Singularities and Intersection Spaces: Theory and Application

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Core topic of this talk

New perspective on Poincaré duality for singular spaces

X^n: compact, oriented pseudomanifold, singular set Σ, only strata of even codimension.
L^2-cohomology (Cheeger):

- $X - \Sigma$ equipped with conical Riemannian metric.
- $\Omega^*_{(2)}(X - \Sigma) = \{\omega \mid \int \omega \wedge *\omega < \infty, \int d\omega \wedge *d\omega < \infty\}$.
- $H^*_{(2)}(X) := H^*(\Omega^*_{(2)}(X - \Sigma), d)$.
- $H^i_{(2)}(X) \otimes H^{n-i}_{(2)}(X) \to \mathbb{R}$ nondegenerate.
- $\Omega^*_{(2)}(X - \Sigma)$ is no DGA w.r.t. $\wedge$.

Intersection homology (Goresky, MacPherson)

- Perversity $\bar{p}$: $\bar{p}(2) = 0$, $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$.
- $IC_*^{\bar{p}}(X) \subset C_*(X)$: The larger the codim. k of a stratum, the more a chain is allowed to deviate from transversality.
- $IH_*^{\bar{p}}(X) := H_*(IC_*^{\bar{p}}(X), \partial)$.
- $\bar{p} + \bar{q} = (0, 1, 2, \ldots)$: $IH_i^{\bar{p}}(X; \mathbb{Q}) \otimes IH_{n-i}^{\bar{q}}(X; \mathbb{Q}) \to \mathbb{Q}$ nondegenerate.
- $IC_{\bar{p}}^*(X)$ has no $\bar{p}$-internal product.

Chain level $\to$ Space level

- E_* homology theory (Eilenberg-Steenrod.)
- Burdick, Conner, Floyd:
  E_* = homology of a chain functor $\Rightarrow$ E_* trivial.
- Exple: Bordism does not come from a chain functor.

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- X^n compact, oriented pseudomfd., perversity $\bar{p}$.
- **Program**:

$$\underset{\text{Space}}{X} \quad \rightsquigarrow \quad \underset{\text{Space}}{I^{\bar{p}}X} \quad \rightsquigarrow \quad HI_*^{\bar{p}}(X) := H_*(I^{\bar{p}}X)$$

- Poincaré Duality: $\widetilde{H}_i(I^{\bar{p}}X;\mathbb{Q}) \otimes \widetilde{H}_{n-i}(I^{\bar{q}}X;\mathbb{Q}) \to \mathbb{Q}$ nondegenerate.

Call the $I^{\bar{p}}X$ *intersection spaces* of X. (IX for $\bar{p}$ = middle.)

Construction of Intersection Spaces

$X^n = M^n \cup_{\partial M = L} \text{cone}(L)$.

- **Def.** A *stage k Moore approximation* of a CW-complex L is a CW-complex $L_{<k}$ with a map $f : L_{<k} \to L$ such that $f_* : H_r(L_{<k}) \cong H_r(L)$ for $r < k$ and $H_r(L_{<k}) = 0$ for $r \geq k$.
- P. Hilton: Exists if L is simply connected.
- Set $k = n - 1 - \bar{p}(n)$.
- $I^{\bar{p}}X := \text{cone}(L_{<k} \xrightarrow{f} L = \partial M \hookrightarrow M)$.
- Attempt fiberwise Moore approximation for nonisolated singularities ($\to$ obstructions).

Present Existence and Duality Results

- **Thm.** (-) $I^{\bar{p}}X$ with duality exists for X with isolated singularities and simply connected links.
- **Thm.** (-) $I^{\bar{p}}X$ with duality exists for depth 1 nonisolated singularities with trivializable link bundle and simply conn. links.
  Exple. Buoncristiano-Rourke-Sanderson's *framified sets* (Stone stratification, all block bundles trivialized).
- **Thm.** (Florian Gaisendrees) $I^{\bar{p}}X$ with duality exists for depth 1 twisted link bundles, spherical singular sets, simply conn. links without odd-dimensional homology, and cellular action of structure group.
- **Thm.** (-) $HI^{*}_{\bar{p}}(X;\mathbb{R})$ exists for depth 1 flat link bundles with structure group acting isometrically.

Internal Algebraic Structure

- Trivially have DGA $(C^*(I^{\bar{p}}X), d, \cup)$.
- For every $\bar{p}$: $HI_{\bar{p}}{}^i(X) \otimes HI_{\bar{p}}{}^j(X) \to HI_{\bar{p}}{}^{i+j}(X)$.
- Consequently: $HI^* \ncong IH^*$!
- Squaring operations $\mathrm{Sq}^i : HI_{\bar{p}}^j(X; \mathbb{Z}/_2) \to HI_{\bar{p}}^{j+i}(X; \mathbb{Z}/_2)$.

Theorem

For $\dim X = n \equiv 0(4)$, the intersection forms on $HI_{n/2}(X)$ and $IH_{n/2}(X)$ agree in the Witt-group $W(\mathbb{Q})$ of the rationals (symmetr. nondegenerate bilinear forms).

Generalized homology theories and intersection spaces

$$E \text{ spectrum} \rightsquigarrow EI^{\bar{p}}_*(X) := E_*(I^{\bar{p}}X).$$

Theorem (J. F. Adams)

Let E be a ring spectrum and M be a closed, E^*-oriented manifold with orientation class $[M]_E \in E^n(M \times M, M \times M - \Delta)$. Then

$$E_i(M) \cong E^{n-i}(M), \; x \mapsto [M]_E/x.$$

Theorem (M. Spiegel)

Let K be complex K-theory and X a compact, K^*-oriented pseudomanifold with isolated singularities. If $H_*(\text{Links})$ is torsion-free, then

$$KI^{\bar{p}}_i(X) \cong KI^{n-i}_{\bar{q}}(X) \qquad (\textit{integrally}).$$

Fails for $\operatorname{Tors} H_*(\text{Links}) \neq 0$, and for KO.

De Rham Description

- $X \supset \Sigma$ (2 strata, oriented).
- Assumptions: link bundle $L^m \to E \to \Sigma$ flat, L Riemannian, structure group acts isometrically.
- Flat link bundles arise in:
  - Foliated stratified spaces (M. Saralegi-Aranguren, R. A. Wolak),
  - Reductive Borel-Serre compactifications of locally symmetric spaces (nilmanifold fibrations).
- Codifferential $d^* : \Omega^k(L) \to \Omega^{k-1}(L)$.
- Cotruncation $\tau_{\geq k}\Omega^*(L) = \cdots \to 0 \to \ker d^* \to \Omega^{k+1}(L) \to \Omega^{k+2}(L) \to \cdots$.

De Rham Description

- Use fiberwise cotruncation.
- $U \subset \Sigma$ open, small, $U \overset{\pi_1}{\leftarrow} U \times L \overset{\pi_2}{\rightarrow} L$, $k = m - \bar{p}(m+1)$.

Definition

$\Omega I^*_{\bar{p}}(X) \subset \Omega^*(X - \Sigma)$: Near the end of $X - \Sigma$, ω looks locally in the boundary direction like

$$\sum \pi_1^* \eta_i \wedge \pi_2^* \gamma_i,$$

$\eta_i \in \Omega^*(U)$, $\gamma_i \in \tau_{\geq k} \Omega^*(L)$.

- Invariantly defined by flatness, isometric action.
- Set $HI^*_{\bar{p},dR}(X) = H^*(\Omega I^*_{\bar{p}}(X))$.

De Rham Description and DGA structure

Theorem (-)

1. *For* $\Sigma = \mathrm{pt}$, $HI^*_{\bar{p},dR}(X) \cong \widetilde{HI}^*_{\bar{p}}(X;\mathbb{R})$.
2. *Poincaré Duality:*

$$HI^i_{\bar{p},dR}(X) \times HI^{n-i}_{\bar{q},dR}(X) \to \mathbb{R}, \ (\omega,\eta) \mapsto \int_{X-\Sigma} \omega \wedge \eta,$$

   is nondegenerate.

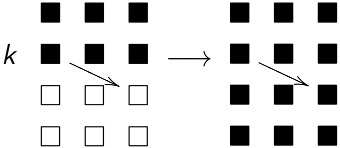
3. $(\Omega I^*_{\bar{p}}(X), d, \wedge)$ *is a DGA.*

For 3, observe that $(\tau_{\geq k}\Omega^*(L), \wedge)$ is a subalgebra of $(\Omega^*(L), \wedge)$, but $\tau_{\leq k}\Omega^*(L)$ is not.

Application: Leray-Serre Spectral Sequence of Flat Bundles

$F \to E \to B$ flat, F Riemannian and orientable, structure group acts isometrically.
$\Omega^*_{\mathrm{MS}} E \subset \Omega^* E$: multiplicatively structured forms as above, H^*-isom.
$\mathrm{ft}_{\geq k}\, \Omega^*_{\mathrm{MS}} E \subset \Omega^*_{\mathrm{MS}} E$: fiberwise cotruncated forms as above.

E_2 :



Theorem (-)

The Leray-Serre spectral sequence with $\mathbb{R}$ coefficients of a flat, smooth, isometrically structured fiber bundle of smooth, closed manifolds collapses at E_2.

Flatness alone does not suffice! (Counterexamples, flat sphere bundles with nontrivial Euler class, $\to$ Milnor).

Milnor: On the Existence of a Connection with Curvature Zero

As a consequence to the previous theorem, we obtain:

Corollary (Milnor 1958)

The sphere bundle with structure group $\mathrm{SO}(n)$ *over a smooth, closed manifold* B *induced by any homomorphism* $\pi_1(B) \to \mathrm{SO}(n)$ *has trivial Euler class with real coefficients.*

Though these results are of a topological nature, Milnor's proof relies on Chern-Weil theory.

Application: Equivariant Cohomology

Theorem (-)

- *M be an oriented, closed, Riemannian manifold,*
- *G a discrete group, whose $K(G,1)$ may be taken to be a closed, smooth manifold,*
- *isometric action of G on M,*
- *Then:*

$$H^k_G(M;\mathbb{R}) \cong \bigoplus_{p+q=k} H^p(G;\mathbf{H}^q(M;\mathbb{R})).$$

Exples. $G = \mathbb{Z}^n$, π_1 of closed manifolds with non-positive sectional curvature, surfaces other than $\mathbb{R}P^2$, infinite π_1 of irreducible, closed, orientable 3-manifolds, torsionfree discrete subgroups of almost connected Lie groups, certain groups arising from Gromov's hyperbolization technique.

Rem. Action need not be proper, M is usually not a G-CW-complex.

Analytic Description

An analytic description of HI^* remains to be found. Shall indicate a partial result.

- $X^n = M^n \cup_{\partial M} \text{cone}(\partial M)$.
- x a boundary-defining function on M, h a metric on ∂M.
- A Riemannian metric g on the interior N of M is a *scattering metric* if near ∂M it has the form

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2}.$$

- $L^2\mathcal{H}^*(N, g) :=$ Hodge cohomology space of L^2-harmonic forms on N.

Analytic Description

Theorem (Melrose, Hausel-Hunsicker-Mazzeo)

If g is a scattering metric, then there are natural isomorphisms

$$L^2\mathcal{H}^k(N,g) \longrightarrow \begin{cases} H^k(M,\partial M), & k < n/2, \\ \mathrm{Im}(H^k(M,\partial M) \to H^k(M)), & k = n/2, \\ H^k(M), & k > n/2. \end{cases}$$

Proposition (-, Hunsicker)

If N is endowed with a scattering metric g and the restriction map $H^{n/2}(M) \to H^{n/2}(\partial M)$ is zero (a "Witt-type" condition), then

$$HI^*(X) \cong L^2\mathcal{H}^*(N,g).$$

Algebraic Geometry

- Consider the Calabi-Yau quintic

$$V_s = \{z_0^5 + \ldots + z_4^5 - 5(1+s)z_0 \cdots z_4 = 0\} \subset \mathbb{P}^4,$$

  depending on a complex structure parameter s.
- V_s is smooth for small $s \neq 0$.
- $V = V_0$ has 125 isolated singularities.
-

i	rk $H_i(V_s)$	rk $H_i(V)$	rk $IH_i(V)$
2	1	1	25
3	204	103	2
4	1	25	25

- The table shows that neither ordinary homology nor intersection homology are stable under the smoothing of V.
- But:

$$\text{rk } HI_2(V) = 1, \;\; \text{rk } HI_3(V) = 204, \;\; \text{rk } HI_4(V) = 1.$$

The Stability Theorem

Theorem (-, L. Maxim)

- *V a complex algebraic projective hypersurface, $n = \dim_{\mathbb{C}} \neq 2$, one isolated singularity.*
- *V_s a nearby smooth deformation of V.*
- *Then:*
  1. *For all $i < 2n$, $i \neq n$, $H^i(V_s; \mathbb{Q}) \cong HI^i(V; \mathbb{Q})$.*
  2. *There is an isomorphism*

$$H^n(V_s; \mathbb{Q}) \cong HI^n(V; \mathbb{Q})$$

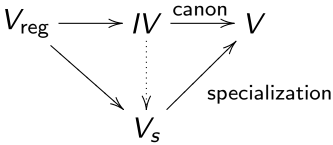
     iff the monodromy operator acting on H^* of the Milnor fiber is trivial.
  3. *Regardless of monodromy,*

$$\max\{\operatorname{rk} IH^n(V), \operatorname{rk} H^n(V_{\text{reg}}), \operatorname{rk} H^n(V)\} \leq \operatorname{rk} HI^n(V) \leq \operatorname{rk} H^n(V_s)$$

     and these bounds are sharp.

Mixed Hodge Structure

At least if $H_{n-1}(L;\mathbb{Z})$ is torsionfree, there is a map $IV \to V_s$ such that



commutes. Induces the stability. $\Rightarrow$ Ring-isomorphism.

Theorem (-,L. Maxim)

For trivial local monodromy, $HI^*(V;\mathbb{Q})$ can be endowed with mixed Hodge structures, so that $IV \to V$ induces homomorphisms of mixed Hodge structures $H^*(V;\mathbb{Q}) \to HI^*(V;\mathbb{Q})$.

Universality of Monodromy Condition

- $\mathcal{C}$ a collection of pseudomanifolds containing all manifolds and cones on closed manifolds, closed under taking boundary.
- $\mathcal{H} : \mathcal{C} \to \mathbb{R}\text{-}\mathrm{MOD}_*$ any deformation stable homology theory satisfying

$$\dim \mathcal{H}_i(\operatorname{cone} X) \leq \dim \mathcal{H}_i(X), \;\; X \in \mathcal{C} \text{ a closed manifold.}$$

- Then: If $X \in \mathcal{C}$ is a complex algebraic projective hypersurface of dimension at least 2 with one isolated singularity, then the monodromy operator of the singularity of X is trivial.

String theory

- World sheet $\to$ target space $= M^4 \times X^6$.
- X should be a Calabi-Yau space. But which one?
- **Conifold Transition** is a means of navigating between Calabi-Yau mfds.
- "It appears that all Calabi-Yau vacua may be connected by conifold transitions."
  [J. Polchinski]

Definition

A *conifold* is (topologically) a compact oriented 6-dim. pseudomfd. S, that has only isolated singularities with link $S^2 \times S^3$.

Conifold Transition

1. *Deformation of the complex structure:*

- X_ϵ CY 3-fold, whose complex structure depends on a small complex parameter ϵ.
- For small $\epsilon \neq 0$: X_ϵ smooth.
- $\epsilon \to 0$: singular conifold S.
- All singularities are nodes; links $\cong S^2 \times S^3$.
- Topologically: S^3-shaped cycles in X_ϵ are collapsed.

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• Topologically: S^3-shaped cycles in X_ϵ are collapsed.

2. *Small resolution:*
• $Y \to S$ replaces each node in S by $\mathbb{C}P^1$.
• Y is a Calabi-Yau mfd.

Conifold Transition:

$$X_\epsilon \rightsquigarrow S \rightsquigarrow Y.$$

Massless D-Branes.

- Z : 3-cycle in X_ϵ, which collapses to a node in S.
- In type IIB string theory: exists a charged 3-Brane that wraps around Z.
- Mass (3-Brane) $\propto$ Vol(Z).
- $\Rightarrow$ 3-Brane becomes **massless** in S.

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- $\mathbb{C}P^1$: 2-cycle in Y, which collapses to a node in S.
- In type IIA string theory: exists a charged 2-Brane that wraps around $\mathbb{C}P^1$.
- Mass (2-Brane) $\propto$ Vol($\mathbb{C}P^1$).
- $\Rightarrow$ 2-Brane becomes **massless** in S.

Cohomology and massless states

Rule: Cohomology classes on X are manifested in 4 dimensions as massless particles.

- ω differential form on $T = M^4 \times X$.
- Necessary condition for ω to be physically realistic:

$$d^* d\omega = 0 \text{ (“Maxwell equation”)},$$

$$d^* \omega = 0 \text{ (“Lorentz gauge condition”)}.$$

- In particular, $\Delta_T \omega = 0$, $\Delta_T = dd^* + d^* d$ Hodge-de Rham Laplace operator on T.
- Decomposition

$$\Delta_T = \Delta_M + \Delta_X.$$

Cohomology and massless states

- Wave equation
$$(\Delta_M + \Delta_X)\omega = 0.$$
- Interpretation: Δ_X ist a kind of “mass”-operator for 4-dimensional fields, whose eigenvalues are masses in $4D$.
- (Klein-Gordon equation $(\Box_M + m^2)\omega = 0$ for a free particle.)
- For the zero-modes of Δ_X (the harmonic forms on X), one sees in the 4-dim. reduction massless fields.

Physical Requirements for Cohomology Theories

- IIA string theory: $\mathcal{H}^*_{\mathrm{IIA}}(-)$
  - Should contain the aforementioned massless 2-Branes as cycles,
  - Poincaré Duality.
- IIB string theory: $\mathcal{H}^*_{\mathrm{IIB}}(-)$
  - Should contain the aforementioned massless 3-Branes as cycles,
  - Poincaré Duality.

Theorem (-)

$\mathcal{H}^*_{\mathrm{IIA}}(-) = IH^*$ *and* $\mathcal{H}^*_{\mathrm{IIB}}(-) = HI^*$ *are solutions.*

Mirror Symmetry

- Mirror-map: Calabi-Yau $S \mapsto$ Calabi-Yau S°.
- IIB string theory on $\mathbb{R}^4 \times S \leftrightarrow$ IIA string theory onf $\mathbb{R}^4 \times S^\circ$.
- For nonsingular S, S°,

$$b_3(S^\circ) = (b_2 + b_4)(S) + 2, \ \ b_3(S) = (b_2 + b_4)(S^\circ) + 2.$$

- Conjecture [Morrison]: The mirror of a conifold transition is again a conifold transition.

Theorem

If a singular Calabi-Yau 3-fold S arises in the course of a conifold transition $X \rightsquigarrow S \rightsquigarrow Y$ and if its mirror S° sits in the reverse conifold transition $Y^\circ \rightsquigarrow S^\circ \rightsquigarrow X^\circ$, then

$$\begin{aligned}
\operatorname{rk} IH_3(S) &= \operatorname{rk} HI_2(S^\circ) + \operatorname{rk} HI_4(S^\circ) + 2, \\
\operatorname{rk} IH_3(S^\circ) &= \operatorname{rk} HI_2(S) + \operatorname{rk} HI_4(S) + 2, \\
\operatorname{rk} HI_3(S) &= \operatorname{rk} IH_2(S^\circ) + \operatorname{rk} IH_4(S^\circ) + 2, \ \ \textit{and} \\
\operatorname{rk} HI_3(S^\circ) &= \operatorname{rk} IH_2(S) + \operatorname{rk} IH_4(S) + 2.
\end{aligned}$$