An Introduction to Hyperplane Arrangements Topology and Geometry – Lecture 1

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Workshop and Summer School on Hyperplane Arrangements and Related Topics

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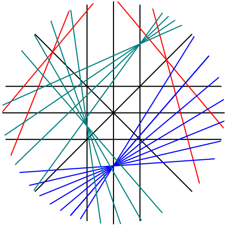

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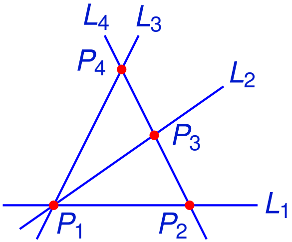
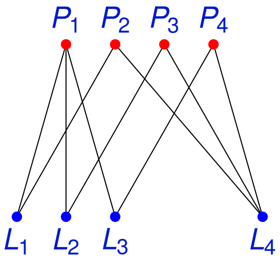
Complement and intersection lattice

- An *arrangement of hyperplanes* is a finite collection $\mathcal{A}$ of codimension 1 linear (or affine) subspaces in $V = \mathbb{C}^d$.
- *Intersection lattice* $L(\mathcal{A})$: the poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, with $\vee$ and $\wedge$, and rank function $\operatorname{rank}(X) = \operatorname{codim}(X)$. It is a geometric lattice.





- *Complement*: $M(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}} H$. It is a connected, smooth, quasi-projective variety.

- We may assume that $\mathcal{A}$ is essential, i.e., $\bigcap_{H\in\mathcal{A}} H = \{0\}$.
- Fix an ordering $\mathcal{A} = \{H_1, \dots, H_n\}$, and choose linear forms $f_i \colon \mathbb{C}^d \to \mathbb{C}$ with $\ker(f_i) = H_i$. Define an injective linear map

$$\iota\colon \mathbb{C}^d \to \mathbb{C}^n, \quad z \mapsto (f_1(z), \dots, f_n(z)).$$

- This map restricts to an inclusion $\iota\colon M(\mathcal{A}) \hookrightarrow (\mathbb{C}^*)^n$. Hence, $M(\mathcal{A}) = \iota(\mathbb{C}^d) \cap (\mathbb{C}^*)^n$ is a Stein manifold.
- Therefore, $M = M(\mathcal{A})$ has the homotopy type of a connected, finite cell complex of dimension d.
- In fact, M has a minimal cell structure. Consequently, $H_*(M, \mathbb{Z})$ is torsion-free.
- Let $U(\mathcal{A}) = \mathbb{P}(M(\mathcal{A})) = \mathbb{CP}^d \setminus \bigcup_{H\in\mathcal{A}} \mathbb{P}(H)$ be the projectivized complement. Then $M(\mathcal{A}) \cong U(\mathcal{A}) \times \mathbb{C}^*$.

Classes of arrangements

Example (The Boolean arrangements)

- $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0, 1\}^n$.
- $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n \simeq K(\mathbb{Z}^n, 1)$.

Example (The braid arrangements)

- $\mathcal{A}_n$: all diagonal hyperplanes $H_{ij} = \{z_i - z_j = 0\}$ in $\mathbb{C}^n$.
- $L(\mathcal{A}_n)$: lattice of partitions of $[n] := \{1, \dots, n\}$, ordered by refinement.
- $M(\mathcal{A}_n)$: configuration space of n ordered points in $\mathbb{C}$, a classifying space for P_n, the pure braid group on n strings.

Example (Supersolvable arrangements)

- A flat $X \in L(\mathcal{A})$ is *modular* if for any other flat Y,
$$\operatorname{rank}(X) + \operatorname{rank}(Y) = \operatorname{rank}(X \vee Y) + \operatorname{rank}(X \wedge Y).$$
- $\mathcal{A}$ is *supersolvable* if $L(\mathcal{A})$ contains a maximal chain of modular elements, $V = X_0 > X_1 > \cdots > X_r = \{0\}$, where $r = \operatorname{rank}(\mathcal{A})$.
- Equivalently, $\mathcal{A}$ is supersolvable (or, fiber-type) if it admits a filtration $\emptyset = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_r = \mathcal{A}$, where each $\mathcal{A}_i$ has $\operatorname{rank}(\mathcal{A}_i) = i$ and $\exists\ M(\mathcal{A}_i) \to M(\mathcal{A}_{i-1})$ a bundle map with fiber $\mathbb{C}\backslash\{d_i \text{ points}\}$ that is the restriction of a linear projection $\mathbb{C}^i \to \mathbb{C}^{i-1}$.
- The complement $M(\mathcal{A})$ is a $K(\pi, 1)$ and its fundamental group is an iterated semidirect product of finitely generated free groups.
- The braid arrangement $\mathcal{A}_n$ is fiber-type. Each projection $M(\mathcal{A}_i) \to M(\mathcal{A}_{i-1})$ has fiber $\mathbb{C}\backslash\{i-1 \text{ points}\}$, and gives rise to split extension $1 \to F_{i-1} \to P_i \to P_{i-1} \to 1$. Hence,
$$P_n = F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1.$$

Example (Graphic arrangements)

- Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a finite simplicial graph with vertex set $\mathsf{V} = [n] := \{1, \dots, n\}$ and edge set $\mathsf{E} \subset 2^{[n]}$.
- The *graphic arrangement* $\mathcal{A}_\Gamma$ consists of the hyperplanes $H_e = \{z_i - z_j = 0\}$ in $\mathbb{C}^n$ indexed by the edges $e = \{i, j\}$ of E.
- If $\Gamma = K_n$, the complete graph on n vertices, then $\mathcal{A}_{K_n} = \mathcal{A}_n$.
- Thus, any graphic arrangement $\mathcal{A}_\Gamma$ can be viewed as a sub-arrangement of the braid arrangement $\mathcal{A}_n$, where $n = |\mathsf{V}|$.
- Stanley: $\mathcal{A}_\Gamma$ is supersolvable if and only if Γ is *chordal* (i.e., every cycle of four or more vertices has a chord).

- Let $\mathcal{A}$ be central arrangement in $\mathbb{C}^d$. For each $H \in \mathcal{A}$, let $f_H \colon \mathbb{C}^d \to \mathbb{C}$ linear map such that $H = \ker(f_H)$.
- The logarithmic 1-forms $\omega_H = \frac{1}{2\pi \mathrm{i}}\, d \log f_H \in \Omega_{\mathrm{dR}}(M)$ are closed.
- Let E be the $\mathbb{Z}$-exterior algebra on the degree 1 cohomology classes $e_H = [\omega_H]$ dual to the meridians x_H around $H \in \mathcal{A}$.
- Let $\partial \colon E^* \to E^{*-1}$ be the differential given by $\partial(e_H) = 1$, and set $e_X = \prod_{H \supseteq X} e_H$ for each $X \in L(A)$.
- The cohomology ring $A(\mathcal{A}) = H^*(M; \mathbb{Z})$ is isomorphic to the Orlik–Solomon algebra E/I, where $I = \langle \partial e_X : \operatorname{rank}(X) < |X| \rangle$.
- Hence, $A(\mathcal{A})$ is determined by $L(\mathcal{A})$.

- The *localization* of an arrangement $\mathcal{A}$ at a flat $X \in L(\mathcal{A})$ is the sub-arrangement $\mathcal{A}_X := \{H \in \mathcal{A} : H \supset X\}$.
- The inclusion $\mathcal{A}_X \subset \mathcal{A}$ gives rise to an inclusion of complements, $j_X \colon M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$. The inclusions $\{j_X\}_{X \in L(\mathcal{A})}$ assemble into a map $j \colon M(\mathcal{A}) \to \prod_{X \in L(\mathcal{A})} M(\mathcal{A}_X)$.
- Brieskorn: the homomorphism induced in cohomology by j is an isomorphism in each degree $k \geqslant 0$. Moreover, the groups $H^k(M(\mathcal{A}_X); \mathbb{Z})$ are torsion-free, and so, by the Künneth formula, we have isomorphisms

$$H^k(M(\mathcal{A}); \mathbb{Z}) \cong \bigoplus_{X \in L_k(\mathcal{A})} H^k(M(\mathcal{A}_X); \mathbb{Z})$$

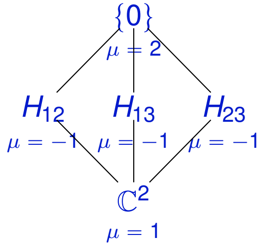
- Likewise, for each $k \geqslant 2$, the degree k piece of the Orlik–Solomon ideal, $I^k(\mathcal{A}_X)$, decomposes as the direct sum of the groups $I^k(\mathcal{A}_X)$, taken over all $X \in L_k(\mathcal{A})$.

- It follows that all the homology groups of $H_k(M, \mathbb{Z})$ are torsion-free. Their ranks are the Betti numbers $b_k(M)$, given by

$$\mathrm{Poin}(M, t) := \sum_{k=0}^{d} b_k(M) t^k = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\mathrm{rank}(X)},$$

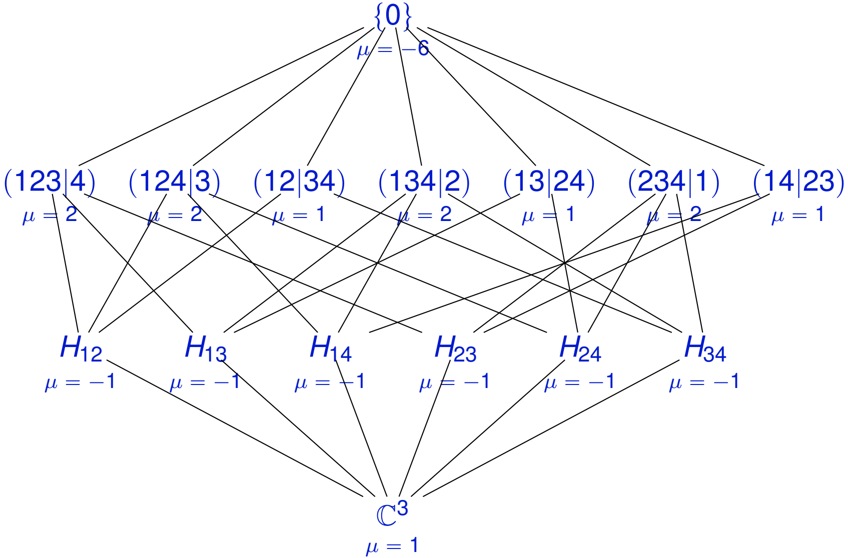
where $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ is the Möbius function, defined recursively by $\mu(\mathbb{C}^d) = 1$ and $\mu(X) = -\sum_{Y \supsetneq X} \mu(Y)$.

- $L(\mathcal{A}_3)$, with Möbius function:

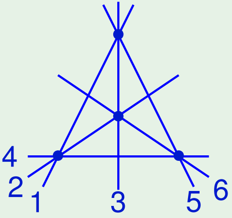


- Since $H^*(M; \mathbb{Z})$ is torsion-free (as a group) and generated in degree 1 (as an algebra), the Hurewicz homomorphism $h\colon \pi_i(M) \to H_i(M; \mathbb{Z})$ is the zero map, for each $2 \leqslant i \leqslant d$.

$L(\mathcal{A}_4)$ and Möbius function



Example



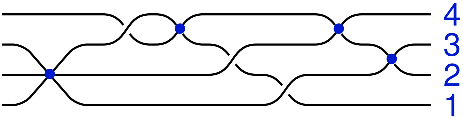
- $\mathcal{A} = \mathcal{A}_4$ braid arrangement
- $E = \bigwedge(e_1, \dots, e_6)$
- $I = \langle (e_1 - e_4)(e_2 - e_4), (e_1 - e_5)(e_3 - e_5), (e_2 - e_6)(e_3 - e_6), (e_4 - e_6)(e_5 - e_6) \rangle$
- $\mathrm{Poin}(M, t) = 1 + 6t + 11t^2 + 6t^3 = (1+t)(1+2t)(1+3t)$

Example (Supersolvable arrangements (continued))

- Let $1 = d_1, d_2, \dots, d_r$, be the *exponents* of a supersolvable arrangement $\mathcal{A}$. Then the Poincaré polynomial of the complement factors completely: $\mathrm{Poin}(M(\mathcal{A}), t) = \prod_{i=1}^{r}(1 + d_i t)$.
- Björner–Ziegler: $\mathcal{A}$ is supersolvable if and only if the OS-ideal $I(\mathcal{A})$ admits a quadratic Gröbner basis, in which case $A = A(\mathcal{A})$ is a Koszul algebra, i.e., $\mathrm{Ext}^i_A(\Bbbk, \Bbbk)_j = 0$ for $i \neq j$.

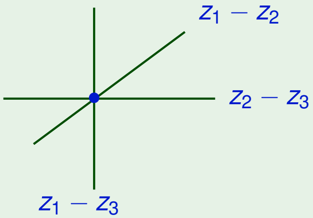
- Question (Yuzvinsky): If $A(\mathcal{A})$ is Koszul, is $\mathcal{A}$ supersolvable?

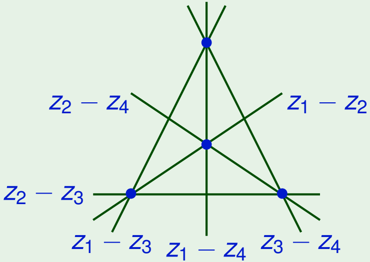
- Let $\mathcal{A}' = \{H \cap \mathbb{C}^2\}_{H \in \mathcal{A}}$ be a generic planar section of $\mathcal{A}$. Then the arrangement group, $G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$, is isomorphic to $\pi_1(M(\mathcal{A}'))$.
- So let $\mathcal{A}$ be an arrangement of n affine lines in $\mathbb{C}^2$. Taking a generic projection $\mathbb{C}^2 \to \mathbb{C}$ yields the braid monodromy $\alpha = (\alpha_1, \dots, \alpha_s)$, where $s = \#\{\text{multiple points}\}$ and the braids $\alpha_r \in P_n$ can be read off an associated braided wiring diagram,



- The group $G(\mathcal{A})$ has a presentation with meridional generators $x_1, \dots, x_n$ and commutator relators $x_i \alpha_j(x_i)^{-1}$.

Example





$G = P_3 \cong F_2 \times \mathbb{Z}$ $\qquad$ $G = P_4 \cong F_3 \rtimes P_3$

Polynomial covers

- Let X be a path-connected space. A *simple Weierstrass polynomial* of degree n on X is a map $f\colon X \times \mathbb{C} \to \mathbb{C}$ given by

$$f(x,z) = z^n + \sum_{i=1}^{n} a_i(x) z^{n-i},$$

  with continuous coefficient maps $a_i\colon X \to \mathbb{C}$, and with no multiple roots for any $x \in X$.

- Let $E = E(f) = \{(x,z) \in X \times \mathbb{C} \mid f(x,z) = 0\}$.

- The restriction of $\mathrm{pr}_1\colon X \times \mathbb{C} \to X$ to E defines an n-fold cover $\pi = \pi_f\colon E \to X$, the *polynomial covering map* associated to f.

$$\begin{array}{ccc} E & \hookrightarrow & X \times \mathbb{C} \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & X & \end{array}$$

Configuration spaces

- Let $\mathrm{Conf}_n(\mathbb{C}) = \{z \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$ and $\mathrm{UConf}_n(\mathbb{C}) = \mathrm{Conf}_n(\mathbb{C})/S_n$.
- Since $f\colon X \times \mathbb{C} \to \mathbb{C}$ has no multiple roots, the *coefficient map* $a = (a_1, \dots, a_n)\colon X \to \mathbb{C}^n$ takes values in

$$B^n := \mathbb{C}^n \backslash \Delta_n = \mathrm{UConf}_n(\mathbb{C}).$$

- Over $\mathrm{UConf}_n(\mathbb{C})$, there is a canonical n-fold polynomial covering map, $\pi_n\colon E(f_n) \to \mathrm{UConf}_n(\mathbb{C})$, determined by the W-polynomial

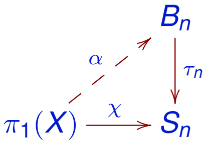
$$f_n(x, z) = z^n + \sum_{i=1}^{n} x_i z^{n-i}.$$

- We get a pullback diagram of covers,

$$\begin{array}{ccc} E(f) & \longrightarrow & E(f_n) \\ \Big\downarrow{\scriptstyle \pi_f} & & \Big\downarrow{\scriptstyle \pi_n} \\ X & \xrightarrow{\ a\ } & B^n \end{array}$$

Braid groups

- Let B_n be the Artin braid group on n strands. Then $B_n = \pi_1(\text{UConf}_n(\mathbb{C}))$.
- We let $\psi_n \colon B_n \hookrightarrow \text{Aut}(F_n)$ be the Artin representation.
- The *coefficient homomorphism*, $\alpha = a_* \colon \pi_1(X) \to B_n$, is well-defined up to conjugacy.
- Polynomial covers are those covers $\pi \colon E \to X$ for which the characteristic homomorphism $\chi \colon \pi_1(X) \to S_n$ factors through the canonical surjection $\tau_n \colon B_n \twoheadrightarrow S_n$,



The root map

- Now assume that the W-polynomial f completely factors as
$$f(x,z)=\prod_{i=1}^{n}(z-b_i(x)),$$
with continuous roots $b_i\colon X\to\mathbb{C}$.

- Since f is simple, the *root map* $b=(b_1,\dots,b_n)\colon X\to\mathbb{C}^n$ takes values in $\mathrm{Conf}_n(\mathbb{C})$.

- Over $\mathrm{Conf}_n(\mathbb{C})$, there is a canonical n-fold cover, $\pi_{Q_n}\colon E(Q_n)\to\mathrm{Conf}_n(\mathbb{C})$, where
$$Q_n(w,z)=(z-w_1)\cdots(z-w_n).$$

- We get a pullback diagram of covers,
$$\begin{array}{ccc} E(f) & \longrightarrow & E(Q_n) \\ {\scriptstyle \pi_f}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{Q_n}} \\ X & \xrightarrow{\;b\;} & \mathrm{Conf}_n(\mathbb{C}) \end{array}$$

Braid bundles

- Let $P_n = \ker(\tau_n \colon B_n \twoheadrightarrow S_n)$ be the pure braid group. Then $P_n = \pi_1(\mathrm{Conf}_n(\mathbb{C}))$.
- The map $\beta = b_* \colon \pi_1(X) \to P_n$ is well-defined up to conjugacy.
- The polynomial covers which are trivial covers are precisely those for which $\alpha = \iota_n \circ \beta$, where $\iota_n \colon P_n \hookrightarrow B_n$ is the inclusion map.

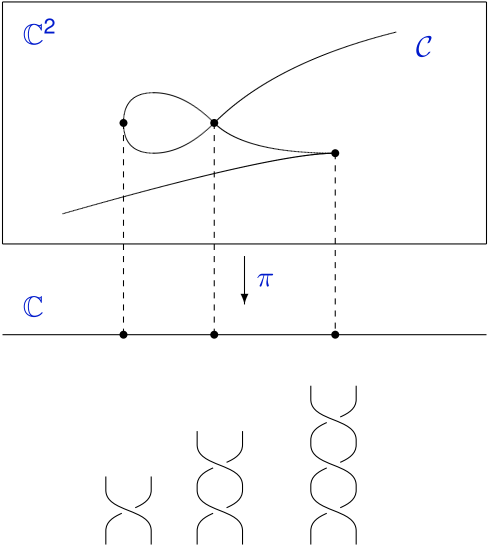
Theorem (Cohen–S. 1997)

Let $f \colon X \times \mathbb{C} \to \mathbb{C}$ be a simple W-polynomial. Let $Y = X \times \mathbb{C} \setminus E(f)$ and let $p \colon Y \to X$ be the restriction of $\mathrm{pr}_1 \colon X \times \mathbb{C} \to X$ to Y.

- *The map $p \colon Y \to X$ is a locally trivial bundle, with structure group B_n and fiber $\mathbb{C}_n = \mathbb{C} \setminus \{n \text{ points}\}$. Upon identifying $\pi_1(\mathbb{C}_n)$ with F_n, the monodromy of this bundle is $\psi_n \circ \alpha \colon \pi_1(X) \to \mathrm{Aut}(F_n)$.*
- *If f completely factors into linear factors, the structure group reduces to P_n, and the monodromy factors as $\psi_n \circ \iota_n \circ \beta$.*

BRAID MONODROMY OF PLANE ALGEBRAIC CURVES

- Let $\mathcal{C}$ be a reduced algebraic curve in $\mathbb{C}^2$, defined by a polynomial $f = f(z_1, z_2)$ of degree n.
- Let $\pi\colon \mathbb{C}^2 \to \mathbb{C}$ be a linear projection, and let $\mathcal{Y} = \{y_1, \dots, y_s\}$ be the set of points in $\mathbb{C}$ for which the fibers of π contain singular points of $\mathcal{C}$, or are tangent to $\mathcal{C}$.
- WLOG, we may assume that $\pi = \mathrm{pr}_1$ is generic with respect to $\mathcal{C}$. That is, for each k, the line $\mathcal{L}_k = \pi^{-1}(y_k)$ contains at most one singular point v_k of $\mathcal{C}$ and does not belong to the tangent cone of $\mathcal{C}$ at v_k, and, moreover, all tangencies are simple.
- Let $\mathcal{L} = \bigcup \mathcal{L}_k$.


$\mathbb{C}^2$
$\mathcal{C}$
π
$\mathbb{C}$

- In the chosen coordinates, the defining polynomial f of $\mathcal{C}$ may be written as $f(x,z) = z^n + \sum_{i=1}^n a_i(x) z^{n-i}$.

- Since $\mathcal{C}$ is reduced, for each $x \notin \mathcal{Y}$, the equation $f(x,z) = 0$ has n distinct roots. Thus, f is a simple W-polynomial over $\mathbb{C} \backslash \mathcal{Y}$, and

$$\pi = \pi_f \colon \mathcal{C} \backslash (\mathcal{C} \cap \mathcal{L}) \to \mathbb{C} \backslash \mathcal{Y}$$

  is the associated polynomial n-fold cover.

- Note that $Y(f) = ((\mathbb{C} \backslash \mathcal{Y}) \times \mathbb{C}) \backslash (\mathcal{C} \backslash (\mathcal{C} \cap \mathcal{L})) = \mathbb{C}^2 \backslash (\mathcal{C} \cup \mathcal{L})$.

- Thus, the restriction of pr_1 to $Y(f)$,

$$p \colon \mathbb{C}^2 \backslash (\mathcal{C} \cup \mathcal{L}) \to \mathbb{C} \backslash \mathcal{Y},$$

  is a bundle map, with structure group B_n, fiber $\mathbb{C}_n$, and monodromy homomorphism $\alpha = a_* \colon \pi_1(\mathbb{C} \backslash \mathcal{Y}) \to B_n$.

Braid monodromy presentation

- The homotopy exact sequence of fibration $p\colon \mathbb{C}^2\backslash(\mathcal{C}\cup\mathcal{L}) \to \mathbb{C}\backslash\mathcal{Y}$:

$$1 \longrightarrow \pi_1(\mathbb{C}_n) \longrightarrow \pi_1(\mathbb{C}^2\backslash(\mathcal{C}\cup\mathcal{L})) \xrightarrow{p_*} \pi_1(\mathbb{C}\backslash\mathcal{Y}) \longrightarrow 1\,.$$

- This sequence is split exact, with action given by the braid monodromy homomorphism $\alpha\colon \pi_1(\mathbb{C}\backslash\mathcal{Y}) \to \operatorname{Aut}(\pi_1(\mathbb{C}_n))$.
- Order the points of $\mathcal{Y}$ by decreasing real part, and pick the basepoint y_0 in $\mathbb{C}\backslash\mathcal{Y}$ with $\operatorname{Re}(y_0) > \max\{\operatorname{Re}(y_k)\}$.
- Choose loops $\xi_k\colon [0,1] \to \mathbb{C}\backslash\mathcal{Y}$ based at y_0, and going around y_k.
- Setting $x_k = [\xi_k]$, identify $\pi_1(\mathbb{C}\backslash\mathcal{Y}, y_0)$ with $F_s = \langle x_1, \dots, x_s\rangle$. Similarly, identify $\pi_1(\mathbb{C}_n, \hat{y}_0)$ with $F_n = \langle t_1, \dots, t_n\rangle$.
- Then $\pi_1(\mathbb{C}^2\backslash(\mathcal{C}\cup\mathcal{L}), \hat{y}_0) = F_n \rtimes_\alpha F_s$.

- The corresponding presentation is

$$\pi_1(\mathbb{C}^2 \backslash (\mathcal{C} \cup \mathcal{L})) = \langle t_1, \dots t_n, x_1 \dots, x_s \mid x_k^{-1} t_i x_k = \alpha(x_k)(t_i) \rangle.$$

- The group $\pi_1(\mathbb{C}^2 \backslash \mathcal{C})$ is the quotient of $\pi_1(\mathbb{C}^2 \backslash (\mathcal{C} \cup \mathcal{L}))$ by the normal closure of $F_s = \langle x_1, \dots, x_s \rangle$. Thus,

$$\pi_1(\mathbb{C}^2 \backslash \mathcal{C}) = \langle t_1, \dots, t_n \mid t_i = \alpha(x_k)(t_i) \rangle.$$

- This presentation can be simplified by Tietze-II moves to eliminate redundant relations. This yields the *braid monodromy presentation*

$$\pi_1(\mathbb{C}^2 \backslash \mathcal{C}) = \langle t_1, \dots, t_n \mid t_i = \alpha(x_k)(t_i), i = j_1, \dots, j_{m_k - 1}; k = 1, \dots, s \rangle.$$

  where m_k is the multiplicity of the singular point y_k.

- (Libgober 1986) The 2-complex modeled on this presentation is homotopy equivalent to $\mathbb{C}^2 \backslash \mathcal{C}$.

- Let $\mathcal{A}$ be a central hyperplane arrangement in $\mathbb{C}^d$.
- For each $H \in \mathcal{A}$, let $f_H \colon \mathbb{C}^d \to \mathbb{C}$ be a linear form with kernel H.
- For each choice of multiplicities $m = (m_H)_{H\in\mathcal{A}}$ with $m_H \in \mathbb{N}$, let

$$Q_m := Q_m(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H^{m_H},$$

  a homogeneous polynomial of degree $N = \sum_{H\in\mathcal{A}} m_H$.
- The map $Q_m \colon \mathbb{C}^d \to \mathbb{C}$ restricts to a map $Q_m \colon M(\mathcal{A}) \to \mathbb{C}^*$.
- This is the projection of a smooth, locally trivial bundle, known as the *Milnor fibration* of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \longrightarrow M(\mathcal{A}) \xrightarrow{\;Q_m\;} \mathbb{C}^*.$$

- The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the *Milnor fiber* of the multi-arrangement.
- $F_m(\mathcal{A})$ is a Stein manifold. It has the homotopy type of a finite cell complex, with $\gcd(m)$ connected components, of dim $d-1$.
- The *(geometric) monodromy* is the diffeomorphism

$$h\colon F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi i/N} z.$$

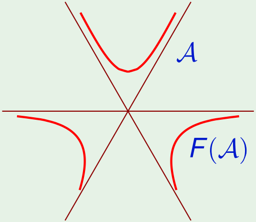
- If all $m_H = 1$, the polynomial $Q = Q(\mathcal{A})$ is the usual defining polynomial, and $F(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$.

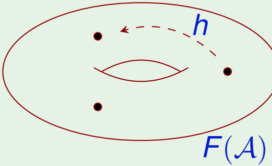
Example

Let $\mathcal{A}$ be the single hyperplane $\{0\}$ inside $\mathbb{C}$. Then $M(\mathcal{A}) = \mathbb{C}^*$, $Q_m(\mathcal{A}) = z^m$, and $F_m(\mathcal{A}) = \{m\text{-roots of } 1\}$.

Example

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and h is an automorphism of order 3:





More generally, if $\mathcal{A}$ is a pencil of n lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\binom{n-1}{2}$, with n punctures.

- Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q_m(\mathcal{B}_n) = z_1^{m_1} \cdots z_n^{m_n}$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and

$$F_m(\mathcal{B}_n) = \ker(Q_m) \cong (\mathbb{C}^*)^{n-1} \times \mathbb{Z}_{\gcd(m)}.$$

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential arrangement. The inclusion $\iota\colon M(\mathcal{A}) \to M(\mathcal{B}_n)$ restricts to a bundle map

$$\begin{array}{ccccc} F_m(\mathcal{A}) & \longrightarrow & M(\mathcal{A}) & \xrightarrow{Q_m(\mathcal{A})} & \mathbb{C}^* \\ \downarrow & & \downarrow{\scriptstyle\iota} & & \| \\ F_m(\mathcal{B}_n) & \longrightarrow & M(\mathcal{B}_n) & \xrightarrow{Q_m(\mathcal{B}_n)} & \mathbb{C}^* \end{array}$$

- Thus,

$$F_m(\mathcal{A}) = M(\mathcal{A}) \cap F_m(\mathcal{B}_n).$$

THE BOUNDARY MANIFOLD OF AN ARRANGEMENT

- Let $\mathcal{A}$ be a (central) arrangement of hyperplanes in $\mathbb{C}^d$ ($d \geqslant 2$).
- Let $\mathbb{P}(\mathcal{A}) = \{\mathbb{P}(H)\}_{H\in\mathcal{A}}$, and let $\nu(V)$ be a regular neighborhood of the algebraic hypersurface $V = \bigcup_{H\in\mathcal{A}} \mathbb{P}(H)$ inside $\mathbb{CP}^{d-1}$.
- Let $\overline{U} = \mathbb{CP}^{d-1} \setminus \operatorname{int}(\nu(V))$ be the *exterior* of $\mathbb{P}(\mathcal{A})$.
- The *boundary manifold* of $\mathcal{A}$ is $\partial\overline{U} = \partial\nu(V)$: a compact, orientable, smooth manifold of dimension $2d-3$.

EXAMPLE

Let $\mathcal{A}$ be a pencil of n hyperplanes in $\mathbb{C}^d$, defined by $Q = z_1^n - z_2^n$. If $n = 1$, then $\partial\overline{U} = S^{2d-3}$. If $n > 1$, then $\partial\overline{U} = \sharp^{n-1} S^1 \times S^{2(d-2)}$.

EXAMPLE

Let $\mathcal{A}$ be a near-pencil of n planes in $\mathbb{C}^3$, defined by $Q = z_1(z_2^{n-1} - z_3^{n-1})$. Then $\partial\overline{U} = S^1 \times \Sigma_{n-2}$, where $\Sigma_g = \sharp^g S^1 \times S^1$.

- By Lefschetz duality: $H_q(\partial\overline{U},\mathbb{Z}) \cong H_q(U,\mathbb{Z}) \oplus H_{2d-q-3}(U,\mathbb{Z})$
- Let $A = H^*(U,\mathbb{Z})$; then $\check{A} = \operatorname{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$ is an A-bimodule, with $(a\cdot f)(b) = f(ba)$ and $(f\cdot a)(b) = f(ab)$.

THEOREM (COHEN–S. 2006)

The ring $\widehat{A} = H^*(\partial\overline{U};\mathbb{Z})$ is the "double" of A, that is: $\widehat{A} = A \oplus \check{A}$, with $(a,f)\cdot(b,g) = (ab, ag+fb)$, and grading $\widehat{A}^q = A^q \oplus \check{A}^{2d-q-3}$.

- When $d=3$, the boundary manifold $\partial\overline{U}$ is a 3-dimensional graph-manifold W_Γ, where
  - Γ is the incidence graph of $\mathcal{A}$, with $V(\Gamma) = L_1(\mathcal{A}) \cup L_2(\mathcal{A})$ and $E(\Gamma) = \{(L,P) \mid P \in L\}$.
  - Vertex manifolds $W_v = S^1 \times \big(S^2 \setminus \bigcup_{\{v,w\}\in E(\Gamma)} D^2_{v,w}\big)$ are glued along edge manifolds $W_e = S^1 \times S^1$ via flip maps.
- $b_1(W_\Gamma) = |\mathcal{A}| + b_1(\Gamma) - 1$.

THEOREM (JIANG–YAU 1993)

$U(\mathcal{A}) \cong U(\mathcal{A}') \Rightarrow W_\Gamma \cong W_{\Gamma'} \Rightarrow \Gamma \cong \Gamma' \Rightarrow L(\mathcal{A}) \cong L(\mathcal{A}')$.

THE BOUNDARY OF THE MILNOR FIBER

- Let $(\mathcal{A}, m)$ be a multi-arrangement in $\mathbb{C}^d$.
- Define $\overline{F}_m(\mathcal{A}) = F_m(\mathcal{A}) \cap D^{2d}$ to be the *closed Milnor fiber* of $(\mathcal{A}, m)$. Clearly, $F_m(\mathcal{A})$ deform-retracts onto $\overline{F}_m(\mathcal{A})$.
- The *boundary of the Milnor fiber* of $(\mathcal{A}, m)$ is the compact, smooth, orientable, $(2d-3)$-manifold $\partial\overline{F}_m(\mathcal{A}) = F_m(\mathcal{A}) \cap S^{2d-1}$.
- The pair $(\overline{F}_m, \partial\overline{F}_m)$ is $(d-2)$-connected. In particular, if $d \geqslant 2$, then $\partial\overline{F}_m$ is connected, and $\pi_1(\partial\overline{F}_m) \to \pi_1(\overline{F}_m)$ is surjective.

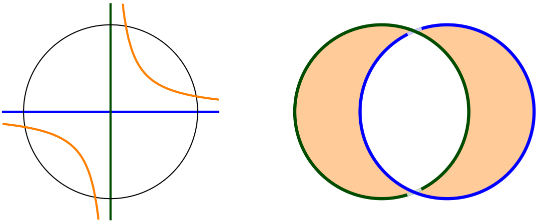


FIGURE: Closed Milnor fiber for $Q(\mathcal{A}) = xy$

Example

- Let $\mathcal{B}_n$ be the Boolean arrangement in $\mathbb{C}^n$. Recall $F = (\mathbb{C}^*)^{n-1}$. Hence, $\overline{F} = T^{n-1} \times D^{n-1}$, and so $\partial\overline{F} = T^{n-1} \times S^{n-2}$.
- Let $\mathcal{A}$ be a near-pencil of n planes in $\mathbb{C}^3$. Then $\partial\overline{F} = S^1 \times \Sigma_{n-2}$.

- The Hopf fibration $\pi\colon \mathbb{C}^d \backslash \{0\} \to \mathbb{CP}^{d-1}$ restricts to regular, cyclic n-fold covers, $\pi\colon \overline{F} \to \overline{U}$ and $\pi\colon \partial\overline{F} \to \partial\overline{U}$, which fit into the ladder

$$
\begin{array}{ccccccccc}
\mathbb{Z}_n & = & \mathbb{Z}_n & = & \mathbb{Z}_n & \longrightarrow & \mathbb{C}^* & = & \mathbb{C}^* \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\partial\overline{F} & \longrightarrow & \overline{F} & \xrightarrow{\simeq} & F & \longrightarrow & M & \longrightarrow & \mathbb{C}^d\backslash\{0\} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\partial\overline{U} & \longrightarrow & \overline{U} & \xrightarrow{\simeq} & U & = & U & \longrightarrow & \mathbb{CP}^{d-1}
\end{array}
$$