Math 754
Chapter III: Fiber bundles. Classifying spaces. Applications

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Contents

1 Fiber bundles

Let G be a topological group (i.e., a topological space endowed with a group structure so that the group multiplication and the inversion map are continuous), acting continuously (on the left) on a topological space F. Concretely, such a continuous action is given by a continuous map $\rho\colon G \times F \to F$, $(g, m) \mapsto g \cdot m$, which satisfies the conditions $(gh) \cdot m = g \cdot (h \cdot m))$ and $e_G \cdot m = m$, for e_G the identity element of G.

Any continuous group action ρ induces a map

$$\mathrm{Ad}_\rho : G \longrightarrow \mathrm{Homeo}(F)$$

given by $g \mapsto (f \mapsto g \cdot f)$, with $g \in G$, $f \in F$. Note that Ad_ρ is a group homomorphism since $(\mathrm{Ad}\,\rho)(gh)(f) := (gh) \cdot f = g \cdot (h \cdot f) = \mathrm{Ad}_\rho(g)(\mathrm{Ad}_\rho(h)(f))$. Note that for nice spaces F (such as CW complexes), if we give $\mathrm{Homeo}(F)$ the compact-open topology, then $\mathrm{Ad}_\rho\colon G \to \mathrm{Homeo}(F)$ is a continuous group homomorphism, and any such continuous group homomorphism $G \to \mathrm{Homeo}(F)$ induces a continuous group action $G \times F \to F$.

We assume from now on that ρ is an *effective* action, i.e., that Ad_ρ is injective.

Definition 1.1 (Atlas for a fiber bundle with group G and fiber F)**.** *Given a continuous map $\pi\colon E \to B$, an* atlas *for the structure of a fiber bundle with group G and fiber F on π consists of the following data:*

a) an open cover $\{U_\alpha\}_\alpha$ of B,

b) homeomorphisms (called trivializing charts *or* local trivializations*) $h_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$ for each α so that the diagram*

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\;h_\alpha\;} & U_\alpha \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mathrm{pr}_1} & \\ & U_\alpha & \end{array}$$

commutes,

c) continuous maps (called transition functions*) $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ so that the horizontal map in the commutative diagram*

$$\begin{array}{ccc} & \pi^{-1}(U_\alpha \cap U_\beta) & \\ {\scriptstyle h_\alpha}\swarrow & & \searrow{\scriptstyle h_\beta} \\ (U_\alpha \cap U_\beta) \times F & \xrightarrow[h_\beta \circ h_\alpha^{-1}]{} & (U_\alpha \cap U_\beta) \times F \end{array}$$

is given by

$$(x, m) \mapsto (x, g_{\beta\alpha}(x) \cdot m).$$

(By the effectivity of the action, if such maps $g_{\alpha\beta}$ exist, they are unique.)

Definition 1.2. *Two atlases $\mathcal{A}$ and $\mathcal{B}$ on π are* compatible *if $\mathcal{A} \cup \mathcal{B}$ is an atlas.*

Definition 1.3 (Fiber bundle with group G and fiber F)**.** *A* structure of a fiber bundle with group G and fiber F *on $\pi\colon E \to B$ is a maximal atlas for $\pi\colon E \to B$.*

Example 1.4.

1. When $G = \{e_G\}$ is the trivial group, $\pi\colon E \to B$ has the structure of a fiber bundle if and only if it is a trivial fiber bundle. Indeed, the local trivializations h_α of the atlas for the fiber bundle have to satisfy $h_\beta \circ h_\alpha^{-1}\colon (x, m) \mapsto (x, e_G \cdot m) = (x, m)$, which implies $h_\beta \circ h_\alpha^{-1} = id$, so $h_\beta = h_\alpha$ on $U_\alpha \cap U_\beta$. This allows us to glue all the local trivializations h_α together to obtain a global trivialization $h\colon \pi^{-1}(B) = E \cong B \times F$.

2. When F is discrete, Homeo(F) is also discrete, so G is discrete by the effectiveness assumption. So for the atlas of $\pi\colon E \to B$ we have $\pi^{-1}(U_\alpha) \cong U_\alpha \times F = \bigcup_{m \in F} U_\alpha \times \{m\}$, so π is in this case a covering map.

3. A locally trivial fiber bundle, as introduced in earlier chapters, is just a fiber bundle with structure group Homeo(F).

Lemma 1.5. *The transition functions $g_{\alpha\beta}$ satisfy the following properties:*

(a) $g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$*, for all* $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

(b) $g_{\beta\alpha}(x) = g_{\alpha\beta}^{-1}(x)$*, for all* $x \in U_\alpha \cap U_\beta$.

(c) $g_{\alpha\alpha}(x) = e_G$.

Proof. On $U_\alpha \cap U_\beta \cap U_\gamma$, we have: $(h_\alpha \circ h_\beta^{-1}) \circ (h_\beta \circ h_\gamma^{-1}) = h_\alpha \circ h_\gamma^{-1}$. Therefore, since Ad_ρ is injective (i.e., ρ is effective), we get that $g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Note that $(h_\alpha \circ h_\beta^{-1}) \circ (h_\beta \circ h_\alpha^{-1}) = id$, which translates into

$$(x, g_{\alpha\beta}(x)g_{\beta\alpha}(x) \cdot m) = (x, m).$$

So, by effectiveness, $g_{\alpha\beta}(x)g_{\beta\alpha}(x) = e_G$ for all $x \in U_\alpha \cap U_\beta$, whence $g_{\beta\alpha}(x) = g_{\alpha\beta}^{-1}(x)$.

Take $\gamma = \alpha$ in Property (a) to get $g_{\alpha\beta}(x)g_{\beta\alpha}(x) = g_{\alpha\alpha}(x)$. So by Property (b), we have $g_{\alpha\alpha}(x) = e_G$. □

Transition functions determine a fiber bundle in a unique way, in the sense of the following theorem.

Theorem 1.6. *Given an open cover $\{U_\alpha\}$ of B and continuous functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ satisfying Properties (a)-(c), there is a unique structure of a fiber bundle over B with group G, given fiber F, and transition functions $\{g_{\alpha\beta}\}$.*

Proof Sketch. Let $\widetilde{E} = \bigsqcup_\alpha U_\alpha \times F \times \{\alpha\}$, and define an equivalence relation $\sim$ on $\widetilde{E}$ by

$$(x, m, \alpha) \sim (x, g_{\alpha\beta}(x) \cdot m, \beta),$$

for all $x \in U_\alpha \cap U_\beta$, and $m \in F$. Properties (a)-(c) of $\{g_{\alpha\beta}\}$ are used to show that $\sim$ is indeed an equivalence relation on $\widetilde{E}$. Specifically, symmetry is implied by property (b), reflexivity follows from (c) and transitivity is a consequence of the cycle property (a).

Let

$$E = \widetilde{E}/\sim$$

be the set of equivalence classes in E, and define $\pi : E \to B$ locally by $[(x, m, \alpha)] \mapsto x$ for $x \in U_\alpha$. Then it is clear that π is well-defined and continuos (in the quotient topology), and the fiber of π is F.

It remains to show the local triviality of π. Let $p : \widetilde{E} \to E$ be the quotient map, and let $p_\alpha := p|_{U_\alpha \times F \times \{\alpha\}} : U_\alpha \times F \times \{\alpha\} \to \pi^{-1}(U_\alpha)$. It is easy to see that p_α is a homeomorphism. We define the local trivializations of π by $h_\alpha := p_\alpha^{-1}$. □

Example 1.7.

1. Fiber bundles with fiber $F = \mathbb{R}^n$ and group $G = GL(n, \mathbb{R})$ are called *rank n real vector bundles*. For example, if M is a differentiable real n-manifold, and TM is the set of all tangent vectors to M, then $\pi : TM \to M$ is a real vector bundle on M of rank n. More precisely, if $\varphi_\alpha : U_\alpha \overset{\cong}{\to} \mathbb{R}^n$ are trivializing charts on M, the transition functions for TM are given by $g_{\alpha\beta}(x) = d(\varphi_\alpha \circ \varphi_\beta^{-1})_{\varphi_\beta(x)}$.

2. If $F = \mathbb{R}^n$ and $G = O(n)$, we get real vector bundles with a *Riemannian structure*.

3. Similarly, one can take $F = \mathbb{C}^n$ and $G = GL(n, \mathbb{C})$ to get *rank n complex vector bundles*. For example, if M is a complex manifold, the tangent bundle TM is a complex vector bundle.

4. If $F = \mathbb{C}^n$ and $G = U(n)$, we get real vector bundles with a *hermitian structure*.

We also mention here the following fact:

Theorem 1.8. *A fiber bundle has the homotopy lifting property with respect to all CW complexes (i.e., it is a Serre fibration). Moreover, fiber bundles over paracompact spaces are fibrations.*

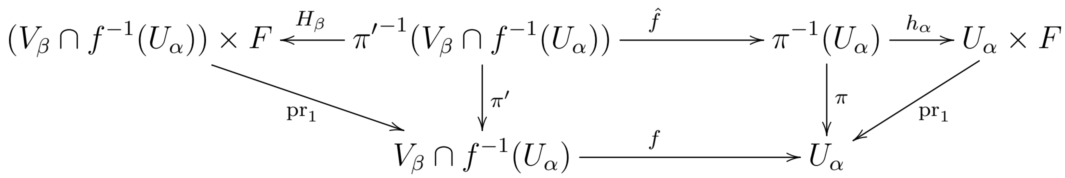
Definition 1.9 (Bundle homomorphism)**.** *Fix a topological group G acting effectively on a space F. A homomorphism between bundles $E' \xrightarrow{\pi'} B'$ and $E \xrightarrow{\pi} B$ with group G and fiber F is a pair $(f, \hat{f})$ of continuous maps, with $f : B' \to B$ and $\hat{f} : E' \to E$, such that:*

1. *the diagram*

$$\begin{array}{ccc} E' & \xrightarrow{\hat{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

commutes, i.e., $\pi \circ \hat{f} = f \circ \pi'$.

2. *if* $\{(U_\alpha, h_\alpha)\}_\alpha$ *is a trivializing atlas of* π *and* $\{(V_\beta, H_\beta)\}_\beta$ *is a trivializing atlas of* π'*, then the following diagram commutes:*



and there exist functions $d_{\alpha\beta} : V_\beta \cap f^{-1}(U_\alpha) \to G$ *such that for* $x \in V_\beta \cap f^{-1}(U_\alpha)$ *and* $m \in F$ *we have:*

$$h_\alpha \circ \hat{f}_| \circ H_\beta^{-1}(x, m) = (f(x), d_{\alpha\beta}(x) \cdot m).$$

An isomorphism *of fiber bundles is a bundle homomorphism* $(f, \hat{f})$ *which admits a map* $(g, \hat{g})$ *in the reverse direction so that both composites are the identity.*

Remark 1.10. *Gauge transformations* of a bundle $\pi : E \to B$ are bundle maps from π to itself over the identity of the base, i.e., corresponding to continuous map $g : E \to E$ so that $\pi \circ g = \pi$. By definition, such g restricts to an isomorphism given by the action of an element of the structure group on each fiber. The set of all gauge transformations forms a group.

Proposition 1.11. *Given functions* $d_{\alpha\beta} : V_\beta \cap f^{-1}(U_\alpha) \to G$ *and* $d_{\alpha'\beta'} : V_{\beta'} \cap f^{-1}(U_{\alpha'}) \to G$ *as in (2) above for different trivializing charts of* π *and resp.* π'*, then for any* $x \in V_\beta \cap V_{\beta'} \cap f^{-1}(U_\alpha \cap U_{\alpha'}) \neq \emptyset$*, we have*

$$d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(f(x))\ d_{\alpha\beta}(x)\ g_{\beta\beta'}(x) \tag{1.1}$$

in G*, where* $g_{\alpha'\alpha}$ *are transition functions for* π *and* $g_{\beta\beta'}$ *are transition functions for* π'*,*

Proof. Exercise. □

The functions $\{d_{\alpha\beta}\}$ determine bundle maps in the following sense:

Theorem 1.12. *Given a map* $f : B' \to B$ *and bundles* $E \xrightarrow{\pi} B$*,* $E' \xrightarrow{\pi'} B'$*, a map of bundles* $(f, \hat{f}) : \pi' \to \pi$ *exists if and only if there exist continuous maps* $\{d_{\alpha\beta}\}$ *as above, satisfying (1.1).*

Proof. Exercise. □

Theorem 1.13. *Every bundle map* $\hat{f}$ *over* $f = \mathrm{id}_B$ *is an isomorphism. In particular, gauge transformations are automorphisms.*

Proof Sketch. Let $d_{\alpha\beta} : V_\beta \cap U_\alpha \to G$ be the maps given by the bundle map $\hat{f} : E' \to E$. So, if $d_{\alpha'\beta'} : V_{\beta'} \cap U_{\alpha'} \to G$ is given by a different choice of trivializing charts, then (1.1) holds on $V_\beta \cap V_{\beta'} \cap U_\alpha \cap U_{\alpha'} \neq \emptyset$, i.e.,

$$d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(x)\ d_{\alpha\beta}(x)\ g_{\beta\beta'}(x) \tag{1.2}$$

in G, where $g_{\alpha'\alpha}$ are transition functions for π and $g_{\beta\beta'}$ are transition functions for π'. Let us now invert (1.2) in G, and set

$$\overline{d_{\beta\alpha}}(x) = d^{-1}_{\alpha\beta}(x)$$

to get:

$$\overline{d_{\beta'\alpha'}}(x) = g_{\beta'\beta}(x)\ \overline{d_{\beta\alpha}}(x)\ g_{\alpha\alpha'}(x).$$

So $\{\overline{d_{\beta\alpha}}\}$ are as in Definition 1.9 and satisfy (1.1). Theorem 1.12 implies that there exists a bundle map $\hat{g} : E \to E'$ over id_B.

We claim that $\hat{g}$ is the inverse $\hat{f}^{-1}$ of $\hat{f}$, and this can be checked locally as follows:

$$(x, m) \overset{\hat{f}}{\mapsto} (x, d_{\alpha\beta}(x) \cdot m) \overset{\hat{g}}{\mapsto} (x, \overline{d_{\beta\alpha}}(x) \cdot (d_{\alpha\beta}(x) \cdot m)) = (x, \underbrace{\overline{d_{\beta\alpha}}(x) d_{\alpha\beta}(x)}_{e_G} \cdot m) = (x, m).$$

So $\hat{g} \circ \hat{f} = id_{E'}$. Similarly, $\hat{f} \circ \hat{g} = id_E$ □

One way in which fiber bundle homomorphisms arise is from the pullback (or the induced bundle) construction.

Definition 1.14 (Induced Bundle)**.** *Given a bundle $E \xrightarrow{\pi} B$ with group G and fiber F, and a continuous map $f : X \to B$, we define*

$$f^*E := \{(x, e) \in X \times E \mid f(x) = \pi(e)\},$$

with projections $f^*\pi : f^*E \to X$, $(x, e) \mapsto x$, and $\hat{f} : f^*E \to E$, $(x, e) \mapsto e$, so that the following diagram commutes:

$$\begin{array}{ccccc} f^*E & \longrightarrow & E & & e \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\ X & \xrightarrow[f]{} & B & & \\ x & \longmapsto & & & f(x) \end{array}$$

$f^*\pi$ is called the induced bundle under f *or the* pullback of π by f, *and as we show below it comes equipped with a bundle map $(f, \hat{f}) : f^*\pi \to \pi$.*

The above definition is justified by the following result:

Theorem 1.15.

(a) $f^*\pi : f^*E \to X$ is a fiber bundle with group G and fiber F.

(b) $(f, \hat{f}) : f^*\pi \to \pi$ is a bundle map.

Proof Sketch. Let $\{(U_\alpha, h_\alpha)\}_\alpha$ be a trivializing atlas of π, and consider the following commutative diagram:

$$\begin{array}{ccccc}
(f^*\pi)^{-1}(f^{-1}(U_\alpha)) & \longrightarrow & \pi^{-1}(U_\alpha) & \xrightarrow{h_\alpha} & U_\alpha \times F \\
\downarrow & & \downarrow & \swarrow & \\
f^{-1}(U_\alpha) & \xrightarrow[f]{} & U_\alpha & &
\end{array}$$

We have

$$(f^*\pi)^{-1}(f^{-1}(U_\alpha)) = \{(x,e) \in f^{-1}(U_\alpha) \times \underbrace{\pi^{-1}(U_\alpha)}_{\cong U_\alpha \times F} \mid f(x) = \pi(e)\}.$$

Define

$$k_\alpha : (f^*\pi)^{-1}(f^{-1}(U_\alpha)) \longrightarrow f^{-1}(U_\alpha) \times F$$

by

$$(x,e) \mapsto (x, \mathrm{pr}_2(h_\alpha(e))).$$

Then it is easy to check that k_α is a homeomorphism (with inverse $k_\alpha^{-1}(x,m) = (x, h_\alpha^{-1}(f(x), m))$, and in fact the following assertions hold:

(i) $\{(f^{-1}(U_\alpha), k_\alpha)\}_\alpha$ is a trivializing atlas of $f^*\pi$.

(ii) the transition functions of $f^*\pi$ are $f^*g_{\alpha\beta} := g_{\alpha\beta} \circ f$, i.e., $f^*g_{\alpha\beta}(x) = g_{\alpha\beta}(f(x))$ for any $x \in f^{-1}(U_\alpha \cap U_\beta)$.

□

Remark 1.16. It is easy to see that $(f \circ g)^*\pi = g^*(f^*\pi)$ and $(id_B)^*\pi = \pi$. Moreover, the pullback of a trivial bundle is a trivial bundle.

As we shall see later on, the following important result holds:

Theorem 1.17. *Given a fibre bundle $\pi : E \to B$ with group G and fiber F, and two homotopic maps $f \simeq g : X \to B$, there is an isomorphism $f^*\pi \cong g^*\pi$ of bundles over X. (In short, induced bundles under homotopic maps are isomorphic.)*

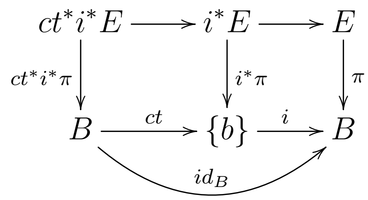
As a consequence, we have:

Corollary 1.18. *A fiber bundle over a contractible space B is trivial.*

Proof. Since B is contractible, id_B is homotopic to the constant map ct. Let

$$b := \mathrm{Image}(ct) \overset{i}{\hookrightarrow} B,$$

so $i \circ ct \simeq id_B$. We have a diagram of maps and induced bundles:

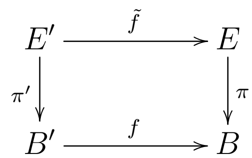


Theorem 1.17 then yields:

$$\pi \cong (id_B)^*\pi \cong ct^*i^*\pi.$$

Since any fiber bundle over a point is trivial, we have that $i^*\pi \cong \{b\} \times F$ is trivial, hence $\pi \cong ct^*i^*\pi \cong B \times F$ is also trivial. □

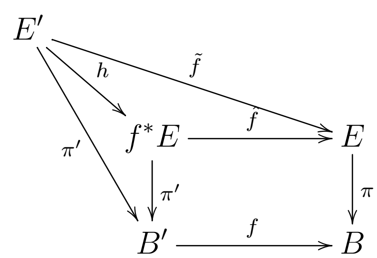
Proposition 1.19. *If*



is a bundle map, then $\pi' \cong f^*\pi$ *as bundles over* B'.

Proof. Define $h : E' \to f^*E$ by $e' \mapsto (\pi'(e'), \tilde{f}(e')) \in B' \times E$. This is well-defined, i.e., $h(e') \in f^*E$, since $f(\pi'(e')) = \pi(\tilde{f}(e'))$.

It is easy to check that h provides the desired bundle isomorphism over B'.



□

Example 1.20. We can now show that the set of isomorphism classes of bundles over S^n with group G and fiber F is isomorphic to $\pi_{n-1}(G)$. Indeed, let us cover S^n with two contractible sets U_+ and U_- obtained by removing the south, resp., north pole of S^n. Let $i_\pm : U_\pm \hookrightarrow S^n$ be the inclusions. Then any bundle π over S^n is trivial when restricted to $U_\pm$, that is, $i_\pm^*\pi \cong U_\pm \times F$. In particular, $U_\pm$ provides a trivializing cover (atlas) for π, and any such bundle π is completely determined by the transition function $g_\pm : U_+ \cap U_- \simeq S^{n-1} \to G$, i.e., by an element in $\pi_{n-1}(G)$.

More generally, we aim to "classify" fiber bundles on a given topological space. Let $\mathcal{B}(X, G, F, \rho)$ denote the isomorphism classes (over id_X) of fiber bundles on X with group G

and fiber F, and G-action on F given by ρ. If $f : X' \to X$ is a continuous map, the pullback construction defines a map

$$f^* : \mathcal{B}(X, G, F, \rho) \longrightarrow \mathcal{B}(X', G, F, \rho)$$

so that $(id_X)^* = id$ and $(f \circ g)^* = g^* \circ f^*$.

2 Principle Bundles

As we will see later on, the fiber F doesn't play any essential role in the classification of fiber bundle, and in fact it is enough to understand the set

$$\mathcal{P}(X, G) := \mathcal{B}(X, G, G, m_G)$$

of fiber bundles with group G and fiber G, where the action of G on itself is given by the multiplication m_G of G. Elements of $\mathcal{P}(X, G)$ are called *principal G-bundles*. Of particular importance in the classification theory of such bundles is the *universal principal G-bundle* $G \hookrightarrow EG \to BG$, with contractible total space EG.

Example 2.1. Any regular cover $p : E \to X$ is a principal G-bundle, with group $G = {\pi_1(X)}\big/{p_*\pi_1(E)}$. Here G is given the discrete topology. In particular, the universal covering $\widetilde{X} \to X$ is a principal $\pi_1(X)$-bundle.

Example 2.2. Any free (right) action of a finite group G on a (Hausdorff) space E gives a regular cover and hence a principal G-bundle $E \to E/G$.

More generally, we have the following:

Theorem 2.3. *Let $\pi : E \to X$ be a principal G-bundle. Then G acts freely and transitively on the right of E so that ${E}\big/{G} \cong X$. In particular, π is the quotient (orbit) map.*

Proof. We will define the action locally over a trivializing chart for π. Let U_α be a trvializing open in X with trivializing homeomorphism $h_\alpha : \pi^{-1}(U_\alpha) \overset{\cong}{\to} U_\alpha \times G$. We define a right action on G on $\pi^{-1}(U_\alpha)$ by

$$\begin{aligned} \pi^{-1}(U_\alpha) \times G &\to \pi^{-1}(U_\alpha) \cong U_\alpha \times G \\ (e, g) &\mapsto e \cdot g := h_\alpha^{-1}(\pi(e), \mathrm{pr}_2(h_\alpha(e)) \cdot g) \end{aligned}$$

Let us show that this action can be globalized, i.e., it is independent of the choice of the trivializing open U_α. If (U_β, h_β) is another trivializing chart in X so that $e \in \pi^{-1}(U_\alpha \cap U_\beta)$, we need to show that $e \cdot g = h_\beta^{-1}(\pi(e), \mathrm{pr}_2(h_\beta(e)) \cdot g)$, or equivalently,

$$h_\alpha^{-1}(\pi(e), \mathrm{pr}_2(h_\alpha(e)) \cdot g) = h_\beta^{-1}(\pi(e), \mathrm{pr}_2(h_\beta(e)) \cdot g). \tag{2.1}$$

After applying h_α and using the transition function $g_{\alpha\beta}$ for $\pi(e) \in U_\alpha \cap U_\beta$, (2.1) becomes

$$(\pi(e), \mathrm{pr}_2(h_\alpha(e)) \cdot g) = h_\alpha h_\beta^{-1}(\pi(e), \mathrm{pr}_2(h_\beta(e)) \cdot g) = (\pi(e), g_{\alpha\beta}(\pi(e)) \cdot (\mathrm{pr}_2(h_\beta(e)) \cdot g)), \tag{2.2}$$

which is guaranteed by the definition of an atlas for π.

It is easy to check locally that the action is free and transitive. Moreover, E/G is locally given as $U_\alpha \times G/G \cong U_\alpha$, and this local quotient globalizes to X. $\square$

The converse of the above theorem holds in some important cases.

Theorem 2.4. *Let E be a compact Hausdorff space and G a compact Lie group acting freely on E. Then the orbit map $E \to E/G$ is a principal G-bundle.*

Corollary 2.5. *Let G be a Lie group, and let $H < G$ be a compact subgroup. Then the projection onto the orbit space $\pi : G \to G/H$ is a principal H-bundle.*

Let us now fix a G-space F. We define a map

$$\mathcal{P}(X, G) \to \mathcal{B}(X, G, F, \rho)$$

as follows. Start with a principal G bundle $\pi : E \to X$, and recall from the previous theorem that G acts freely on the right on E. Since G acts on the left on F, we have a left G-action on $E \times F$ given by:

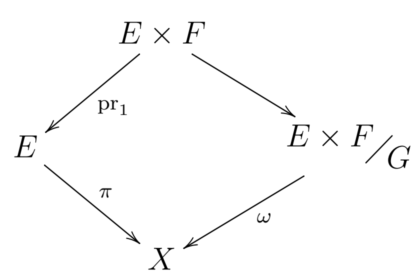
$$g \cdot (e, f) \mapsto (e \cdot g^{-1}, g \cdot f).$$

Let

$$E \times_G F := E \times F/G$$

be the corresponding orbit space, with projection map $\omega : E \times_G F \to E/G \cong X$ fitting into a commutative diagram

(2.3)



Definition 2.6. *The projection $\omega := \pi \times_G F : E \times_G F \to X$ is called the associated bundle with fiber F.*

The terminology in the above definition is justified by the following result.

Theorem 2.7. *$\omega : E \times_G F \to X$ is a fiber bundle with group G, fiber F, and having the same transition functions as π. Moreover, the assignment $\pi \mapsto \omega := \pi \times_G F$ defines a one-to-one correspondence $\mathcal{P}(X, G) \to \mathcal{B}(X, G, F, \rho)$.*

Proof. Let $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ be a trivializing chart for π. Recall that for $e \in \pi^{-1}(U_\alpha)$, $f \in F$ and $g \in G$, if we set $h_\alpha(e) = (\pi(e), h) \in U_\alpha \times G$, then G acts on the right on $\pi^{-1}(U_\alpha)$ by acting on the right on $h = pr_2(h_\alpha(e))$. Then we have by the diagram (2.3) that

$$\begin{aligned} \omega^{-1}(U_\alpha) &\cong \pi^{-1}(U_\alpha) \times F \big/ (e,f) \sim (e \cdot g^{-1}, g \cdot f) \\ &\cong U_\alpha \times G \times F \big/ (u,h,f) \sim (u, hg^{-1}, g \cdot f). \end{aligned}$$

Let us define

$$k_\alpha : \omega^{-1}(U_\alpha) \to U_\alpha \times F$$

by

$$[(u,h,f)] \mapsto (u, h \cdot f).$$

This is a well-defined map since $[(u, hg^{-1}, g \cdot f)] \mapsto (u, hg^{-1}g \cdot f) = (u, h \cdot f)$. It is easy to check that k_al is a trivializing chart for ω with inverse induced by $U_\alpha \times F \to U_\alpha \times G \times F$, $(u,f) \mapsto (u, id_G, f)$. It is clear that ω and π have the same transition functions as they have the same trivializing opens. □

The associated bundle construction is easily seen to be functorial in the following sense.

Proposition 2.8. *If*

$$\begin{array}{ccc} E' & \xrightarrow{\widehat{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

is a map of principal G-bundles (so $\widehat{f}$ is a G-equivariant map, i.e., $\widehat{f}(e \cdot g) = \widehat{f}(e) \cdot g$), then there is an induced map of associated bundles with fiber F,

$$\begin{array}{ccc} E' \times_G F & \xrightarrow{\widehat{f} \times_G id_F} & E \times_G F \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

Example 2.9. Let $\pi : S^1 \to S^1$, $z \mapsto z^2$ be regarded as a principal $\mathbb{Z}/2$-bundle, and let $F = [-1,1]$. Let $\mathbb{Z}/2 = \{1, -1\}$ act on F by multiplication. Then the bundle associated to π with fiber $F = [-1,1]$ is the Möbius strip $S^1 \times_{\mathbb{Z}/2} [-1,1] = S^1 \times [-1,1] \big/ (x,t) \sim (a(x), -t)$, with $a : S^1 \to S^1$ denoting the antipodal map. Similarly, the bundle associated to π with fiber $F = S^1$ is the Klein bottle.

Let us now get back to proving the following important result.

Theorem 2.10. *Let $\pi : E \to Y$ be a fiber bundle with group G and fiber F, and let $f \simeq g : X \to Y$ be two homotopic maps. Then $f^*\pi \cong g^*\pi$ over id_X.*

It is of course enough to prove the theorem in the case of principal G-bundles. The idea of proof is to construct a bundle map over id_X between $f^*\pi$ and $g^*\pi$:

$$\begin{array}{ccc} f^*E & \overset{?}{\dashrightarrow} & g^*E \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

So we first need to understand maps of principal G-bundles, i.e., to solve the following problem: given two principal G-bundles bundles $E_1 \xrightarrow{\pi_1} X$ and $E_2 \xrightarrow{\pi_2} Y$, describe the set $maps(\pi_1, \pi_2)$ of bundle maps

$$\begin{CD} E_1 @>\hat{f}>> E_2 \\ @V\pi_1VV @VV\pi_2V \\ X @>f>> Y \end{CD}$$

Since G acts on the right of E_1 and E_2, we also get an action on the left of E_2 by $g{\cdot}e_2 := e_2{\cdot}g^{-1}$. Then we get an associated bundle of π_1 with fiber E_2, namely

$$\omega := \pi_1 \times_G E_2 : E_1 \times_G E_2 \longrightarrow X.$$

We have the following result:

Theorem 2.11. *Bundle maps from π_1 to π_2 are in one-to-one correspondence to sections of ω.*

Proof. We work locally, so it suffices to consider only trivial bundles.

Given a bundle map $(f, \widehat{f}) : \pi_1 \mapsto \pi_2$, let $U \subset Y$ open, and $V \subset f^{-1}(U)$ open, so that the following diagram commutes (this is the bundle maps in trivializing charts)

$$\begin{CD} V \times G @>\widehat{f}>> U \times G \\ @V\pi_1VV @VV\pi_2V \\ V @>f>> U \end{CD}$$

We define a section σ in

$$\begin{CD} (V \times G) \times_G (U \times G) \\ @A\sigma AA @VV\omega V \\ V \end{CD}$$

as follows. For $e_1 \in V \times G$, with $x = \pi_1(e_1) \in V$, we set

$$\sigma(x) = [e_1, \widehat{f}(e_1)].$$

This map is well-defined, since for any $g \in G$ we have:

$$[e_1 \cdot g, \widehat{f}(e_1 \cdot g)] = [e_1 \cdot g, \widehat{f}(e_1) \cdot g] = [e_1 \cdot g, g^{-1} \cdot \widehat{f}(e_1)] = [e_1, \widehat{f}(e_1)].$$

Now, it is an exercise in point-set topology (using the local definition of a bundle map) to show that σ is continuous.

Conversely, given a section of $E_1 \times_G E_2 \overset{\omega}{\mapsto} X$, we define a bundle by $(f, \widehat{f})$ by

$$\widehat{f}(e_1) = e_2,$$

where $\sigma(\pi_1(e_1)) = [(e_1, e_2)]$. Note that this is an equivariant map because

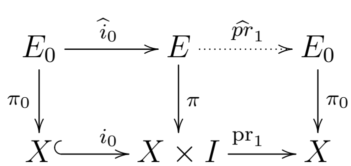
$$[e_1 \cdot g, e_2 \cdot g] = [e_1 \cdot g, g^{-1} \cdot e_2] = [e_1, e_2],$$

hence $\widehat{f}(e_1 \cdot g) = e_2 \cdot g = \widehat{f}(e_1) \cdot g$. Thus $\widehat{f}$ descends to a map $f : X \to Y$ on the orbit spaces. We leave it as an exercise to check that $(f, \widehat{f})$ is indeed a bundle map, i.e., to show that locally $\widehat{f}(v, g) = (f(v), d(v)g)$ with $d(v) \in G$ and $d : V \to G$ a continuous function. □

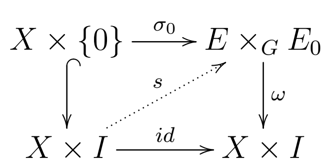
The following result will be needed in the proof of Theorem 2.10.

Lemma 2.12. *Let $\pi : E \to X \times I$ be a bundle, and let $\pi_0 := i_0^* \pi : E_0 \to X$ be the pullback of π under $i_0 : X \to X \times I$, $x \mapsto (x, 0)$. Then $\pi \cong (pr_1)^* \pi_0 \cong \pi_0 \times id_I$, where $pr_1 : X \times I \to X$ is the projection map.*

Proof. It suffices to find a bundle map $(pr_1, \widehat{pr}_1)$ so that the following diagram commutes



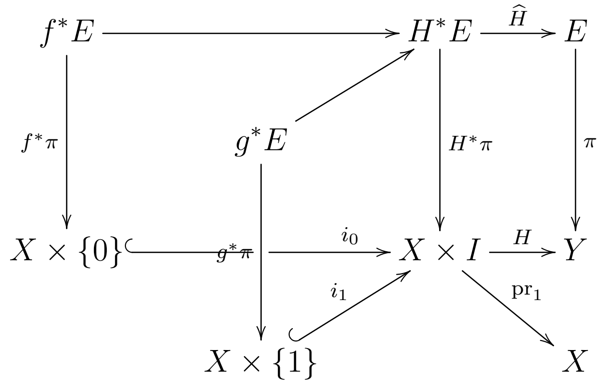
By Theorem 2.11, this is equivilant to the existence of a section σ of $\omega : E \times_G E_0 \to X \times I$. Note that there exists a section σ_0 of $\omega_0 : E_0 \times_G E_0 \to X = X \times \{0\}$, corresponding to the bundle map $(id_X, id_{E_0}) : \pi_0 \to \pi_0$. Then composing σ_0 with the top inclusion arrow, we get the following diagram



Since ω is a fibration, by the homotopy lifting property one can extend $s\sigma_0$ to a section σ of ω. □

We can now finish the proof of Theorem 2.10.

Proof of Theorem 2.10. Let $H : X \times I \to Y$ be a homotopy between f and g, with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Consider the induced bundle $H^*\pi$ over $X \times I$. Then we have the following diagram.

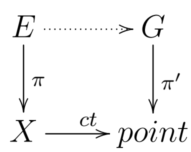


Since $f = H(-, 0)$, we get $f^*\pi = i_0^* H^*\pi$. By Lemma 2.12, $H^*\pi \cong pr_1^*(f^*\pi) \cong pr_1^*(g^*\pi)$, and thus $f^*\pi = i_0^* H^*\pi = i_0^* \operatorname{pr}_1^* g^*\pi = g^*\pi$. ☐

We conclude this section with the following important consequence of Theorem 2.11

Corollary 2.13. *A principle G-bundle $\pi : E \to X$ is trivial if and only if π has a section.*

Proof. The bundle π is trivial if and only if $\pi = ct^*\pi'$, with $ct : X \to point$ the constant map, and $\pi' : G \to point$ the trvial bundle over a point space. This is equivalent to saying that there is a bundle map



or, by Theorem 2.11, to the existence of a section of the bundle $\omega : E \times_G G \to X$. On the other hand, $\omega \cong \pi$, since $E \times_G G \to X$ looks locally like

$$\pi^{-1}(U_\alpha) \times G \big/ \sim \;\cong U_\alpha \times G \times G \big/ (u, g_1, g_2) \sim (u, g_1 g^{-1}, g g_2) \cong U\alpha \times G,$$

with the last homeomorphism defined by $[(u, g_1, g_2)] \mapsto (u, g_1 g_2)$.

Altogether, π is trivial if and only if $\pi : E \mapsto X$ has a section. ☐

3 Classification of principal G-bundles

Let us assume for now that there exists a principal G-bundle $\pi_G : EG \to BG$, with contractible total space EG. As we will see below, such a bundle plays an essential role in the classification theory of principal G-bundles. Its base space BG turns out to be unique up to homotopy, and it is called the *classifying space for principal G-bundles* due to the following fundamental result:

Theorem 3.1. *If X is a CW-complex, there exists a bijective correspondence*

$$\Phi : \mathcal{P}(X,G) \xrightarrow{\cong} [X,BG]$$
$$f^*\pi_G \leftarrow\!\shortmid f$$

Proof. By Theorem 2.10, Φ is well-defined.

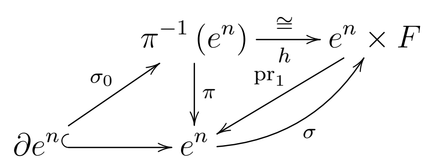
Let us next show that Φ is onto. Let $\pi \in \mathcal{P}(X,G)$, $\pi : E \to X$. We need to show that $\pi \cong f^*\pi_G$ for some map $f : X \to BG$, or equivalently, that there is a bundle map $(f,\widehat{f}) : \pi \to \pi_G$. By Theorem 2.11, this is equivalent to the existence of a section of the bundle $E \times_G EG \to X$ with fiber EG. Since EG is contractible, such a section exists by the following:

Lemma 3.2. *Let X be a CW complex, and $\pi : E \to X \in \mathcal{B}(X,G,F,\rho)$ with $\pi_i(F) = 0$ for all $i \geq 0$. If $A \subseteq X$ is a subcomplex, then every section of π over A extends to a section defined on all of X. In particular, π has a section. Moreover, any two sections of π are homotopic.*

Proof. Given a section $\sigma_0 : A \to E$ of π over A, we extend it to a section $\sigma : X \to E$ of π over X by using induction on the dimension of cells in $X - A$. So it suffices to assume that X has the form

$$X = A \cup_\phi e^n,$$

where e^n is an n-cell in $X - A$, with attaching map $\phi : \partial e^n \to A$. Since e^n is contractible, π is trivial over e^n, so we have a commutative diagram



with $h : \pi^{-1}(e^n) \to e^n \times F$ the trivializing chart for π over e^n, and σ to be defined. After composing with h, we regard the restriction of σ_0 over ∂e^n as given by

$$\sigma_0(x) = (x, \tau_0(x)) \in e^n \times F,$$

with $\tau_0 : \partial e^n \cong S^{n-1} \to F$. Since $\pi_{n-1}(F) = 0$, τ_0 extends to a map $\tau : e^n \to F$ which can be used to extend σ_0 over e^n by setting

$$\sigma(x) = (x, \tau(x)).$$

After composing with h^{-1}, we get the desired extension of σ_0 over e^n.

Let us now assume that σ and σ' are two sections of π. To find a homotopy between σ and σ', it suffices to construct a section Σ of $\pi \times id_I : E \times I \to X \times I$. Indeed, if such Σ exists, then $\Sigma(x,t) = (\sigma_t(x), t)$, and σ_t provides the desired homotopy. Now, by regarding σ as a section of $\pi \times id_I$ over $X \times \{0\}$, and σ' as a section of $\pi \times id_I$ over $X \times \{1\}$, the question reduces to constructing a section of $\pi \times id_I$, which extends the section over $X \times \{0,1\}$ defined by (σ, σ'). This can be done as in the first part of the proof. $\square$

In order to finish the proof of Theorem 3.1, it remains to show that Φ is a one-to-one map. If $\pi_0 = f^*\pi_G \cong g^*\pi_G = \pi_1$, we will show that $f \simeq g$. Note that we have the following commutative diagrams:

$$\begin{array}{ccc} E_0 = f^*E_G & \xrightarrow{\widehat{f}} & E_G \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_G} \\ X = X \times \{0\} & \xrightarrow{f} & B_G \end{array}$$

$$\begin{array}{ccc} E_0 \cong E_1 = g^*E_G & \xrightarrow{\widehat{g}} & E_G \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_G} \\ X = X \times \{1\} & \xrightarrow{g} & B_G \end{array}$$

where we regard $\widehat{g}$ as defined on E_0 via the isomorphism $\pi_0 \cong \pi_1$. By putting together the above diagrams, we have a commutative diagram

$$\begin{array}{ccccc} E_0 \times I & \xleftarrow{\hookleftarrow} & E_0 \times \{0,1\} & \xrightarrow{\widehat{\alpha}=(\widehat{f},0)\cup(\widehat{g},1)} & E_G \\ \downarrow{\scriptstyle \pi_0 \times Id} & & \downarrow{\scriptstyle \pi_0 \times \{0,1\}} & & \downarrow{\scriptstyle \pi_G} \\ X \times I & \xleftarrow{\hookleftarrow} & X \times \{0,1\} & \xrightarrow{\alpha=(f,0)\cup(g,1)} & B_G \end{array}$$

Therefore, it suffices to extend $(\alpha, \widehat{\alpha})$ to a bundle map $(H, \widehat{H}) : \pi_0 \times Id \to \pi_G$, and then H will provide the desired homotopy $f \simeq g$.

By Theorem 2.11, such a bundle map $(H, \widehat{H})$ corresponds to a section σ of the fiber bundle

$$\omega : (E_0 \times I) \times_G E_G \to X \times I.$$

On the other hand, the bundle map $(\alpha, \widehat{\alpha})$ already gives a section σ_0 of the fiber bundle

$$\omega_0 : (E_0 \times \{0,1\}) \times_G E_G \to X \times \{0,1\},$$

which under the obvious inclusion $(E_0 \times \{0,1\}) \times_G E_G \subseteq (E_0 \times I) \times_G E_G$ can be regarded as a section of ω over the subcomplex $X \times \{0,1\}$. Since EG is contractible, Lemma 3.2 allows us to extend σ_0 to a section σ of ω defined on $X \times I$, as desired. $\square$

Example 3.3. We give here a more conceptual reasoning for the assertion of Example 1.20. By Theorem 3.1, we have

$$\mathcal{B}(S^n, G, F, \rho) \cong \mathcal{P}(S^n, G) \cong [S^n, BG] = \pi_n(BG) \cong \pi_{n-1}(G),$$

where the last isomorphism follows from the homotopy long exact sequence for π_G, since EG is contractible.

Back to the universal principal G-bundle, we have the following

Theorem 3.4. *Let G be a locally compact topological group. Then a universal principal G-bundle $\pi_G : EG \to BG$ exists (i.e., satisfying $\pi_i(EG) = 0$ for all $i \geq 0$), and the construction is functorial in the sense that a continuous group homomorphism $\mu : G \to H$ induces a bundle map $(B\mu, E\mu) : \pi_G \to \pi_H$. Moreover, the classifying space B_G is unique up to homotopy.*

Proof. To show that BG is unique up to homotopy, let us assume that $\pi_G : E_G \to B_G$ and $\pi'_G : E'_G \to B'_G$ are universal principal G-bundles. By regarding π_G as the universal principal G-bundle for π'_G, we get a map $f : B'_G \to B_G$ such that $\pi'_G = f^*\pi_G$, i.e., a bundle map:

$$\begin{array}{ccc} E'_G & \xrightarrow{\hat{f}} & E_G \\ \downarrow{\scriptstyle \pi'_G} & & \downarrow{\scriptstyle \pi_G} \\ B'_G & \xrightarrow{f} & B_G \end{array}$$

Similarly, y regarding π'_G as the universal principal G-bundle for π_G, there exists a map $g : B_G \to B'_G$ such that $\pi_G = g^*\pi'_G$. Therefore,

$$\pi_G = g^*\pi'_G = g^*f^*\pi_G = (f \circ g)^*\pi_G.$$

On the other hand, we have $\pi_G = (id_{B_G})^*\pi_G$, so by Theorem 3.1 we get that $f \circ g \simeq id_{B_G}$. Similarly, we get $g \circ f \simeq id_{B'_G}$, and hence $f : B'_G \to B_G$ is a homotopy equivalence.

We will not discuss the existence of the universal bundle here, instead we will indicate the universal G-bundle, as needed, in specific examples. □

Example 3.5. Recall that we have a fiber bundle

$$O(n) \hookrightarrow V_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty), \tag{3.1}$$

with $V_n(\mathbb{R}^\infty)$ contractible. In particular, the uniqueness part of Theorem 3.4 tells us that $BO(n) \simeq G_n(\mathbb{R}^\infty)$ is the classifying space for rank n real vector bundles. Similarly, there is a fiber bundle

$$U(n) \hookrightarrow V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty), \tag{3.2}$$

with $V_n(\mathbb{C}^\infty)$ contractible. Therefore, $BU(n) \simeq G_n(\mathbb{C}^\infty)$ is the classifying space for rank n complex vector bundles.

Before moving to the next example, let us mention here without proof the following useful result:

Theorem 3.6. *Let G be an abelian group, and let X be a CW complex. There is a natural bijection*

$$T : [X, K(G,n)] \longrightarrow H^n(X, G)$$
$$[f] \mapsto f^*(\alpha)$$

where $\alpha \in H^n(K(G,n), G) \cong \mathrm{Hom}(H_n(K(G,n), \mathbb{Z}), G)$ is given by the inverse of the Hurewicz isomorphism $G = \pi_n(K(G,n)) \to H_n(K(G,n), \mathbb{Z})$.

Example 3.7 (Classification of real line bundles)**.** Let $G = \mathbb{Z}/2$ and consider the principal $\mathbb{Z}/2$-bundle $\mathbb{Z}/2 \hookrightarrow S^\infty \to \mathbb{R}P^\infty$. Since S^∞ is contractible, the uniqueness of the universal bundle yields that $B\mathbb{Z}/2 \cong \mathbb{R}P^\infty$. In particular, we see that $\mathbb{R}P^\infty$ classifies the real line (i.e., rank-one) bundles. Since we also have that $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$, we get:

$$\mathcal{P}(X, \mathbb{Z}/2) = [X, B\mathbb{Z}/2] = [X, K(\mathbb{Z}/2, 1)] \cong H^1(X, \mathbb{Z}/2)$$

for any CW complex X, where the last identification follows from Theorem 3.6. Let now π be a real line bundle on a CW complex X, with classifying map $f_\pi : X \to \mathbb{R}P^\infty$. Since $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[w]$, with w a generator of $H^1(\mathbb{R}P^\infty, \mathbb{Z}/2)$, we get a well-defined degree one cohomology class

$$w_1(\pi) := f_\pi^*(w)$$

called the *first Stiefel-Whitney class of* π. The bijection $\mathcal{P}(X, \mathbb{Z}/2) \xrightarrow{\cong} H^1(X, \mathbb{Z}/2)$ is then given by $\pi \mapsto w_1(\pi)$, so real line bundles on X are classified by their first Stiefel-Whitney classes.

Example 3.8 (Classification of complex line bundles)**.** Let $G = S^1$ and consider the principal S^1-bundle $S^1 \hookrightarrow S^\infty \to \mathbb{C}P^\infty$. Since S^∞ is contractible, the uniqueness of the universal bundle yields that $BS^1 \cong \mathbb{C}P^\infty$. In particular, as $S^1 = GL(1, \mathbb{C})$, we see that $\mathbb{C}P^\infty$ classifies the complex line (i.e., rank-one) bundles. Since we also have that $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, we get:

$$\mathcal{P}(X, S^1) = [X, BS^1] = [X, K(\mathbb{Z}, 2)] \cong H^2(X, \mathbb{Z})$$

for any CW complex X, where the last identification follows from Theorem 3.6. Let now π be a complex line bundle on a CW complex X, with classifying map $f_\pi : X \to \mathbb{C}P^\infty$. Since $H^*(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}[c]$, with c a generator of $H^2(\mathbb{C}P^\infty, \mathbb{Z})$, we get a well-defined degree two cohomology class

$$c_1(\pi) := f_\pi^*(c)$$

called the *first Chern class of* π. The bijection $\mathcal{P}(X, S^1) \xrightarrow{\cong} H^2(X, \mathbb{Z})$ is then given by $\pi \mapsto c_1(\pi)$, so complex line bundles on X are classified by their first Chern classes.

Remark 3.9. If X is any orientable closed oriented surface, then $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, so Example 3.8 shows that isomorphism classes of complex line bundles on X are in bijective correspondence with the set of integers. On the other hand, if X is a non-orientable closed surface, then $H^2(X, \mathbb{Z}) \cong \mathbb{Z}/2$, so there are only two isomorphism classes of complex line bundles on such a surface.

4 Exercises

1. Let $p : S^2 \to \mathbb{R}P^2$ be the (oriented) double cover of $\mathbb{R}P^2$. Since $\mathbb{R}P^2$ is a non-orientable surface, we know by Remark 3.9 that there are only two isomorphism classes of complex line bundles on $\mathbb{R}P^2$: the trivial one, and a non-trivial complex line bundle which we denote by

$\pi : E \to \mathbb{R}P^2$. On the other hand, since S^2 is a closed orientable surface, the isomorphism classes of complex line bundles on S^2 are in bijection with $\mathbb{Z}$. Which integer corresponds to complex line bundle $p^*\pi : p^*E \to S^2$ on S^2?

2. Consider a locally trivial fiber bundle $S^2 \hookrightarrow E \xrightarrow{\pi} S^2$. Recall that such π can be regarded as a fiber bundle with structure group $G = Homeo(S^2) \cong SO(3)$. By the classification Theorem 3.1, $SO(3)$-bundles over S^2 correspond to elements in

$$[S^2, BSO(3)] = \pi_2(BSO(3)) \cong \pi_1(SO(3)).$$

(a) Show that $\pi_1(SO(3)) \cong \mathbb{Z}/2$. (Hint: Show that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.)

(b) What is the non-trivial $SO(3)$-bundle over S^2?

3. Let $\pi : E \to X$ be a principal S^1-bundle over the simply-connected space X. Let $a \in H^1(S^1, \mathbb{Z})$ be a generator. Show that

$$c_1(\pi) = d_2(a),$$

where d_2 is the differential on the E_2-page of the Leray-Serre spectral sequence associated to π, i.e., $E_2^{p,q} = H^p(X, H^q(S^1)) \Rightarrow H^{p+q}(E, \mathbb{Z})$.

4. By the classification Theorem 3.1, (isomorphism classes of) S^1-bundles over S^2 are given by

$$[S^2, BS^1] = \pi_2(BS^1) \cong \pi_1(S^1) \cong \mathbb{Z}$$

and this correspondence is realized by the first Chern class, i.e., $\pi \mapsto c_1(\pi)$.

(a) What is the first Chern class of the Hopf bundle $S^1 \hookrightarrow S^3 \to S^2$?

(b) What is the first Chern class of the sphere (or unit) bundle of the tangent bundle TS^2?

(c) Construct explicitely the S^1-bundle over S^2 corresponding to $n \in \mathbb{Z}$. (Hint: Think of lens spaces, and use the above Exercise 3.)