

REMARKS ON THE TOTAL CUT LOCUS

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ABSTRACT. Any two distinct points in a closed compact connected Riemannian manifold M can be joined by a geodesic whose length is the distance between the end points. For a fixed $x \in M$ most other points in M can be joined by a unique such geodesic. The closure of the set of points which can be joined by two or more such geodesics is called the *cut locus* of x and is denoted $\check{c}(x)$. Two standard results are that $y \in \check{c}(x)$ if and only if $x \in \check{c}(y)$ and that $x \notin \check{c}(x)$ for all $x \in M$. A pair of points $\{x, y\}$ is called a *cut pair* if $y \in \check{c}(x)$.

The *total cut locus*, denoted $\check{C} \subset M \times M$, is the set of all (x, y) with $y \in \check{c}(x)$. The main technical result in this talk is the observation that $M \times M - \text{diagonal}$ deformation retracts to $\check{C}$ and $M \times M - \check{C}$ deformation retracts to the diagonal. Both deformations are over p_1 , the projection to the first coordinate.

There are three general sorts of applications and the talk will endeavor to give examples of each. The first sort are rifts on “ $\check{c}(x)$ varies continuously in x ”. One example is that $p_1: \check{C} \rightarrow M$ is a fibration. To describe another example, let $\mathcal{H}(M)$ denote the set of all non-empty closed subsets of M endowed with the Hausdorff metric. Then the map $\kappa: M \rightarrow \mathcal{H}(M)$ defined by $\kappa(x) = \check{c}(x)$ is a one-to-one map.

A second application involves restrictions on the behavior of maps $f: M^n \rightarrow X$ where M^n has dimension n . One example here is the Riemannian Borsuk-Ulam Theorem, a version of which says that for any map $f: M^n \rightarrow \mathbb{R}^n$ there exists a cut pair $\{x, y\}$ with $f(x) = f(y)$. When M^n a round sphere this is the classical Borsuk-Ulam Theorem.

A third sort involves restrictions on the behavior of maps $f: X \rightarrow M$. Given two such maps, f_0 and f_1 , let the *far from set* of f_0 and f_1 , denoted $\mathcal{F}(f_0, f_1)$, be the set of all $x \in X$ such that $\{f_0(x), f_1(x)\}$ is a cut pair. If f_0 is not homotopic to f_1 , then $\mathcal{F}(f_0, f_1)$ is non-empty. If X is also a closed compact Riemannian manifold such that f_0 and f_1 are not homotopic when restricted to $X - \{x\}$ for some point x , then $\mathcal{F}(f_0, f_1)$ is uncountable.

Time permitting, additional examples of these phenomena will be explored.

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