# Chern classes of hyperplane arrangements

Paolo Aluffi

Florida State University

Hefei—July 25th, 2011

Reference:

- Chern classes of hyperplane arrangements
  arXiv:1103.2777.

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- Mustaţă-Schenck: *The module of logarithmic p-forms of a locally free arrangement,* J. Algebra, 241(2):699-719, 2001

Soundbite (slightly imprecise):

**Theorem**

*If $A \subseteq \mathbb{P}^n$ is a free arrangement, then $c(\Omega^1_{\mathbb{P}^n}(\log A)^\vee) \cap [\mathbb{P}^n]$ equals the Chern-Schwartz-MacPherson class of the complement $\mathbb{P}^n \smallsetminus A$.*

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(Work by Xia Liao on this question.)

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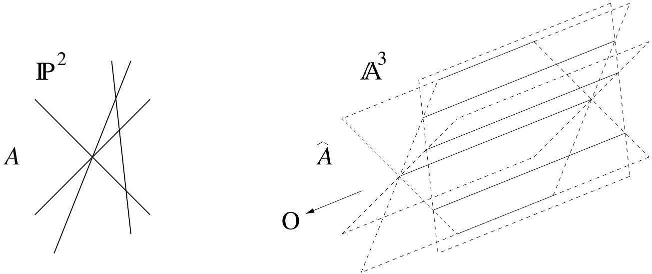
Also, useful to consider corresponding affine, central arrangements $\widehat{A}$, $\widehat{\mathscr{A}}$ in $\mathbb{A}^{n+1}$.



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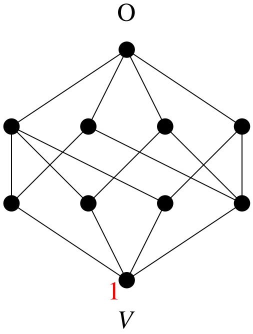

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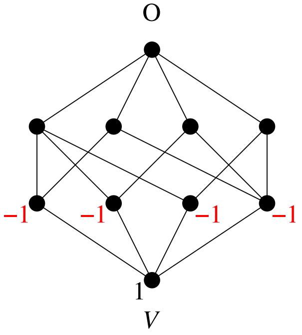

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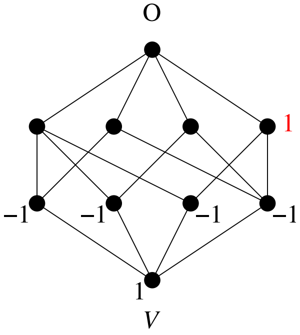

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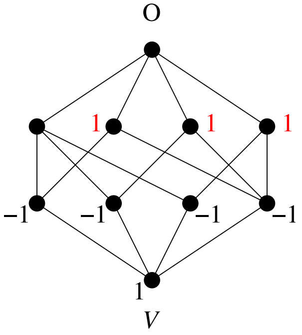

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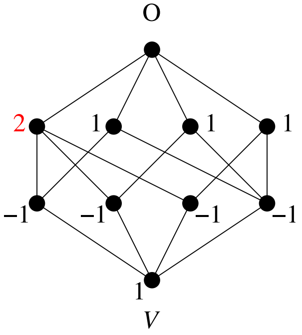

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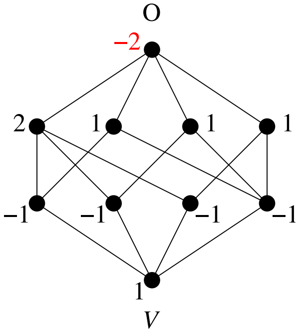

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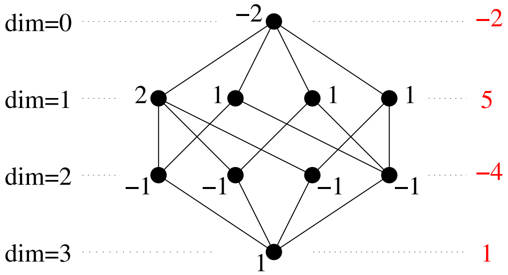


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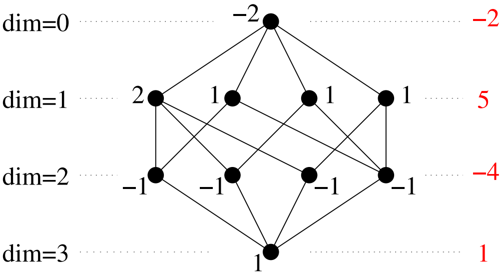


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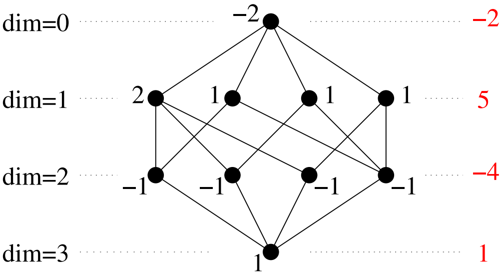


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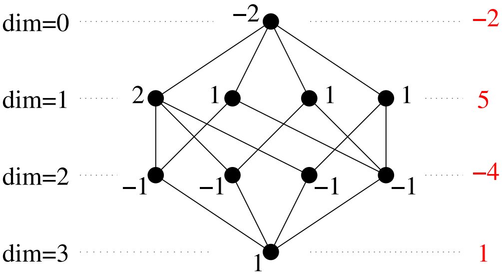


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Plan: sometime during this talk, this should become obvious to everybody (even algebraic geometers!).

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a polynomial of degree $\leq n$ in hyperplane class $h$.

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Locally defined by $f(x_1, \dots, x_n) = 0$, local parameters $x_i$.
$\rightsquigarrow$ subscheme of $V$ locally defined by $(f, \partial f/\partial x_1, \dots, \partial f/\partial x_n)$ (Jacobian ideal)
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Example: $X$ with isolated singularities $\rightsquigarrow \int s(JX, V) =$sum of Milnor numbers.

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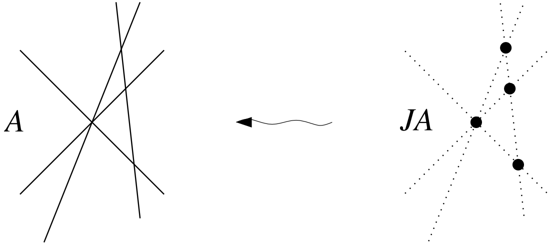

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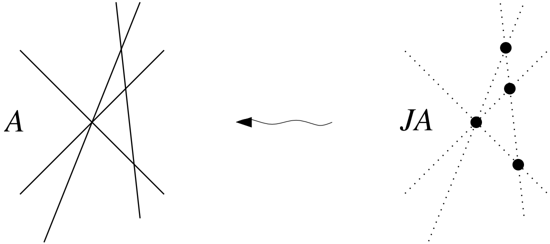


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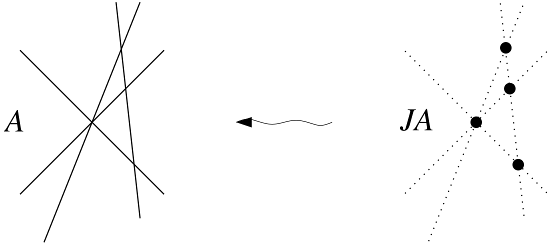


What about $s(JA, \mathbb{P}^2)$?

In this case: $\int s(JA, \mathbb{P}^2) =$ sum of Milnor numbers $= 7$.
This is the only information left! What can we do with it?

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Question: For what hypersurfaces in what nonsingular varieties and in what sense does something like this hold?

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Write $[\mathbb{P}^n] - s(JA, \mathbb{P}^n) = \sum_{i=0}^n \sigma_i h^i \cap [\mathbb{P}^n]$; then

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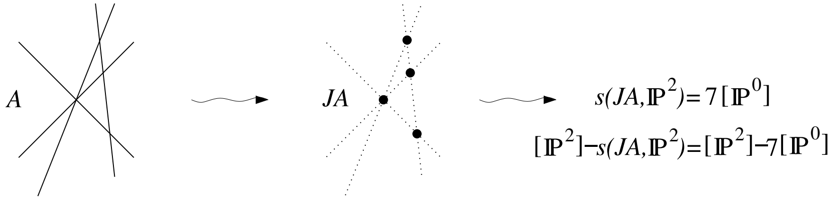

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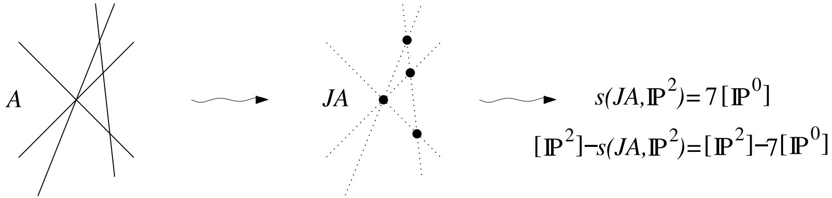


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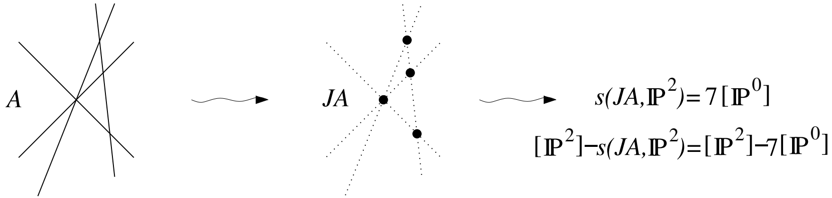


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$$\sum_{k=0}^{n} \left( \sum_{i=0}^{k} \binom{k}{i} (d-1)^{k-i} \sigma_i \right) t^k = 1 + ((4-1)+0)t + ((4-1)^2 + 0 - 7)t^2$$

$$= 1 + 3t + 2t^2 \quad .$$

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That's the next topic.

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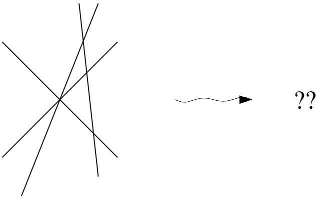

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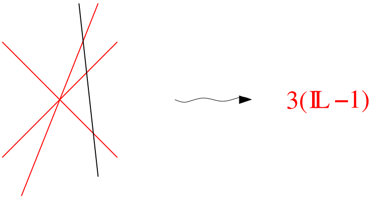

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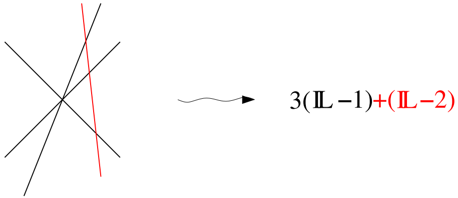

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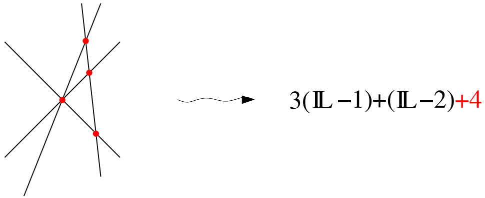

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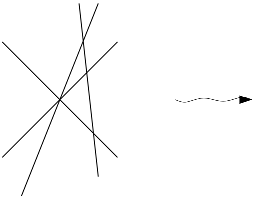

$3(\mathbb{L}-1)+(\mathbb{L}-2)+4$
$= 4\mathbb{T}+3$

## Theorem (—, Marcolli '08)

*Let $X \subseteq \mathbb{P}^n$ be obtained by fin. many set theoretic operations on linear subspaces. Then $c_{\mathrm{SM}}(X) \in H_*(\mathbb{P}^n, \mathbb{Q})$ and $[X] \in K(\mathrm{Var})$ carry the same information.*

More precise statement:

Write $[X] = \sum_{i=0}^n a_i \mathbb{T}^i$, where $\mathbb{T} = \mathbb{L} - 1 = [\mathbb{C}^*]$.

Then $c_{\mathrm{SM}}(X) = \sum_{i=0}^n a_i [\mathbb{P}^i]$.

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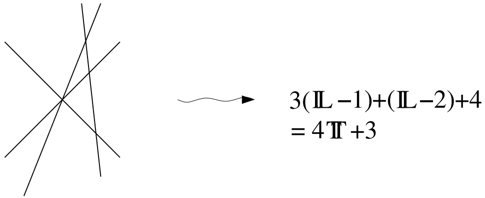

$4\mathbb{T} + 3$ in $K(\mathrm{Var})$, hence $c_{\mathrm{SM}}(A) = 4[\mathbb{P}^1] + 3[\mathbb{P}^0]$.

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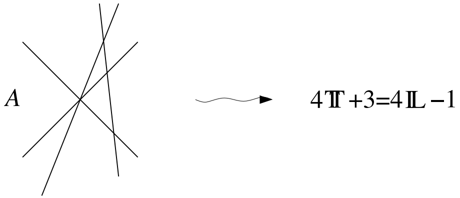

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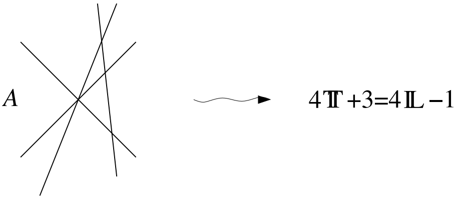


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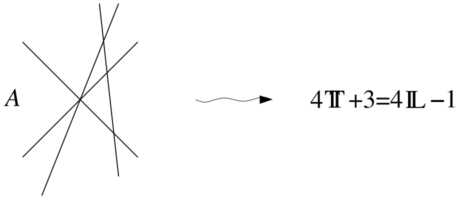


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In general?

Theorem (—)

$[\mathbb{P}^n \smallsetminus A] = \underline{\chi_{\widehat{\mathscr{A}}}}(\mathbb{L})$

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- The result follows. □

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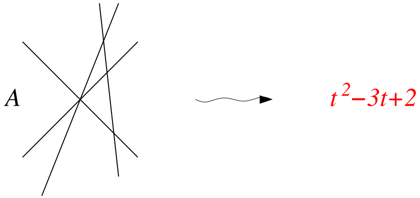

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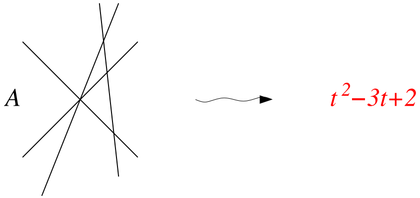


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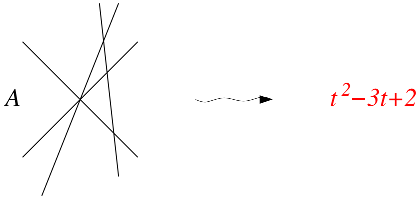


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Fact: The mixed Hodge structure of $\mathbb{P}^n \smallsetminus A$ is pure (Shapiro).
Therefore. . .

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As promised, it is now (nearly) obvious.

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General case would follow from good blow-up formula for logarithmic differential.

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- done, as $\underline{\chi_{\widehat{\mathscr{A}}}}(t+1) = c_{\mathrm{SM}}(\mathbb{P}^n \smallsetminus A)$ as we saw earlier.

Details in arXiv:1103.2777

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Maybe this should be taken as counter-evidence for the conjecture?

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Challenge: Characterize hyperplane arrangements with effective Chern-Schwartz-MacPherson class.

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$A$: projective arrangement; $\mu$: corresponding Möbius function. Then $c_{\mathrm{SM}}(A)$ is effective $\iff$ all coefficients of $-\sum_{x\neq 0}\mu(x)(t+1)^{\dim x}$ are $\geq 0$.

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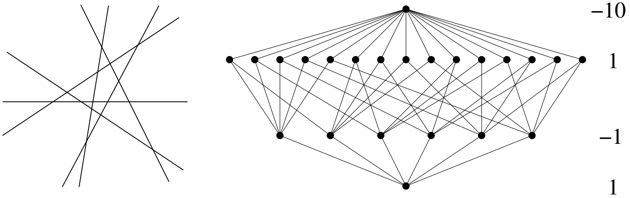
Example (smallest non-effective generic)




$$-\sum_{x\neq 0}\mu(x)(t+1)^{\dim x} = -(6\cdot(-1)\cdot(t+1)^2 + 15\cdot 1\cdot(t+1) - 10)$$
$$= 6t^2 - 3t + 1.$$

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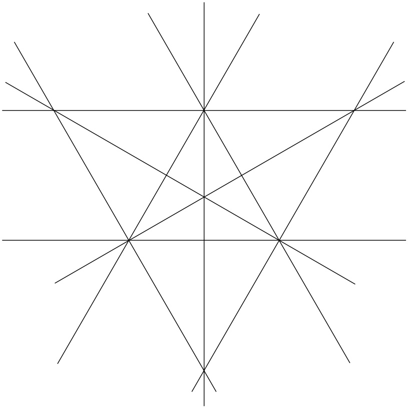
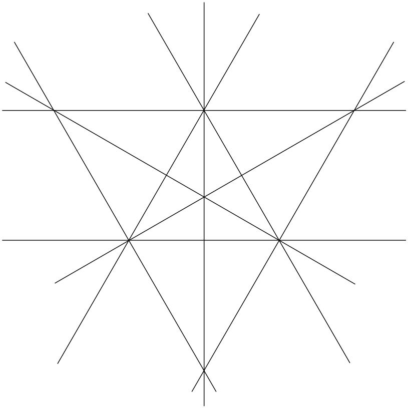
If only I were a physicist...



$\mathscr{A}$ = cone in $\mathbb{P}^9$ over this projective arrangement.



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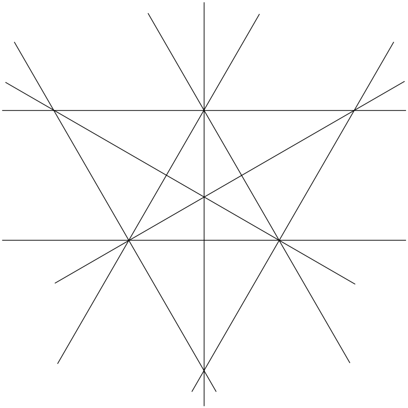

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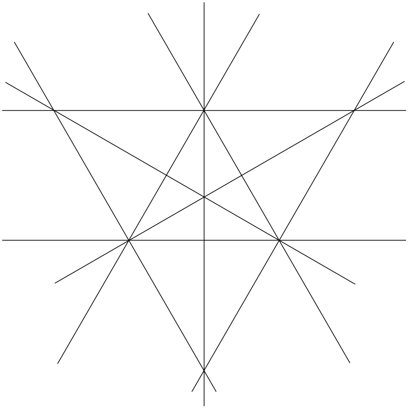

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- It is free
- $d \leq n$
- $c_{\mathrm{SM}}(A)$ is not effective!

Thanks for your attention!