Motivic Infinite Cyclic Covers

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Motivation

- *topological* Milnor fiber of a hypersurface singularity germ $f : (\mathbb{C}^{d+1}, x) \to (\mathbb{C}, 0)$ yields an example of *infinite cyclic cover*.
- Denef-Loeser defined a *motivic* Milnor fiber, from which one can recover Hodge-theoretic invariants of the topological Milnor fiber (such as Hodge spectrum).
- **Aim:** extend Denef-Loeser's definition to arbitrary infinite cyclic covers, and give a *topological* construction/perspective (*without* arc spaces) of Denef-Loeser's motivic Milnor fiber.

1. What are infinite cyclic covers?

Definition

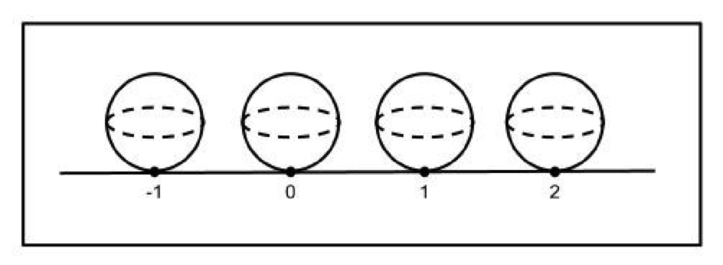
An infinite cyclic cover (icc, for short) is a cover $\widetilde{X} \xrightarrow{p} X$, with fiber $p^{-1}(x) = \mathbb{Z}$, for all $x \in X$.

Construction

If X is a connected CW complex, and $\alpha : \pi_1(X) \to \mathbb{Z}$ is a homomorphism, the cover $\widetilde{X}$ of X defined by $\ker(\alpha)$ is an icc with $[\mathbb{Z} : \mathrm{Im}(\alpha)]$ connected components.

Example

If $X = S^1 \vee S^2$, $\alpha = id_{\mathbb{Z}} : \pi_1(X) \cong \mathbb{Z} \to \mathbb{Z}$, *then* $\widetilde{X} \simeq \bigvee_{k \in \mathbb{Z}} S^2_k$.



In particular, the icc $\widetilde{X}$ *is not of finite type.*

- If $\widetilde{X}$ is an icc of X, the covering group of $\widetilde{X}$ is $\mathbb{Z}$, acting by a covering homeomorphism h.
- So $C_i(\widetilde{X};\mathbb{Q})$, $H_i(\widetilde{X};\mathbb{Q})$, $H^i_{(c)}(\widetilde{X};\mathbb{Q})$ become $\mathbb{Q}[\mathbb{Z}] \cong \mathbb{Q}[t^{\pm 1}]$ - modules, where multiplication by t corresponds to the action induced by h.
- Hence, $C_i(\widetilde{X};\mathbb{Q})$, $H_i(\widetilde{X};\mathbb{Q})$, $H^i_{(c)}(\widetilde{X};\mathbb{Q})$ have a structure of:
    - $\mathbb{Q}$-vector space (*not* finite dim'l in general)
    - $\mathbb{Q}[t^{\pm 1}]$-module (*not* torsion in general)
- In Example, $H_2(\widetilde{X};\mathbb{Q}) \cong \mathbb{Q}[t^{\pm 1}]$, with all S^2's identified via h.

2. Milnor fiber of a hypersurface singularity germ

- $f : \mathbb{C}^{d+1} \to \mathbb{C}$, $x \in f^{-1}(0)$.
- for $0 < \delta \ll \epsilon$ small enough, there exists a locally trivial *Milnor fibration* at x,

$$B_{\epsilon,\delta}(x) := B_\epsilon(x) \cap f^{-1}(D^*_\delta) \stackrel{\pi(=f)}{\longrightarrow} D^*_\delta$$

  with monodromy homeo h acting on the *Milnor fiber* $M_{f,x}$.
- $H^i_{(c)}(M_{f,x};\mathbb{Q})$ has a mixed Hodge structure, compatible with the semi-simple part of the monodromy.
- Basic Problem: compute invariants of $M_{f,x}$, such as Betti numbers, Steenbrink-Hodge spectrum, etc.

Remark

$M_{f,x}$ has the homotopy type of an infinite cyclic cover.
Indeed, the right-hand vertical map of the fiber product

$$\begin{array}{ccc} B_{\epsilon,\delta}(x) \times_{D^*_\delta} \mathbb{R} & \xrightarrow{p_2} & \mathbb{R} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \exp} \\ B_{\epsilon,\delta}(x) & \xrightarrow[\pi]{} & D^*_\delta \simeq S^1. \end{array}$$

is an icc, hence so is the left-hand vertical arrow.
For $r \in \mathbb{R}$, have: $p_2^{-1}(r) \cong \pi^{-1}(\exp(r)) = M_{f,x}$.
But $\mathbb{R}$ is contractible, so $p_2^{-1}(r) \simeq B_{\epsilon,\delta}(x) \times_{D^*_\delta} \mathbb{R}$.
Altogether, $M_{f,x} \simeq B_{\epsilon,\delta}(x) \times_{D^*_\delta} \mathbb{R}$ (which is an icc), compatible with the monodromy and resp. covering group action.

3. Equivariant motivic Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$

Good actions

- μ_n = group of all n-th roots of unity.
- the groups μ_n form a *projective system* with respect to the maps $\mu_{d \cdot n} \to \mu_n$, $\alpha \mapsto \alpha^d$.
- let $\hat{\mu} := \lim \mu_n$ be the projective limit of the μ_n.
- a good μ_n-action on a complex algebraic variety X is an algebraic action $\mu_n \times X \to X$, s.t. each orbit is contained in an affine subvariety of X. (This last condition is automatically satisfied if X is quasi-projective.)
- A good $\hat{\mu}$-action on X is a $\hat{\mu}$-action which factors through a good μ_n-action, for some n.

The Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$ of the category $\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}}$ of complex algebraic varieties endowed with a good $\hat{\mu}$-action is generated by classes $[Y, \sigma]$ of isomorphic varieties endowed with good $\hat{\mu}$-actions, modulo the following relations:

- $[Y, \sigma] = [Y \setminus Y', \sigma_{|_{Y \setminus Y'}}] + [Y', \sigma_{|_{Y'}}]$, if Y' is a closed σ-invariant subset of Y.
- $[Y \times Y', (\sigma, \sigma')] = [Y, \sigma][Y', \sigma']$.
- $[Y \times \mathbb{A}^1_{\mathbb{C}}, \sigma] = [Y \times \mathbb{A}^1_{\mathbb{C}}, \sigma']$, if σ and σ' are two affine liftings of the same $\mathbb{C}^*$-action on Y.

We denote by $\mathbb{L} = [\mathbb{C}]$ the class in $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$ of the affine line, with the trivial $\hat{\mu}$-action.

4. Motivic Milnor fiber of Denef-Loeser

Definition (Denef-Loeser)

For a non constant morphism $f : \mathbb{C}^{d+1} \to \mathbb{C}$ and $x \in f^{-1}(0)$, the local motivic Milnor fibre $\mathcal{S}_{f,x}$ of f at x is defined as:

$$\mathcal{S}_{f,x} = - \lim_{T \to +\infty} \sum_{n \geq 1} [\mathcal{X}_{n,1}] \mathbb{L}^{(d+1)n} T^n \in K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})[\mathbb{L}^{-1}],$$

where

$$\mathcal{X}_{n,1} := \{(n+1) - \textit{jets } \varphi \textit{ of } \mathbb{C}^{d+1} \textit{ at } x \mid f \circ \varphi = t^n + \dots\}$$

with μ_n-action given by $\lambda \times \varphi \mapsto \varphi(\lambda \cdot t)$.

Remark

$\mathcal{S}_{f,x}$ can be computed in terms of a log-resolution (X, E) of the pair $(\mathbb{C}^{d+1}, f^{-1}(0))$.

Hodge realization

Let $K_0(HS^{mon})$ be the Grothendieck group of *monodromic Hodge structures* (i.e., Hodge structures endowed with a finite order automorphism).

Definition (Hodge realization)

$$\chi_{Hodge} : K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})[\mathbb{L}^{-1}] \to K_0(HS^{mon})$$

$$[Y, \sigma] \mapsto [H_c^*(Y;\mathbb{Q}), \sigma^*] := \sum_{i \geq 0} (-1)^i [H_c^i(Y;\mathbb{Q}), \sigma^*].$$

Theorem (Denef-Loeser)

$$\chi_{Hodge}(\mathcal{S}_{f,x}) = [H_c^*(M_{f,x}), h_{ss}^*] \in K_0(HS^{mon})$$

5. Infinite cyclic covers of finite type

- infinite cyclic covers are generally *not* of finite type.
- in order to talk about "motives" or "Betti/Hodge realizations" of an icc, we need to impose some finiteness assumptions.

Setup

- $X \rightsquigarrow$ smooth q-projective complex variety
- $E = \sum_{i \in J} E_j \rightsquigarrow$ reduced simple normal crossing divisor in X
- $T^*_{X,E} := T_{X,E} \setminus E \rightsquigarrow$ *link of E* (i.e. punctured regular neighborhood of E in X)
- holonomy: $\Delta : \pi_1(T^*_{X,E}) \to \mathbb{Z}$, with $m_i := \Delta(\delta_i)$, for δ_i the boundary of a small oriented disc transversal at a generic point to E_i.
- $\widetilde{T}^*_{X,E,\Delta} \rightsquigarrow$ infinite cyclic cover of $T^*_{X,E}$ defined by $\ker(\Delta)$.
- $\widetilde{T}^*_{X,E,\Delta}$ is *not an algebraic variety* in general, so we can't assign a "motive" to it.
- But we give an algebro-geometric interpretation of it.

Definition

Say that $\widetilde{T}^*_{X,E,\Delta}$ is an icc of finite type if $m_i \neq 0$, for all $i \in J$.

Theorem (Gonzalez-Villa – M. – Libgober)

If $m_i \neq 0$, for all $i \in J$, then $H^k_{(c)}(\widetilde{T}^*_{X,E,\Delta};\mathbb{Q})$ is a finite dim'l $\mathbb{Q}$-vector space for all k.

Remark

$H^k_{(c)}(\widetilde{X \setminus E};\mathbb{Q})$ are not finite dim'l $\mathbb{Q}$-vector spaces in general, e.g., take $X = \mathbb{P}^1$, $E = 3$ points.

6. Motivic infinite cyclic covers of finite type

- stratify $E = \sum_{i \in J} E_j$ with strata: $\{E_I^\circ, \emptyset \neq I \subseteq J\}$, where:

$$E_I = \bigcap_{i \in I} E_i \quad \text{and} \quad E_I^\circ = E_I \setminus \bigcup_{j \notin I} E_j$$

- $X = \bigcup_{I \subseteq J} E_I^\circ$, $X \setminus E = E_\emptyset^\circ$ and $E = \bigcup_{\emptyset \neq I \subseteq J} E_I^\circ$.
- $$T^*_{X,E} = \bigcup_{\emptyset \neq I \subseteq J} T^*_{E_I^\circ},$$

  where $T^*_{E_I^\circ} \to E_I^\circ$ is a locally trivial fibration with fibre $(\mathbb{C}^*)^{|I|}$.

We use the holonomy $\Delta : \pi_1(T^*_{X,E}) \to \mathbb{Z}$ to define a *finite cyclic cover* $\widetilde{E}^\circ_I \to E^\circ_I$, where $\widetilde{E}^\circ_I$ is an algebraic variety with a good μ_{m_I}-action, for $m_I := \gcd(m_i \mid i \in I)$.

Definition (Motivic infinite cyclic cover of $T^*_{X,E,\Delta}$)

$$S_{X,E,\Delta} := \sum_{\emptyset \neq I \subseteq J} (-1)^{|I|} \cdot [\widetilde{E}^\circ_I](\mathbb{L}-1)^{|I|-1} \in K_0(\mathrm{Var}^{\hat{\mu}}_{\mathbb{C}})$$

Remark

We regard the summation as coming from the inclusion-exclusion principle for $T^*_{X,E} = \bigcup_{\emptyset \neq I \subseteq J} T^*_{E^\circ_I}$*, while* $[\widetilde{E}^\circ_I](\mathbb{L}-1)^{|I|-1}$ *should be thought as the "motive" of* $\widetilde{T}^*_{E^\circ_I}$*. (Justification to follow.)*

Remark

More generally, for any $A \subseteq J$*, define:*

$$S^A_{X,E,\Delta} := \sum_{\emptyset \neq I \subseteq J, A \cap I \neq \emptyset} (-1)^{|I|} \cdots$$

How are the covers $\widetilde{E}_I^\circ$ defined?

The $(\mathbb{C}^*)^{|I|}$-fibration $T^*_{E_I^\circ} \to E_I^\circ$ induces a commutative diagram:

$$\begin{array}{ccccccc}
\mathbb{Z}^{|I|} & \longrightarrow & \pi_1(T^*_{E_I^\circ}) & \longrightarrow & \pi_1(E_I^\circ) & \longrightarrow & 0 \\
\downarrow & & \downarrow \Delta & & \downarrow \Delta_I & & \downarrow \\
m_I\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/m_I\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

for $m_I := \gcd(m_i,\ i \in I)$ and $\Delta_I : \pi_1(E_I^\circ) \to \mathbb{Z}/m_I\mathbb{Z}$ induced by Δ.
Define $\widetilde{E}_I^\circ \to E_I^\circ$ to be the cover of E_I° defined by $\ker(\Delta_I)$.
$\widetilde{E}_I^\circ$ has $n = [\mathbb{Z} : \mathrm{Im}(\Delta)]$ connected components, each being the cyclic cover of E_I° with deck group $n\mathbb{Z}/m_I\mathbb{Z}$.

Remark

The locally trivial $(\mathbb{C}^*)^{|I|}$*-fibration* $T^*_{E_I^\circ} \to E_I^\circ$ *induces a fibration*

$$\widetilde{T}^*_{E_I^\circ} \to \widetilde{E}_I^\circ$$

on the icc $\widetilde{T}^*_{E_I^\circ}$*, with connected fiber* $\widetilde{(\mathbb{C}^*)^{|I|}} \cong (\mathbb{C}^*)^{|I|-1}$*, the infinite cyclic cover of* $(\mathbb{C}^*)^{|I|}$ *defined by* $\ker(\mathbb{Z}^{|I|} \twoheadrightarrow m_I\mathbb{Z})$*.*
This motivates considering $[\widetilde{E}_I^\circ](\mathbb{L}-1)^{|I|-1}$ *as the "motive" of* $\widetilde{T}^*_{E_I^\circ}$ *in the above definition.*

Theorem (Gonzalez-Villa – M. – Libgober)

The motivic infinite cyclic cover $S_{X,E,\Delta}$ is an invariant of the link $T^*_{X,E,\Delta}$ of E, i.e., it is invariant under the following equivalence relation:

$T^*_{X_1,E_1,\Delta_1} \sim T^*_{X_2,E_2,\Delta_2}$ if there is a birational map $\Phi : X_1 \to X_2$ s.t.

- *$\Phi_| : T^*_{X_1,E_1,\Delta_1} \to T^*_{X_2,E_2,\Delta_2}$ is biregular*
- *$\Phi(T^*_{X_1,E_1,\Delta_1})$ and $T^*_{X_2,E_2,\Delta_2}$ are deformation retracts of each other*
- *commutative diagram:*

$$\begin{array}{ccc} \pi_1(T^*_{X_1,E_1,\Delta_1}) & \xrightarrow{(\Phi_|)_*} & \pi_1(T^*_{X_2,E_2,\Delta_2}) \\ \Delta_1 \searrow & & \swarrow \Delta_2 \\ & \mathbb{Z} & \end{array}$$

Remark

By WFT, it suffices to show that $S_{X,E,\Delta}$ is independent under blow-ups along a smooth center in E.

7. Betti realization

Definition (Betti realization)

$$\chi_b : K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}}) \longrightarrow K_0(V_{\mathbb{Q}}^{\mathrm{aut}})$$

$$[Y,\sigma] \mapsto [H_c^*(Y;\mathbb{Q}),\sigma^*] := \sum_{i\geq 0}(-1)^i[H_c^i(Y;\mathbb{Q}),\sigma^*].$$

Theorem (Gonzalez-Villa – M. – Libgober)

$$\chi_b(S_{X,E,\Delta}) = \left[H_c^*(\widetilde{T}^*_{X,E,\Delta};\mathbb{Q}), h^*\right]$$

Remark

This points to the existence of a mixed Hodge structure on $H_c^*(\widetilde{T}^*_{X,E,\Delta};\mathbb{Q})$, whose class in $K_0(HS^{mon})$ is $\chi_{Hodge}(S_{X,E,\Delta})$. Such structures are constructed in special cases by Liu-M.

8. Relation with Denef-Loeser's motivic Milnor fiber

Theorem (Gonzalez-Villa – M. – Libgober)

Let $f : \mathbb{C}^{d+1} \to \mathbb{C}$ be a non-constant morphism with $x \in f^{-1}(0)$, and let (X, E) be a log-resolution of the pair $(\mathbb{C}^{d+1}, f^{-1}(0))$. Then

$$\mathcal{S}_{f,x} = -S^{A}_{X,E,\Delta} \in K_0(\mathrm{Var}^{\hat{\mu}}_{\mathbb{C}})[\mathbb{L}^{-1}],$$

for $A = \{i \in J \mid E_i \subset p^{-1}(x)\}$ and Δ induced by $\alpha \mapsto \int_\alpha \frac{df}{f}$.

9. Concluding remarks

(a) Our construction also applies to *motivic Milnor fibers of rational functions* and *motivic Milnor fibers at infinity*.

(b) More generally, we can define a *motivic zeta function* $Z_{X,E,\Delta}$ for any log-resolution (X, E) of a pair (Y, D) with D a divisor on a smooth variety Y, show that it is an invariant of the link $T^*_{Y,D}$ of D in Y, and formulate a *global monodromy conjecture* relating the poles of this zeta function to the zeros of the (global) Alexander polynomials of $T^*_{Y,D}$.

THANK YOU !!!