Intersection cohomology invariants
of complex algebraic varieties

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Dedicated to Lê Dũng Tráng on His 60th Birthday

ABSTRACT. In this note we use the deep BBDG decomposition theorem in order to give a new proof of the so-called "stratified multiplicative property" for certain intersection cohomology invariants of complex algebraic varieties.

1. Introduction

We study the behavior of intersection cohomology invariants under morphisms of complex algebraic varieties. The main result described here is classically referred to as the "stratified multiplicative property" (cf. [**CS94, S94**]), and it shows how to compute the invariant of the source of a proper algebraic map from its values on various varieties that arise from the singularities of the map. For simplicity, we consider in detail only the case of Euler characteristics, but we will also point out the additions needed in the arguments in order to make the proof work in the Hodge-theoretic setting.

While the study of the classical Euler-Poincaré characteristic in complex algebraic geometry relies entirely on its additivity property together with its multiplicativity under fibrations, the intersection cohomology Euler characteristic is studied in this note with the aid of a deep theorem of Bernstein, Beilinson, Deligne and Gabber, namely the BBDG decomposition theorem for the pushforward of an intersection cohomology complex under a proper algebraic morphism [**BBD, CM05**]. By using certain Hodge-theoretic aspects of the decomposition theorem (cf. [**CM05, CM07**]), the same arguments extend, with minor additions, to the study of all Hodge-theoretic intersection cohomology genera (e.g., the $I\chi_y$-genus or, more generally, the intersection cohomology Hodge-Deligne E-polynomials).

While a functorial approach based on the standard calculus of constructible functions (and Grothendieck groups of contructible sheaves, resp. mixed Hodge modules) can be used for proving these results in even greater generality (see [**CMSa, CMSb**]), we believe that the proof presented here has its own interest, since for example it can be adapted to the setting of algebraic geometry over a field of positive characteristic. This aspect will be discussed in detail elsewhere.

Unless otherwise specified, all (co)homology and intersection (co)homology groups in this paper are those with rational coefficients.

2. Topological Euler-Poincaré characteristic

For a complex algebraic variety X, let $\chi(X)$ denote its topological Euler-Poincaré characteristic. Then $\chi(X)$ equals the compactly supported Euler characteristic, $\chi_c(X)$ (see [**F93**], page 141), and the latter is additive with respect to open and closed inclusions. More precisely, if Z is a Zariski closed subset of X and U denotes the complement, then

$$(2.1) \qquad \chi_c(X) = \chi_c(Z) + \chi_c(U),$$

so the same relation holds for χ. Another important property of the Euler-Poincaré characteristic is its multiplicativity in fibrations, which asserts that if $F \hookrightarrow E \to B$ is a locally trivial fibration of finite CW complexes then

$$(2.2) \qquad \chi(E) = \chi(B) \cdot \chi(F).$$

These two properties can be used for studying the behavior of χ under a proper algebraic map.

Let $f : X \to Y$ be a proper morphism of complex algebraic varieties. Such a map can be stratified with subvarieties as strata, i.e., there exist finite algebraic Whitney stratifications $\mathfrak{X}$ of X and $\mathcal{V}$ of Y, such that for any component V of a stratum of Y, $f^{-1}(V)$ is a union of connected components of strata of $\mathfrak{X}$, each of which is mapping submersively to V. This implies that $f_{|f^{-1}(V)} : f^{-1}(V) \to V$ is a locally trivial map of Whitney stratified spaces. For simplicity, we assume that Y is irreducible, so that f is smooth over the dense open stratum in Y (with respect to a Whitney stratification), which we denote by S. For $V, W \in \mathcal{V}$ we write $V \leq W$ if and only if $V \subset \bar{W}$. We denote by F the general fiber of f, i.e. the fiber over S, and by F_V the fiber of f above the singular stratum $V \in \mathcal{V} \setminus \{S\}$. Then the Euler-Poincaré characteristic satisfies the "stratified multiplicative property", i.e. the following holds:

PROPOSITION 2.1. ([**CMSa**], *Proposition 2.4*) *Let* $f : X \to Y$ *be a proper algebraic morphism of (possibly singular) complex algebraic varieties, with* Y *irreducible. Let* $\mathcal{V}$ *be the set of components of strata of* Y *in a stratification of* f. *For each* $V \in \mathcal{V} \setminus \{S\}$, *define* $\hat{\chi}(\bar{V})$ *inductively by the formula:*

$$\hat{\chi}(\bar{V}) = \chi(\bar{V}) - \sum_{W < V} \hat{\chi}(\bar{W}).$$

Then:

$$(2.3) \qquad \chi(X) = \chi(Y) \cdot \chi(F) + \sum_{V < S} \hat{\chi}(\bar{V}) \cdot (\chi(F_V) - \chi(F)).$$

REMARK 2.2. A Hodge-theoretic analogue of Proposition 2.1 was obtained in [[**CLMSa**], Proposition 2.11] for the motivic (additive) Hodge genus $\chi_y^c(-)$, and more generally, for the Hodge-Deligne E-polynomial $E_c(-; u, v)$ defined by means of (the Hodge-Deligne numbers of) compactly supported cohomology,

$$(2.4) \qquad E_c(X; u, v) = \sum_{p,q} \left(\sum_i (-1)^i \cdot h^{p,q} \left(H_c^i(X; \mathbb{C}) \right) \right) \cdot u^p v^q \in \mathbb{Z}[u, v].$$

For a complex algebraic variety X we have that

$$(2.5) \qquad \chi_y^c(X) = E_c(X; -y, 1)$$

and

$$(2.6) \qquad \chi(X) = \chi_c(X) = \chi_{-1}^c(X).$$

The major difference from χ is that the Hodge-Deligne polynomial E_c is not in general multiplicative under locally trivial (in the complex topology) fibrations of complex algebraic varieties (cf. [**CLMSa**], Example 2.9). However, both the multiplicativity in fibrations and the stratified multiplicative property for E_c can be recovered under the assumption of trivial monodromy along strata (e.g., if all strata $V \in \mathcal{V}$ are simply-connected)[1]; for complete details, see [**CLMSa**], §2.4-5.

3. Intersection homology Euler characteristics

Let X be a complex algebraic variety of pure (complex) dimension n. Denote by $D_c^b(X)$ the bounded derived category of constructible sheaves of $\mathbb{Q}$-vector spaces on X, endowed with the t-structure associated to the middle-perversity, and let $Perv(X)$ be the abelian category of perverse sheaves on X; $Perv(X) \subset D_c^b(X)$ is the heart of the middle-perversity t-structure. Let

$$^p\mathcal{H}^i : D_c^b(X) \to Perv(X)$$

denote the associated cohomological functors.

Let IC_X^{top} be the sheaf complex defined by

$$(IC_X^{top})^k(U) := IC_{-k}^{BM}(U)$$

for $U \subset X$ open (cf. [**GM83**]), where $IC_\bullet^{BM}$ is the complex of locally-finite chains with respect to the middle-perversity ([**GM80**]). Let

$$IC_X := IC_X^{top}[-n].$$

If X is non-singular, then $IC_X = \mathbb{Q}_X[n]$, where $\mathbb{Q}_X$ denotes the constant sheaf with stalk $\mathbb{Q}$ on X. For a non-negative integer k, the middle-perversity intersection cohomology group $IH^k(X; \mathbb{Q})$ is defined in terms of the hypercohomology groups of IC_X, that is,

$$IH^k(X; \mathbb{Q}) := \mathbb{H}^{k-n}(X; IC_X).$$

In general, for a r-dimensional stratified pseudomanifold L (e.g., L can be the link of a stratum in a Whitney stratification of a complex algebraic variety), the intersection cohomology groups are defined by $IH^k(L; \mathbb{Q}) := \mathbb{H}^{k-2r}(L; IC_L^{top})$.

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[1]The special case of *weight polynomials* was considered in [**DL97**], where, perhaps for the first time, it was pointed out the necessity of the trivial monodromy assumption in order to obtain multiplicative properties for Hodge-theoretic invariants.

Since complex algebraic varieties are *compactifiable*, their rational intersection cohomology groups (with either compact or closed support) are finite dimensional (cf. [**B84**], V.10.13), therefore the intersection homology Euler characteristics of complex algebraic varieties are well-defined. We let $I\chi(X)$ denote the intersection homology Euler characteristic of X (with closed support), and $I\beta^k(X) := \dim IH^k(X)$ the k-th intersection cohomology Betti number of X.

The main result of this note asserts that $I\chi$ satisfies the stratified multiplicative property. More precisely:

THEOREM 3.1. *([**CMSa**], Proposition 3.6) Let $f : X \to Y$ be a proper map of complex algebraic varieties, with X pure-dimensional and Y irreducible. Let $\mathcal{V}$ be the set of components of strata of Y in a stratification of f, and assume (for simplicity) that $\pi_1(V) = 0$ for all $V \in \mathcal{V}$. For each stratum $V \in \mathcal{V} \setminus \{S\}$ define inductively*

$$\widehat{I\chi}(\bar{V}) = I\chi(\bar{V}) - \sum_{W < V} \widehat{I\chi}(\bar{W}) \cdot I\chi(c^\circ L_{W,V}),$$

where, for $W < V$, $c^\circ L_{W,V}$ denotes the open cone on the link of W in $\bar{V}$. Then:
(3.1)
$$I\chi(X) = I\chi(Y) \cdot I\chi(F) + \sum_{V < S} \widehat{I\chi}(\bar{V}) \cdot \left(I\chi(f^{-1}(c^\circ L_{V,Y})) - I\chi(F) \cdot I\chi(c^\circ L_{V,Y}) \right).$$

where F is the generic fiber of f and $L_{V,Y}$ is the link of V in Y.

The present proof of Theorem 3.1 remains valid without simple connectivity assumptions provided the relevant local coefficient systems extend to the closures of the strata, see footnote 3. In [**CMSa**] we gave a different proof, replacing integral invariants by a functorial approach based on the standard calculus of constructible functions, which yields Theorem 3.1 with no monodromy assumptions (even for Chern-MacPherson classes). The proof we give here is a direct consequence of the BBDG decomposition theorem, which we recall below.

THEOREM 3.2. *([**BBD**], [**CM05**])*
Let $f : X \to Y$ be a proper map of complex algebraic varieties. Then:
(1) Decomposition: Rf_*IC_X is p-split, i.e. there is a (non-canonical) isomorphism in $D_c^b(Y)$:

(3.2)
$$\phi : \oplus_i {}^p\mathcal{H}^i(f_*IC_X)[-i] \simeq Rf_*IC_X,$$

where ${}^p\mathcal{H}$ is the perverse cohomology functor.[2]
(2) Semi-simplicity: For every i, ${}^p\mathcal{H}^i(f_*IC_X)$ is semisimple; more precisely, if $\mathcal{V}$ is the set of connected components of strata of Y in a stratification of f, there is a canonical isomorphism in $Perv(Y)$:

(3.4)
$${}^p\mathcal{H}^i(f_*IC_X) \simeq \oplus_{V \in \mathcal{V}} IC_{\bar{V}}(\mathcal{L}_{i,V})$$

where the local systems $\mathcal{L}_{i,V} := \mathcal{H}^{-dim_\mathbb{C}(V)}({}^p\mathcal{H}^i(f_*IC_X)|_V)$ on $V \in \mathcal{V}$ are semisimple.

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[2]If the morphism f is projective, then (3.2) is a formal consequence of the Relative Hard Lefschetz Theorem (cf. [**De68**]). The decomposition (3.2) also implies that the perverse Leray spectral sequence for the map f, that is,

(3.3)
$$E_2^{i,j} = \mathbb{H}^i(Y, {}^p\mathcal{H}^j(f_*IC_X)) \Longrightarrow \mathbb{H}^{i+j}(X; IC_X),$$

degenerates at the E_2-term.

REMARK 3.3. Let U_r be the disjoint union of strata $V \in \mathcal{V}$ with $\dim(V) \geq r$. By the condition of (co)support for perverse sheaves on Y, it follows that

$${}^p\mathcal{H}^i(f_*IC_X)_{|U_r} \simeq \tau_{\geq -dim(Y)}\tau_{\leq -r}{}^p\mathcal{H}^i(f_*IC_X)_{|U_r}, \text{ for all } 0 \leq r \leq dim(Y),$$

where $(\tau_{\leq}, \tau_{\geq})$ is the natural t-structure on $D_c^b(Y)$ (cf. [**CM05**], §3.6). Hence, the support condition implies that the sheaf $\mathcal{H}^{-r}({}^p\mathcal{H}^i(f_*IC_X)_{|U_r})$ is supported on the r-dimensional stratum of U_r and is, by constructibility, a local system on each connected component V of this bottom stratum of U_r.

PROOF. (*of Theorem 3.1*)
Let $n = \dim(X)$, $m = \dim(Y)$. With the notations in the statement of the theorem, the following sequence of equalities hold:

$$
\begin{aligned}
I\chi(X) &= \sum_k (-1)^k \cdot I\beta^k(X) \\
&= \sum_k (-1)^k \cdot dim\mathbb{H}^{k-n}(X, IC_X) \\
&= (-1)^n \cdot \chi(X; IC_X) \\
&= (-1)^n \cdot \chi(Y, Rf_*IC_X) \\
&\overset{(a)}{=} \sum_i (-1)^{n+i} \cdot \chi(Y; {}^p\mathcal{H}^i(f_*IC_X)) \\
&\overset{(b)}{=} \sum_{i,V} (-1)^{n+i} \cdot \chi(V; IC_{\bar{V}}(\mathcal{L}_{i,V})) \\
&= \sum_{i,V} (-1)^{n-dim(V)+i} \cdot I\chi(\bar{V}; \mathcal{L}_{i,V}) \\
&\overset{(c)}{=} \sum_V I\chi(\bar{V}) \cdot \left(\sum_i (-1)^{n-dim(V)+i} \cdot rank(\mathcal{L}_{i,V}) \right)
\end{aligned}
$$

where (a) follows from the decomposition isomorphism (3.2), or by using the perverse Leray spectral sequence (3.3), (b) is a consequence of the semi-simplicity part (3.4) of the BBDG theorem, and (c) follows from the fact that if $\mathcal{L}$ is a local coefficient system defined everywhere on a pseudomanifold X, then the intersection homology Euler characteristic of X with coefficients in $\mathcal{L}$ is computed by the formula [3]:

(3.5) $$I\chi(X; \mathcal{L}) = rank(\mathcal{L}) \cdot I\chi(X).$$

It remains to identify the (dimension of) stalks of the local systems appearing in the statement of the semi-simplicity part of the BBDG theorem.

Let us fix for each stratum $V \in \mathcal{V}$ a point $v \in V$ with inclusion $i_v : \{v\} \hookrightarrow V$. The stalk of $\mathcal{L}_{i,V}$ at the point v can be found by applying the stalk cohomology functor $\mathcal{H}^\bullet(-)_v$ to the isomorphism of the BBDG theorem obtained by putting together the decomposition and resp. the semi-simplicity statements, that is,

(3.6) $$Rf_*IC_X \simeq \oplus_{i,V} IC_{\bar{V}}(\mathcal{L}_{i,V})[-i].$$

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[3]The proof of Theorem 3.1 relies in fact only on the assumption that the local systems appearing in the BBDG theorem extend to the closures of strata on which they are defined (this is of course the case if $\pi_1(V) = 0$ for all $V \in \mathcal{V}$). Formula (3.5) is not true in general for local systems that are only partially defined.

We first compute the stalk cohomologies on the left-hand side of the isomorphism (3.6). The (generic) fiber of f over the top dimensional stratum S is (locally) normally nonsingular embedded in X, so there is a quasi-isomorphism (cf. [**GM83**], §5.4.1):

$$IC_X|_F \simeq IC_F[\mathrm{codim}_{\mathbb{C}}F].$$

Thus, for $s \in S$, by proper base change we have that

$$(3.7) \qquad \mathcal{H}^j(Rf_*IC_X)_s \cong H^j(i_s^*Rf_*IC_X) \cong \mathbb{H}^j(F, IC_X|_F) \cong IH^{n+j}(F;\mathbb{Q}).$$

Next, as in [[**CMSa**], Proposition 3.6], by using the fact that the inverse image of a normal slice to a stratum of Y in a stratification of f is (locally) normally non-singular embedded in X, it follows that for a point v in a stratum $V \in \mathcal{V} \setminus \{S\}$ we have that

$$(3.8) \qquad \mathcal{H}^j(Rf_*IC_X)_v \cong IH^{n+j}(f^{-1}(c^{\circ}L_{V,Y});\mathbb{Q}),$$

where $c^{\circ}L_{V,Y}$ is the open cone on the link $L_{V,Y}$ of V in Y.

For any integer r so that $0 \leq r \leq m$, let U_r be the disjoint union of strata $V \in \mathcal{V}$ with $\dim(V) \geq r$, and let $f_r : U_r' = f^{-1}(U_r) \to U_r$ be the corresponding maps. In particular, $U_m = S$, so $\{U_r\}_r$ is an open cover that exhausts Y. Recall that the operation of taking perverse cohomology commutes with restriction to open subsets, and the same is true for the operation of forming intersection cohomology complexes associated with local systems. Now, for a point v in a connected component V of the r-dimensional stratum in Y, we can identify the stalk $(\mathcal{L}_{i,V})_v$ as follows: first replace f by f_r, i.e. by assuming that V is a closed stratum of f, then calculate the stalk cohomology at v for both terms in the isomorphism (3.6) of the decomposition theorem.

First, by restricting the isomorphism (3.6) over the open set S, and by using (3.7), we obtain immediately that for a point $s \in S$,

$$(3.9) \qquad (\mathcal{L}_{i,S})_s \cong IH^{n-m+i}(F;\mathbb{Q}),$$

for F the general fiber of f. In particular, under our assumptions we have that

$$(3.10)\ \ I\chi(X) = I\chi(Y) \cdot I\chi(F) + \sum_{V<S} I\chi(\bar{V}) \cdot \left(\sum_i (-1)^{n-dim(V)+i} \cdot rank(\mathcal{L}_{i,V}) \right).$$

Next, recall that for strata $V, W \in \mathcal{V}$ and a point $w \in W$, we may have the non-vanishing of stalk cohomologies $\mathcal{H}^k(IC_{\bar{V}}(\mathcal{L}))_w \neq 0$ only if $W \leq V$, and if this is the case then:

$$(3.11) \qquad \mathcal{H}^k(IC_{\bar{W}}(\mathcal{L}))_w \cong \begin{cases} \mathcal{L}_w, & \text{if } k = -dim(W); \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(3.12) \qquad \mathcal{H}^k(IC_{\bar{V}}(\mathcal{L}))_w \cong IH^{k+dim(V)}(c^{\circ}L_{W,V};\mathcal{L}), \ \ \text{if } W < V.$$

Therefore, for a stratum $W \in \mathcal{V} \setminus \{S\}$ and an integer j, by applying to (3.6) the stalk cohomology functor $\mathcal{H}^j(-)_w$ at the point $w \in W$, the equations (3.8), (3.11) and (3.12) yield the following isomorphism of $\mathbb{Q}$-vector spaces:

$$(3.13)\ \ IH^{n+j}(f^{-1}(c^{\circ}L_{W,Y})) \cong \oplus_i \oplus_{W \leq V} \mathcal{H}^{j-i}(IC_{\bar{V}}(\mathcal{L}_{i,V}))_w$$

$$\cong (\mathcal{L}_{j+dim(W),W})_w \oplus \left(\oplus_i \oplus_{W<V} IH^{j-i+dim(V)}(c^{\circ}L_{W,V};\mathcal{L}_{i,V}) \right).$$

After reindexing, the latter formula becomes:

$$(3.14) \quad IH^{n-dim(W)+i}(f^{-1}(c^\circ L_{W,Y})) \cong$$

$$\cong (\mathcal{L}_{i,W})_w \oplus \left(\oplus_r \oplus_{W<V} IH^{i-dim(W)-r+dim(V)}(c^\circ L_{W,V}; \mathcal{L}_{r,V}) \right).$$

Therefore, the summands of the right-hand side of equation (3.10) can be computed inductively by the formula:

$$\sum_i (-1)^{n-dim(W)+i} \cdot rank(\mathcal{L}_{i,W}) = \left(I\chi(f^{-1}(c^\circ L_{W,Y})) - I\chi(F) \cdot I\chi(c^\circ L_{W,Y}) \right) -$$

$$- \sum_{W<V<S} I\chi(c^\circ L_{W,V}) \cdot \left(\sum_i (-1)^{n-dim(V)+i} \cdot rank(\mathcal{L}_{i,V}) \right).$$

This finishes the proof of the theorem. $\square$

EXAMPLE 3.4. Let X be obtained from Y by blowing-up a point y, and denote by $f : X \to Y$ the blow-up map. Let $D = f^{-1}(y)$ be the exceptional divisor. Then (3.1) becomes

$$I\chi(X) - I\chi(f^{-1}(c^\circ L_y)) = I\chi(Y) - I\chi(c^\circ L_y)$$

for L_y the link of y in Y. Moreover, if X is smooth, the formula yields:

$$I\chi(Y) - I\chi(c^\circ L_y) = \chi(X) - \chi(D) = \chi(Y \setminus \{y\}).$$

These are additivity-type properties for the intersection homology Euler characteristic, similar to those for the usual topological Euler characteristic.

REMARK 3.5. We conclude this note with a discussion on the Hodge-theoretic aspects of the stratified multiplicative property.

By taking advantage on the canonical mixed Hodge structure on the intersection cohomology groups of a pure-dimensional complex algebraic variety X ([**CM05, Sa89, Sa90**]), one can define intersection homology Hodge genera, $I\chi_y(X)$, that encode the intersection cohomology (mixed) Hodge numbers, and provide an extension of Hirzebruch's χ_y-genus to the singular setting:

$$(3.15) \qquad I\chi_y(X) := \sum_p \left(\sum_{i,q} (-1)^i h^{p,q}(IH^i(X; \mathbb{C})) \right) \cdot (-y)^p,$$

where $h^{p,q}(IH^i(X; \mathbb{C})) = \dim_{\mathbb{C}} Gr_F^p(Gr_{p+q}^W IH^i(X) \otimes \mathbb{C})$, with $F^\bullet$ and $W_\bullet$ the Hodge and respectively the weight filtration of the mixed Hodge structure on $IH^i(X)$. For example, $I\chi_{-1}(X) = I\chi(X)$, and if X is projective then $I\chi_1(X) = \sigma(X)$ is the Goresky-MacPherson signature of X. Similarly, if $\mathcal{L}$ underlies an admissible variation of mixed Hodge structures on a smooth Zariski open and dense subset U of X, then the intersection cohomology groups $IH^*(X; \mathcal{L})$ carry canonical mixed Hodge structures, and the associated Hodge genus $I\chi_y(X; \mathcal{L})$ can be regarded as a Hodge-theoretic generalization of the twisted signature of a Poincaré local system [**BCS**].

In order to avoid "canonicity" issues (which disappear if one works at the level of Grothendieck groups, as in [**CMSb**]), let us assume that f is projective, and let η be a fixed f-ample line bundle on X. The claim is that Theorem 3.1 remains true

if we replace $I\chi$ by $I\chi_y$, as long as we assume trivial monodromy along all strata of f (e.g., if $\pi_1(V) = 0$, for all $V \in \mathcal{V}$) [4].

Before justifying the claim, let us first note that the isomorphisms (3.7), (3.8) and (3.12) induce, via Saito's theory of algebraic mixed Hodge modules [**Sa89, Sa90**], canonical mixed Hodge structures on the intersection cohomology groups of all spaces involved in formula (3.1). Secondly, the Hodge-theoretic analogue of formula (3.5) holds provided that $\mathcal{L}$ underlies a *constant* variation of mixed Hodge structures (e.g., if $\pi_1(U) = 0$), so in this case we have that

$$(3.16) \qquad\qquad I\chi_y(X; \mathcal{L}) = I\chi_y(X) \cdot \chi_y(\mathcal{L}_x),$$

where $\mathcal{L}_x$ is the stalk of $\mathcal{L}$ at a point $x \in X$, and $\chi_y(\mathcal{L}_x)$ is the Hodge genus of the underlied mixed Hodge structure. However, in general (for non-constant variations, even if defined on all of X), one has to take into account the monodromy contributions in order to compute these twisted Hodge-genera (e.g., see [**CLMSa, CLMSb**], and also [[**MS07**], Theorem 4.9]).

The next ingredient for proving our claim is that the perverse Leray spectral sequence (3.3) for f is a spectral sequence in the category of mixed Hodge structures, e.g., see [[**CLMSa**], §2.3]. Lastly, the operation of taking stalk cohomologies in (3.6) yields an isomorphism of mixed Hodge structures. This can be seen as follows. It was shown by M. Saito [**Sa89, Sa90**] that the BBDG theorem can be lifted to $D^b MHM(Y)$, the bounded derived category of mixed Hodge modules on Y [5], and then Theorem 3.2 can be obtained by simply taking the underlying rational complexes (e.g., the semi-simplicity isomorphism (3.4) follows from the decomposition by strict support of a pure Hodge module). Moreover, if f is projective and η is a f-ample line bundle on X, then the Relative Hard Lefschetz theorem holds at the level on mixed Hodge modules (cf. [[**Sa88**], Theorem 5.3.1 and Remark 5.3.12]). Thus, following [**De94**] or as in [**CM07**], the decomposition isomorphism can be chosen "canonically" (it depends only on η) in $D^b MHM(Y)$. It follows that by taking stalk cohomologies on the underlying rational complexes we obtain an isomorphism of mixed Hodge structures (3.14), where we use the fact that over a point mixed Hodge modules are just (graded) polarizable rational mixed Hodge structures.

Therefore all arguments used in the proof of Theorem 3.1 extend to Hodge theory, and the claim follows.

Similar considerations apply to the study of characteristic classes. In [**BSY**], the authors defined a natural transformation MHT_y on the Grothendieck group of algebraic mixed Hodge modules, which, when evaluated at the intersection complex IC_X, yields a homology (total) characteristic class $IT_{y_*}(X)$ (with coefficients in a localized Laurent polynomial ring), whose associated genus for compact X is $I\chi_y(X)$. The definition of the transformation MHT_y uses Saito's theory of algebraic mixed Hodge modules (for complete details on the construction see [**CMSb, CLMSb, MS07**]). If X is non-singular, then $IT_{y_*}(X)$ is just the Poincaré dual of the modified Todd class that appears in the generalized Hirzebruch-Riemann-Roch theorem. Similarly, for $\mathcal{L}$ an admissible variation of mixed Hodge

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[4]This claim was proved in [**CMSb**] by using the calculus of Grothendieck groups of Saito's mixed Hodge modules.

[5]For a quick overview of the basics of Saito's theory of mixed Hodge modules, the reader is advised to consult [**Sa89**], but see also [[**CMSb**], §3.1] and [[**CLMSa**], §2.2].

structures on a smooth Zariski open and dense subset U of X, one defines twisted characteristic classes $IT_{y_*}(X; \mathcal{L})$ by evaluating MHT_y at the twisted chain complex $IC_X(\mathcal{L})$, regarded as a mixed Hodge module.

These characteristic classes also satisfy the stratified multiplicative property[6], in the sense that for a proper algebraic map $f : X \rightarrow Y$ satisfying the trivial monodromy assumption along all the strata, one can express the push-forward $f_* IT_{y_*}(X)$ in terms of the corresponding characteristic classes of closures of strata $V \in \mathcal{V}$ of Y. Indeed, the BBDG theorem can be used again in order to show that $f_* IT_{y_*}(X)$ is a linear combination of twisted characteristic classes $IT_{y_*}(\bar{V}; \mathbb{V})$ corresponding to closures of strata $V \in \mathcal{V}$ and with coefficients in certain admissible variations of mixed Hodge structures. And under the trivial monodromy assumption along each $V \in \mathcal{V}$, the variation $\mathbb{V}$ on V is constant, and we obtain the following characteristic class version of formula (3.16):

$$(3.17) \qquad IT_{y_*}(\bar{V}; \mathbb{V}) = IT_{y_*}(\bar{V}) \cdot \chi_y(\mathbb{V}_v), \quad \text{for } v \in V.$$

The rest follows as in the case of genera.

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[6]See [[**CMSb**], Theorem 4.7] for a functorial approach, also surveyed in [**MS07**], §3.

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