Math 754
Chapter IV: Vector Bundles. Characteristic classes. Cobordism. Applications

Laurenţiu Maxim
Department of Mathematics
University of Wisconsin
maxim@math.wisc.edu

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Contents

1 Chern classes of complex vector bundles **2**

2 Chern classes of complex vector bundles **2**

3 Stiefel-Whitney classes of real vector bundles **5**

4 Stiefel-Whitney classes of manifolds and applications **5**
4.1 The embedding problem 6
4.2 Boundary Problem. 9

5 Pontrjagin classes **11**
5.1 Applications to the embedding problem 14

6 Oriented cobordism and Pontrjagin numbers **15**

7 Signature as an oriented cobordism invariant **17**

8 Exotic 7-spheres **19**

9 Exercises **20**

1 Chern classes of complex vector bundles

2 Chern classes of complex vector bundles

We begin with the following

Proposition 2.1.
$$H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, \cdots, c_n],$$
with $\deg c_i = 2i$

Proof. Recall that $H^*(U(n);\mathbb{Z})$ is a free $\mathbb{Z}$-algebra on odd degree generators $x_1, \cdots, x_{2n-1}$, with $\deg(x_i) = i$, i.e.,
$$H^*(U(n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[x_1, \cdots, x_{2n-1}].$$
Then using the Leray-Serre spectral sequence for the universal $U(n)$-bundle, and using the fact that $EU(n)$ is contractible, yields the desired result.

Alternatively, the functoriality of the universal bundle construction yields that for any subgroup $H < G$ of a topological group G, there is a fibration $G/H \hookrightarrow BH \to BG$. In our case, consider $U(n-1)$ as a subgroup of $U(n)$ via the identification $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Hence, there exists fibration
$$U(n)/U(n-1) \cong S^{2n-1} \hookrightarrow BU(n-1) \to BU(n).$$
Then the Leray-Serre spectral sequence and induction on n gives the desired result, where we use the fact that $BU(1) \simeq \mathbb{C}P^\infty$ and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[c]$ with $\deg c = 2$. □

Definition 2.2. *The generators $c_1, \cdots, c_n$ of $H^*(BU(n);\mathbb{Z})$ are called the universal Chern classes of $U(n)$-bundles.*

Recall from the classification theorem that, given $\pi : E \to X$ a principal $U(n)$-bundle, there exists a "classifying map" $f_\pi : X \to BU(n)$ such that $\pi \cong f_\pi^* \pi_{U(n)}$.

Definition 2.3. *The i-th Chern class of the $U(n)$-bundle $\pi : E \to X$ with classifying map $f_\pi : X \to BU(n)$ is defined as*
$$c_i(\pi) := f_\pi^*(c_i) \in H^{2i}(X;\mathbb{Z}).$$

Remark 2.4. Note that if π is a $U(n)$-bundle, then by definition we have that $c_i(\pi) = 0$, if $i > n$.

Let us now discuss important properties of Chern classes.

Proposition 2.5. *If $\mathcal{E}$ denotes the trivial $U(n)$-bundle on a space X, then $c_i(\mathcal{E}) = 0$ for all $i > 0$.*

Proof. Indeed, the trivial bundle is classified by the constant map $ct : X \to BU(n)$, which induces trivial homomorphisms in positive degree cohomology. $\square$

Proposition 2.6 (Functoriality of Chern classes)**.** *If $f : Y \to X$ is a continuous map, and $\pi : E \to X$ is a $U(n)$-bundle, then $c_i(f^*\pi) = f^*c_i(\pi)$, for any i.*

Proof. We have a commutative diagram

$$\begin{array}{ccccc} f^*E & \xrightarrow{\widehat{f}} & E & \longrightarrow & EU(n) \\ \downarrow{\scriptstyle f^*\pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{U(n)}} \\ Y & \xrightarrow{f} & X & \xrightarrow{f_\pi} & BU(n) \end{array}$$

which shows that $f_\pi \circ f$ classifies the $U(n)$-bundle $f^*\pi$ on Y. Therefore,

$$\begin{aligned} c_i(f^*\pi) &= (f_\pi \circ f)^* c_i \\ &= f^* \left(f_\pi^* c_i \right) \\ &= f^* c_i (\pi) . \end{aligned}$$

$\square$

Definition 2.7. *The total Chern class of a $U(n)$-bundle $\pi : E \to X$ is defined by*

$$c(\pi) = c_0(\pi) + c_1(\pi) + \cdots c_n(\pi) = 1 + c_1(\pi) + \cdots c_n(\pi) \in H^*(X; \mathbb{Z}),$$

as an element in the cohomology ring of the base space.

Definition 2.8 (Whitney sum)**.** *Let $\pi_1 \in \mathcal{P}(X, U(n))$, $\pi_2 \in \mathcal{P}(X, U(m))$. Consider the product bundle $\pi_1 \times \pi_2 \in \mathcal{P}(X \times X, U(n) \times U(m))$, which can be regarded as a $U(n+m)$-bundle via the canonical inclusion $U(n) \times U(m) \hookrightarrow U(n+m)$, $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. The Whitney sum of the bundles π_1 and π_2 is defined as:*

$$\pi_1 \oplus \pi_2 := \Delta^*(\pi_1 \times \pi_2),$$

where $\Delta : X \to X \times X$ is the diagonal map given by $x \mapsto (x, x)$.

Remark 2.9. The Whitney sum $\pi_1 \oplus \pi_2$ of π and π_s is the $U(n+m)$-bundle on X with transition functions (in a common refinement of the trvialization atlases for π_1 and π_2) given by $\left(\begin{array}{c|c} g^1_{\alpha\beta} & 0 \\ \hline 0 & g^2_{\alpha\beta} \end{array} \right)$ where $g^i_{\alpha\beta}$ are the transition function of π_i, $i = 1, 2$.

Proposition 2.10 (Whitney sum formula)**.** *If $\pi_1 \in \mathcal{P}(X, U(n))$ and $\pi_2 \in \mathcal{P}(X, U(m))$, then*

$$c(\pi_1 \oplus \pi_2) = c(\pi_1) \cup c(\pi_2).$$

Equivalently, $c_k(\pi_1 \oplus \pi_2) = \sum_{i+j=k} c_i(\pi_1) \cup c_j(\pi_2)$

Proof. First note that

$$B(U(n) \times U(m)) \simeq BU(n) \times BU(m). \tag{2.1}$$

Indeed, by taking the product of the universal bundles for $U(n)$ and $U(m)$, we get a $U(n) \times U(m)$-bundle over $BU(n) \times BU(m)$, with total space $EU(n) \times EU(m)$:

$$U(n) \times U(m) \hookrightarrow EU(n) \times EU(m) \to BU(n) \times BU(m). \tag{2.2}$$

Since $\pi_i(EU(n) \times EU(m)) \cong \pi_i(EU(n)) \times \pi_i(EU(m)) \cong 0$ for all i, it follows that (2.2) is the universal bundle for $U(n) \times U(m)$, thus proving (2.1).

Next, the inclusion $U(n) \times U(m) \hookrightarrow U(n+m)$ yields a map

$$\omega : B(U(n) \times U(m)) \simeq BU(n) \times BU(m) \longrightarrow BU(n+m).$$

By using the Künneth formula, one can show (e.g., see Milnor's book, p.164) that:

$$\omega^* c_k = \sum_{i+j=k} c_i \times c_j.$$

Therefore,

$$\begin{aligned} c_k(\pi_1 \oplus \pi_2) &= c_k(\Delta^*(\pi_1 \times \pi_2)) \\ &= \Delta^* c_k(\pi_1 \times \pi_2) \\ &= \Delta^*(f^*_{\pi_1 \times \pi_2}(c_k)) \\ &= \Delta^*(f^*_{\pi_1} \times f^*_{\pi_2})(\omega^* c_k) \\ &= \sum_{i+j=k} \Delta^*(f^*_{\pi_1}(c_i) \times f^*_{\pi_2}(c_j)) \\ &= \sum_{i+j=k} \Delta^*(c_i(\pi_1) \times c_j(\pi_2)) \\ &= \sum_{i+j=k} c_i(\pi_1) \cup c_j(\pi_2). \end{aligned}$$

Here, we use the fact that the classifying map for $\pi_1 \times \pi_2$, regarded as a $U(n+m)$-bundle is $\omega \circ (f_{\pi_1} \times f_{\pi_2})$. □

Since the trivial bundle has trivial Chern classes in positive degrees, we get

Corollary 2.11 (Stability of Chern classes)**.** *Let $\mathcal{E}^1$ be the trivial $U(1)$-bundle. Then*

$$c(\pi \oplus \mathcal{E}^1) = c(\pi).$$

3 Stiefel-Whitney classes of real vector bundles

As in Proposition 2.1, one easily obtains the following

Proposition 3.1.

$$H^* (BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2 [w_1, \cdots, w_n],$$

with $\deg w_i = i$.

Proof. This can be easily deduced by induction on n from the Leray-Serre spectral sequence of the fibration

$$O(n)/O(n-1) \cong S^{n-1} \hookrightarrow BO(n-1) \to BO(n),$$

by using the fact that $BO(1) \simeq \mathbb{R}P^\infty$ and $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1]$. □

Definition 3.2. *The generators* $w_1, \cdots, w_n$ *of* $H^*(BO(n); \mathbb{Z}/2)$ *are called the universal Stiefel-Whitney classes of* $O(n)$*-bundles.*

Recall from the classification theorem that, given $\pi : E \to X$ a principal $O(n)$-bundle, there exists a "classifying map" $f_\pi : X \to BO(n)$ such that $\pi \cong f_\pi^* \pi_{U(n)}$.

Definition 3.3. *The* i*-th Stiefel-Whitney class of the* $O(n)$*-bundle* $\pi : E \to X$ *with classifying map* $f_\pi : X \to BO(n)$ *is defined as*

$$w_i(\pi) := f_\pi^*(w_i) \in H^i(X; \mathbb{Z}/2).$$

The total Stiefel-Whitney class of π *is defined by*

$$w(\pi) = 1 + w_1(\pi) + \cdots w_n(\pi) \in H^*(X; \mathbb{Z}/2),$$

as an element in the cohomology ring with $\mathbb{Z}/2$*-coefficients.*

Remark 3.4. If π is a $O(n)$-bundle, then by definition we have that $w_i(\pi) = 0$, if $i > n$. Also, since the trivial bundle is classified by the constant map, it follows that the positive-degree Stiefel-Whitney classes of a trivial $O(n)$-bundle are all zero.

Stiefel-Whitney classes of $O(n)$-bundles enjoy similar properties as the Chern classes.

Proposition 3.5. *The Stiefel-Whitney classes satisfy the functoriality property and the Whitney sum formula.*

4 Stiefel-Whitney classes of manifolds and applications

If M is a smooth manifold, its tangent bundle TM can be regarded as an $O(n)$-bundle.

Definition 4.1. *The Stiefel-Whitney classes of a smooth manifold* M *are defined as*

$$w_i(M) := w_i(TM).$$

Theorem 4.2 (Wu). *Stiefel-Whitney classes are homotopy invariants, i.e., if $h : M_1 \to M_2$ is a homotopy equivalence then $h^*w_i(M_2) = w_i(M_1)$, for any $i \geq 0$.*

Characteristic classes are particularly useful for solving a wide range of topological problems, including the following:

(a) Given an n-dimensional smooth manifold M, find the minimal integer k such that M can be embedded/immersed in $\mathbb{R}^{n+k}$.

(b) Given an n-dimensional smooth manifold M, is there an $(n+1)$-dimensional smooth manifold W such that $\partial W = M$?

(c) Given a topological manifold M, classify/find exotic smooth structures on M.

4.1 The embedding problem

Let $f : M^m \to N^{m+k}$ be an embedding of smooth manifolds. Then

$$f^*TN = TM \oplus \nu, \tag{4.1}$$

where ν is the normal bundle of M in N. In particular, ν is of rank k, hence $w_i(\nu) = 0$ for all $i > k$. The Whitney product formula for Stiefel-Whitney classes, together with (4.1), yields that

$$f^*w(N) = w(M) \cup w(\nu). \tag{4.2}$$

Note that $w(M) = 1 + w_1(M) + \cdots$ is invertible in $H^*(M; \mathbb{Z}/2)$, hence

$$w(\nu) = w(M)^{-1} \cup f^*w(N).$$

In particular, if $N = \mathbb{R}^{m+k}$, one gets $w(\nu) = w(M)^{-1}$.

The same considerations apply in the case when $f : M^m \to N^{m+k}$ is required to be only an immersion. In this case, the existence of the normal bundle ν is guaranteed by the following simple result:

Lemma 4.3. *Let*

$$\begin{array}{ccc} E_1 & \xrightarrow{\quad i \quad} & E_2 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & X & \end{array}$$

be a linear monomorphism of vector bundles, i.e., in local coordinates, i is given by $U \times \mathbb{R}^n \to U \times \mathbb{R}^m$ $(n \leq m)$, $(u, v) \mapsto (u, \ell(u)v)$, where $\ell(u)$ is a linear map of rank n for all $u \in U$. Then there exists a vector bundle $\pi_1^\perp : E_1^\perp \to X$ so that $\pi_2 \cong \pi_1 \oplus \pi_1^\perp$.

To summarize, we showed that if $f : M^m \to N^{m+k}$ is an embedding or an immersion of smooth manifolds, than one can solve for $w(\nu)$ in (4.2), where ν is the normal bundle of M in N. Moreover, since ν has rank k, we must have that $w_i(\nu) = 0$ for all $i > k$.

The following result of Whitney states that one can always solve for $w(\nu)$ if the codimension k is large enough. More precisely, we have:

Theorem 4.4 (Whitney)**.** *Any smooth map $f : M^m \to N^{m+k}$ is homotopic to an embedding for $k \geq m+1$.*

Let us now consider the problem of embedding (or immersing) $\mathbb{R}P^m$ into $\mathbb{R}^{m+k}$. If ν is the corresponding normal bundle of rank k, we have that $w(\nu) = w(\mathbb{R}P^m)^{-1}$.

We need the following calculation:

Theorem 4.5.

$$w(\mathbb{R}P^m) = (1+x)^{m+1}, \tag{4.3}$$

where $x \in H^1(\mathbb{R}P^m; \mathbb{Z}/2)$ is a generator.

Before proving Theorem 4.5, let us discuss some examples.

Example 4.6. Let us investigate constraints on the codimension k of an embedding of $\mathbb{R}P^9$ into $\mathbb{R}^{9+k}$. By Theorem 4.5, we have:

$$w(\mathbb{R}P^9) = (1+x)^{10} = (1+x)^8(1+x)^2 = (1+x^8)(1+x^2) = 1 + x^2 + x^8,$$

since $x^{10} = 0$ in $H^*(\mathbb{R}P^9; \mathbb{Z}/2)$. Therefore,

$$w(\mathbb{R}P^9)^{-1} = 1 + x^2 + x^4 + x^6.$$

If an embedding (or immersion) f of $\mathbb{R}P^9$ into $\mathbb{R}^{9+k}$ exists, then $w(\nu) = w^{-1}(\mathbb{R}P^9)$, where ν is the corresponding rank k normal bundle. In particular, $w_6(\nu) \neq 0$. Since we must have $w_i(\nu) = 0$ for $i > k$, we conclude that $k \geq 6$. For example, this shows that $\mathbb{R}P^9$ cannot be embedded into $\mathbb{R}^{14}$.

Example 4.7. Similarly, if $m = 2^r$ then

$$w(\mathbb{R}P^{2^r}) = (1+x)^{2^r+1} = (1+x)^{2^r}(1+x) = 1 + x + x^{2^r}.$$

If there exists an embedding or immersion $\mathbb{R}P^{2^r} \hookrightarrow \mathbb{R}^{2^r+k}$ with normal bundle ν, then

$$w(\nu) = w(\mathbb{R}P^{2^r})^{-1} = 1 + x + x^2 + \cdots + x^{2^r-1},$$

hence $k \geq 2^r - 1 = m - 1$. In particular, $\mathbb{R}P^8$ cannot be immersed in $\mathbb{R}^{14}$. In this case, one can actually construct an immersion of $\mathbb{R}P^{2^r}$ into $\mathbb{R}^{2^r+k}$ for any $k \geq 2^r - 1$, due to the following result:

Theorem 4.8 (Whitney)**.** *An m-dimensional smooth manifold can be embedded in $\mathbb{R}^{2m}$ and immersed in $\mathbb{R}^{2m-1}$.*

Definition 4.9. *A smooth manifold is parallelizable if its tanget bundle TM is trivial.*

Example 4.10. Lie groups, hence in particular S^1, S^3 and S^7, are parallelizable.

Theorem 4.5 can be used to prove the following:

Theorem 4.11. *$w(\mathbb{R}P^m) = 1$ if and only if $m + 1 = 2^r$ for some r. In particular, if $\mathbb{R}P^m$ is parallelizable, then $m + 1 = 2^r$ for some r.*

Proof. Note that if $\mathbb{R}P^m$ is parallelizable, then $w(\mathbb{R}P^m) = 1$ since $T\mathbb{R}P^m$ is a trivial bundle. If $m + 1 = 2^r$, then $w(\mathbb{R}P^m) = (1 + x)^{2^r} = 1 + x^{2^r} = 1 + x^{m+1} = 1$. On the other hand, if $m + 1 = 2^r k$, where $k > 1$ is an odd integer, we have

$$w(\mathbb{R}P^m) = [(1 + x)^{2^r}]^k = (1 + x^{2^r})^k = 1 + kx^{2^r} + \cdots \neq 1,$$

since $x^{2^r} \neq 0$ (indeed, $2^r < 2^r k = m + 1$). □

In fact, the following result holds:

Theorem 4.12 (Adams)**.** *$\mathbb{R}P^m$ is parallelizable if and only if $m \in \{1, 3, 7\}$.*

Let us now get back to the proof of Theorem 4.5

Proof of Theorem 4.5. The idea is to find a splitting of (a stabilization of) $T\mathbb{R}P^m$ into line bundles, then to apply the Whitney sum formula.

Recall that $O(1)$-bundles on $\mathbb{R}P^m$ are classified by

$$[\mathbb{R}P^m, BO(1)] = [\mathbb{R}P^m, K(\mathbb{Z}/2, 1)] \cong H^1(\mathbb{R}P^m; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

We'll denote by $\mathcal{E}^1$ the trivial $O(1)$-bundle, and let π be the non-trivial $O(1)$-bundle on $\mathbb{R}P^m$. Since $O(1) \cong \mathbb{Z}/2$, $O(1)$-bundles are regular double coverings. It is then clear that π corresponds to the 2-fold cover $S^m \to \mathbb{R}P^m$.

We have $w(\mathcal{E}^1) = 1 \in H^*(\mathbb{R}P^n; \mathbb{Z}/2)$. To calculate $w(\pi)$, we notice that the inclusion map $i : \mathbb{R}P^n \to \mathbb{R}P^\infty$ classifies the bundle π, as the universal bundle $S^\infty \to \mathbb{R}P^\infty$ pulls back under the inclusion to $S^m \to \mathbb{R}P^m$. In particular,

$$w_1(\pi) = i^* w_1 = i^* x = x,$$

where x is the generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2) = H^1(\mathbb{R}P^m; \mathbb{Z}/2)$. Therefore,

$$w(\pi) = 1 + x.$$

We next show that

$$T\mathbb{R}P^m \oplus \mathcal{E}^1 \cong \underbrace{\pi \oplus \cdots \oplus \pi}_{m+1 \text{ times}}, \tag{4.4}$$

from which the computation of $w(\mathbb{R}P^m)$ follows by an application of the Whitney sum formula.

To prove (4.4), start with $S^m \hookrightarrow \mathbb{R}^{m+1}$ with (rank one) normal bundle denoted by $\mathcal{E}_\nu$. Note that $\mathcal{E}_\nu$ is a trivial line bundle on S^m, as it has a global non-zero section (mapping $y \in S^m$ to the normal vector ν_y at y). We then have

$$TS^m \oplus \mathcal{E}_\nu \cong T\mathbb{R}^{m+1}|_{S^m} = \mathcal{E}^{m+1} \cong \underbrace{\mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^1}_{m+1 \text{ times}},$$

with $\mathcal{E}^{m+1}$ the trivial bundle of rank $n+1$ on S^m, i.e., the Whitney sum of $m+1$ trivial line bundles $\mathcal{E}^1$ on S^m, each of which is generated by the global non-zero section $y \mapsto \frac{d}{dx_i}|_y$, $i = 1, \cdots, m+1$.

Let $a : S^m \to S^m$ be the antipodal map, with differential $da : TS^m \to TS^m$. Let $\gamma : (-\epsilon, \epsilon) \to S^m$, $\gamma(0) = y$, $v = \gamma'(0) \in T_y S^m$. Then $da(v) = \frac{d}{dt}(a \circ \gamma(t))|_{t=0} = -\gamma'(0) = -v \in T_{a(y)} S^m$. Therefore da is an involution on TS^m, commuting with a, and hence

$$TS^m/da = T\mathbb{R}P^m.$$

Next note that the normal bundle $\mathcal{E}_\nu$ on S^m is invariant under the antipodal action (as $da(\nu_y) = \nu_{a(y)}$), so it descends to the trivial line bundle on $\mathbb{R}P^m$, i.e.,

$$\mathcal{E}_\nu/da \cong \mathcal{E}^1.$$

Finally,

$$S^m \times \mathbb{R}/da \cong S^m \times \mathbb{R}/(y, t\frac{d}{dx_i}) \sim (-y, -t\frac{d}{dx_i}) \cong S^n \times_{\mathbb{Z}/2} \mathbb{R},$$

which is the associated bundle of π with fiber $\mathbb{R}$. So,

$$\mathcal{E}^1/da \cong \pi.$$

This concludes the proof of (4.4) and of the theorem. □

Remark 4.13. Note that $\mathbb{R}P^3 \cong SO(3)$ is a Lie group, so its tangent bundle is trivial. In this case, once can conclude directly that $w(\mathbb{R}P^3) = 1$, but this fact can also be seen from formula (4.3).

4.2 Boundary Problem.

For a closed manifold M^n, let μ_M be the fundamental class in $H_n(M, \mathbb{Z}/2)$. We will associate to M certain $\mathbb{Z}/2$-invariants, called its *Stiefel-Whitney numbers*.

Definition 4.14. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a tuple of non-negative integers such that $\sum_{i=1}^n i\alpha_i = n$. Set*

$$w^{[\alpha]}(M) := w_1(M)^{\alpha_1} \cup \cdots \cup w_n(M)^{\alpha_n} \in H^n(M; \mathbb{Z}/2).$$

The Stiefel-Whitney number of M with index α is defined as

$$w_{(\alpha)}(M) := \langle w^{[\alpha]}(M), \mu_M \rangle \in \mathbb{Z}/2,$$

where $\langle -, - \rangle : H^n(M; \mathbb{Z}/2) \times H_n(M; \mathbb{Z}/2) \to \mathbb{Z}/2$ is the Kronecker evaluation pairing.

We have the following result:

Theorem 4.15 (Pontrjagin-Thom)**.** *A closed n-dimensional manifold M is the boundary of a smooth compact $(n+1)$-dimensional manifold W if and only if all Stiefel-Whitney numbers of M vanish.*

Proof. We only show here one implication (due to Pontrjagin), namely that if $M = \partial W$ then $w_{(\alpha)}(M) = 0$, for any $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\sum_{i=1}^n i\alpha_i = n$.

If $i : M \hookrightarrow W$ denotes the boundary embedding, then

$$i^*TW \cong TM \oplus \nu^1,$$

where ν^1 is the rank-one normal bundle of M in W.

Assume that TW has a Euclidean metric. Then the normal bundle ν^1 is trivialized by picking the inward unit normal vector at every point in M. Hence

$$i^*TW \cong TM \oplus \mathcal{E}^1,$$

where $\mathcal{E}^1$ is the trivial line bundle on M. In particular, the Whitney sum formula yields that

$$w_k(M) = i^* w_k(W),$$

for $k = 1, \cdots, n$, so $w^{[\alpha]}(M) = i^* w^{[\alpha]}(W)$ for any α as above.

Let μ_W be the fundamental class of (W, M) i.e., the generator of $H_{n+1}(W, M; \mathbb{Z}/2)$, and let μ_M be the fundamental class of M as above. From the long exact homology sequence for the pair (W, M) and Poincaré duality, we have that

$$\partial(\mu_W) = \mu_M.$$

Let $\delta : H^n(M; \mathbb{Z}/2) \to H^{n+1}(W, M; \mathbb{Z}/2)$ be the map adjoint to ∂. The naturality of the cap product yields the identity:

$$\langle y, \mu_M \rangle = \langle y, \partial \mu_W \rangle = \langle \delta y, \mu_W \rangle$$

for any $y \in H^n(M; \mathbb{Z}/2)$. Putting it all together we have:

$$\begin{aligned} w_{(\alpha)}(M) &= \langle w^{[\alpha]}(M), \mu_M \rangle \\ &= \langle i^* w^{[\alpha]}(W), \partial \mu_W \rangle \\ &= \langle \delta(i^* w^{[\alpha]}(W)), \mu_W \rangle \\ &= \langle 0, \mu_W \rangle \\ &= 0, \end{aligned}$$

since $\delta \circ i^* = 0$, as can be seen from the long exact cohomology sequence for the pair (W, M). □

Example 4.16. Suppose $M = X \sqcup X$, i.e., M is the disjoint union of two copies of a closed n-dimensional manifold X. Then for any α, $w_{(\alpha)}(M) = 2w_{(\alpha)}(X) = 0$. This is consistent with the fact that $X \sqcup X$ is the boundary of the cylinder $X \times [0, 1]$.

Example 4.17. Every $\mathbb{R}P^{2k-1}$ is a boundary. Indeed, the total Stiefel-Whitney class of $\mathbb{R}P^{2k-1}$ is $(1+x)^{2k} = (1+x^2)^k$, with x the generator of $H^1(\mathbb{R}P^{2k-1}; \mathbb{Z}/2)$. Thus, all the odd degree Stiefel-Whitney classes of $\mathbb{R}P^{2k-1}$ are 0. Since every monomial in the Stiefel-Whitney classes of $\mathbb{R}P^{2k-1}$ of total degree $2k-1$ must contain a factor w_j with j odd, all Stiefel-Whitney numbers of $\mathbb{R}P^{2k-1}$ vanish. This implies the claim by the Pontrjagin-Thom Theorem 4.15.

Example 4.18. The real projective space $\mathbb{R}P^{2k}$ is not a boundary, for any integer $k \geq 0$. Indeed, the total Stiefel-Whitney class of $\mathbb{R}P^{2k}$ is

$$\begin{aligned} w(\mathbb{R}P^{2k}) = (1+x)^{2k+1} &= 1 + \binom{2k+1}{1} x + \cdots + \binom{2k+1}{2k} x^{2k} \\ &= 1 + x + \cdots + x^{2k} \end{aligned}$$

In particular, $w_{2k}(\mathbb{R}P^{2k}) = x^{2k}$. It follow that for $\alpha = (0, 0, \ldots, 1)$ we have

$$w_{(\alpha)}(\mathbb{R}P^{2k}) = 1 \neq 0.$$

5 Pontrjagin classes

In this section, unless specified, we use the symbol π to denote real vector bundles (or $O(n)$-bundles), and use ω for complex vector bundles (or $U(n)$-bundles) on a topological space X.

Given a real vector bundle π, we can consider its *complexification* $\pi \otimes \mathbb{C}$, i.e., the complex vector bundle with same transition functions as π:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(n) \subset U(n),$$

and fiber $\mathbb{R}^n \otimes \mathbb{C} \cong \mathbb{C}^n$.

Given a complex vector bundle ω, we can consider its *realization* $\omega_{\mathbb{R}}$, obtained by forgeting the complex structure, i.e., with transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(n) \hookrightarrow O(2n).$$

Given a complex vector bundle ω, its *conjugation* $\overline{\omega}$ is defined by transition functions

$$\overline{g_{\alpha\beta}} : U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} U(n) \xrightarrow{\bar{\cdot}} U(n),$$

with the second homomorphism given by conjugation. $\overline{\omega}$ has the same underlying real vector bundle as ω, but the opposite complex structure on its fibers.

Lemma 5.1. *If ω is a complex vector bundle, then*

$$\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}.$$

Proof. Let $\jmath$ be the linear transformation on $F_{\mathbb{R}} \otimes \mathbb{C}$ given by multiplication by i. Here F is the fiber of complex vector bundle ω, and $F_{\mathbb{R}}$ is the fiber of its realization $\omega_{\mathbb{R}}$. Then $\jmath^2 = -id$, so we have

$$F_{\mathbb{R}} \otimes \mathbb{C} \cong Eigen(i) \oplus Eigen(-i),$$

where $\jmath$ acts as multiplication by i on $Eigen(i)$, and it acts as multiplication by $-i$ on $Eigen(-i)$. Moreover, we have $F \subseteq Eigen(i)$ and $\overline{F} \subseteq Eigen(-i)$. By a dimension count we then get that $F_{\mathbb{R}} \otimes \mathbb{C} \cong F \oplus \overline{F}$. ∎

Lemma 5.2. *Let π be a real vector bundle. Then*

$$\overline{\pi \otimes \mathbb{C}} \cong \pi \otimes \mathbb{C}.$$

Proof. Indeed, since the transition functions of $\pi \otimes \mathbb{C}$ are real-values (same as those of π), they are also the transition functions for $\overline{\pi \otimes \mathbb{C}}$. □

Lemma 5.3. *If ω is a rank n complex vector bundle, the Chern classes of its conjugate $\overline{\omega}$ are computed by*

$$c_k(\overline{\omega}) = (-1)^k \cdot c_k(\omega),$$

for any $k = 1, \cdots, n$.

Proof. Recall that one way to define (universal) Chern classes is by induction by using the fibration

$$S^{2k-1} \hookrightarrow BU(k-1) \to BU(k).$$

In fact,

$$c_k = d_{2k}(a),$$

where a is the generator of $H^{2k-1}(S^{2k-1}; \mathbb{Z})$.

The complex conjugation on the fiber S^{2k-1} of the above fibration is a map of degree $(-1)^k$ (it keeps k out of $2k$ real basis vectors invariant, and it changes the sign of the other k; each sign change is a reflection and it has degree -1). In particular, the homomorphism $H^{2k-1}(S^{2k-1}; \mathbb{Z}) \to H^{2k-1}(S^{2k-1}; \mathbb{Z})$ induced by conjugation is defined by $a \mapsto (-1)^k \cdot a$. Altogether, this gives $c_k(\overline{\omega}) = (-1)^k \cdot c_k(\omega)$. □

Combining the results from Lemma 5.2 and Lemma 5.3, we have the following:

Corollary 5.4. *For any real vector bundle π,*

$$c_k(\pi \otimes \mathbb{C}) = c_k(\overline{\pi \otimes \mathbb{C}}) = (-1)^k c_k(\pi \otimes \mathbb{C}).$$

In particular, for any odd integer k, $c_k(\pi \otimes \mathbb{C})$ is an integral cohomology class of order 2.

Definition 5.5 (Pontryagin classes of real vector bundles)**.** *Let $\pi : E \to X$ be a real vector bundle of rank n. The i-th Pontrjagin class of π is defined as:*

$$p_i(\pi) := (-1)^i c_{2i}(\pi \otimes \mathbb{C}) \in H^{4i}(X; \mathbb{Z}).$$

If ω a complex vector bundle of rank n, we define its i-th Pontryagin class as

$$p_i(\omega) := p_i(\omega_{\mathbb{R}}) = (-1)^i c_{2i}(\omega \oplus \overline{\omega}).$$

Remark 5.6. Note that $p_i(\pi) = 0$ for all $i > \frac{n}{2}$.

Definition 5.7. *If π is a real vector bundle on X, its total Pontrjagin class is defined as*

$$p(\pi) = p_0 + p_1 + \cdots \in H^*(X; \mathbb{Z}).$$

Theorem 5.8 (Product formula)**.** *If π_1 and π_2 are real vector bundles on X, then*

$$p(\pi_1 \oplus \pi_2) = p(\pi_1) \cup p(\pi_2) \text{ mod 2-torsion.}$$

Proof. We have $(\pi_1 \oplus \pi_2) \otimes \mathbb{C} \cong (\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C})$. Therefore,

$$\begin{aligned} p_i(\pi_1 \oplus \pi_2) &= (-1)^i c_{2i}((\pi_1 \oplus \pi_2) \otimes \mathbb{C}) \\ &= (-1)^i \sum_{k+l=2i} c_k(\pi_1 \otimes \mathbb{C}) \cup c_l(\pi_2 \otimes \mathbb{C}) \\ &= (-1)^i \sum_{a+b=i} c_{2a}(\pi_1 \otimes \mathbb{C}) \cup c_{2b}(\pi_2 \otimes \mathbb{C}) + \{\text{elements of order } 2\} \\ &= \sum_{a+b=i} p_a(\pi_1) \cup p_b(\pi_2) + \{\text{elements of order } 2\}, \end{aligned}$$

thus proving the claim. □

Definition 5.9. *If M is a real smooth manifold, we define*

$$p(M) := p(TM).$$

If M is a complex manifold, we define

$$p(M) := p((TM)_{\mathbb{R}}).$$

Here TM is the tangent bundle of the manifold M.

In order to give applications of Pontrjagin classes, we need the following computational result:

Theorem 5.10 (Chern and Pontrjagin classes of complex projective space)**.** *The total Chern and Pontrjagin classes of the complex projective space $\mathbb{C}P^n$ are computed by:*

$$c(\mathbb{C}P^n) = (1+c)^{n+1}, \tag{5.1}$$

$$p(\mathbb{C}P^n) = (1+c^2)^{n+1}, \tag{5.2}$$

where $c \in H^2(\mathbb{C}P^n; \mathbb{Z})$ is a generator.

Proof. The arguments involved in the computation of $c(\mathbb{C}P^n)$ are very similar to those of Theorem 4.5. Indeed, one first shows that there is a splitting

$$T\mathbb{C}P^n \oplus \mathcal{E}^1 = \underbrace{\gamma \oplus \cdots \oplus \gamma}_{n+1 \text{ times}},$$

were $\mathcal{E}^1$ is the trivial complex line bundle on $\mathbb{C}P^n$ and γ is the complex line bundle associated to the principle S^1-bundle $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$. Then γ is classified by the inclusion

$$\begin{array}{ccc} S^{2n+1} & \hookrightarrow & S^\infty \\ \downarrow & & \downarrow \\ \mathbb{C}P^n & \hookrightarrow & \mathbb{C}P^\infty = BU(1) \end{array}$$

and hence $c_1(\gamma) = c$, the generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z}) = H^2(\mathbb{C}P^n;\mathbb{Z})$. The Whitney sum formula for Chern classes then yields:

$$c(\mathbb{C}P^n) = c(T\mathbb{C}P^n) = c(\gamma)^{n+1} = (1+c)^{n+1}.$$

By conjugation, one gets

$$c(\overline{T\mathbb{C}P^n}) = (1-c)^{n+1}.$$

Therefore,

$$c((T\mathbb{C}P^n)_{\mathbb{R}} \otimes \mathbb{C}) = c(T\mathbb{C}P^n \oplus \overline{T\mathbb{C}P^n}) = c(T\mathbb{C}P^n) \cup c(\overline{T\mathbb{C}P^n}) = (1-c^2)^{n+1},$$

from which one can readily deduce that $p(\mathbb{C}P^n) = (1+c^2)^{n+1}$. □

5.1 Applications to the embedding problem

After forgetting the complex structure, $\mathbb{C}P^n$ is a $2n$-dimensional real smooth manifold. Suppose that there is an embedding

$$\mathbb{C}P^n \hookrightarrow \mathbb{R}^{2n+k},$$

and we would like to find contraints on the embedding codimension k by means of Pontrjagin classes.

Let $(T\mathbb{C}P^n)_{\mathbb{R}}$ be the realization of the tangent bundle for $\mathbb{C}P^n$. Then the existence of an embedding as above implies that there exists a normal (real) bundle ν^k of rank k such that

$$(T\mathbb{C}P^n)_{\mathbb{R}} \oplus \nu^k \cong T\mathbb{R}^{2n+k}|_{\mathbb{C}P^n} \cong \mathcal{E}^{2n+k}, \tag{5.3}$$

with $\mathcal{E}^{2n+k}$ denoting the trivial real vector bundle of rank $2n+k$.

By applying the Pontrjagin class p to (5.3) and using the product formula of Theorem 5.8 together with the fact that there are no elements of order 2 in $H^*(\mathbb{C}P^n;\mathbb{Z})$, we have

$$p(\mathbb{C}P^n) \cdot p(\nu^k) = 1.$$

Therefore, we get

$$p(\nu^k) = p(\mathbb{C}P^n)^{-1}. \tag{5.4}$$

And by the definition of the Pontryagin classes, we know that if $p_i(\nu^k) \neq 0$, then $i \leq \frac{k}{2}$.

Example 5.11. *In this example, we use Pontrjagin classes to show that $\mathbb{C}P^2$ does not embed in $\mathbb{R}^5$. First,*

$$p(\mathbb{C}P^2) = (1+c^2)^3 = 1 + 3c^2,$$

with $c \in H^2(\mathbb{C}P^2;\mathbb{Z})$ a generator (hence $c^3 = 0$). If there is an embedding $\mathbb{C}P^2 \hookrightarrow \mathbb{R}^{4+k}$ with normal bundle ν^k, then

$$p(\nu^k) = p(\mathbb{C}P^2)^{-1} = 1 - 3c^2.$$

Hence $p_1(\nu^k) \neq 0$, which implies that $k \geq 2$.

6 Oriented cobordism and Pontrjagin numbers

If M is a smooth oriented manifold, we denote by $-M$ the same manifold but with the opposite orientation.

Definition 6.1. *Let M^n and N^n be smooth, closed, oriented real manifolds of dimension n. We say M and N are oriented cobordant if there exists a smooth, compact, oriented $(n+1)$-dimensional manifold W^{n+1}, such that $\partial W = M \sqcup (-N)$.*

Remark 6.2. Let us say a word of convention about orienting a boundary. For any $x \in \partial W$, there exist an inward normal vector $\nu_+(x)$ and an outward normal vector $\nu_-(x)$ to the boundary at x. By using a partition of unity, one can construct an inward/outward normal vector field $\nu_\pm : \partial W \to TW|_{\partial W}$. By convention, a frame $\{e_1, \cdots, e_n\}$ on $T_x(\partial W)$ is positive if $\{e_1, \cdots, e_n, \nu_-(x)\}$ is a positive frame for $T_x W$.

Lemma 6.3. *Oriented cobordism is an equivalence relation.*

Proof. M and $-M$ are clearly oriented cobordant because $M \sqcup (-M)$ is diffeomorphic to the boundary of $M \times [0,1]$. Hence oriented cobordism is reflexive. The symmetry can be deduced from the fact that, if $M \sqcup (-N) \simeq \partial W$, then $N \sqcup (-M) \simeq \partial(-W)$. Finally, if $M_1 \sqcup (-M_2) \simeq \partial W$, and $M_2 \sqcup (-M_3) \simeq \partial W'$, then we can glue W and W' along the common boundary and get a new manifold with boundary $M_1 \sqcup (-M_3)$. Hence oriented cobordism is also transitive. □

Definition 6.4. *Let Ω_n be the set of cobordism classes of closed, oriented, smooth n-manifolds.*

Corollary 6.5. *The set Ω_n is an abelian group with the disjoint union operation.*

Proof. This is an immediate consequence of Lemma 6.3. The zero element in Ω_n is the class of $\emptyset$, or equivalently, $[M] = 0 \in \Omega_n$ if and only if $M = \partial W$, for some compact manifold W. The inverse of $[M]$ is $[-M]$, since $M \sqcup (-M)$ is a boundary. □

A natural problem to investigate is to describe the group Ω_n by generators and relations. For example, both $[\mathbb{C}P^4]$ and $[\mathbb{C}P^2 \times \mathbb{C}P^2]$ are elements of Ω_8. Do they represent the same element, i.e., are $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ oriented cobordant? A lot of insight is gained by using *Pontrjagin numbers*.

Definition 6.6. *Let M^n be a smooth, closed, oriented real n-manifold, with fundamental class $\mu_M \in H_n(M;\mathbb{Z})$. Let $k = [\frac{n}{4}]$ and choose a partition $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{Z}^k$ such that $\sum_{j=1}^k 4j\alpha_j = n$. The Pontrjagin number of M associated to the partition α is defined as:*

$$p_{(\alpha)}(M) = \langle p_1(M)^{\alpha_1} \cup \cdots \cup p_k(M)^{\alpha_k}, \mu_M \rangle \in \mathbb{Z}.$$

Remark 6.7. If n is not divisible by 4, then $p_{(\alpha)}(M) = 0$ by definition.

Theorem 6.8. *For $n = 4k$, each $p_{(\alpha)}$ defines a homomorphism $\Omega_n \longrightarrow \mathbb{Z}$, $[M] \mapsto p_{(\alpha)}(M)$. Hence oriented cobordant manifolds have the same Pontrjagin numbers. In particular, if $M^n = \partial W^{n+1}$, then $p_{(\alpha)}(M) = 0$ for any partition α.*

Proof. If $M = M_1 \sqcup M_2$, then $[M] = [M_1] + [M_2] \in \Omega_n$ and $\mu_M = \mu_{M_1} + \mu_{M_2} \in H_n(M; \mathbb{Z})$. It follows readily that $p_{(\alpha)}(M) = p_{(\alpha)}(M_1) + p_{(\alpha)}(M_2)$.

If $M = \partial N$, then it can be shown as in the proof of Theorem 4.15 that $p_{(\alpha)}(M) = 0$ for any partition α. □

Example 6.9. By Theorem 5.10, we have that $p(\mathbb{C}P^{2n}) = (1+c^2)^{2n+1}$, where c is a generator of $H^2(\mathbb{C}P^{2n}; \mathbb{Z})$. Hence $p_i(\mathbb{C}P^{2n}) = \binom{2n+1}{i} c^{2i}$. For the partition $\alpha = (0, \dots, 0, 1)$, we find that $p_{(\alpha)}(\mathbb{C}P^{2n}) = \left\langle \binom{2n+1}{n} c^{2n}, \mu_{\mathbb{C}P^{2n}} \right\rangle = \binom{2n+1}{n} \neq 0$. We conclude that $\mathbb{C}P^{2n}$ is not an oriented boundary.

Remark 6.10. If we reverse the orientation of a manifold M of real dimension $n = 4k$, the Pontrjagin classes remain unchanged, but the fundamental class μ_M changes sign. Therefore, all Pontrjagin numbers $p_{(\alpha)}(M)$ change sign. This shows that, if some Pontrjagin number $p_{(\alpha)}(M)$ is nonzero, then M cannot have any orientation-reversing diffeomorphism.

Example 6.11. The above remark and Example 6.9 show that $\mathbb{C}P^{2n}$ does not have any orientation-reversing diffeomorphism. However, $\mathbb{C}P^{2n+1}$ has an orientation-reversing diffeomorphism induced by complex conjugation.

Example 6.12. Let us consider Ω_4. As $\mathbb{C}P^2$ is not an oriented boundary by Example 6.9, we have $[\mathbb{C}P^2] \neq 0 \in \Omega_n$. Recall that $p(\mathbb{C}P^2) = 1 + 3c^2$, so $p_1(\mathbb{C}P^2) = 3c^2$. For the partition $\alpha = (1)$, we then get that $p_{(1)}(\mathbb{C}P^2) = 3$. So

$$\Omega_4 \xrightarrow{p_{(1)}} 3\mathbb{Z} \longrightarrow 0$$

is exact, thus $\operatorname{rank}(\Omega_4) \geq 1$.

Example 6.13. We next consider Ω_8. The partitions to work with in this case are $\alpha_1 = (2, 0)$ and $\alpha_2 = (0, 1)$, and Theorem 6.8 yields a homomorphism

$$\Omega_8 \xrightarrow{(p_{(\alpha_1)}, p_{(\alpha_2)})} \mathbb{Z} \oplus \mathbb{Z}.$$

We aim to show that

$$\operatorname{rank}(\Omega_8) = \dim_{\mathbb{Q}}(\Omega_8 \otimes \mathbb{Q}) \geq 2.$$

We start by noting that both $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ are compact oriented 8-manifolds which are not boundaries. We calculate the Pontrjagin numbers of these two spaces. First,

$$p(\mathbb{C}P^4) = (1 + c^2)^5 = 1 + 5c^2 + 10c^4,$$

where c is a generator of $H^2(\mathbb{C}P^4; \mathbb{Z})$. Hence, $p_1(\mathbb{C}P^4) = 5c^2$ and $p_2(\mathbb{C}P^4) = 10c^4$. The Pontrjagin numbers of $\mathbb{C}P^4$ corresponding to the partitions $\alpha_1 = (2, 0)$ and $\alpha_2 = (0, 1)$ are given as:

$$p_{(\alpha_1)}(\mathbb{C}P^4) = \langle p_1(\mathbb{C}P^4)^2, \mu_{\mathbb{C}P^4} \rangle = 25, \quad p_{(\alpha_2)}(\mathbb{C}P^4) = \langle p_2(\mathbb{C}P^4), \mu_{\mathbb{C}P^4} \rangle = 10.$$

In order to compute the corresponding Pontrjagin numbers for $\mathbb{C}P^2 \times \mathbb{C}P^2$, let $pr_i : \mathbb{C}P^2 \times \mathbb{C}P^2 \to \mathbb{C}P^2$, $i = 1, 2$, be the projections on factors. Then

$$T(\mathbb{C}P^2 \times \mathbb{C}P^2) \cong pr_1^* T(\mathbb{C}P^2) \oplus pr_2^* T(\mathbb{C}P^2),$$

so Theorem 5.8 yields that

$$p(\mathbb{C}P^2 \times \mathbb{C}P^2) = pr_1^* p(\mathbb{C}P^2) \cup pr_2^* p(\mathbb{C}P^2) = p(\mathbb{C}P^2) \times p(\mathbb{C}P^2),$$

where $\times$ denotes the external product. Let c_1 and c_2 denote the generators of the second integral cohomology of the two $\mathbb{C}P^2$ factors. Then:

$$p(\mathbb{C}P^2 \times \mathbb{C}P^2) = (1 + c_1^2)^3 \cdot (1 + c_2^2)^3 = (1 + 3c_1^2) \cdot (1 + 3c_2^2) = 1 + 3c_1^2 + 3c_2^2 + 9c_1^2 c_2^2.$$

Hence, $p_1(\mathbb{C}P^2 \times \mathbb{C}P^2) = 3(c_1^2 + c_2^2)$ and $p_2(\mathbb{C}P^2 \times \mathbb{C}P^2) = 9c_1^2 c_2^2$. Therefore, the Pontrjagin numbers of $\mathbb{C}P^2 \times \mathbb{C}P^2$ corresponding to the partitions α_1 and α_2 are computed by (here we use the fact that $c_1^4 = 0 = c_2^4$):

$$p_{(\alpha_1)}(\mathbb{C}P^2 \times \mathbb{C}P^2) = 18, \quad p_{(\alpha_2)}(\mathbb{C}P^2 \times \mathbb{C}P^2) = 9.$$

The values of the homomorphism $(p_{(\alpha_1)}, p_{(\alpha_2)}) : \Omega_8 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ on $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ fit into the 2×2 matrix $\begin{bmatrix} 25 & 18 \\ 10 & 9 \end{bmatrix}$ with nonzero determinant. Hence $\mathrm{rank}(\Omega_8) \geq 2$.

More generally, we the following qualitative description of Ω_n, which we mention here without proof.

Theorem 6.14 (Thom). *The oriented cobordism group Ω_n is finitely generated of rank $|I|$, where I is the collection of partitions α satisfying $\sum_j 4j\alpha_j = n$. In fact, modulo torsion, Ω_n is generated by products of even complex projective spaces. Moreover, $\bigoplus_{\alpha \in I} p_{(\alpha)} : \Omega_n \to \mathbb{Z}^{|I|}$ is an injective homomorphism onto a subgroup of the same rank.*

Example 6.15. Our computations from Examples 6.12 and 6.13 together with Theorem 6.14 yield that in fact we have: $\mathrm{rank}(\Omega_4) = 1$ and $\mathrm{rank}(\Omega_8) = 2$.

7 Signature as an oriented cobordism invariant

Recall that if M^{4k} is a closed, oriented manifold of real dimension $n = 4k$, then we can define its *signature* $\sigma(M)$ as the signature of the bilinear symmetric pairing

$$H^{2k}(M; \mathbb{Q}) \times H^{2k}(M; \mathbb{Q}) \to \mathbb{Q},$$

which is non-degenerate by Poincaré duality. Recall also that if M is an oriented boundary then $\sigma(M) = 0$. This suffices to deduce the folowing result:

Theorem 7.1 (Thom). *$\sigma : \Omega_{4k} \to \mathbb{Z}$ is a homomorphism.*

It follows from Theorems 6.14 and 7.1 that the *signature is a rational combination of Pontrjagin numbers*, i.e.,

$$\sigma = \sum_{\alpha \in I} a_\alpha p_{(\alpha)} \tag{7.1}$$

for some coefficients $a_\alpha \in \mathbb{Q}$. The *Hirzebruch signature theorem* provides an explicit formula for these coefficients a_α. In what follows we compute by hand the coefficients a_α in the cases of Ω_4 and Ω_8.

Example 7.2. On closed oriented 4-manifolds, the signature is computed by

$$\sigma = ap_{(1)}, \tag{7.2}$$

with $a \in \mathbb{Q}$ to be determined. Since a is the same for any $[M] \in \Omega_4$, we will determine it by performing our calculations on $M = \mathbb{C}P^2$. Recall that $\sigma(\mathbb{C}P^2) = 1$, and if $c \in H^2(\mathbb{C}P^2; \mathbb{Z})$ is a generator then $p_1(\mathbb{C}P^2) = 3c^2$. Hence $p_{(1)}(\mathbb{C}P^2) = 3$, and (7.2) implies that $1 = 3a$, or $a = \frac{1}{3}$. Therefore, for any closed oriented 4-manifold M^4 we have that

$$\sigma(M) = \langle \frac{1}{3} p_1(M), \mu_M \rangle = \frac{1}{3} p_{(1)}(M) \in \mathbb{Z}.$$

Example 7.3. On closed oriented 8-manifolds, the signature is computed by (7.1) as

$$\sigma = a_{(2,0)} p_{(2,0)} + a_{(0,1)} p_{(0,1)}, \tag{7.3}$$

with $a_{(2,0)}, a_{(0,1)} \in \mathbb{Q}$ to be determined. Since Ω_8 is generated rationally by $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$, we can calculate $a_{(2,0)}$ and $a_{(0,1)}$ by evaluating (7.3) on $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$. Using our computations from Example 6.13, we have:

$$1 = \sigma(\mathbb{C}P^4) = 25a_{(2,0)} + 10a_{(0,1)}, \tag{7.4}$$

and

$$1 = \sigma(\mathbb{C}P^2 \times \mathbb{C}P^2) = 18a_{(2,0)} + 9a_{(0,1)}. \tag{7.5}$$

Solving for $a_{(2,0)}$ and $a_{(0,1)}$ in (7.4) and (7.5), we get:

$$a_{(2,0)} = -\frac{1}{45}, \quad a_{(0,1)} = \frac{7}{45}.$$

Altogether, the signature of a closed oriented manifold M^8 is computed by the following formula:

$$\sigma(M^8) = \frac{1}{45} \langle 7p_2(M) - p_1(M)^2, \mu_M \rangle. \tag{7.6}$$

8 Exotic 7-spheres

Now we turn to the construction of exotic 7-spheres. Start with M a smooth, 3-connected orientable 8-manifold. Up to homotopy, $M \simeq (S^4 \vee \cdots \vee S^4) \cup_f e^8$. Assume further that $\beta_4(M) = 1$, i.e., $M \simeq S^4 \cup_f e^8$, for some map $f : S^7 \to S^4$. By Whitney's embedding theorem, there is a smooth embedding $S^4 \hookrightarrow M$. Let E be a tubular neighborhood of this embedded S^4 in M; in other words, E is a D^4-bundle on S^4 inside M. Such D^4-bundles on S^4 are classified by

$$\pi_3(SO(4)) \cong \pi_3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

(Here we use the fact that $S^3 \times S^3$ is a 2-fold covering of $SO(4)$.) That means that E corresponds to a pair of integers (i, j).

Let X^7 be the boundary of E, so X is a S^3-bundle over S^4. If X is diffeomorphic to a 7-sphere, one can recover M from E by attaching an 8-cell to $X = \partial E$. So the question to investigate is: *for which pairs of integers (i, j) is X diffeomorphic to S^7?*

One can show the following:

Lemma 8.1. *X is homotopy equivalent to S^7 if and only if $i + j = \pm 1$.*

Suppose $i + j = 1$. Then for each choice of i, we get an S^3-bundle over S^4, namely $X = \partial E$, which has the homotopy type of S^7. If X is in fact diffeomorphic to S^7, then we can recover M by attaching an 8-cell to X, and in this case the signature of M is computed by

$$\sigma(M) = \frac{1}{45}\left(7p_{(0,1)}(M) - p_{(2,0)}(M)\right).$$

Moreover, one can show that:

Lemma 8.2. $p_{(2,0)}(M) = 4(i - j)^2 = 4(2i - 1)^2$.

Note that $\sigma(M) = \pm 1$ since $H^4(M; \mathbb{Z}) = \mathbb{Z}$, and let us fix the orientation according to which $\sigma(M) = 1$. Our assumption that X was diffeomorphic to S^7 leads now to a contradiction, since

$$p_{(0,1)}(M) = \frac{4(2i - 1)^2 + 45}{7}$$

is by definition an integer for all i, which is contradicted e.g., for $i = 2$.

So far (for $i = 2$ and $j = -1$), we constructed a space X which is homotopy equivalent to S^7, but which is not diffeomorphic to S^7. In fact, one can further show the following:

Lemma 8.3. *X is homeomorphic to S^7, so in fact X is an exotic 7-sphere.*

This latest fact can be shown by constructing a Morse function $g : X \to \mathbb{R}$ with only two nondegenerate critical points (a maximum and a minimum). An application of Reeb's theorem then yields that X is homeomorphic to S^7.

9 Exercises

1. Construct explicitly the bounding manifold for $\mathbb{R}P^3$.

2. Let ω be a rank n complex vector bundle on a topological space X, and let $c_i(\omega) \in H^{2i}(X;\mathbb{Z})$ be its i-th Chern class. Via $\mathbb{Z} \to \mathbb{Z}/2$, $c_i(\omega)$ determines a cohomology class $\bar{c}_i(\omega) \in H^{2i}(X;\mathbb{Z}/2)$. By forgetting the complex structure on the fibers of ω, we obtain the realization $\omega_{\mathbb{R}}$ of ω, as a rank $2n$ real vector bundle on X.

Show that the Stiefel-Whitney classes of $\omega_{\mathbb{R}}$ are computed as follows:

(a) $w_{2i}(\omega_{\mathbb{R}}) = \bar{c}_i(\omega)$, for $0 \leq i \leq n$.

(b) $w_{2i+1}(\omega_{\mathbb{R}}) = 0$ for any integer i.

3. Let M be a $2n$-dimensional smooth manifold with tangent bundle TM. Show that, if M admits a complex structure, then $w_{2i}(M)$ is the mod 2 reduction of an integral class for any $0 \leq i \leq n$, and $w_{2i+1}(M) = 0$ for any integer i. In particular, Stiefel-Whitney classes give obstructions to the existence of a complex structure on an even-dimensional real smooth manifold.

4. Show that a real smooth manifold M is orientable if and only if $w_1(M) = 0$.

5. Show that $\mathbb{C}P^3$ does not embed in $\mathbb{R}^7$.

6. Show that $\mathbb{C}P^4$ does not embed in $\mathbb{R}^{11}$.

7. Example 6.9 shows that $\mathbb{C}P^2$ is not the boundary on an oriented compact 5-manifold. Can $\mathbb{C}P^2$ be the boundary on some non-orientable compact 5-manifold?

8. Show that $\mathbb{C}P^{2n+1}$ is the boundary of a compact manifold.