

ASPHERICAL MANIFOLDS, MELLIN TRANSFORMATION AND A QUESTION OF BOBADILLA-KOLLÁR

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ABSTRACT. In their 2012 paper, Bobadilla and Kollár studied topological conditions which guarantee that a proper map of complex algebraic varieties is a topological or differentiable fibration. They also asked whether a certain finiteness property on the relative covering space can imply that a proper map is a fibration. In this paper, we answer positively the integral homology version of their question in the case of abelian varieties, and the rational homology version in the case of compact ball quotients. We also propose several conjectures in relation to the Singer-Hopf conjecture in the complex projective setting.

1. INTRODUCTION

A CW-complex X is called *aspherical* if it is connected and all its higher homotopy groups vanish, i.e., $\pi_i(X)$ is trivial for all $i \geq 2$. The vanishing of higher-homotopy groups is equivalent to the fact that the universal covering \tilde{X} of X is contractible. The homotopy type of an aspherical CW complex depends only of its fundamental group.

Interesting examples of aspherical spaces are the closed Riemannian manifolds with non-positive sectional curvature, compact ball quotients or abelian varieties.

There are several prominent open conjectures concerning aspherical manifolds. For instance, a conjecture of Borel asserts that two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic. The Borel conjecture is proved in many important cases, e.g., it is true in dimensions $\neq 3, 4$ for all non-positively curved closed Riemannian manifolds, see [FJ]. Another important conjecture was made by Hopf (and later on strengthened by Singer) on the sign of the topological Euler characteristic of an aspherical closed manifold.

Conjecture 1.1. (Singer-Hopf) *If X^{2n} is a closed, aspherical manifold of real dimension $2n$, then*

$$(-1)^n \chi(X^{2n}) \geq 0.$$

More recently, inspired by work of Kollár and Pardon [KP], Bobadilla and Kollár [BK] used aspherical manifolds in their search for homotopy/homology fiber bundles, which in turn are conjectured to be differentiable fibre bundles. In order to state their conjecture, we make the following definition.

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Definition 1.2 ([BK, Definition 1]). Let X and Y be complex manifolds.¹ A proper holomorphic map $f : X \rightarrow Y$ is said to be a *homotopy fiber bundle* if Y has an open cover $Y = \bigcup_j U_j$ such that for every j and for every $y \in U_j$ the inclusion

$$f^{-1}(y) \hookrightarrow f^{-1}(U_j) \text{ is a homotopy equivalence.}$$

Similarly, given a commutative ring \mathbb{A} , the map $f : X \rightarrow Y$ is called an \mathbb{A} -*homology fiber bundle* if

$$H_*(f^{-1}(y), \mathbb{A}) \rightarrow H_*(f^{-1}(U_j), \mathbb{A}) \text{ is an isomorphism.}$$

Let X and Y be smooth projective varieties and $f : X \rightarrow Y$ a surjective morphism. Let $\tilde{Y} \rightarrow Y$ denote the universal cover. By pull-back we obtain a map $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$. In [BK], Bobadilla and Kollár asked the following question, which also appears as Question 26 in [KP] (see also [KP, Question 4] for a broader statement).

Question 1.3. *Assume that \tilde{Y} is contractible and \tilde{X} is homotopy equivalent to a finite CW complex. Does this imply that f is a topological or differentiable fiber bundle?*

The above question can be divided into two parts. The first part is more topological:

Question 1.4. ([BK, Question 4.2]) *Assume that \tilde{Y} is contractible and \tilde{X} is homotopy equivalent to a finite CW complex. Does this imply that f is a homotopy or \mathbb{Z} -homology fiber bundle?*

We will refer to the homological part of Question 1.4 as the *integral Bobadilla-Kollár question*. If we replace “ \mathbb{Z} -homology fiber bundle” by “ \mathbb{Q} -homology fiber bundle” in the above question, we call it the *rational Bobadilla-Kollár question*.

The second part is more geometric, and is formulated as a conjecture in [BK].

Conjecture 1.5. ([BK, Conjecture 3]) *Let $f : X \rightarrow Y$ be a proper map of smooth complex algebraic varieties. If f is a homotopy or \mathbb{Z} -homology fiber bundle, then it is a differentiable fiber bundle.*

In this paper we answer positively the homological versions of the Bobadilla-Kollár question 1.4 in the case of aspherical projective manifolds with ample cotangent bundles (e.g., compact ball quotients) and abelian varieties. More precisely, we show the following (see Theorem 5.3 and 5.17).

Theorem 1.6. *The rational Bobadilla-Kollár question is true if Y is an aspherical projective manifold with ample cotangent bundle (e.g., a compact ball quotient). Moreover, the integral Bobadilla-Kollár question is true if Y is an abelian variety.*

As a concrete application, we get the following.

Corollary 1.7. *Let X be a projective manifold, and denote by X^{ab} the universal free abelian cover of X , i.e., the covering associated to the homomorphism $\pi_1(X) \rightarrow H_1(X, \mathbb{Z})/\text{torsion}$. If X^{ab} is homotopy equivalent to a finite CW-complex, then the Albanese map of X is a \mathbb{Z} -homology fiber bundle.*

¹In [BK], the authors considered more generally complex spaces. In this paper, we will restrict ourselves to smooth manifolds/varieties.

Our main results should be compared to [KP, Theorem 16, Theorem 20] by Kollár and Pardon, where special cases of Question 1.3 are addressed.

Our approach to proving Theorem 1.6 relies on the theory of perverse sheaves and derived calculus (for an introduction to these techniques, see, e.g., [Di, Ma]). Let us mention here the main ideas of the proof. First note that a proper map $f : X \rightarrow Y$ of smooth complex algebraic varieties is a \mathbb{Z} -homology fiber bundle if and only if the higher derived pushforwards $R^i f_* \mathbb{Z}_X$ are locally constant sheaves on Y . If this is the case, we say that the constructible complex $Rf_* \mathbb{Z}_X$ is locally constant. (Similar considerations apply to rational coefficients.) Such complexes are introduced and characterized in Section 4. To answer the homological version of Question 1.4, in Section 2 we first introduce and study the properties of a nonabelian version of the Mellin transformation considered in [LMW2, LMW3], see Definition 2.1. A positive answer to the rational Bobadilla–Kollár question can be given for any aspherical projective manifold with an ample cotangent bundle (e.g., a compact ball quotient) by using the decomposition theorem and positivity results for Chern classes of ample vector bundles (cf. Section 3). For the integral version, we have to also work with fields of positive characteristics. A positive answer to the integral Bobadilla–Kollár question for abelian varieties is given in Section 5, and it relies on a key non-vanishing property of the Mellin transformation (see Proposition 5.6), which is proved using characteristic cycles and the geometry of abelian varieties.

Finally, in Section 6 we speculate around the Singer–Hopf conjecture 1.1 in the complex algebraic setting. We propose various generalizations, also in relation to the Shafarevich conjecture. For instance, we prove the following result (see Corollary 6.8).

Theorem 1.8. *Let Y be an aspherical projective manifold. Then the Shafarevich conjecture implies that the universal cover of Y is Stein.*

We also conjecture that the cotangent bundle of a projective manifold with a Stein universal cover is nef (see Conjecture 6.3). Together with semi-positivity results for nef vector bundles from [DPS], this would then imply the Singer–Hopf Conjecture 1.1 in the complex projective setting.

As a convention, in this paper all varieties and manifolds are assumed to be connected.

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2. NONABELIAN MELLIN TRANSFORMATION

In this section we introduce the Mellin transformation for complex manifolds and study its immediate properties.

Let Y be a complex manifold of dimension d with fundamental group $\pi_1(Y) = G$. We denote the universal cover of Y by \tilde{Y} . Let \mathbb{A} be a commutative ring, and let \mathcal{F} be a bounded \mathbb{A} -constructible complex on Y .

Definition 2.1. We define the *Mellin transformation* of \mathcal{F} on Y as

$$\mathcal{M}_*(Y, \mathcal{F}) := R\tilde{q}_!(p^* \mathcal{F}) \in D^b(\mathbb{A}[G]),$$

where $p : \tilde{Y} \rightarrow Y$ is the universal covering map, $\tilde{q} : \tilde{Y} \rightarrow \text{pt}$ is the projection to a point, and $D^b(\mathbb{A}[G])$ is the derived category of right $\mathbb{A}[G]$ -modules. When there is no risk of confusion, we will simply write $\mathcal{M}_*(\mathcal{F})$ instead of $\mathcal{M}_*(Y, \mathcal{F})$.

Note that since $p^*\mathcal{F}$ is a right G -equivariant complex and \tilde{q} is a right G -equivariant map, $R\tilde{q}_!(p^*\mathcal{F})$ admits a natural right G -action. By definition, the Mellin transformation

$$\mathcal{M}_*(Y, -) : D_c^b(Y, \mathbb{A}) \rightarrow D^b(\mathbb{A}[G])$$

is a functor of triangulated categories.

Let \mathcal{L}_G be the rank one $\mathbb{A}[G]$ -local system on Y whose monodromy action is given by multiplication on the right. Note that $\mathcal{L}_G \cong Rp_!\mathbb{A}_{\tilde{Y}}$ as sheaves of right $\mathbb{A}[G]$ -modules. Then an equivalent description of the Mellin transformation is given by the following

Proposition 2.2.

$$(2.1) \quad \mathcal{M}_*(Y, \mathcal{F}) = Rq_!(\mathcal{F} \otimes_{\mathbb{A}} \mathcal{L}_G) \in D^b(\mathbb{A}[G]),$$

where $q : Y \rightarrow \text{pt}$ is the projection to a point.

Proof. Indeed, we have:

$$R\tilde{q}_!(p^*\mathcal{F}) \cong Rq_!Rp_!(p^*\mathcal{F} \otimes_{\mathbb{A}} \mathbb{A}_{\tilde{Y}}) \cong Rq_!(\mathcal{F} \otimes_{\mathbb{A}} Rp_!\mathbb{A}_{\tilde{Y}}) \cong Rq_!(\mathcal{F} \otimes_{\mathbb{A}} \mathcal{L}_G),$$

where the second isomorphism uses the projection formula. \square

Remark 2.3. Here we made a choice that the equivariant G -actions are on the right. This essentially depends on how we let G act on \tilde{Y} . We consider points in \tilde{Y} as homotopy classes of paths from the base point y_0 to an arbitrary point y on Y . Then the natural action of G on \tilde{Y} is on the right.

Example 2.4. Suppose that Y is an aspherical complex manifold of complex dimension d and L is an \mathbb{A} -local system on Y . Then $\mathcal{M}_*(L) \cong V[-2d]$, where V is the $\mathbb{A}[G]$ -module associated to the monodromy representation of L .

Let V be a finitely generated free \mathbb{A} -module. Fix a representation $\rho : G \rightarrow \text{Aut}_{\mathbb{A}}(V)$, which induces a left $\mathbb{A}[G]$ -module structure on V . We write V_ρ to emphasize the $\mathbb{A}[G]$ -module structure on V . We denote by L_ρ the \mathbb{A} -local system on Y whose stalks are isomorphic to V and whose monodromy action is equal to ρ .

Proposition 2.5. *Given any ρ as above and $\mathcal{F} \in D_c^b(Y, \mathbb{A})$, there is a canonical isomorphism*

$$(2.2) \quad H_c^i(Y, \mathcal{F} \otimes_{\mathbb{A}} L_\rho) \cong H^i(\mathcal{M}_*(Y, \mathcal{F}) \otimes_{\mathbb{A}[G]}^L V_\rho)$$

where $\otimes_{\mathbb{A}[G]}^L$ denotes the derived tensor product of right and left $\mathbb{A}[G]$ -modules.

Proof. By projection formula, we have

$$Rq_!(\mathcal{F} \otimes_{\mathbb{A}} \mathcal{L}_G) \otimes_{\mathbb{A}[G]}^L V_\rho \cong Rq_!(\mathcal{F} \otimes_{\mathbb{A}} \mathcal{L}_G \otimes_{\mathbb{A}[G]}^L q^*V_\rho) \cong Rq_!(\mathcal{F} \otimes_{\mathbb{A}} L_\rho).$$

Formula (2.2) follows by taking i -th cohomology on both sides. \square

Corollary 2.6. *Let Y be a smooth complex algebraic variety or a compact complex manifold. Given a field \mathbb{K} , if $\mathcal{F} \in D_c^b(Y, \mathbb{K})$ has the property that its Euler characteristic $\chi(Y, \mathcal{F})$ is not zero, then $\mathcal{M}_*(Y, \mathcal{F}) \neq 0$.*

Proof. Suppose that $\mathcal{M}_*(Y, \mathcal{F}) = 0$. Applying the above proposition to the case when V_ρ is the trivial rank one G -representation, we get that $H_c^k(Y, \mathcal{F}) = 0$ for all $k \in \mathbb{Z}$. Then $\chi(Y, \mathcal{F}) = \chi_c(Y, \mathcal{F}) = 0$, contradicting our assumptions. For the equality of the Euler characteristics with and without compact support in the algebraic case, e.g., see [Di, Proposition 4.1.23]. \square

Corollary 2.7. *Let \mathcal{F} be a \mathbb{K} -constructible complex on an abelian variety A . Then $\mathcal{M}_*(\mathcal{F}) = 0$ if and only if $H^i(A, \mathcal{F} \otimes_{\mathbb{K}} L) = 0$ for any i and any rank one $\overline{\mathbb{K}}$ -local system L , where $\overline{\mathbb{K}}$ is the algebraic closure of \mathbb{K} .*

Proof. Notice that the group ring $\mathbb{K}[\pi_1(A)]$ is isomorphic to a Laurent polynomial ring. Over a finitely generated \mathbb{K} -algebra, a module is zero if and only if its restriction to every $\overline{\mathbb{K}}$ -point is zero. So the assertion follows from Proposition 2.5. \square

3. EULER CHARACTERISTIC OF PERVERSE SHEAVES AND CHARACTERISTIC CYCLES

In this section, we recall several (semi-)positivity results for ample (resp., nef) vector bundles on projective manifolds. We use such results to deduce (semi-)positivity statements for the Euler characteristics of perverse sheaves on complex projective manifolds with ample (resp., nef) cotangent bundles. For perverse sheaves we use field coefficients.

First, let us recall the definition of ample and, resp., nef vector bundles.

Definition 3.1. If E is a vector bundle on a projective manifold X , denote by $\mathbf{P}(E)$ the projective bundle of hyperplanes in the fibers of E . A vector bundle E on X is called *ample* (resp. *nef*) if the line bundle $\mathcal{O}_E(1)$ on $\mathbf{P}(E)$ is ample (resp. nef).

In [FL], Fulton and Lazarsfeld studied the positivity of Chern classes of ample vector bundles, and proved the following result.

Theorem 3.2. ([FL, Theorem II]) *Let X be a projective manifold and let E be a rank r ample vector bundle on X . For any r -dimensional conic subvariety C of E , that is a subvariety that is invariant under the \mathbb{C}^* -action on E , the intersection number satisfies*

$$\langle C, Z_E \rangle_E > 0$$

where Z_E is the zero section of E .

Together with Kashiwara's index theorem, we have the following positivity result on the Euler characteristics of perverse sheaves.

Proposition 3.3. *Let X be a projective manifold with ample cotangent bundle, and let \mathcal{P} be a nonzero perverse sheaf on X . Then $\chi(X, \mathcal{P}) > 0$.*

Proof. Kashiwara's global index theorem [Ka] computes the Euler characteristic of any bounded constructible complex \mathcal{P} on X by the formula:

$$(3.1) \quad \chi(X, \mathcal{P}) = \langle CC(\mathcal{P}), [X] \rangle_{T^*X},$$

that is, the intersection index in the cotangent bundle T^*X , of the characteristic cycle of \mathcal{P} with the zero section of T^*X . Recall that the characteristic cycle $CC(\mathcal{P})$ is a formal \mathbb{Z} -linear combination of irreducible conic Lagrangian cycles $T_Z^*X := \overline{T_{Z_{\text{reg}}}^*X}$ in T^*X given by the conormal spaces of certain irreducible closed subvarieties $Z \subseteq X$.

Since the characteristic cycle of a perverse sheaf is known to be effective (e.g., see [Di, Corollary 5.2.24]), the positivity of $\chi(X, \mathcal{P})$ follows immediately from the ampleness of the cotangent bundle of X together with Theorem 3.2. \square

Since compact ball quotients have ample cotangent bundles (e.g., see [Laz, Construction 6.3.36]), we get the following:

Corollary 3.4. *If X is a compact ball quotient and \mathcal{P} is a nonzero perverse sheaf on X , then $\chi(X, \mathcal{P}) > 0$.*

The analogous result of Fulton-Lazarsfeld for nef vector bundles is proved by Demailly-Peternell-Schneider [DPS].

Theorem 3.5. ([DPS, Proposition 2.3]) *Let X be a projective manifold and let E be a rank r nef vector bundle on X . For any r -dimensional conic subvariety C of E , the intersection number satisfies*

$$\langle C, Z_E \rangle_E \geq 0$$

where Z_E is the zero section of E .

The following analog of Proposition 3.3 can be proved by the same arguments.

Proposition 3.6. *Let X be a projective manifold with nef cotangent bundle, and let \mathcal{P} be a nonzero perverse sheaf on X . Then $\chi(X, \mathcal{P}) \geq 0$. In particular, $(-1)^{\dim X} \chi(X) \geq 0$.*

Since the intersection complex IC_Z of any pure-dimensional complex algebraic variety Z is a perverse sheaf and since $\mathrm{IH}^k(Z) \cong H^{k+\dim Z}(Z, \mathrm{IC}_Z)$, we have the following.

Corollary 3.7. *Let X be a projective manifold with nef cotangent bundle, and let Z be an irreducible closed subvariety of X . Then the intersection cohomology Euler characteristics of Z , that is,*

$$\chi_{\mathrm{IH}}(Z) := \sum_{0 \leq k \leq 2 \dim Z} (-1)^k \dim \mathrm{IH}^k(Z)$$

satisfies

$$(-1)^{\dim Z} \chi_{\mathrm{IH}}(Z) \geq 0.$$

Example 3.8. The class of complex projective manifolds whose cotangent bundles are nef is closed under taking products, subvarieties and finite unramified covers, and it includes smooth subvarieties of abelian varieties. It should also be noted that if A is an abelian variety of dimension g , and $X \subset A$ is a smooth subvariety of dimension n and codimension $g-n < n$, then the cotangent bundle of X is not ample (see [De, Sc], and also, [Laz, Example 7.2.3]). On the other hand, for an arbitrary smooth m -dimensional projective variety M and each $n \leq m/2$, there exist plenty of smooth n -dimensional subvarieties $X \subset M$ with ample cotangent bundle, e.g., complete intersection of sections of M by general hypersurfaces of sufficiently high degrees in the ambient projective space, see [BD, Xi].

4. LOCALLY CONSTANT CONSTRUCTIBLE COMPLEXES

We should note here that a proper map $f : X \rightarrow Y$ of smooth complex algebraic varieties is a \mathbb{Z} -homology fiber bundle if and only if the higher derived pushforwards $R^i f_* \mathbb{Z}_X$ are locally constant sheaves on Y . If this is the case, we say that the constructible complex $Rf_* \mathbb{Z}_X$ is locally constant.

This motivates the following.

Definition 4.1. Let M be a complex manifold, \mathbb{A} a commutative ring, and let \mathcal{F} be a bounded \mathbb{A} -constructible complex. We say that \mathcal{F} is *locally constant* if for any contractible open subset $U \subset M$, the restriction map

$$R\Gamma(U, \mathcal{F}) \rightarrow i_x^* \mathcal{F}$$

is a quasi-isomorphism for every point $x \in U$, where $i_x : \{x\} \hookrightarrow M$ is the point inclusion and $R\Gamma(U, -) : D_c^b(M, \mathbb{A}) \rightarrow D^-(\mathbb{A})$ is the derived functor of taking sections over U .

Lemma 4.2. *The mapping cone of any morphism of locally constant bounded \mathbb{A} -constructible complexes is also locally constant.*

Proof. Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of bounded constructible complexes, and let $C(f)$ be its mapping cone. Let U be a contractible open subset of M , and $x \in U$. Since the restriction map is functorial, we have the following map of long exact sequences

$$\begin{array}{ccccccccc} H^k(U, \mathcal{F}|_U) & \longrightarrow & H^k(U, \mathcal{G}|_U) & \longrightarrow & H^k(U, C(f)|_U) & \longrightarrow & H^{k+1}(U, \mathcal{F}|_U) & \longrightarrow & H^{k+1}(U, \mathcal{G}|_U) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^k(i_x^* \mathcal{F}) & \longrightarrow & H^k(i_x^* \mathcal{G}) & \longrightarrow & H^k(i_x^* C(f)) & \longrightarrow & H^k(i_x^* \mathcal{F}) & \longrightarrow & H^k(i_x^* \mathcal{G}). \end{array}$$

By definition, all the vertical arrows are isomorphisms except the middle one. By the 5-lemma, the middle vertical arrow is also an isomorphism. This implies that the restriction map $R\Gamma(U, C(f)) \rightarrow i_x^* C(f)$ is a quasi-isomorphism. \square

Locally constant complexes are described by the following result.

Proposition 4.3. *Let \mathcal{F} be a bounded \mathbb{A} -constructible complex on a complex manifold M . The following conditions are equivalent:*

- (1) \mathcal{F} is locally constant;
- (2) there exists a covering $\{U_\lambda\}_{\lambda \in I}$ of M by contractible open subsets such that the restriction map $R\Gamma(U_\lambda, \mathcal{F}) \rightarrow i_x^* \mathcal{F}$ is an isomorphism for all $\lambda \in I$ and $x \in U_\lambda$.
- (3) the cohomology sheaves $\mathcal{H}^k(\mathcal{F})$ are local systems for all k ;
- (4) the perverse cohomology sheaves ${}^p\mathcal{H}^k(\mathcal{F})$ are shifts of local systems for all k .

If \mathbb{A} is a field, then the above conditions are also equivalent to

- (5) the Verdier dual $\mathbb{D}(\mathcal{F})$ is locally constant.

Proof. First, we prove (2) \Rightarrow (3). If $\mathcal{F} = 0$, there is nothing to prove. Otherwise, let $\mathcal{H}^l(\mathcal{F})$ be the lowest degree nonzero sheaf cohomology. Using the hypercohomology spectral sequence

$$E_2^{ij} = H^i(U_\lambda, \mathcal{H}^j(\mathcal{F})) \implies H^{i+j}(U_\lambda, \mathcal{F}),$$

we get that

$$H^0(U_\lambda, \mathcal{H}^l(\mathcal{F})) \cong H^l(U_\lambda, \mathcal{F}).$$

Since the restriction map $R\Gamma(U_\lambda, \mathcal{F}) \rightarrow i_x^* \mathcal{F}$ is a quasi-isomorphism, we have an isomorphism of cohomology groups

$$H^l(U_\lambda, \mathcal{F}) \cong H^l(i_x^* \mathcal{F}).$$

Since taking sheaf cohomology commutes with taking stalk, we also have a natural isomorphism

$$H^l(i_x^* \mathcal{F}) \cong i_x^*(\mathcal{H}^l(\mathcal{F})).$$

Combining the above three isomorphism, we have

$$H^0(U_\lambda, \mathcal{H}^l(\mathcal{F})) \cong i_x^*(\mathcal{H}^l(\mathcal{F}))$$

for every U_λ and any $x \in U_\lambda$. Thus, $\mathcal{H}^l(\mathcal{F})$ is a local system. Since $\mathcal{H}^l(\mathcal{F})$ is the lowest nonzero sheaf cohomology of \mathcal{F} , there is an isomorphism $\mathcal{H}^l(\mathcal{F})[-l] \cong \tau^{\leq l}\mathcal{F}$, which induces a natural morphism

$$g : \mathcal{H}^l(\mathcal{F})[-l] \rightarrow \mathcal{F}$$

of constructible complexes. Here, $(\tau^{\leq}, \tau^{\geq})$ denotes the canonical (or standard) truncation on $D_c^b(M)$. Applying the proof of Lemma 4.2 for the map g over each U_λ , we get that the mapping cone $C(g)$ also satisfies property (2). By construction, we have that $C(g) \cong \tau^{\geq l+1}\mathcal{F}$, and hence

$$\mathcal{H}^k(C(g)) \cong \begin{cases} 0 & \text{if } k < l + 1 \\ \mathcal{H}^k(\mathcal{F}) & \text{if } k \geq l + 1. \end{cases}$$

Applying the above argument to $C(g)$, we get that $\mathcal{H}^{l+1}(C(g)) \cong \mathcal{H}^{l+1}(\mathcal{F})$ is a local system. Repeating this argument, we can conclude that all $\mathcal{H}^k(\mathcal{F})$ are local systems.

Next, we prove (3) \Rightarrow (4). If $\mathcal{H}^k(\mathcal{F})$ are local systems for all k , then \mathcal{F} is constructible with respect to the trivial stratification of M , that is, the one with a single stratum. With respect to this stratification, the truncations ${}^p\tau^{\leq k}$ (resp. ${}^p\tau^{\geq k}$) and $\tau^{\leq k - \dim M}$ (resp., $\tau^{\geq k - \dim M}$) are equal. Here, $({}^p\tau^{\leq}, {}^p\tau^{\geq})$ denotes the perverse truncation. Thus,

$${}^p\mathcal{H}^k(\mathcal{F}) \cong \mathcal{H}^{k - \dim M}(\mathcal{F})[\dim M]$$

is the shift of a local system.

The implication (4) \Rightarrow (1) follows immediately from Lemma 4.2, and the implication (1) \Rightarrow (2) is obvious. So far, we have proved that the statements (1) - (4) are all equivalent. When \mathbb{A} is a field, we have that $\mathbb{D}(\mathcal{H}^k(\mathcal{F})) \cong \mathcal{H}^{-k}(\mathbb{D}\mathcal{F})$. Hence statement (5) is a consequence of the first four statements. Since $\mathbb{D}(\mathbb{D}\mathcal{F}) \cong \mathcal{F}$, the implication (1) \Rightarrow (5) also implies that (5) \Rightarrow (1). \square

5. THE RATIONAL AND INTEGRAL BOBADILLA-KOLLÁR QUESTIONS

In this section, we answer positively the rational Bobadilla-Kollár question for compact ball quotients and the integral Bobadilla-Kollár question for abelian varieties. The rational version is in general easier thanks to the decomposition theorem. We will first simultaneously answer the rational Bobadilla-Kollár question for both ball quotients and abelian varieties. For the integral version, we have to work with fields of positive characteristics, in which case neither the decomposition theorem nor the stronger generic vanishing theorem as in [BSS] hold. We use characteristic cycles and the geometry of abelian varieties to prove a key non-vanishing property of the Mellin transformation, which is sufficient to answer the integral Bobadilla-Kollár question for abelian varieties.

Let Y be an aspherical compact complex manifold with fundamental group G and of dimension d . Let \mathcal{F} be a \mathbb{K} -constructible complex on Y , where \mathbb{K} is a field, and assume that each cohomology group

$$V^j := H^j(\mathcal{M}_*(\mathcal{F}))$$

is a finite dimensional \mathbb{K} -vector space. Notice that V^j has a natural $\mathbb{K}[G]$ -module structure, and we denote the corresponding \mathbb{K} -local system on Y by L^j .

Lemma 5.1. *Under the above notations, let $V^r = H^r(\mathcal{M}_*(\mathcal{F}))$ be the nonzero cohomology group of the lowest degree. Then there exists a morphism $\phi : L^r[2d - r] \rightarrow \mathcal{F}$ such that the induced morphism $\mathcal{M}_*(\phi) : \mathcal{M}_*(L^r[2d - r]) \rightarrow \mathcal{M}_*(\mathcal{F})$ induces an isomorphism*

$$H^r\left(\mathcal{M}_*(L^r[2d - r])\right) \cong H^r(\mathcal{M}_*(\mathcal{F})).$$

Proof. The canonical truncation on $\mathcal{M}_*(\mathcal{F})$ induces a spectral sequence

$$E_2^{ij} = H^i\left(H^j(\mathcal{M}_*(\mathcal{F})) \otimes_{\mathbb{K}[G]}^L (V^r)^\vee\right) \implies H^{i+j}\left(\mathcal{M}_*(\mathcal{F}) \otimes_{\mathbb{K}[G]}^L (V^r)^\vee\right).$$

In fact, choosing a free resolution F^\bullet of $(V^r)^\vee$ and a complex G^\bullet representing $\mathcal{M}_*(\mathcal{F})$, it is the spectral sequence of the double complex $G^\bullet \otimes_{\mathbb{K}[G]} F^\bullet$.

By assumption, $H^j(\mathcal{M}_*(\mathcal{F})) = 0$ if $j < r$. By Example 2.4 and Proposition 2.5, we have

$$H^i\left(H^j(\mathcal{M}_*(\mathcal{F})) \otimes_{\mathbb{K}[G]}^L (V^r)^\vee\right) = H^i\left(V^j \otimes_{\mathbb{K}[G]}^L (V^r)^\vee\right) \cong H^{i+2d}\left(Y, L^j \otimes_{\mathbb{K}} (L^r)^\vee\right),$$

which vanishes when $i < -2d$. Therefore,

$$E_2^{-2d,r} \cong E_\infty^{-2d,r} \cong H^{r-2d}\left(\mathcal{M}_*(\mathcal{F}) \otimes_{\mathbb{K}[G]}^L (V^r)^\vee\right),$$

that is,

$$H^{-2d}\left(V^r \otimes_{\mathbb{K}[G]}^L (V^r)^\vee\right) \cong H^{r-2d}\left(\mathcal{M}_*(\mathcal{F}) \otimes_{\mathbb{K}[G]}^L (V^r)^\vee\right).$$

Since $\mathcal{M}_*(L^r) \cong V^r[-2d]$ (cf. Example 2.4), by Proposition 2.5 we have canonical isomorphisms

$$H^0\left(Y, L^r \otimes_{\mathbb{K}} (L^r)^\vee\right) \cong H^0\left(\mathcal{M}_*(L^r) \otimes_{\mathbb{K}[G]}^L (V^r)^\vee\right) \cong H^{-2d}\left(V^r \otimes_{\mathbb{K}[G]}^L (V^r)^\vee\right).$$

On the other hand, also by Proposition 2.5, we have canonical isomorphisms

$$\mathrm{Hom}(L^r[2d - r], \mathcal{F}) \cong H^{r-2d}\left(Y, \mathcal{F} \otimes (L^r)^\vee\right) \cong H^{r-2d}\left(\mathcal{M}_*(\mathcal{F}) \otimes_{\mathbb{K}[G]}^L (V^r)^\vee\right).$$

Combining the above three displayed equations, we have a natural isomorphism

$$H^0\left(Y, L^r \otimes_{\mathbb{K}} (L^r)^\vee\right) \cong \mathrm{Hom}(L^r[2d - r], \mathcal{F}).$$

There exists an element in $H^0(Y, L^r \otimes_{\mathbb{K}} (L^r)^\vee)$ associated to the identity map on L^r . We let ϕ be the corresponding element in $\mathrm{Hom}(L^r[2d - r], \mathcal{F})$. Then ϕ satisfies the desired property. \square

Proposition 5.2. *Let Y be an aspherical projective manifold of dimension d , and let \mathbb{K} be any field. Let \mathcal{P} be a simple \mathbb{K} -perverse sheaf on Y . Suppose that $\mathcal{M}_*(\mathcal{P})$ is nonzero and the cohomology of $\mathcal{M}_*(\mathcal{P})$ in every degree is finite dimensional. Then \mathcal{P} is a shift of a local system.*

Proof. As in Lemma 5.1, we let $H^r(\mathcal{M}_*(\mathcal{P}))$ be the nonzero cohomology group of the lowest degree, with corresponding local system on Y denoted by L^r . By Lemma 5.1 there exist a nonzero morphism $\phi : L^r[2d - r] \rightarrow \mathcal{P}$. By definition, $\mathcal{M}_*(\mathcal{P}) = R\tilde{q}_!(p^*\mathcal{P})$, where $p : \tilde{Y} \rightarrow Y$ is the universal covering map and $\tilde{q} : \tilde{Y} \rightarrow pt$ is the projection to a point. Since $p^*\mathcal{P}$ is a perverse sheaf on \tilde{Y} (cf. [Di, Proposition 5.2.13 and Corollary 5.2.15]), the cohomology

$$H^k(R\tilde{q}_!(p^*\mathcal{P})) \cong H_c^k(\tilde{Y}, p^*\mathcal{P})$$

is zero if $k \notin [-d, d]$ (e.g., see [Di, Proposition 5.2.20]). Thus, we have $r \leq d$. On the other hand, since

$$L^r[2d - r] \in {}^pD_c^{\leq r-d}(Y, \mathbb{K}) \quad \text{and} \quad \mathcal{P} \in {}^pD_c^{\geq 0}(Y, \mathbb{K}),$$

the existence of a nonzero morphism $\phi : L^r[2d - r] \rightarrow \mathcal{P}$ implies that $r - d \geq 0$. Combining the two inequalities, we have that $r = d$. Thus, we have a nonzero morphism of perverse sheaves $L^r[d] \rightarrow \mathcal{P}$. Since \mathcal{P} is simple, it must be a quotient of $L^r[d]$ in the category of perverse sheaves. Therefore, \mathcal{P} is the shift of a local system. \square

Theorem 5.3. *Let $f : X \rightarrow Y$ be a morphism of smooth projective varieties. Let \tilde{Y} be the universal cover of Y , and assume that $\tilde{X} := X \times_Y \tilde{Y}$ is homotopy equivalent to a finite CW-complex. Suppose that Y is either an abelian variety or an aspherical projective manifold with an ample cotangent bundle (e.g., a compact ball quotient). Then f is a \mathbb{Q} -homology fiber bundle.*

Proof. By our assumptions, the cohomology groups $H^k(\tilde{X}, \mathbb{Q})$ are finite dimensional \mathbb{Q} -vector spaces. The assertion in the theorem is equivalent to showing that $Rf_*\mathbb{Q}_X$ is locally constant.

Since f is proper, by Poincaré duality and proper base change (e.g., see [Ma, Theorem 5.1.7]) we have

$$H^{2 \dim X - k}(\tilde{X}, \mathbb{Q})^\vee \cong H_c^k(\tilde{X}, \mathbb{Q}) \cong H_c^k(\tilde{Y}, R\tilde{f}_!\mathbb{Q}_{\tilde{X}}) \cong H_c^k(\tilde{Y}, p^*Rf_!\mathbb{Q}_X) \cong H_c^k(\tilde{Y}, p^*Rf_*\mathbb{Q}_X)$$

where $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ is the lifting of $f : X \rightarrow Y$, and $p : \tilde{Y} \rightarrow Y$ is the universal covering map. Thus, the vector spaces $H_c^k(\tilde{Y}, p^*(Rf_*\mathbb{Q}_X))$ are finite dimensional for all k . Applying the decomposition theorem [BBD] for the proper map $f : X \rightarrow Y$ yields a decomposition

$$Rf_*\mathbb{Q}_X \cong \bigoplus_{1 \leq i \leq l} \mathcal{P}_i[n_i]$$

where $n_i \in \mathbb{Z}$ and \mathcal{P}_i are nonzero simple perverse sheaves. Therefore, each \mathcal{P}_i has the property that $H_c^k(\tilde{Y}, p^*(\mathcal{P}_i))$ are finite dimensional for all k .

By the definition of Mellin transformation, we have

$$H_c^k(\tilde{Y}, p^*\mathcal{P}_i) \cong H^k(\mathcal{M}_*(\mathcal{P}_i)).$$

Thus, the cohomology groups of $\mathcal{M}_*(\mathcal{P}_i)$ are all finite dimensional \mathbb{Q} -vector spaces. To apply Proposition 5.2, we need to prove that $\mathcal{M}_*(\mathcal{P}_i)$ are nonzero. When Y is an aspherical projective manifold with an ample cotangent bundle (e.g., a compact ball quotient), this follows from Corollary 2.6 and Proposition 3.3. When Y is an abelian variety, this follows from the Riemann-Hilbert correspondence and the Fourier-Mukai transformation being an equivalence of categories ([Lau] and [Ro]). See [We, Theorem 2] and [Sch, Theorem 7.6] for stronger results.

Now, by Proposition 5.2, each \mathcal{P}_i is a shifted local system, and hence $Rf_*\mathbb{Q}_X \cong \bigoplus_{1 \leq i \leq l} \mathcal{P}_i[n_i]$ is locally constant. \square

In the remainder of this section, we deal with the integral Bobadilla-Kollár question for abelian varieties.

Lemma 5.4. *Let Y be an aspherical projective manifold of complex dimension d , and fix a field \mathbb{K} . Let \mathcal{F} be a locally constant complex with $\mathcal{M}_*(\mathcal{F}) = 0$. Then $\mathcal{F} = 0$.*

Proof. Let $p : \tilde{Y} \rightarrow Y$ be the universal covering map. Then $p^*\mathcal{F}$ is also locally constant. By Proposition 4.3, the Verdier dual $\mathbb{D}(p^*\mathcal{F})$ is also locally constant. Since \tilde{Y} is contractible, Proposition 4.3 yields isomorphisms

$$(5.1) \quad R\Gamma(\tilde{Y}, \mathbb{D}(p^*\mathcal{F})) \cong i_x^*(\mathbb{D}(p^*\mathcal{F}))$$

for all $x \in \tilde{Y}$. For $\tilde{q} : \tilde{Y} \rightarrow \text{pt}$ the projection to a point, we have

$$\mathcal{M}_*(\mathcal{F}) = R\tilde{q}_!(p^*\mathcal{F}) \cong \mathbb{D}R\tilde{q}_*(\mathbb{D}(p^*\mathcal{F})) \cong \mathbb{D}R\Gamma(\tilde{Y}, \mathbb{D}(p^*\mathcal{F})).$$

Since $\mathcal{M}_*(\mathcal{F}) = 0$, we get that $\mathbb{D}R\Gamma(\tilde{Y}, \mathbb{D}(p^*\mathcal{F})) = 0$, and hence $R\Gamma(\tilde{Y}, \mathbb{D}(p^*\mathcal{F})) = 0$. By (5.1), we get that the stalk of $\mathbb{D}(p^*\mathcal{F})$ at every point $x \in \tilde{Y}$ is zero. Thus, $\mathbb{D}(p^*\mathcal{F}) = 0$, and equivalently $p^*\mathcal{F} = 0$. This then also implies that $\mathcal{F} = 0$. \square

Proposition 5.5. *Let Y be an aspherical projective manifold of complex dimension d , and fix a field \mathbb{K} . The following statements are equivalent:*

- (1) *The Mellin transformation of any nonzero \mathbb{K} -constructible complex is nonzero.*
- (2) *For any constructible complex \mathcal{F} on Y , if $H^k(\mathcal{M}_*(\mathcal{F}))$ is finite dimensional for all k , then \mathcal{F} is locally constant.*

Proof. The implication (2) \Rightarrow (1) follows directly from Lemma 5.4.

To prove the implication (1) \Rightarrow (2), we use induction on the total dimension

$$\sigma(\mathcal{F}) := \sum_{k \in \mathbb{Z}} \dim_{\mathbb{K}} H^k(\mathcal{M}_*(\mathcal{F})).$$

If $\sigma(\mathcal{F}) = 0$, that is, $\mathcal{M}_*(\mathcal{F}) = 0$, then statement (1) implies that $\mathcal{F} = 0$. Assume that $\sigma(\mathcal{F}) > 0$. Let $H^r(\mathcal{M}_*(\mathcal{F}))$ be the nonzero cohomology group of the lowest degree. By Lemma 5.1, there exists a morphism $\phi : L^r[2d - r] \rightarrow \mathcal{F}$ such that the mapping cone $C(\phi)$ satisfies

$$H^k(\mathcal{M}_*(C(\phi))) = \begin{cases} 0 & \text{if } k \leq r \\ H^k(\mathcal{M}(\mathcal{F})) & \text{if } k > r. \end{cases}$$

Thus, $\sigma(C(\phi)) < \sigma(\mathcal{F})$. By the inductive hypothesis, $C(\phi)$ is locally constant. By Lemma 4.2 and the distinguished triangle

$$L^r[2d - r] \rightarrow \mathcal{F} \rightarrow C(\phi) \rightarrow L^r[2d - r + 1],$$

we know that \mathcal{F} is also locally constant. \square

We also have the following result, whose proof will be given later on.

Proposition 5.6. *Let A be an abelian variety, and let \mathcal{F} be a nonzero \mathbb{K} -constructible complex. Then the Mellin transformation $\mathcal{M}_*(\mathcal{F})$ is nonzero.*

Combining the above two propositions, we have the following corollary.

Corollary 5.7. *Let A be an abelian variety, and let \mathcal{F} be a \mathbb{K} -constructible complex. If $\mathcal{M}_*(\mathcal{F})$ has finite dimensional cohomology in every degree, then \mathcal{F} is locally constant.*

Lemma 5.8. *Let $f : M^\bullet \rightarrow N^\bullet$ be a homomorphism of bounded complexes of free \mathbb{Z} -modules, with finitely generated cohomology. If $f \otimes_{\mathbb{Z}} \mathbb{K} : M^\bullet \otimes_{\mathbb{Z}} \mathbb{K} \rightarrow N^\bullet \otimes_{\mathbb{Z}} \mathbb{K}$ is a quasi-isomorphism for any field \mathbb{K} , then f is a quasi-isomorphism.*

Proof. Consider the mapping cone $C(f)$ of f . By assumption, $C(f) \otimes_{\mathbb{Z}} \mathbb{K}$ is acyclic for any field \mathbb{K} . Suppose that $C(f)$ is not acyclic. Let $H^k(C(f))$ be a nonzero cohomology group. Since $H^k(C(f))$ is a finitely generated abelian group, we can choose \mathbb{K} to be either \mathbb{Q} or a finite field such that $H^k(C(f)) \otimes_{\mathbb{Z}} \mathbb{K} \neq 0$. By the universal coefficient theorem, we have a short exact sequence

$$0 \rightarrow H^k(C(f)) \otimes_{\mathbb{Z}} \mathbb{K} \rightarrow H^k(C(f) \otimes_{\mathbb{Z}} \mathbb{K}) \rightarrow \mathrm{Tor}(H^{k+1}(C(f)), \mathbb{K}) \rightarrow 0.$$

This contradicts the fact that $C(f) \otimes_{\mathbb{Z}} \mathbb{K}$ is acyclic. Hence $C(f)$ must be acyclic, that is, f is a quasi-isomorphism. \square

Lemma 5.9. *Let \mathcal{F} be a \mathbb{Z} -constructible complex on a complex manifold M . If $\mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K}$ is locally constant for any field \mathbb{K} , then \mathcal{F} is locally constant.*

Proof. First, we can take an open cover $\{U_\lambda\}_{\lambda \in I}$ such that each U_λ is contractible and the cohomology groups of $R\Gamma(U_\lambda, \mathcal{F})$ are finitely generated. The second condition can be achieved by choosing U_λ as sufficiently small balls.

We claim that, for any U_λ , there exists a canonical isomorphism

$$R\Gamma(U_\lambda, \mathcal{F}) \otimes_{\mathbb{Z}}^L \mathbb{K} \cong R\Gamma(U_\lambda, \mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K})$$

in the derived category $D(\mathbb{K})$ of complexes of \mathbb{K} -vector spaces. In fact, taking an injective resolution \mathcal{I}^\bullet of \mathcal{F} and a free resolution P^\bullet of \mathbb{K} , the total complex of $\mathcal{I}^\bullet \otimes_{\mathbb{Z}} P^\bullet$ is a complex of injective sheaves representing $\mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K}$, and hence both sides are isomorphic to the total complex in $D(\mathbb{K})$. Since taking direct limit commutes with taking tensor product, using the same resolutions, we also have a canonical isomorphism

$$i_x^* \mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K} \cong i_x^*(\mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K})$$

in $D(\mathbb{K})$ for any point $x \in M$.

Since $\mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K}$ is locally constant, the restriction map

$$R\Gamma(U_\lambda, \mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K}) \rightarrow i_x^*(\mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K})$$

is an isomorphism. Combining the above three displayed equations, it follows that for any field \mathbb{K} , the restriction map $R\Gamma(U_\lambda, \mathcal{F}) \rightarrow i_x^* \mathcal{F}$ induces isomorphisms

$$R\Gamma(U_\lambda, \mathcal{F}) \otimes_{\mathbb{Z}}^L \mathbb{K} \rightarrow i_x^* \mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K}.$$

Since both $R\Gamma(U_\lambda, \mathcal{F})$ and $i_x^* \mathcal{F}$ are bounded complexes with finitely generated cohomology groups, it follows from Lemma 5.8 that $R\Gamma(U_\lambda, \mathcal{F}) \rightarrow i_x^* \mathcal{F}$ is an isomorphism. \square

Corollary 5.10. *Let A be an abelian variety, and let \mathcal{F} be a \mathbb{Z} -constructible complex. If the cohomology group of $\mathcal{M}_*(\mathcal{F})$ in every degree is a finitely generated abelian group, then \mathcal{F} is locally constant.*

Proof. Since $\mathcal{M}_*(\mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K}) \cong \mathcal{M}_*(\mathcal{F}) \otimes_{\mathbb{Z}}^L \mathbb{K}$, our assumption implies that $\mathcal{M}_*(\mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K})$ has finite dimensional cohomology groups. So $\mathcal{F} \otimes_{\mathbb{Z}}^L \mathbb{K}$ are locally constant by Corollary 5.7. Now the assertion follows from Lemma 5.9. \square

The rest of this section is devoted to proving Proposition 5.6 and to answer the integral homology version of the Bobadilla-Kollár question for abelian varieties. To this end, we will use induction to reduce to the case when A is a simple abelian variety. We first need some preparatory results about simple abelian varieties.

Lemma 5.11. *Let C be an irreducible curve in a simple abelian variety A . There does not exist a nonzero holomorphic 1-form ξ on A such that the restriction $\xi|_{C_{\text{reg}}}$ is zero.*

Proof. Let \tilde{C} be the normalization of C . Denote its Jacobian variety by $J(\tilde{C})$, and the Abel-Jacobi map by $\alpha_{\tilde{C}} : \tilde{C} \rightarrow J(\tilde{C})$. By the universal property of the Abel-Jacobi map, there exists a unique map $\psi : J(\tilde{C}) \rightarrow A$ such that the following diagram commutes

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\alpha_{\tilde{C}}} & J(\tilde{C}) \\ \downarrow & & \downarrow \psi \\ C & \hookrightarrow & A \end{array}$$

where the left vertical map is the normalization map and the bottom horizontal map is the inclusion. Since \tilde{C} admits a nonconstant map to an abelian variety, the genus of \tilde{C} is nonzero and $\alpha_{\tilde{C}}$ is an inclusion. Up to a translate, the image of ψ is an abelian subvariety of A . Since ψ is not constant and A is simple, ψ must be surjective. Thus, the pull-back $\psi^*(\xi)$ of a nonzero holomorphic 1-form ξ on A is a nonzero holomorphic 1-form on $J(\tilde{C})$.

Taking pullback to \tilde{C} defines a canonical isomorphism $\alpha_{\tilde{C}}^* : H^0(J(\tilde{C}), \Omega_{J(\tilde{C})}^1) \rightarrow H^0(\tilde{C}, \Omega_{\tilde{C}}^1)$. Thus, $\alpha_{\tilde{C}}^* \psi^*(\xi)$ is a nonzero holomorphic 1-form on \tilde{C} . Away from the singular points of C , the pullback $\alpha_{\tilde{C}}^* \psi^*(\xi)$ factors through the restriction $\xi|_{C_{\text{reg}}}$. So the restriction cannot be zero. \square

Proposition 5.12. *Let A be a simple abelian variety, and let $Z \subset A$ be a proper irreducible subvariety. For a general holomorphic 1-form η on A , the degeneration locus of the restriction $\eta|_{Z_{\text{reg}}}$ is a nonempty finite set.*

Proof. Using translations, we can identify T^*A with $A \times T_e^*A$, where e is the identity element of A . Using the canonical isomorphism $T_e^*A \cong H^0(A, \Omega_A^1)$, we can also identify T^*A with $A \times H^0(A, \Omega_A^1)$. For a holomorphic 1-form η , we define its image in T^*A by Γ_η . Then Γ_η corresponds to $A \times \{\eta\}$ under the above identification. Let $\mu : T^*A \rightarrow A$ be the projection of the cotangent bundle. Then $\mu(T_{Z_{\text{reg}}}^*A \cap \Gamma_\eta)$ is equal to the set of degeneration points of $\eta|_{Z_{\text{reg}}}$.

Consider the map

$$\nu : T^*A \cong A \times H^0(A, \Omega_A^1) \rightarrow H^0(A, \Omega_A^1),$$

that is the second projection after the above identification. By definition, $\Gamma_\eta = \nu^{-1}(\eta)$ for any holomorphic 1-form η on A . Thus, to prove the proposition, it suffices to show that for a general $\eta \in H^0(A, \Omega_A^1)$, the intersection

$$(5.2) \quad \nu^{-1}(\eta) \cap T_{Z_{\text{reg}}}^*A = \Gamma_\eta \cap T_{Z_{\text{reg}}}^*A$$

is a nonempty finite set. Since $T_{Z_{\text{reg}}}^*A$ and $H^0(A, \Omega_A^1)$ have the same dimension, the cardinality of the set (5.2) is finite and equal to the degree of the map

$$\nu|_{T_{Z_{\text{reg}}}^*A} : T_{Z_{\text{reg}}}^*A \rightarrow H^0(A, \Omega_A^1).$$

So we only need to show that the above map is dominant. Suppose that this is not true. Since Z is a proper subvariety of A , $\nu(T_{Z_{\text{reg}}}^*A)$ has dimension at least one. Choose a general point ξ in $\nu(T_{Z_{\text{reg}}}^*A)$. Then $\nu^{-1}(\xi) \cap T_{Z_{\text{reg}}}^*A$ has dimension at least one. Choose any irreducible curve C' in the above intersection and let C be the closure of $\mu(C')$. Then the curve C and the 1-form ξ give a contradiction to Lemma 5.11. \square

Corollary 5.13. *Let A be a simple abelian variety. Let $\Lambda \subset T^*A$ be an irreducible conic Lagrangian subvariety. If Λ is not the zero section of T^*A , then the intersection number $\langle [\Lambda], [A] \rangle_{T^*A}$ is positive, where $[A]$ denotes the class of the zero section of T^*A .*

Proof. For a general holomorphic 1-form η with image Γ_η in T^*A , Λ intersects Γ_η transversally. Since Γ_η is a deformation of the zero-section of T^*A , the intersection number $\langle [\Lambda], [A] \rangle_{T^*A}$ is equal to the cardinality of the set $\Lambda \cap \Gamma_\eta$, which is strictly positive by Proposition 5.12. \square

Lemma 5.14. *Let \mathcal{P} be a \mathbb{K} -perverse sheaf on a compact complex manifold or a smooth complex algebraic variety M .² If \mathcal{P} is not locally constant, then the characteristic cycle of \mathcal{P} contains a component that is not the zero section of the cotangent bundle T^*M .*

Proof. Since the category of \mathbb{K} -perverse sheaves on M is an artinian abelian category, \mathcal{P} is obtained by extensions of simple perverse sheaves. Denote the decomposition factors of \mathcal{P} by $\mathcal{P}_1, \dots, \mathcal{P}_r$. By Proposition 4.3, if all \mathcal{P}_i are locally constant, then so is \mathcal{P} , a contradiction to the assumption. So at least one of the \mathcal{P}_i 's is not locally constant, which we assume to be \mathcal{P}_1 without loss of generality.

Since the characteristic cycle is additive under a short exact sequence of perverse sheaves, we have

$$CC(\mathcal{P}) = \sum_{1 \leq i \leq r} CC(\mathcal{P}_i).$$

Since the \mathcal{P}_i 's are perverse, each $CC(\mathcal{P}_i)$ is a positive combination of conic Lagrangian cycles on T^*M . This means that there is no cancelation in the above sum. Thus, to prove that \mathcal{P} contains a component that is not the zero-section, it suffices to show that $CC(\mathcal{P}_1)$ contains a component that is not the zero-section.

Since \mathcal{P}_1 is simple, it is the intermediate extension of a local system on a smooth locally closed subvariety. In other words, there exists an irreducible subvariety $Z \subset M$, a Zariski open subset $U \subset Z_{\text{reg}}$, and a local system L_U on U such that \mathcal{P}_1 is isomorphic to the intermediate extension of $L_U[d]$ where $d = \dim Z$. Then T_Z^*M must appear as a component of the characteristic cycle $CC(\mathcal{P}_1)$. If $Z \neq M$, then we are done. From now on, we assume that $Z = M$ and U is a Zariski open subset of M .

Without loss of generality, we can assume U is the largest open set over which \mathcal{P}_1 is the shift of a local system. Since \mathcal{P}_1 is not locally constant on M , we know that $M \setminus U$ is nonempty. Since the intermediate extension of (the shift of) a local system across a subvariety of codimension at least two is again (the shift of) a local system, $D := M \setminus U$ is a divisor of M . Near a smooth point x of D , the intermediate extension of $L_U[d]$ is equal to (the shift of) the push forward $j_{U*}(L_U)[d]$, where $j_U : U \rightarrow M$ is the open embedding. Since we have assumed that U is the maximal open subset where \mathcal{P}_1 is the shift of a local system, the monodromy action of L_U near x is non-trivial. Thus,

$$\dim(j_{U*}(L_U))_x < \text{rank}(L_U),$$

that is, the stalk of $j_{U*}(L_U)$ at x has a smaller rank than at a general point. Therefore, the stalk-wise Euler characteristic of \mathcal{P}_1 is not a constant function near x . Since the characteristic cycle of a constructible complex determines its stalk-wise Euler characteristics, the characteristic cycle $CC(\mathcal{P}_1)$ must have a component supported on the conormal variety of each of the irreducible components of D . \square

²In the algebraic case, we need to assume that \mathcal{P} is algebraically constructible in order to ensure that the category of perverse sheaves is Artinian.

Corollary 5.15. *Let A be a simple abelian variety. Let \mathcal{P} be a \mathbb{K} -perverse sheaf that is not locally constant. Then $\chi(A, \mathcal{P}) > 0$.*

Proof. Write the characteristic cycle $CC(\mathcal{P})$ as

$$CC(\mathcal{P}) = \sum_{j \in J} n_j [\Lambda_j]$$

where J is a finite index set and Λ_j are conic Lagrangian cycles on T^*A . Since \mathcal{P} is a nonzero perverse sheaf, the coefficients n_j are positive. By Kashiwara's index formula,

$$\chi(A, \mathcal{P}) = \sum_{j \in J} n_j \langle [\Lambda_j], [A] \rangle_{T^*A}.$$

Since we can represent $[A]$ by the image of a general holomorphic 1-form, we have that $\langle [\Lambda_j], [A] \rangle_{T^*A} \geq 0$. (The inequality also follows from Theorem 3.5.) By Lemma 5.14 and Corollary 5.13, we know that one of intersection number $\langle [\Lambda_j], [A] \rangle_{T^*A}$ is strictly positive. Thus $\chi(A, \mathcal{P}) > 0$. \square

We also recall a generic vanishing theorem of Bhatt-Schnell-Scholze. (See also [LMW1] for a generalization to semi-abelian varieties.)

Theorem 5.16 ([BSS]). *Let \mathcal{P} be a \mathbb{K} -perverse sheaf on an abelian variety A . Let $\overline{\mathbb{K}}$ be the algebraic closure of \mathbb{K} . For a general rank one $\overline{\mathbb{K}}$ -local system L on A , one has*

$$H^i(A, \mathcal{P} \otimes_{\mathbb{K}} L) = 0 \quad \text{for all } i \neq 0.$$

We now have all the ingredients to complete the proof of Proposition 5.6.

Proof of Proposition 5.6. Let \mathcal{F} be a \mathbb{K} -constructible complex on an abelian variety A such that $\mathcal{M}_*(A, \mathcal{F}) = 0$. We will show that $\mathcal{F} = 0$.

If \mathcal{F} is locally constant, then Lemma 5.4 implies that $\mathcal{F} = 0$.

Let us next assume that A is a simple abelian variety and \mathcal{F} is not locally constant. Then by Proposition 4.3, there exists some perverse cohomology ${}^p\mathcal{H}^l(\mathcal{F})$ that is not locally constant. By Corollary 5.15, we have $\chi(A, {}^p\mathcal{H}^l(\mathcal{F})) > 0$. Let L be a general rank one $\overline{\mathbb{K}}$ -local system on A . Then by Theorem 5.16, we have

$$H^j(A, {}^p\mathcal{H}^i(\mathcal{F}) \otimes_{\mathbb{K}} L) = 0 \quad \text{for all } i \in \mathbb{Z} \text{ and } j \neq 0.$$

Since $\chi(A, {}^p\mathcal{H}^l(\mathcal{F}) \otimes_{\mathbb{K}} L) = \chi(A, {}^p\mathcal{H}^l(\mathcal{F})) > 0$, we have that $H^0(A, {}^p\mathcal{H}^l(\mathcal{F}) \otimes_{\mathbb{K}} L) \neq 0$. Since

$${}^p\mathcal{H}^i(\mathcal{F}) \otimes_{\mathbb{K}} L \cong {}^p\mathcal{H}^i(\mathcal{F} \otimes_{\mathbb{K}} L),$$

the perverse cohomology spectral sequence

$$E_2^{ij} = H^i(A, {}^p\mathcal{H}^j(\mathcal{F} \otimes_{\mathbb{K}} L)) \implies H^{i+j}(A, \mathcal{F} \otimes_{\mathbb{K}} L)$$

degenerates at the E_2 -page. Therefore, $H^l(A, \mathcal{F} \otimes_{\mathbb{K}} L) \neq 0$. By Corollary 2.7, we have that $\mathcal{M}_*(A, \mathcal{F}) \neq 0$, a contradiction to the assumption. So this case cannot occur.

So far, we have proved the proposition when A is a simple abelian variety. Finally, we prove the general case using induction on the dimension of A . Since we are done with the case when A is simple, from now on we can assume that A is not simple. In this case, there exists a short exact sequence of positive dimensional abelian varieties

$$0 \rightarrow A_1 \rightarrow A \xrightarrow{p_2} A_2 \rightarrow 0.$$

In general, the short exact sequence does not split in the category of abelian varieties, but it does split in the category of real Lie groups. Fix such a splitting, and denote the induced projection $A \rightarrow A_1$ by p_1 .

Since $\mathcal{M}_*(A, \mathcal{F}) = 0$, we get by Corollary 2.7 that

$$H^i(A, \mathcal{F} \otimes_{\mathbb{K}} p_1^* L_1 \otimes_{\mathbb{K}} p_2^* L_2) = 0$$

for and $i \in \mathbb{Z}$ and any rank one $\overline{\mathbb{K}}$ -local systems L_1 and L_2 on A_1 and A_2 , respectively. Moreover, by the projection formula, we have

$$0 = H^i(A, \mathcal{F} \otimes_{\mathbb{K}} p_1^* L_1 \otimes_{\mathbb{K}} p_2^* L_2) \cong H^i(A_2, Rp_{2*}(\mathcal{F} \otimes_{\mathbb{K}} p_1^* L_1) \otimes_{\mathbb{K}} L_2).$$

By Corollary 2.7, we have

$$\mathcal{M}_*(A_2, Rp_{2*}(\mathcal{F} \otimes_{\mathbb{K}} p_1^* L_1)) = 0.$$

By the induction hypothesis, the proposition holds for A_2 . So we have that

$$Rp_{2*}(\mathcal{F} \otimes_{\mathbb{K}} p_1^* L_1) = 0 \quad \text{for any rank one } \overline{\mathbb{K}}\text{-local system } L_1 \text{ on } A_1.$$

Choose any point $x \in A_2$. By the base change formula, we have

$$0 = i_x^* Rp_{2*}(\mathcal{F} \otimes_{\mathbb{K}} p_1^* L_1) \cong Rp_{2*}((\mathcal{F} \otimes_{\mathbb{K}} p_1^* L_1)|_{p_2^{-1}(x)}) \cong Rp_{2*}(\mathcal{F}|_{p_2^{-1}(x)} \otimes_{\mathbb{K}} p_1^* L_1|_{p_2^{-1}(x)}),$$

or, equivalently,

$$H^i(p_2^{-1}(x), \mathcal{F}|_{p_2^{-1}(x)} \otimes_{\mathbb{K}} p_1^* L_1|_{p_2^{-1}(x)}) = 0$$

for any $i \in \mathbb{Z}$ and any rank one $\overline{\mathbb{K}}$ -local system L_1 on A_1 .

Notice that $p_2^{-1}(x)$ is isomorphic to A_1 and as L_1 varies through all rank one $\overline{\mathbb{K}}$ -local systems on A_1 , $p_1^* L_1|_{p_2^{-1}(x)}$ varies through all rank one $\overline{\mathbb{K}}$ -local systems on $p_2^{-1}(x)$. Thus, by Corollary 2.7, we have

$$\mathcal{M}_*(p_2^{-1}(x), \mathcal{F}|_{p_2^{-1}(x)}) = 0.$$

Again, by the induction hypothesis, this implies that $\mathcal{F}|_{p_2^{-1}(x)} = 0$. Since x is an arbitrary point on A_2 , we conclude that $\mathcal{F} = 0$. \square

We can now prove the integral Bobadilla-Kollár question for abelian varieties.

Theorem 5.17. *Let $f : X \rightarrow Y$ be a morphism from a smooth projective variety to an abelian variety. Let \tilde{Y} be the universal cover of Y , and assume that $\tilde{X} := X \times_Y \tilde{Y}$ is homotopy equivalent to a finite CW-complex. Then f is a \mathbb{Z} -homology fiber bundle.*

Proof. It follows by our assumptions, as in the proof of Theorem 5.3, that the Mellin transformation $\mathcal{M}_*(Rf_*\mathbb{Z}_X)$ has finitely generated cohomology groups. By Corollary 5.10, we get that $Rf_*\mathbb{Z}_X$ is locally constant. By definition, this is equivalent to the fact that the map $X \rightarrow Y$ is a \mathbb{Z} -homology fiber bundle. \square

Remark 5.18. The same proof also works for a compact complex torus. Notice that for a non-simple abelian variety, we never used the fact that, up to an isogeny, it is the product of two smaller abelian varieties. We only used the fact that it is the extension of two smaller abelian varieties, which also holds in the category of compact complex tori.

Remark 5.19. In [GL], Gabber and Loeser proved a t-exactness theorem about the Mellin transformation on a complex affine torus and for arbitrary field coefficients, see also [LMW1]. This result can be used to give a positive answer to the integral Bobadilla-Kollár question when Y is an affine torus. However, it is not clear how to combine the results for abelian varieties and affine torus to deduce an answer for semi-abelian varieties. We leave this case to further investigation.

Applying Theorem 5.17 to the Albanese map of a complex projective manifold, we have the following.

Corollary 5.20. *Let X be a projective manifold. Let X^{ab} be the universal free abelian cover of X , that is, the covering space of X associated to the group homomorphism $\pi_1(X) \rightarrow H_1(X, \mathbb{Z})/\text{torsion}$. If X^{ab} is homotopy equivalent to a finite CW-complex, then the Albanese map of X is a \mathbb{Z} -homology fiber bundle.*

6. ASPHERICAL PROJECTIVE MANIFOLDS AND THE SINGER-HOPF CONJECTURE

This paper is motivated in part by the following long-standing conjecture (e.g., see [Gu, Conjecture 25.1]):

Conjecture 6.1. (Singer-Hopf) *Suppose X^{2n} is a closed, aspherical manifold of real dimension $2n$. Then*

$$(-1)^n \chi(X^{2n}) \geq 0.$$

The conjecture is true for $n = 1$ (i.e., real dimension 2) since the only closed surfaces with positive Euler characteristic are S^2 and $\mathbb{R}P^2$, and they are the only non-aspherical ones. In the special case when X^{2n} is a Riemannian manifold with non-positive sectional curvature, this conjecture is attributed to Hopf and Chern. (The fact that a Riemannian manifold with non-positive sectional curvature is aspherical is a consequence of Hadamard's Theorem.) It was strengthened to the aspherical case by Singer, and it also asserts the vanishing of all L^2 -Betti numbers of the universal cover, except possibly the middle one.

Jost and Zuo [JZ] proved Conjecture 6.1 for X a compact Kähler manifold with non-positive sectional curvature. Their techniques rely on analytic arguments introduced by Gromov [Gr], who confirmed the Singer-Hopf Conjecture for Kähler hyperbolic manifolds (these include Kähler manifolds with negative and pinched sectional curvature).

We propose the following natural generalization of Conjecture 6.1 in the projective context:

Conjecture 6.2. *If X is an aspherical projective manifold and \mathcal{P} is a perverse sheaf on X , then the Euler characteristic of \mathcal{P} is semipositive, that is, $\chi(X, \mathcal{P}) \geq 0$.*

By Proposition 3.6, Conjecture 6.2 is a consequence of the following two conjectures.

Conjecture 6.3. *Let Y be a projective manifold. If the universal cover of Y is a Stein manifold, then the cotangent bundle of Y is nef.*

A weaker version of this conjecture is proved in [Kr], namely, if the universal cover of Y is a bounded domain in a Stein manifold, then the cotangent bundle of Y is nef.

Conjecture 6.4. *If Y is an aspherical projective manifold, then the universal cover is Stein.*

Remark 6.5. Corollary 3.7 suggests that one may formulate a generalization of the Singer-Hopf conjecture to singular varieties. For example, one may conjecture that if the universal cover of a (possibly singular) complex projective variety X is a Stein space, then $(-1)^{\dim X} \chi_{\text{IH}}(X) \geq 0$.

We show below that Conjecture 6.4 follows from the following Shafarevich conjecture (see [Ey] for an introduction) on the universal cover of projective manifolds.

Conjecture 6.6. (Shafarevich conjecture) *The universal cover of any projective manifold is holomorphically convex.*

Proposition 6.7. *Suppose Y is an aspherical compact projective manifold. Then its universal cover \tilde{Y} does not contain any positive dimensional compact analytic subvariety.*

Proof. Suppose that \tilde{Y} does contain a positive dimensional compact analytic subvariety Z . By taking intersections with the preimage of general hyperplane sections and taking irreducible components, we can assume that Z is one-dimensional and irreducible. Let Z' be the normalization of Z . Then Z' is a compact Riemann surface. Consider the composition

$$Z' \rightarrow Z \hookrightarrow \tilde{Y} \rightarrow Y,$$

where the first map is the normalization map, the second is the inclusion map and the third is the universal covering map. The composition is a nonconstant holomorphic map, whose image is an irreducible 1-cycle in Y . In a projective manifold, any positive cycle corresponds to a nonzero element in the homology groups. So the above composition is not null-homotopic. However, since it factors through a contractible space \tilde{Y} , it must be null-homotopic, a contradiction. \square

Corollary 6.8. *The Shafarevich conjecture implies Conjecture 6.4.*

Proof. Let Y be an aspherical projective manifold with universal cover \tilde{Y} . The Shafarevich conjecture implies that \tilde{Y} is holomorphically convex. By the Cartan-Remmert reduction, there exists a proper surjective holomorphic map $f : \tilde{Y} \rightarrow Z$ to a Stein space with connected fibers such that $f_*(\mathcal{O}_{\tilde{Y}}) \cong \mathcal{O}_Z$. Since \tilde{Y} does not contain any positive dimensional compact analytic subvariety, the map f must be a bijection, and hence a biholomorphic map. \square

If the fundamental group of Y admits a faithful finite-dimensional linear representation, the Shafarevich conjecture is proved in the recent breakthrough [EKPR] by Eyssidieux-Katzarkov-Pantev-Ramachandran. We are not aware of any example of an aspherical projective manifold whose fundamental group does not admit a finite-dimensional faithful representation. This leads us to the following question.

Question 6.9. *Does the fundamental group of an aspherical projective manifold always admit a finite-dimensional faithful representation?*

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