$$M_\Gamma := V^{\mathbb{C}} \setminus \bigcup_{s \in R} H_s^{\mathbb{C}} .$$

$$W_\Gamma \overset{\text{freely}}{\curvearrowright} M_\Gamma$$

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$$\boxed{\text{Thm (Brieskorn 70')} \quad \pi_1(M_\Gamma / W_\Gamma) \cong A_\Gamma .}$$

6. KZ connections for Hecke algebras.

$\mathcal{A}$: arrangement in $V \cong \mathbb{C}^n$. $M_{\mathcal{A}} := V \setminus \bigcup_{i \in I} H_i$

$\overset{\shortparallel}{\{H_i\}}_{i \in I}$. $\forall i \in I$, choose ~~linear~~ deg 1 function α_i on V.

s.t: $H_i = \ker(\alpha_i)$

set: $\omega_i = d\alpha_i / \alpha_i$: closed, holomorphic 1-form on $M_{\mathcal{A}}$.

$E \cong \mathbb{C}^m$, $\forall i$. $f_i \in \mathrm{End}(E)$.

$\omega = \sum_i f_i \omega_i$: connection on $M_{\mathcal{A}} \times E$.

Thm (Kohno 80'). ω is flat iff:
$\forall$ codim 2 edge L, Let $\{H_{i_1}, \dots, H_{i_k}\} = \{H_i \mid i \in I, H_i \supset L\}$
then: $[f_{i_1} + f_{i_2} + \dots + f_{i_k}, f_{i_\nu}] = 0 \quad \forall \nu$.

when:

$$\pi_1(M_{\mathcal{A}}) \xrightarrow{\text{monodromy}} \mathrm{Aut}(E).$$

Now. Let $\mathcal{A} = \{H_s\}_{s \in R}$, Let (E, ρ) be a finite dimensional representation of W_Γ. We have a connection:

$$\boxed{\overline{\omega} = k \sum_{s \in R} \rho(s)\, \omega_s}$$

Cherednik.

on $M_\Gamma \times E$.

Thm (Cherednik). $\overline{\omega}$ is flat and W_Γ-invariant. So it further induces a flat connection ω on the quotient bundle: $M_\Gamma \times_{W_\Gamma} E$. the monodromy of ω factor through $H_\Gamma(q)$.

$$\begin{tikzcd}
A_\Gamma \cong \pi_1(M_\Gamma) \arrow[rr] \arrow[dr] & & \mathrm{Aut}(E) \\
& H_\Gamma(q) \arrow[ur, dashed] &
\end{tikzcd}$$

Formally:

$$\omega_n := \kappa \sum_{s \in R} s \cdot \omega_s .$$

7. KZ connection for BMW algebras.

$$M_\Gamma = \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \} =: X_n .$$

$$s_{ij} \longmapsto \omega_{ij} = \frac{dz_i - dz_j}{z_i - z_j} .$$

$$\boxed{\widetilde{\omega}_n := k \sum_{1 \le i < j \le n} (s_{ij} - e_{ij})\, \omega_{ij}}$$ (Ivan Marin)

- $\widetilde{\omega}_n$ can deform every rep of $B_n(\tau)$ to a rep of $B_n(\tau, l)$ similarly.

- Flatness:

$$[\, s_{ik} - e_{ik},\ s_{ij} - e_{ij} + s_{jk} - e_{jk} \,]$$

$$= (e_{ik} e_{ij} - e_{ij} e_{ik} + e_{ik} e_{jk} - e_{jk} e_{ik}$$
$$- e_{ik} s_{ij} + s_{ij} e_{ik} - e_{ik} s_{jk} + s_{jk} e_{ik} .$$

$$= (e_{ik} e_{ij} - e_{ik} s_{jk}) - (e_{ij} e_{ik} - s_{jk} e_{ik}) + (e_{ik} e_{jk} - e_{ik} s_{ij}) - (e_{jk} e_{ik} - s_{ij} e_{ik})$$

Hypothesis I. Let $\Gamma = (m_{ij})_{n\times n}$ be any finite type coxeter Matrix. Let R be the set of reflections in W_Γ. then $B_\Gamma(\tau)$ is generated by $R \cup \{e_s\}_{s\in R}$ such that the formal connection:

$$\boxed{\widetilde{\omega}_\Gamma = \kappa \sum_{s\in R} (s - e_s)\,\omega_s}$$

is flat and W_Γ-invariant.

The theorem will give some relations between $R \cup \{e_s\}_{s\in R}$. but they are too coarse.

9. Lawrence-Krammer representations.

Let Y_n be configuration space of n different points on $\mathbb{C}$.

$\bigcup_{\bar{z} \in Y_n} U_{\bar{z}}$ ← certan local system homology $\mathbb{C}$ groups of discriminantal arrangement complements.

$$B_n = \pi_1(Y_n) \xrightarrow{\text{monodromy}} \mathrm{Aut}(U_{\bar{z}_0})$$

- $\dim U_{\bar{z}} = \frac{n(n-1)}{2} = \# R$ of S_n;
- Krammer proved it is faithful (Garside theory) also proved by Bigelow by using Geometric methods.

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explicit form: Basis of $U_{\bar{z}}$: $\{ x_{ij} \}_{1 \le i < j \le n}$.

$$\boxed{\begin{aligned}
&\sigma_K x_{K,K+1} = tq^2 x_{K,K+1} \\
&\sigma_K x_{i,K} = (1-q) x_{iK} + q x_{i,K+1} && i<K \\
&\sigma_K x_{i,K+1} = x_{iK} + tq^{K-i+1}(q-1) x_{K,K+1} && i<K \\
&\sigma_K x_{Kj} = tq(q-1) x_{K,K+1} + q x_{K+1,j}. && K+1<j. \\
&\sigma_K x_{K+1,j} = x_{Kj} + (1-q) x_{K+1,j}, && K+1<j. \\
&\sigma_K x_{ij} = x_{ij} && i<j<K \text{ or } K+1<i<j; \\
&\sigma_K x_{ij} = x_{ij} + tq^{K-i}(q-1)^2 x_{K,K+1}. && i<K<K+1<j.
\end{aligned}}$$

Let Γ be a simply-laced finite type Coxeter matrix. Cohen-Wales. Digne defined generalized LK representations for A_Γ.

LK_Γ.

- $\dim LK_\Gamma = \# R$ of W_Γ;
- Krammer's arguements applies to LK_Γ, so prove the faithfulness of LK_Γ, and linearity of A_Γ.

Let Γ be any simply-laced Coxeter matrix. L. Paris defined generalized LK representations for A_Γ:

- LK_Γ has a basis in 1-1 correspondence with reflections of W_Γ;
- Krammer's arguements still applies to give:

$$A_\Gamma^+ \hookrightarrow A_\Gamma.$$

11. Marin's infinitesimal Vesion and generalization.

Let W_Γ be any finite reflection group. acting on $V \cong \mathbb{C}^n$. Set R, M_Γ as before, also ω_s.

For $s, u \in R$. set:

$$\boxed{\alpha(s,u) = \#\{y \in R \mid yuy = s\}.}$$

Let $U \cong \mathbb{C}\langle v_s \rangle_{s\in R}$. For $s, u \in R$, define $t_s \in \mathrm{End}(U)$ as:

$$\boxed{\begin{aligned} t_s \cdot v_s &= m v_s \ ; \\ t_s \cdot v_u &= v_{sus} - \alpha(s,u) v_s . \quad \text{for } s \neq u. \end{aligned}}$$

Thm (Marin 2007') The connection

$$\omega_{LK} := k \sum_{s\in R} t_s \cdot \omega_s$$

on $M_\Gamma \times U$ is flat and W_Γ invariant, when Γ is simply-Laced, the monodromy gives Cohen-Wales. Digues generalized LK representation.

~~H7~~ LK and BMW:

2000. Zinno : LK is a sub rep of $B_n(\tau, \ell)$;

2005 Cohen. Wales..: simply laced LK is a sub rep of $B_\Gamma(\tau, \ell)$.

Hypothesis 2: For any Γ. $B_\Gamma(\tau)$ can be deformed to certain $B_\Gamma(\tau, \ell)$, and it contain Marin's generalized LK representation, In infinitesimal level, it is:

notice: Let (U, ρ) be the natural representation of W_Γ on U, then Marin's operator:

$$\boxed{t_s \cdot v_u = \rho(s) \cdot v_u - \underline{\underline{P_s}} \cdot v_u} \qquad t_s = \rho(s) - P_s.$$

where: $P_s \cdot v_s = (1-m) v_s$;

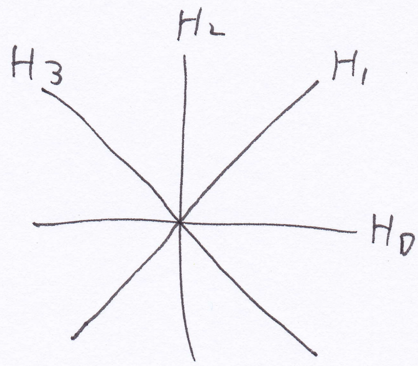
$P_s \cdot v_u = \alpha(s,u) v_s \quad (u \neq s)$

is a projector to $\mathbb{C} v_s$

$s \longmapsto \rho(s)$,
$e_s \longmapsto P_s$ should extends to a representation of $B_\Gamma(\tau)$.

Case: $\overset{4}{\circ\!\!-\!\!\!-\!\!\!-\!\!\circ}$.

Arrangement:



Flatness condition:

$$[s_0 - e_0,\ s_1 - e_1 + s_2 - e_2 + s_3 - e_3].$$

$$= e_0e_1 - e_1e_0 + e_0e_2 - e_2e_0 + e_0e_3 - e_3e_0 = 0$$
$$- e_0s_1 + s_1e_0 - e_0s_2 + s_2e_0 - e_0s_3 + s_3e_0$$

$$\|$$

$$(e_0e_1) - (e_1e_0) + (e_0e_2 - e_0s_1 - e_0s_3)$$
$$-(e_2e_0 - s_1e_0 - s_3e_0) + (e_0e_3) - (e_3e_0) = 0.$$

Notice: $H_0 \xrightarrow{NO} H_1$; $H_0 \xrightarrow{s_1, s_3} H_2$

$H_0 \xrightarrow{NO} H_3$;

Above can be written as:

$$e_{s_i} \cdot e_{s_j} = \Big(\sum_{s_k : s_k s_j s_k = s_i} s_k \Big) e_{s_j} = e_{s_i} \Big(\sum_{s_k : s_k s_j s_k = s_i} s_k \Big). \quad (*)$$

(*) is an identity making sense for any W.

If add relation (*) to $R \cup \{e_s\}_{s \in R}$, then Hypothesis 1, 2 are satisfied for any W.

Definition of $B_\Gamma(\tau)$:

generated by $R \cup \{e_s\}_{s \in R}$;

relations: (*) ;

$$\left. \begin{array}{l} e_s^2 = \tau e_s ; \\ s \cdot e_s = e_s \cdot s = e_s \end{array} \right\} \text{naturally relations.}$$

......

properties of $B_\Gamma(\tau)$:

1. For finite type Γ. $\dim B_\Gamma(\tau) < +\infty$.
2. For simply laced Γ. $B_\Gamma(\tau)$ is isomorphic to Cohen-Wales's generalized Brauer algebra.
3. For Rank 2 Γ and H_3, $B_\Gamma(\tau)$ is genericly semi-simple and cellular.

4. We have defined algebra $B_\Gamma(\tau, l)$ s.t.:

for rank 2 Γ, $\dim B_\Gamma(\tau, l) = \dim B_\Gamma(\tau)$, and $B_\Gamma(\tau, l) \cong B_\Gamma(\tau)$ for generic data;

(so rank 2 $B_\Gamma(\tau)$'s are deformable.)

for finite type Γ, $\dim B_\Gamma(\tau, l) < +\infty$.

5. By definition, $B_\Gamma(\tau, l)$ contains Marin's LK rep.

$B_\Gamma(\tau)$ and $B_\Gamma(\tau, l)$ also have canonical presentations with generating set $\{s_1, e_1, s_2, e_2, \dots, s_n, e_n\}$.

Thanks.