# CHARACTERISTIC CLASSES OF SINGULAR TORIC VARIETIES

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Abstract. We introduce a new approach for the computation of characteristic classes of singular toric varieties and, as an application, we obtain generalized Pick-type formulae for lattice polytopes. Many of our results (e.g., lattice point counting formulae) hold even more generally, for closed algebraic torus-invariant subspaces of toric varieties. In the simplicial case, by combining this new computation method with the Lefschetz-Riemann-Roch theorem, we give new proofs of several characteristic class formulae originally obtained by Cappell and Shaneson in the early $1990$s.

A $d$-dimensional *toric variety* $X$ is an irreducible normal variety on which the complex $d$-torus acts with an open orbit, e.g., see [11, 12, 15, 23]. Toric varieties arise from combinatorial objects called *fans*, which are collections of cones in a lattice. Toric varieties are of interest both in their own right as algebraic varieties, and for their application to the theory of convex polytopes. For instance, Danilov [12] used the Hirzebruch-Riemann-Roch theorem to establish a direct connection between the problem of counting the number of lattice points in a convex polytope and the Todd classes of toric varieties. Thus, the problem of finding explicit formulae for characteristic classes of toric varieties is of interest not only to topologists and algebraic geometers, but also to combinatorists, programmers, etc.

Let $X = X_\Sigma$ be the toric variety associated to a fan $\Sigma$ in a $d$-dimensional lattice $N$ of $\mathbb{R}^d$. We say that $X$ is *simplicial* if the fan is simplicial, i.e., each cone $\sigma$ of $\Sigma$ is spanned by linearly independent elements of the lattice $N$. If, moreover, each cone $\sigma$ of $\Sigma$ is smooth (i.e., generated by a subset of a $\mathbb{Z}$-basis for the lattice $N$), the corresponding toric variety $X_\Sigma$ is *smooth*. Denote by $\Sigma(i)$ the set of $i$-dimensional cones of $\Sigma$. One-dimensional cones $\rho \in \Sigma(1)$ are called *rays*. For each ray $\rho \in \Sigma(1)$, we denote by $u_\rho$ the unique generator of the semigroup $\rho \cap N$. The $\{u_\rho\}_{\rho \in \sigma(1)}$ are the *generators* of $\sigma$. To each cone $\sigma \in \Sigma$ there corresponds a subvariety $V_\sigma$ of $X_\Sigma$, which is the closure of the orbit $O_\sigma$ of $\sigma$ under the torus action. Each $V_\sigma$ is itself a toric variety of dimension $d - \dim(\sigma)$. The singular locus of $X_\Sigma$ is $\bigcup V_\sigma$, the union being taken over all singular cones $\sigma \in \Sigma_{\rm sing}$ in the fan $\Sigma$.

In this note, we announce the computation of several characteristic classes of *(possibly singular) toric varieties* (see [20] for more details and proofs). More concretely, we indicate how to compute the *motivic Chern class* $mC_y(X_\Sigma)$, and the *homology Hirzebruch classes*, both the normalized $\widehat{T}_{y*}(X_\Sigma)$ and the un-normalized $T_{y*}(X_\Sigma)$ [5], of such a toric variety $X_\Sigma$. These characteristic classes depending on a parameter $y$ are valued in $G_0(X)[y]$ and resp. $H_*(X_\Sigma) \otimes \mathbb{Q}[y]$, where $G_0(X)$ is the Grothendieck group of coherent sheaves, and $H_*(-)$ denotes either the Chow group $CH_*$ or the even degree Borel-Moore homology group $H^{BM}_{2*}$.

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By using work of Ishida [18], the motivic Chern class of a toric variety $X_\Sigma$ can be computed as

$$mC_y(X_\Sigma) = \sum_{p=0}^{d} [\widetilde{\Omega}^p_{X_\Sigma}] \cdot y^p \in G_0(X)[y], \tag{1}$$

where $\widetilde{\Omega}^p_{X_\Sigma}$ denotes the sheaf of Zariski differential $p$-forms. In fact, this formula holds even for a torus-invariant closed algebraic subset $X := X_{\Sigma'} \subseteq X_\Sigma$ (i.e., a closed union of torus-orbits) corresponding to a star-closed subset $\Sigma' \subseteq \Sigma$, with the sheaves of differential $p$-forms $\widetilde{\Omega}^p_X$ as introduced by Ishida [18]. Most of our results below apply to this more general context.

The *un-normalized homology Hirzebruch class* of such a torus-invariant closed algebraic subset $X := X_{\Sigma'}$ is defined as

$$T_{y*}(X) := td_* \circ mC_y(X) = \sum_{p=0}^{d'} td_*([\widetilde{\Omega}^p_X]) \cdot y^p, \tag{2}$$

with $d' = \dim(X)$ and $td_* : G_0(-) \to H_*(-) \otimes \mathbb{Q}$ the Todd class transformation of [3]. The *normalized Hirzebruch class* $\widehat{T}_{y*}(X)$ is obtained from $T_{y*}(X)$ by multiplying by $(1+y)^{-k}$ on the degree $k$-part $T_{y,k}(X) \in H_k(X)$, $k \geq 0$.

The degree of these characteristic classes for compact $X := X_{\Sigma'}$ recovers the $\chi_y$-genus of $X$ defined in terms of the Hodge filtration of Deligne's mixed Hodge structure on $H^*_c(X, \mathbb{C})$:

$$\sum_{j,p} (-1)^j \dim_{\mathbb{C}} \mathrm{Gr}^p_F H^j_c(X, \mathbb{C}) \, (-y)^p =: \chi_y(X) = \sum_{p \geq 0} \chi(X, \widetilde{\Omega}^p_X) y^p = \int_X T_{y*}(X) = \int_X \widehat{T}_{y*}(X). \tag{3}$$

The homology Hirzebruch classes, defined by Brasselet-Schürmann-Yokura in [5], have the virtue of unifying several other known characteristic class theories for singular varieties. For example, by specializing the parameter $y$ in the normalized Hirzebruch class $\widehat{T}_{y*}$ to the value $y = -1$, our results for Hirzebruch classes translate into formulae for the (rational) MacPherson-Chern classes $c_*(X) = \widehat{T}_{-1*}(X)$. By letting $y = 0$, we get formulae for the Baum-Fulton-MacPherson Todd classes $td_*(X) = \widehat{T}_{0*}(X)$ of a torus-invariant closed algebraic subset $X$ of a toric variety. Moreover, for simplicial projective toric varieties or compact toric manifolds $X_\Sigma$, we have for $y = 1$ the following identification of the Thom-Milnor $L$-classes

$$L_*(X_\Sigma) = \widehat{T}_{1*}(X_\Sigma).$$

Conjecturally, the last equality should hold for a complete simplicial toric variety. The Thom-Milnor $L$-classes $L_*(X) \in H_{2*}(X, \mathbb{Q})$ are only defined for a compact rational homology manifold $X$, i.e., in our context for a complete simplicial toric variety.

We present below two different perspectives for the computation of these characteristic classes of toric varieties and their torus invariant closed algebraic subsets.

**Motivic Chern and Hirzebruch classes via orbit decomposition.**
First, we take advantage of the torus-orbit decomposition of a toric variety together with the motivic properties of these characteristic classes to express the motivic Chern and resp. homology Hirzebruch classes in terms of dualizing sheaves and resp. the (dual) Todd classes of closures of orbits. We prove the following result:

**Theorem 1.** *Let $X_\Sigma$ be the toric variety defined by the fan $\Sigma$, and $X := X_{\Sigma'} \subseteq X_\Sigma$ a torus-invariant closed algebraic subset defined by a star-closed subset $\Sigma' \subseteq \Sigma$. For each cone $\sigma \in \Sigma$ with corresponding orbit $O_\sigma$, denote by $k_\sigma : V_\sigma \hookrightarrow X$ the inclusion of the orbit closure. Then the motivic Chern class $mC_*(X)$ is computed by:*

$$mC_y(X) = \sum_{\sigma \in \Sigma'} (1+y)^{\dim(O_\sigma)} \cdot (k_\sigma)_*[\omega_{V_\sigma}], \tag{4}$$

*and, similarly, for the un-normalized homology Hirzebruch class $T_{y*}(X_\Sigma)$ we have:*

$$\begin{aligned} T_{y*}(X) &= \sum_{\sigma\in\Sigma'} (1+y)^{\dim(O_\sigma)} \cdot (k_\sigma)_* td_*([\omega_{V_\sigma}]) \\ &= \sum_{\sigma\in\Sigma'} (-1-y)^{\dim(O_\sigma)} \cdot (k_\sigma)_* \left(td_*(V_\sigma)^\vee\right). \end{aligned} \tag{5}$$

*Here $\omega_{V_\sigma}$ is the canonical sheaf of $V_\sigma$ and the duality $(-)^\vee$ acts by multiplication by $(-1)^i$ on the $i$-th degree homology part $H_i(-)$. The normalized Hirzebruch class $\widehat{T}_{y*}(X)$ is given by:*

$$\widehat{T}_{y*}(X) = \sum_{\sigma\in\Sigma',i} (-1-y)^{\dim(O_\sigma)-i} \cdot (k_\sigma)_* td_i(V_\sigma). \tag{6}$$

These results should be seen as characteristic class versions of the well-known formula

$$\chi_y(X) = \sum_{\sigma\in\Sigma'} (-1-y)^{\dim(O_\sigma)} \tag{7}$$

for the $\chi_y$-genus of a toric variety, resp. its torus-invariant closed algebraic subsets, which one recovers for compact $X$ by taking the degree in (5) or (6) above, as $\int_{V_\sigma} td_0(V_\sigma) = 1$.

Letting, moreover, $y = -1, 0, 1$ in (5) and (6), we obtain as special cases the following:

**Corollary 2.** *(Chern, Todd and $L$-classes)*
*Let $X_\Sigma$ be the toric variety defined by the fan $\Sigma$, and $X := X_{\Sigma'} \subseteq X_\Sigma$ a torus-invariant closed algebraic subset defined by a star-closed subset $\Sigma' \subseteq \Sigma$. For each cone $\sigma \in \Sigma$ with corresponding orbit $O_\sigma$, denote by $k_\sigma : V_\sigma \hookrightarrow X$ the inclusion of the orbit closure.*
$(a)$ *(Ehler's formula* [1, 4]*) The (rational) MacPherson-Chern class $c_*(X)$ is computed by:*

$$c_*(X) = \sum_{\sigma\in\Sigma'} (k_\sigma)_* td_{\dim(O_\sigma)}(V_\sigma) = \sum_{\sigma\in\Sigma'} (k_\sigma)_*[V_\sigma]. \tag{8}$$

$(b)$ *The Todd class $td_*(X)$ of $X$ is computed by:*

$$td_*(X) = \sum_{\sigma\in\Sigma'} (k_\sigma)_* td_*([\omega_{V_\sigma}]) = \sum_{\sigma\in\Sigma'} (-1)^{\dim(O_\sigma)} \cdot (k_\sigma)_* \left(td_*(V_\sigma)^\vee\right). \tag{9}$$

$(c)$ *The classes $T_{1*}(X)$ and $\widehat{T}_{1*}(X)$ are computed by:*

$$T_{1*}(X) = \sum_{\sigma\in\Sigma'} 2^{\dim(O_\sigma)} \cdot (k_\sigma)_* td_*([\omega_{V_\sigma}]) = \sum_{\sigma\in\Sigma'} (-2)^{\dim(O_\sigma)} \cdot (k_\sigma)_* \left(td_*(V_\sigma)^\vee\right) \tag{10}$$

*and*

$$\widehat{T}_{1*}(X) = \sum_{\sigma\in\Sigma',i} (-2)^{\dim(O_\sigma)-i} \cdot (k_\sigma)_* td_i(V_\sigma). \tag{11}$$

*Here $T_{1*}(X_\Sigma) = L_*^{AS}(X_\Sigma)$ is the Atiyah-Singer $L$-class for $X_\Sigma$ a compact toric manifold, and $\widehat{T}_{1*}(X_\Sigma) = L_*(X_\Sigma)$ is the Thom-Milnor $L$-class for $X_\Sigma$ a compact toric manifold or a simplicial projective toric variety.*

Taking the degree in (11), we get the following description of the signature $sign(X_\Sigma)$ of a compact toric manifold or a simplicial projective toric variety $X_\Sigma$ (compare also with [22], and in the smooth case also with [23][Thm.3.12]):

$$sign(X_\Sigma) = \chi_1(X_\Sigma) = \sum_{\sigma\in\Sigma} (-2)^{\dim(O_\sigma)}. \tag{12}$$

Moreover, in the case of a toric variety $X_\Sigma$ (i.e., $\Sigma' = \Sigma$), one gets from (9) and (10) by using a "duality argument" also the following formula:

$$td_*([\omega_{X_\Sigma}]) = \sum_{\sigma\in\Sigma} (-1)^{\mathrm{codim}(O_\sigma)} \cdot (k_\sigma)_* td_*(V_\sigma) \tag{13}$$

and in the simplicial context:

$$T_{1*}(X_\Sigma) = (-1)^{\dim(X_\Sigma)} \cdot (T_{1*}(X_\Sigma))^\vee = \sum_{\sigma \in \Sigma} (-1)^{\dim(X_\Sigma)} \cdot (-2)^{\dim(O_\sigma)} \cdot (k_\sigma)_* td_*(V_\sigma). \tag{14}$$

The above formulae calculate the homology Hirzebruch classes (and, in particular, the Chern, Todd and $L$-classes, respectively) of singular toric varieties in terms of the Todd classes of closures of torus orbits. As it will be discussed below, in the simplicial context these Todd classes can also be computed by using the Lefschetz-Riemann-Roch theorem of [13] for geometric quotients.

**Application: Weighted lattice point counting.**
Let $M$ be a lattice and $P \subset M_\mathbb{R} \cong \mathbb{R}^d$ be a full-dimensional lattice polytope with associated projective toric variety $X_P$ and ample Cartier divisor $D_P$. By the classical work of Danilov [12], the (dual) Todd classes of $X_P$ can be used for counting the number of lattice points in (the interior of) a lattice polytope $P$:

$$|P \cap M| = \int_{X_P} ch(\mathcal{O}_{X_P}(D_P)) \cap td_*(X_P) = \sum_{k \geq 0} \frac{1}{k!} \int_{X_P} [D_P]^k \cap td_k(X_P),$$

$$|\mathrm{Int}(P) \cap M| = \int_{X_P} ch(\mathcal{O}_{X_P}(D_P)) \cap td_*([\omega_{X_P}]) = \sum_{k \geq 0} \frac{(-1)^d}{k!} \int_{X_P} [-D_P]^k \cap td_k(X_P),$$

as well as the coefficients of the *Ehrhart polynomial of* $P$, $\mathrm{Ehr}_P(\ell)$, counting the number of lattice points in the dilated polytope $\ell P := \{\ell \cdot u \mid u \in P\}$ for a positive integer $\ell$:

$$\mathrm{Ehr}_P(\ell) := |\ell P \cap M| = \sum_{k=0}^{d} a_k \ell^k,$$

with

$$a_k = \frac{1}{k!} \int_{X_P} [D_P]^k \cap td_k(X_P).$$

In the special case of a lattice polygon $P \subset \mathbb{R}^2$, the Todd class of the corresponding surface $X_P$ is given by the well-known formula (e.g., see [15][p.130])

$$td_*(X_P) = [X_P] + \frac{1}{2} \sum_{\rho \in \Sigma_P(1)} [V_\rho] + [\mathrm{pt}], \tag{15}$$

where $\Sigma_P$ denotes the fan associated to $P$. So by evaluating the right-hand side of the above lattice point counting formula for $P$, one gets the classical *Pick's formula*:

$$|P \cap M| = \mathrm{Area}(P) + \frac{1}{2} |\partial P \cap M| + 1, \tag{16}$$

where $\mathrm{Area}(P)$ is the usual Euclidian area of $P$.

As a direct application of Theorem 1, we show that the un-normalized homology Hirzebruch classes are useful for lattice point counting with certain weights, reflecting the face decomposition

$$P = \bigcup_{Q \preceq P} \mathrm{Relint}(Q).$$

In particular, we obtain the following *generalized Pick-type formulae*:

**Theorem 3.** *Let $M$ be a lattice of rank $d$ and $P \subset M_\mathbb{R} \cong \mathbb{R}^d$ be a full-dimensional lattice polytope with associated projective toric variety $X_P$ and ample Cartier divisor $D_P$. In addition, let $X := X_{P'}$ be a torus-invariant closed algebraic subset of $X_P$ corresponding to a polytopal subcomplex $P' \subseteq P$ (i.e., a closed union of faces of $P$). Then the following formula holds:*

$$\sum_{Q \preceq P'} (1+y)^{\dim(Q)} \cdot |\mathrm{Relint}(Q) \cap M| = \int_X ch(\mathcal{O}_{X_P}(D_P)|_X) \cap T_{y*}(X), \tag{17}$$

*where the summation on the left is over the faces $Q$ of $P'$, $|-|$ denotes the cardinality of sets and $\mathrm{Relint}(Q)$ the relative interior of a face $Q$.*

**Corollary 4.** *The Ehrhart polynomial of the polytopal subcomplex $P' \subseteq P$ as above, counting the number of lattice points in the dilated complex $\ell P' := \{\ell \cdot u \mid u \in P'\}$ for a positive integer $\ell$, is computed by:*

$$\mathrm{Ehr}_{P'}(\ell) := |\ell P' \cap M| = \sum_{k=0}^{d'} a_k \ell^k,$$

*with*

$$a_k = \frac{1}{k!} \int_X [D_P|_X]^k \cap td_k(X),$$

*and $X := X_{P'}$ the corresponding $d'$-dimensional torus-invariant closed algebraic subset of $X_P$.*

Note that $\mathrm{Ehr}_{P'}(0) = a_0 = \int_X td_*(X)$ can be computed on the combinatorial side by formula (7) as the Euler characteristic of $P'$, while on the algebraic geometric side it is given by the arithmetic genus of $X := X_{P'}$:

$$\chi(P') := \sum_{Q \preceq P'} (-1)^{\dim Q} = \chi_0(X) = \int_X td_*(X) = \chi(X, \mathcal{O}_X). \tag{18}$$

Moreover, by using the duality formula (14), we also have:

**Theorem 5.** *If $X_P$ is the simplicial projective toric variety associated to a simple polytope $P$, with corresponding ample Cartier divisor $D_P$, then the following formula holds:*

$$\sum_{Q \preceq P} \left(-\frac{1}{2}\right)^{\mathrm{codim}(Q)} \cdot |Q \cap M| = \int_{X_P} ch(\mathcal{O}_{X_P}(D_P)) \cap \left(\frac{1}{2}\right)^{\dim(X_P)} \cdot T_{1*}(X_P). \tag{19}$$

Formula (19) extends the result of [14][Corollary 5] for Delzant lattice polytopes to the more general case of simple lattice polytopes.

**Generalized toric Hirzebruch-Riemann-Roch.**
By using a well-known "module property" of the Todd class transformation $td_*$, we obtain the following:

**Theorem 6.** *(generalized toric Hirzebruch-Riemann-Roch)*
*Let $X_\Sigma$ be the toric variety defined by the fan $\Sigma$, and $X := X_{\Sigma'} \subseteq X_\Sigma$ be a torus-invariant closed algebraic subset of dimension $d'$ defined by a star-closed subset $\Sigma' \subseteq \Sigma$. Let $D$ be a fixed Cartier divisor on $X$. Then the (generalized) Hirzebruch polynomial of $D$:*

$$\chi_y(X, \mathcal{O}_X(D)) := \sum_{p=0}^{d'} \chi(X, \widetilde{\Omega}_X^p \otimes \mathcal{O}_X(D)) \cdot y^p, \tag{20}$$

*with $\widetilde{\Omega}_X^p$ the corresponding Ishida sheaf of $p$-forms, is computed by the formula:*

$$\chi_y(X, \mathcal{O}_X(D)) = \int_X ch(\mathcal{O}_X(D)) \cap T_{y*}(X). \tag{21}$$

By combining the above formula with Theorem 3, we get the following combinatorial description (see also [19] for another approach to this result in the special case $\Sigma' = \Sigma$):

**Corollary 7.** *Let $M$ be a lattice of rank $d$ and $P \subset M_\mathbb{R} \cong \mathbb{R}^d$ be a full-dimensional lattice polytope with associated projective simplicial toric variety $X = X_P$ and ample Cartier divisor $D_P$. In addition, let $X := X_{P'}$ be a torus-invariant closed algebraic subset of $X_P$ corresponding to a polytopal subcomplex $P' \subseteq P$. Then the following formula holds:*

$$\chi_y(X, \mathcal{O}_X(D_P|_X)) = \sum_{Q \preceq P'} (1+y)^{\dim(Q)} \cdot |\mathrm{Relint}(Q) \cap M|. \tag{22}$$

**Hirzebruch classes of simplicial toric varieties via Lefschetz-Riemann-Roch.**
It is known that simplicial toric varieties have an intersection theory, provided that $\mathbb{Q}$-coefficients are used. The generators of the rational cohomology (or Chow) ring are the classes $[V_\rho]$ defined by the $\mathbb{Q}$-Cartier divisors corresponding to the rays $\rho \in \Sigma(1)$ of the fan $\Sigma$. So it natural to try to express all characteristic classes in the simplicial context in terms of these generators. This brings us to our second computational method, which employs the Lefschetz-Riemann-Roch theorem for the geometric quotient realization $X_\Sigma = W/G$ of such simplicial toric varieties. This approach was already used by Edidin-Graham [13] for the computation of Todd classes of simplicial toric varieties, extending a corresponding result of Brion-Vergne [6] for complete simplicial toric varieties.

We enhance the calculation of Edidin-Graham [13] to the homology Hirzebruch classes, by using their description (2) in terms of the sheaves of Zariski $p$-forms, to prove the following:

**Theorem 8.** *The normalized Hirzebruch class of a simplicial toric variety $X = X_\Sigma$ is given by:*

$$\widehat{T}_{y*}(X) = \left( \sum_{g \in G_\Sigma} \prod_{\rho \in \Sigma(1)} \frac{[V_\rho] \cdot \left(1 + y \cdot a_\rho(g) \cdot e^{-[V_\rho](1+y)}\right)}{1 - a_\rho(g) \cdot e^{-[V_\rho](1+y)}} \right) \cap [X]\,. \tag{23}$$

Here $G_\Sigma \subset G$ is the set of group elements having fixed-points in the Cox quotient realization $X_\Sigma = W/G$, and $a_\rho$ is a character (coordinate projection) of a $|\Sigma(1)|$-torus containing $G$.

**Example 9.** *Weighted projective spaces*
Consider a $d$-dimensional weighted projective space $X_\Sigma = \mathbb{P}(q_0, \ldots, q_d) := (\mathbb{C}^{d+1} \setminus \{0\})/G$, with $G \simeq \mathbb{C}^*$ acting with reduced weights $(q_0, \ldots, q_d)$, so that this quotient realization is the Cox construction. The normalized Hirzebruch class is then computed by:

$$\widehat{T}_{y*}(X_\Sigma) = \left( \sum_{\lambda \in G_\Sigma} \prod_{j=0}^{d} \frac{q_j T \cdot \left(1 + y \cdot \lambda^{q_j} \cdot e^{-q_j(1+y)T}\right)}{1 - \lambda^{q_j} \cdot e^{-q_j(1+y)T}} \right) \cap [X_\Sigma]\,, \tag{24}$$

with

$$G_\Sigma := \bigcup_{j=0}^{d} \{\lambda \in \mathbb{C}^* \mid \lambda^{q_j} = 1\}$$

and $T = \frac{[V_j]}{q_j} \in H^2(X_\Sigma; \mathbb{Q})$, $j = 0, \ldots, d$, the distinguished generator of the rational cohomology ring $H^*(X_\Sigma; \mathbb{Q}) \simeq \mathbb{Q}[T]/(T^{d+1})$. The un-normalized Hirzebruch class of $X_\Sigma$ is obtained from the above formula (24) by substituting $T$ for $(1+y)T$ and $(1+y)^d \cdot [X]$ in place of $[X]$. The resulting formula recovers Moonen's calculation in [21][p.176] (see also [7][Ex.1.7]).

By specializing to $y = 0$, one recovers from (23) exactly the result of Edidin-Graham [13] for the Todd class $td_*(X) = \widehat{T}_{0*}(X)$ of $X$. Moreover, in the case of a smooth toric variety one has $G_\Sigma = \{id\}$, with all $a_\rho(id) = 1$, so we also obtain formulae for the Hirzebruch classes of smooth toric varieties, in which the generators $[V_\rho]$ ($\rho \in \Sigma(1)$) behave like the Chern roots of the tangent bundle of $X_\Sigma$:

**Corollary 10.** *The normalized Hirzebruch class of a smooth toric variety $X = X_\Sigma$ is given by:*

$$\widehat{T}_{y*}(X) = \left( \prod_{\rho \in \Sigma(1)} \frac{[V_\rho] \cdot \left(1 + y \cdot e^{-[V_\rho](1+y)}\right)}{1 - e^{-[V_\rho](1+y)}} \right) \cap [X]\,. \tag{25}$$

**Mock Hirzebruch classes.**
Following terminology from [8, 24], we define the (normalized) *mock Hirzebruch class* $\widehat{T}^{(m)}_{y*}(X)$ of a simplicial toric variety $X = X_\Sigma$ to be the class given by the right hand side of formula (25). Similarly, by taking the appropriate polynomials in the $[V_\rho]$'s, one can define mock Todd classes, mock Chern classes

and mock $L$-classes, all of which are specializations of the mock Hirzebruch class obtained by evaluating the parameter $y$ at $-1$, $0$, and $1$, respectively.

The following result shows that the difference $\widehat{T}_{y*}(X) - \widehat{T}^{(m)}_{y*}(X)$ between the actual (normalized) Hirzebruch class and the mock Hirzebruch class of a simplicial toric variety $X = X_\Sigma$ is localized on the singular locus, and the contribution of each singular cone $\sigma \in \Sigma_{\mathrm{sing}}$ to this difference is identified explicitly:

**Theorem 11.** *The normalized homology Hirzebruch class of a simplicial toric variety $X = X_\Sigma$ is computed by the formula:*

$$\widehat{T}_{y*}(X) = \widehat{T}^{(m)}_{y*}(X) + \sum_{\sigma\in\Sigma_{\mathrm{sing}}} \mathcal{A}_y(\sigma)\cdot\left((k_\sigma)_*\widehat{T}^{(m)}_{y*}(V_\sigma)\right)\,, \tag{26}$$

*with $k_\sigma : V_\sigma \hookrightarrow X$ denoting the orbit closure inclusion, and*

$$\mathcal{A}_y(\sigma) := \frac{1}{\mathrm{mult}(\sigma)}\cdot\sum_{g\in G^\circ_\sigma}\prod_{\rho\in\sigma(1)}\frac{1+y\cdot a_\rho(g)\cdot e^{-[V_\rho](1+y)}}{1-a_\rho(g)\cdot e^{-[V_\rho](1+y)}}. \tag{27}$$

*Here, the numbers $a_\rho(g)$ are roots of unity of order $\mathrm{mult}(\sigma)$, different from $1$. In particular, since $X$ is smooth in codimension one, we get:*

$$\widehat{T}_{y*}(X) = [X] + \frac{1-y}{2}\sum_{\rho\in\Sigma(1)}[V_\rho] + \text{lower order homological degree terms.} \tag{28}$$

The corresponding mock Hirzebruch classes $\widehat{T}^{(m)}_{y*}(V_\sigma)$ of orbit closures in (26) can be regarded as *tangential data* for the fixed point sets of the action in the geometric quotient description of $X_\Sigma$, whereas the coefficient $\mathcal{A}_y(\sigma)$ encodes the *normal data* information.

It should be noted that in the above coefficient $\mathcal{A}_y(\sigma)$ only the cohomology classes $[V_\rho]$ depend on the fan $\Sigma$, but the multiplicity $\mathrm{mult}(\sigma)$ and the character $a_\rho : G_\sigma \to \mathbb{C}^*$ depend only on the rational simplicial cone $\sigma$ (and not on the fan $\Sigma$ nor the group $G$ from the Cox quotient construction). In fact, these can be given directly as follows. For a $k$-dimensional rational simplicial cone $\sigma$ generated by the rays $\rho_1,\dots,\rho_k$ one defines the finite abelian group $G_\sigma := N_\sigma/(u_1,\dots,u_k)$ as the quotient of the sublattice $N_\sigma$ of $N$ spanned by the points in $\sigma\cap N$ modulo the sublattice $(u_1,\dots,u_k)$ generated by the ray generators $u_j$ of $\rho_j$. Then $|G_\sigma| = \mathrm{mult}(\sigma)$ is just the multiplicity of $\sigma$, with $\mathrm{mult}(\sigma)=1$ exactly in case of a smooth cone. Let $m_i \in M_\sigma$ for $1\le i\le k$ be the unique primitive elements in the dual lattice $M_\sigma$ of $N_\sigma$ satisfying $\langle m_i,u_j\rangle = 0$ for $i\neq j$ and $\langle m_i,u_i\rangle > 0$, so that the dual lattice $M'_\sigma$ of $(u_1,\dots,u_k)$ is generated by the elements $\frac{m_j}{\langle m_j,u_j\rangle}$. Then $a_{\rho_j}(g)$ for $g = n + (u_1,\dots,u_k) \in G_\sigma$ is defined as

$$a_{\rho_j}(g) := \exp\left(2\pi i\cdot\gamma_{\rho_j}(g)\right)\,,\quad \text{with}\quad \gamma_{\rho_j}(g) := \frac{\langle m_j,n\rangle}{\langle m_j,u_j\rangle}\,.$$

Moreover,

$$G^\circ_\sigma := \{g\in G_\sigma \,|\, a_{\rho_j}(g)\neq 1 \text{ for all } 1\le j\le k\}.$$

Finally, using the parallelotopes

$$P_\sigma := \{\sum_{i=1}^k \lambda_i u_i |\ 0\le\lambda_i<1\}\quad\text{and}\quad P^\circ_\sigma := \{\sum_{i=1}^k \lambda_i u_i |\ 0<\lambda_i<1\},$$

we also have $G^\circ_\sigma \simeq N\cap P^\circ_\sigma$ via the natural bijection $G_\sigma \simeq N\cap P_\sigma$.

Note that for $y=-1$ one gets by $\mathcal{A}_{-1}(\sigma) = \frac{|G^\circ_\sigma|}{\mathrm{mult}(\sigma)}$ an easy relation between Ehler's formula (8) for the MacPherson Chern class and the mock Chern class of a toric variety which is given by

$$c^{(m)}_*(X_\Sigma) := \left(\prod_{\rho\in\Sigma(1)}(1+[V_\rho])\right)\cap[X_\Sigma] = \sum_{\sigma\in\Sigma}\frac{1}{\mathrm{mult}(\sigma)}\cdot[V(\sigma)].$$

Similarly, the top-dimensional contribution of a singular cone $\sigma \in \Sigma_{\rm sing}$ in formula (26) is computed by

$$\frac{1}{\mathrm{mult}(\sigma)} \cdot \sum_{g \in G_\sigma^\circ} \left( \prod_{\rho \in \sigma(1)} \frac{1 + y \cdot a_\rho(g)}{1 - a_\rho(g)} \right) \cdot [V_\sigma], \tag{29}$$

which for an isolated singularity $V_\sigma = \{pt\}$ is already the full contribution. For a singular cone of smallest codimension, we have that $G_\sigma^\circ = G_\sigma \backslash \{id\}$ and $G_\sigma$ is cyclic of order $\mathrm{mult}(\sigma)$, so that $a_\rho(g)$ $(g \in G_\sigma^\circ)$ are primitive roots of unity. Specializing further to $y = 0$ (resp. $y = 1$) we recover the correction factors for Todd (and resp. $L$-) class formulae of toric varieties appearing in [2, 6, 16, 24, 25]) (resp. [27] in the case of weighted projective spaces) also in close relation to (generalized) Dedekind (resp. cotangent) sums.

Moreover, for $y = 1$, we get for a projective simplicial toric variety $X_\Sigma$ the correction terms of the singular cones for the difference between the Thom-Milnor and resp. the mock $L$-classes of $X_\Sigma$. In particular, the equality

$$sign(X_\Sigma) = \int_{X_\Sigma} L_*^{(m)}(X_\Sigma)$$

holds in general only for a smooth projective toric variety.

**Example 12.** *Weighted projective spaces, revisited.*
Consider the example of a $d$-dimensional weighted projective space $X_\Sigma = \mathbb{P}(1, \dots, 1, m)$ with weights $(1, \dots, 1, m)$ and $m > 1$. This is a projective simplicial toric variety which has exactly one isolated singular point for $d \geq 2$. Formula (26) becomes in this case:

$$\widehat{T}_{y*}(X_\Sigma) = \widehat{T}_{y*}^{(m)}(X_\Sigma) + \frac{1}{m} \sum_{\lambda^m = 1, \lambda \neq 1} \left( \frac{1 + \lambda y}{1 - \lambda} \right)^d \cdot [pt] \,. \tag{30}$$

In particular, for $y = 1$, we get:

$$sign(X_\Sigma) = \int_{X_\Sigma} L_*^{(m)}(X_\Sigma) + \frac{1}{m} \sum_{\lambda^m = 1, \lambda \neq 1} \left( \frac{1 + \lambda}{1 - \lambda} \right)^d ,$$

and it can be easily seen that the correction factor does not vanish in general, e.g., for $d = 2$, the last sum is given by

$$\sum_{\lambda^m = 1, \lambda \neq 1} \left( \frac{1 + \lambda}{1 - \lambda} \right)^2 = -\frac{1}{3}(m - 1)(m - 2),$$

see [17][p.7]. Similarly, for $y = 0$, we obtain for the arithmetic genus

$$\chi_0(X_\Sigma) = \int_{X_\Sigma} td_*^{(m)}(X_\Sigma) + \frac{1}{m} \sum_{\lambda^m = 1, \lambda \neq 1} \left( \frac{1}{1 - \lambda} \right)^d ,$$

where the last sum is given for $d = 2$ by (see [17][p.4]):

$$\sum_{\lambda^m = 1, \lambda \neq 1} \left( \frac{1}{1 - \lambda} \right)^2 = -\frac{(m - 1)(m - 5)}{12}.$$

**The characteristic class formulae of Cappell-Shaneson.**
We next combine the two computational methods described above for proving several formulae originally obtained by Cappell and Shaneson in the early 1990s, see [8, 26]. The following formula (in terms of $L$-classes) appeared first in Shaneson's $1994$ ICM proceeding paper, see [26][(5.3)].

**Theorem 13.** *Let the $T$-class of a $d$-dimensional simplicial toric variety $X$ be defined as*

$$T_*(X) := \sum_{k=0}^{d} 2^{d-k} \cdot td_k(X). \tag{31}$$

*For any cone $\sigma \in \Sigma$, let $k_\sigma : V_\sigma \hookrightarrow X$ be the inclusion of the corresponding orbit closure. Then the following holds:*

$$T_*(X) = \sum_{\sigma\in\Sigma} (k_\sigma)_* \widehat{T}_{1*}(V_\sigma). \tag{32}$$

Recall here that the identification $L_*(X_\Sigma) = \widehat{T}_{1*}(X_\Sigma)$ holds for a simplicial projective toric variety or a compact toric manifold.

While Cappell-Shaneson's method of proof of the above result uses mapping formulae for Todd and $L$-classes in the context of resolutions of singularities, our approach relies on Theorem 1, i.e., on the additivity properties of the motivic Hirzebruch classes.

Theorem 13 suggests the following definition of mock $T$-classes (cf. [8, 26]) in terms of mock $L$-classes $L_*^{(m)}(X) = \widehat{T}_{1*}^{(m)}(X)$ of a simplicial toric variety $X = X_\Sigma$ (and similarly for the orbit closures $V_\sigma$):

$$T_*^{(m)}(X) := \sum_{\sigma\in\Sigma} (k_\sigma)_* L_*^{(m)}(V_\sigma). \tag{33}$$

Then we have as in [8][Theorem 4] the following expression for the $T$-classes (which are, in fact, suitable renormalized Todd classes) in terms of mock $T$- and resp. mock $L$-classes, with $\mathcal{A}_y(\sigma)$ as in (27):

**Theorem 14.** *Let $X = X_\Sigma$ be a simplicial toric variety. Then in the above notations we have that:*

$$T_*(X) = T_*^{(m)}(X) + \sum_{\sigma\in\Sigma_{\mathrm{sing}}} \mathcal{A}_1(\sigma)\cdot (k_\sigma)_* T_*^{(m)}(V_\sigma)\,. \tag{34}$$

While Cappell-Shaneson's approach to the above result uses induction and Atiyah-Singer (resp. Hirzebruch-Zagier) type results for $L$-classes in the *local* orbifold description of toric varieties, we use the specialization of Theorem 11 to the value $y = 1$, which is based on Edidin-Graham's Lefschetz-Riemann-Roch theorem in the context of the Cox *global* geometric quotient construction.

Finally, the following renormalization of the result of Theorem 14 in terms of coefficients

$$\begin{aligned}\alpha(\sigma) &:= \frac{1}{\mathrm{mult}(\sigma)}\cdot \sum_{g\in G_\sigma^\circ} \prod_{\rho\in\sigma(1)} \frac{1 + a_\rho(g)\cdot e^{-[V_\rho]}}{1 - a_\rho(g)\cdot e^{-[V_\rho]}} \\ &= \frac{1}{\mathrm{mult}(\sigma)}\cdot \sum_{g\in G_\sigma^\circ} \prod_{\rho\in\sigma(1)} \coth\left(\pi i\cdot \gamma_\rho(g) + \frac{1}{2}[V_\rho]\right)\end{aligned} \tag{35}$$

fits better with the corresponding Euler-MacLaurin formulae (see [9][Thm.2], [26][Sect.6]), with $\alpha(\{0\}) := 1$ and $\alpha(\sigma) := 0$ for any other smooth cone $\sigma \in \Sigma$.

**Corollary 15.** *Let $X = X_\Sigma$ be a simplicial toric variety. Then we have:*

$$td_*(X) = \sum_{\sigma\in\Sigma} \alpha(\sigma)\cdot \left( \sum_{\{\tau\mid\sigma\preceq\tau\}} \mathrm{mult}(\tau) \prod_{\rho\in\tau(1)} \frac{1}{2}[V_\rho] \prod_{\rho\notin\tau(1)} \frac{\frac{1}{2}[V_\rho]}{\tanh(\frac{1}{2}[V_\rho])} \right) \cap [X]. \tag{36}$$

The relation with the corresponding Euler-MacLaurin formulae will be discussed elsewhere.

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