Singularities through the lens of characteristic classes

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Lecture plan

♣ **Lecture I**:
Introduction to characteristic classes for singular varieties

♣ **Lecture 2**:
Characteristic classes of hypersurfaces via specialization

♣ **Lecture 3**:
Spectral classes. Applications to rational and du Bois singularities

Lecture I.

Introduction to characteristic classes for singular varieties

Motivation. Overview

Manifolds: genera and characteristic classes

Definition

A multiplicative genus ϕ is a ring homomorphism

$$\phi : \Omega_*^G \to R,$$

where

- Ω_*^G=cobordism ring of closed ($G = O$) and oriented ($G = SO$) or stably almost complex manifolds ($G = U$).
- R=commutative, unital $\mathbb{Q}$-algebra.

♣ Here we focus on $G = U$.

♣ Hirzebruch: There is a one-to-one correspondence between:

- genera $\phi_f : \Omega_*^U \to R$;
- normalized power series f in the variable c^1;
- normalized and multiplicative cohomology characteristic classes cl_f^* over a finite-dim. base space X,

$$cl_f^* : (K(X), \oplus) \to (H^*(X) \otimes R, \cup)$$

(with $H^*(X) = H^{2*}(X; \mathbb{Z})$, and $K(X)$ the Grothendieck group of $\mathbb{C}$-vector bundles on X), s.t.

$$cl_f^*(L) = f(c^1(L)), \text{ if } L \text{ is a complex line bundle}$$

♣ Given f a normalized ($f(0) = 1$) power series as above, with corresponding class cl_f^*, the associated genus ϕ_f is defined by:

$$\phi_f(X) = deg(cl_f^*(X)) := \langle cl_f^*(T_X), [X] \rangle =: \int_X cl_f^*(T_X) \cap [X]$$

♣ Every multiplicative genus is completely determined by its values on all complex projective spaces, since:

- Milnor: $\Omega_*^U \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^1, \mathbb{CP}^2, \mathbb{CP}^3, \cdots]$

Example: Hirzebruch's χ_y-genus

Hirzebruch χ_y-genus of a complex manifold X:

$$\chi_y(X) := \sum_j \chi(X, \Omega_X^j) y^j$$

- genus $\chi_y : \Omega_*^U \to \mathbb{Q}[y]$, with $\chi_y(\mathbb{CP}^n) = \sum_{i=0}^n (-y)^i$.
- χ_y comes from the power series in $z = c^1$:

$$f_y(z) = \frac{z(1+y)}{1 - e^{-z(1+y)}} - zy \in \mathbb{Q}[y][[z]]$$

- associated characteristic class: Hirzebruch class $\widehat{T}_y^*$
- correspondence: generalized Hirzebruch-Riemann-Roch:

$$\chi_y(X) = \langle \widehat{T}_y^*(T_X), [X] \rangle \qquad \text{(g-HRR)}$$

- $y = 0$: arithmetic genus, Todd class, and Riemann-Roch.

♣ The value $\phi(X)$ of a genus $\phi : \Omega_*^G \to \mathbb{Q}$ on a closed manifold X is called a characteristic number of X. Characteristic numbers are used to classify manifolds up to cobordism, e.g.,

- Milnor-Novikov: Two closed stably almost complex manifolds are cobordant $\iff$ all their Chern numbers are the same.

Remark

Singular spaces do not usually have tangent bundles, so cohomology characteristic classes and genera cannot be defined as in the manifold case. Instead, one works with homology characteristic classes defined via suitable natural transformations.

Functorial characteristic classes for singular varieties

♣ A functorial characteristic class theory of singular complex algebraic varieties is a covariant transformation

$$cl_* : A(-) \to H_*(-) \otimes R,$$

with $A(-)$ a covariant theory depending on cl_*, and $H_*(-) = H_*^{BM}(-;\mathbb{Z})$.
♣ For any X, there is a distinguished element $\alpha_X \in A(X)$.
♣ the characteristic class of the singular space X is:

$$cl_*(X) := cl_*(\alpha_X)$$

♣ cl_* satisfies the normalization property: if X is smooth and $cl^*(T_X)$ is the corresponding cohomology class of X, then:

$$cl_*(\alpha_X) = cl^*(T_X) \cap [X] \in H_*(X) \otimes R$$

♣ A characteristic number of a *compact* singular variety X is defined by:

$$\#(X) := deg(cl_*(\alpha_X)) := const_*(cl_*(\alpha_X))$$

for $const : X \to point$ the constant map.

♣ If X is smooth, get by normalization that

$$\#(X) = \langle cl^*(T_X), [X] \rangle,$$

so $\#(X)$ is a singular extension of the notion of characteristic numbers of manifolds.

Example (Euler characteristic)

The topological Euler characteristic

$$\chi(X) := \sum_i (-1)^i b_i(X)$$

is a characteristic number via the singular version of Gauss-Bonnet-Chern theorem:

$$\chi(X) := deg(c_*^{SM}(X)),$$

with $c_*^{SM}(X) := c_*(1_X)$ the CSM class of X, for

$$c_* : CF(X) \to H_*(X)$$

the MacPherson-Chern class transformation on X, defined on the group $CF(X)$ of constructible functions on X.

Example (Arithmetic genus)

The arithmetic genus of a compact complex algebraic variety,

$$\chi_a(X) := \chi(X, \mathcal{O}_X) = \sum_i (-1)^i \dim_{\mathbb{C}} H^i(X; \mathcal{O}_X)$$

is a characteristic number via the singular Riemann-Roch:

$$\chi_a(X) := deg(td_*([\mathcal{O}_X])),$$

for

$$td_* : K_0(X) := K_0(Coh(X)) \to H_*(X) \otimes \mathbb{Q}$$

the Baum-Fulton-MacPherson Todd class transformation.

Example (Hirzebruch polynomial)

$$\chi_y(X) := \sum_{i,p} (-1)^i \dim_{\mathbb{C}} Gr_F^p H^i(X;\mathbb{C}) \cdot (-y)^p$$

is a characteristic number of X via the singular (g-HRR):

$$\chi_y(X) = deg(T_{y*}([\mathbb{Q}_X^H])) = deg(\widehat{T}_{y*}([\mathbb{Q}_X^H])),$$

for

$$T_{y*}, \widehat{T}_{y*} : K_0(\mathrm{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}]$$

the Brasselet-Schürmann-Yokura Hirzebruch class transformations, defined on the Grothendieck group of mixed Hodge modules on X.

Hirzebruch class transformations

Mixed Hodge modules. Examples

♣ X - complex algebraic variety.
♣ MHM(X)= *algebraic mixed Hodge modules on X*
♣ If $X = pt$ is a point, then

$$\mathrm{MHM}(pt) = \mathrm{MHS}^p = \text{ (polarizable) } \mathbb{Q}\mathrm{MHS}$$

♣ If X is *smooth*, then $\mathrm{MHM}(X) \ni M = ((\mathcal{M}, F, W), (K, W))$, with

- $(\mathcal{M}, F)$ a regular holonomic filtered (left) $\mathcal{D}_X$-module, with F a *good* filtration.
- K a perverse sheaf
- isomorphism $\alpha : \mathrm{DR}(\mathcal{M})^{an} \simeq K \otimes_{\mathbb{Q}_X} \mathbb{C}_X$ compatible with W.

♣ If X is *singular*, use suitable local embeddings into manifolds and filtered $\mathcal{D}$-modules supported on X.

Basic example: *(good) variations of MHS*

♣ X - complex algebraic *manifold* of pure complex dimension n.

♣ (L, F, W) – *good* (i.e., admissible, with quasi-unipotent monodromy at infinity) variation of $\mathbb{Q}$-MHS on X.

♣ $(\mathcal{L} := L \otimes_{\mathbb{Q}_X} \mathcal{O}_X, \nabla)$ is a holonomic (left) $\mathcal{D}_X$-module.

♣ Hodge filtration F on L induces by Griffiths' transversality a good filtration $F_p\mathcal{L} := F^{-p}\mathcal{L}$ on $\mathcal{L}$ as a filtered $\mathcal{D}_X$-module.

♣ Perverse sheaf: $L[n]$.

♣ $\alpha : \mathrm{DR}(\mathcal{L})^{an} \simeq L[n]$, with shifted de Rham complex

$$\mathrm{DR}(\mathcal{L}) := [\mathcal{L} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{L} \otimes_{\mathcal{O}_X} \Omega^n_X]$$

with $\mathcal{L}$ in degree $-n$.

♣ α is compatible with the induced filtration W defined by $W^i(L[n]) := W^{i-n}L[n]$ and $W^i(\mathcal{L}) := (W^{i-n}L) \otimes_{\mathbb{Q}_X} \mathcal{O}_X$

♣ This data defines a mixed Hodge module $L^H[n]$ on X.

Grothendieck groups of MHM

♣ $K_0(\mathrm{MHM}(X)) \simeq K_0(D^b\mathrm{MHM}(X))$ – Grothendieck group of (complexes of) MHM on X

♣ $K_0(\mathrm{MHM}(X))$ is generated by $f_*[j_*L^H]$ (or, alternatively, by $f_*[j_!L^H]$), with:

- $f : Y \to X$ a proper morphism from a complex algebraic manifold Y,
- $j : U \hookrightarrow Y$ the inclusion of a Zariski open and dense subset U with complement D a sncd, and
- L a good variation of mixed Hodge structures on U.

(This follows by induction from resolution of singularities and from the existence of a standard distinguished triangle associated to a closed inclusion.)

Filtered de Rham complexes

Theorem (Saito)

For any variety X, there is a functor of triangulated categories

$$Gr_p^F DR : D^b MHM(X) \longrightarrow D^b_{\mathrm{coh}}(X)$$

commuting with proper pushforward, with $Gr_p^F DR(M) = 0$ for almost all p and M fixed.

(a) *If X is a (pure) n-dimensional complex algebraic manifold, and $M \in MHM(X)$, then $Gr_p^F DR(M)$ is the complex associated to the de Rham complex of the underlying algebraic left $\mathcal{D}_X$-module $\mathcal{M}$ with its integrable connection ∇:*

$$DR(\mathcal{M}) = [\mathcal{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X^n]$$

with $\mathcal{M}$ in degree $-n$, filtered by

$$F_p DR(\mathcal{M}) = [F_p \mathcal{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} F_{p+n} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X^n]$$

Theorem (Filtered de Rham complexes, cont'd)

(b) *$\bar{X}$ – smooth partial compactification of the algebraic manifold X with complement D a sncd, and open inclusion $j : X \hookrightarrow \bar{X}$. For a good variation (L, F, W) of MHS on X, $(DR(j_* L^H), F)$ is filtered quasi-isomorphic to the logarithmic de Rham complex*

$$DR_{\log}(\mathcal{L}) := [\overline{\mathcal{L}} \xrightarrow{\overline{\nabla}} \cdots \xrightarrow{\overline{\nabla}} \overline{\mathcal{L}} \otimes_{\mathcal{O}_{\bar{X}}} \Omega^n_{\bar{X}}(\log(D))]$$

with increasing filtration $F_{-p} := F^p$ given by

$$F^p DR_{log}(\mathcal{L}) = [F^p \overline{\mathcal{L}} \xrightarrow{\overline{\nabla}} \cdots \xrightarrow{\overline{\nabla}} F^{p-n} \overline{\mathcal{L}} \otimes_{\mathcal{O}_{\bar{X}}} \Omega^n_{\bar{X}}(\log(D))]$$

where $\overline{\mathcal{L}}$ is the canonical Deligne extension of $\mathcal{L} := L \otimes_{\mathbb{Q}_X} \mathcal{O}_X$. In particular, $Gr^F_{-p} DR(j_* L^H)$ is quasi-isomorphic to

$$Gr^p_F DR_{log}(\mathcal{L}) = [Gr^p_F \overline{\mathcal{L}} \xrightarrow{Gr\, \overline{\nabla}} \cdots \xrightarrow{Gr\, \overline{\nabla}} Gr^{p-n}_F \overline{\mathcal{L}} \otimes_{\mathcal{O}_{\bar{X}}} \Omega^n_{\bar{X}}(\log(D))]$$

(c) *For $(DR(j_! L^H), F)$, consider instead the log de Rham complex associated to the Deligne extension $\overline{\mathcal{L}} \otimes \mathcal{O}_{\bar{X}}(-D)$ of $\mathcal{L}$.*

Hodge–Chern classes

The transformations $Gr_p^F \mathrm{DR}$ induce group homomorphisms

$$Gr_p^F \mathrm{DR} : K_0(\mathrm{MHM}(X)) \longrightarrow K_0(X) \simeq K_0(D^b_{\mathrm{coh}}(X))$$

Definition (Brasselet–Schürmann–Yokura)

♣ The *Hodge–Chern class transformation* of a variety X is:

$$\mathrm{DR}_y : K_0(\mathrm{MHM}(X)) \longrightarrow K_0(X) \otimes \mathbb{Z}[y^{\pm 1}]$$

$$\mathrm{DR}_y([M]) := \sum_{i,p} (-1)^i \left[\mathcal{H}^i Gr^F_{-p} \mathrm{DR}(M)\right] \cdot (-y)^p$$

$$= \sum_{p} \left[Gr^F_{-p} \mathrm{DR}(M)\right] \cdot (-y)^p$$

♣ The *Hodge–Chern class* of a complex algebraic variety X is:

$$\mathrm{DR}_y(X) := \mathrm{DR}_y([\mathbb{Q}^H_X])$$

Hirzebruch classes of mixed Hodge modules

Definition (Brasselet–Schürmann–Yokura)

♣ The *un-normalized Hirzebruch class transformation* is:

$$T_{y*} := td_* \circ \mathrm{DR}_y : K_0(\mathrm{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}]$$

with $td_* : K_0(X) \to H_*(X) \otimes \mathbb{Q}$ the *Todd class transformation* of the *singular (G-R-R) thm* of Baum-Fulton-MacPherson, linearly extended over $\mathbb{Z}[y^{\pm 1}]$, and $H_*(X) := H_{2*}^{BM}(X)$.

♣ The *normalized Hirzebruch class transformation* is:

$$\widehat{T}_{y*} := td_{(1+y)*} \circ \mathrm{DR}_y : K_0(\mathrm{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}\left[y, \tfrac{1}{y(y+1)}\right]$$

where

$$td_{(1+y)*} : K_0(X) \otimes \mathbb{Z}[y^{\pm 1}] \to H_*(X) \otimes \mathbb{Q}\left[y, \tfrac{1}{y(y+1)}\right]$$

is the scalar extension of td_* together with the multiplication by $(1+y)^{-k}$ on the degree k component.

Homology Hirzebruch characteristic classes of varieties

Definition (Brasselet-Schürmann-Yokura)

Homology Hirzebruch characteristic classes of a complex algebraic variety X are defined by evaluating at the (class of the) constant Hodge module $\mathbb{Q}^H_X$:

$$T_{y*}(X) := T_{y*}([\mathbb{Q}^H_X]), \ \widehat{T}_{y*}(X) := \widehat{T}_{y*}([\mathbb{Q}^H_X]) \in H_*(X) \otimes \mathbb{Q}[y].$$

♣ The classes $T_{y*}(X)$ and $\widehat{T}_{y*}(X)$ are "motivic", i.e., they are images of $[id_X]$ under natural transformations (motivic lifts):

$$T_{y*}, \widehat{T}_{y*} : K_0(var/X) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}],$$

where $K_0(var/X)$ is generated by isomorphism classes $[f : Y \to X]$ and the scissor relation.

Properties

♣ The transformations DR_y and (by Riemann-Roch) T_{y*} and $\widehat{T}_{y*}$ *commute with proper pushforward*.

♣

$$\widehat{T}_{y*}([M]) \in H_*(X) \otimes \mathbb{Q}[y^{\pm 1}],$$

and for $y = -1$:

$$\widehat{T}_{-1*}([M]) = c_*([\mathrm{rat}(M)]) \in H_*(X) \otimes \mathbb{Q}$$

is the *MacPherson-Chern class* of the constructible complex $\mathrm{rat}(M)$ (i.e., the MacPherson-Chern class of the *constructible function* defined by taking stalkwise the Euler characteristic).

♣ If X is *Du Bois* (e.g., rational), i.e., the canonical map

$$\mathcal{O}_X \xrightarrow{\sim} Gr_F^0 \mathrm{DR}(\mathbb{Q}_X^H) \in D^b_{\mathrm{coh}}(X)$$

is a quasi-isomorphism (cf. *Saito*), then

$$T_{0*}(X) = \widehat{T}_{0*}(X) = td_*([\mathcal{O}_X]) =: td_*(X)$$

for td_* the Todd class transformation.

Normalization and degree

♣ Normalization: if X is smooth, then

$$\mathrm{DR}_y(X) := \mathrm{DR}_y([\mathbb{Q}_X^H]) = \Lambda_y(T_X^*),$$

where for a vector bundle V on X define its Λ-class by

$$\Lambda_y(V) = \sum_{p\geq 0}[\Lambda^p V]y^p \in K^0(X)[y].$$

$$T_{y*}(X) = T_y^*(T_X) \cap [X]\,, \quad \widehat{T}_{y*}(X) = \widehat{T}_y^*(T_X) \cap [X]$$

with $T_y^*(T_X)$ and $\widehat{T}_y^*(T_X)$ defined by power series

$$Q_y(\alpha) := \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}}\,, \quad \widehat{Q}_y(\alpha) := \frac{\alpha(1+ye^{-\alpha(1+y)})}{1-e^{-\alpha(1+y)}} \in \mathbb{Q}[y][[\alpha]]$$

♣ Degree: If X is compact:

$$\begin{aligned} deg(T_{y*}(X)) = deg(\widehat{T}_{y*}(X)) &= \sum_{j,p}(-1)^j \dim \mathrm{Gr}_F^p H^j(X;\mathbb{C}) \cdot (-y)^p \\ &= \chi_y(X) \end{aligned}$$

Example: $\bar{X}$ – smooth partial compactification of the algebraic manifold X with complement D a sncd, and open inclusion $j : X \hookrightarrow \bar{X}$.

- *Recall*: if (L, F, W) is a good variation of MHS on X, then
$$\left(\mathrm{DR}(j_* L^H), F_{-\bullet}\right) \simeq \left(\mathrm{DR}_{log}(\mathcal{L}), F^{\bullet}\right)$$
with $F_{-p} := F^p$ induced by Griffiths' transversality.
- Define a *cohomological Hodge-Chern class*
$$\mathrm{DR}^y(Rj_* L) := \sum_p [Gr_F^p(\overline{\mathcal{L}})] \cdot (-y)^p \in K^0(\bar{X})[y^{\pm 1}],$$
with $K^0(\bar{X})$= Grothendieck group of algebraic vector bundles
- Get
$$\mathrm{DR}_y([j_* L^H]) = \mathrm{DR}^y(Rj_* L) \cap \left(\Lambda_y\left(\Omega^1_{\bar{X}}(\log(D))\right) \cap [\mathcal{O}_{\bar{X}}]\right).$$
- Similarly, for
$$\mathrm{DR}^y(j_! L) := \sum_p [\mathcal{O}_{\bar{X}}(-D) \otimes Gr_F^p(\overline{\mathcal{L}})] \cdot (-y)^p \in K^0(\bar{X})[y^{\pm 1}],$$
get
$$\mathrm{DR}_y([j_! L^H]) = \mathrm{DR}^y(j_! L) \cap \left(\Lambda_y\left(\Omega^1_{\bar{X}}(\log(D))\right) \cap [\mathcal{O}_{\bar{X}}]\right)$$
- For $j = id : X \to X$, get the *Atiyah-Meyer type formula*:
$$\mathrm{DR}_y([L^H]) = DR^y(L) \cap (\Lambda_y(T^*_X) \cap [\mathcal{O}_X]) \in K_0(X)[y^{\pm 1}]$$

Additivity of Hodge–Chern and Hirzebruch classes

Proposition

For a complex variety X, fix $M \in D^b MHM(X)$ with $K := \mathrm{rat}(M)$. Let $\mathcal{S} = \{S\}$ be a complex algebraic stratification of X so that for any $S \in \mathcal{S}$: S is smooth, $\bar{S} \setminus S$ is a union of strata, and the sheaves $L_{S,\ell} := \mathcal{H}^\ell K|_S$ are local systems on S for any ℓ.

If $j_S : S \overset{i_{S,\bar{S}}}{\hookrightarrow} \bar{S} \overset{i_{\bar{S},X}}{\hookrightarrow} X$ is the inclusion map of a stratum $S \in \mathcal{S}$, then:

$$[M] = \sum_{S,\ell} (-1)^\ell \left[(j_S)_! L^H_{S,\ell}\right] = \sum_{S,\ell} (-1)^\ell \, (i_{\bar{S},X})_* \left[(i_{S,\bar{S}})_! L^H_{S,\ell}\right]$$

In particular,

$$DR_y([M]) = \textstyle\sum_{S,\ell} (-1)^\ell \, (i_{\bar{S},X})_* DR_y \left[(i_{S,\bar{S}})_! L^H_{S,\ell}\right]$$

$$T_{y*}(M) = \textstyle\sum_{S,\ell} (-1)^\ell \, (i_{\bar{S},X})_* T_{y*}\left((i_{S,\bar{S}})_! L^H_{S,\ell}\right).$$

Explicit computation of summands $DR_y\left[(i_{S,\bar{S}})_! L^H_{S,\ell}\right]$

Theorem (M.-Saito-Schürmann)

Let L be a good variation of MHS on a stratum S and $i_{S,Z} : S \hookrightarrow Z$ a smooth partial compactification of S so that $D := Z \setminus S$ is a sncd and $i_{S,\bar{S}} = \pi_Z \circ i_{S,Z}$ for a proper morphism $\pi_Z : Z \to \bar{S}$. Then:

$$DR_y(\left[(i_{S,\bar{S}})_! L^H\right]) = (\pi_Z)_* \left[DR^y((i_{S,Z})_! L^H) \cap \Lambda_y \left(\Omega^1_Z(\log(D)\right)\right].$$

In particular, if $\overline{\mathcal{L}}$ is the canonical Deligne extension on Z of $\mathcal{L} := L \otimes_{\mathbb{Q}_S} \mathcal{O}_S$, then:

$$T_{y*}((i_{S,\bar{S}})_! L^H) = \sum_{p,q} (-1)^q (\pi_Z)_* td_* \left[\mathcal{O}_Z(-D) \otimes Gr^p_F \overline{\mathcal{L}} \otimes \Omega^q_Z(\log D)\right](-y)^{p+q}.$$

Application: Hirzebruch classes of toric varieties

Theorem (M.-Schürmann)

Let X_Σ be the toric variety defined by the fan Σ. For any cone $\sigma \in \Sigma$, with orbit O_σ and inclusion $i_\sigma : O_\sigma \hookrightarrow \overline{O}_\sigma = V_\sigma$, have:

$$DR_y\left(\left[(i_\sigma)_! \mathbb{Q}^H_{O_\sigma}\right]\right) = (1+y)^{\dim(O_\sigma)} \cdot [\omega_{V_\sigma}],$$

where ω_{V_σ} is the canonical sheaf on the toric variety V_σ.

Corollary

Let X_Σ be the toric variety defined by the fan Σ. Then:

$$DR_y(X_\Sigma) = \sum_{\sigma \in \Sigma} (1+y)^{\dim(O_\sigma)} \cdot [\omega_{V_\sigma}].$$

$$T_{y*}(X_\Sigma) = \sum_{\sigma \in \Sigma} (1+y)^{\dim(O_\sigma)} \cdot td_*([\omega_{V_\sigma}]).$$

$$\widehat{T}_{y*}(X_\Sigma) = \sum_{\sigma, k} (1+y)^{\dim(O_\sigma)-k} \cdot td_k([\omega_{V_\sigma}]).$$

Corollary

(*a*) (*Ehler's formula*) *The (rational) MacPherson-Chern class* $c_*(X_\Sigma) := c_*([\mathbb{Q}_{X_\Sigma}])$ *of a toric variety* X_Σ *is computed by:*

$$c_*(X_\Sigma) = \widehat{T}_{-1*}(X_\Sigma) = \sum_{\sigma \in \Sigma} [V_\sigma].$$

(*b*) *The Todd class* $td_*(X_\Sigma)$ *of a toric variety is computed by:*

$$td_*(X_\Sigma) = T_{0*}(X_\Sigma) = \sum_{\sigma \in \Sigma} td_*([\omega_{V_\sigma}]).$$

Application: Weighted lattice point counting

Corollary (*Generalized Pick's formula*)

If X_P is the projective toric variety associated to a full-dimensional lattice polytope $P \subset M_{\mathbb{R}} \cong \mathbb{R}^n$, and $\ell \in \mathbb{Z}_{>0}$ then:

$$\sum_{Q \preceq P} (1+y)^{\dim(Q)} \cdot \#\left(\mathrm{Relint}(\ell Q) \cap M\right) = \int_{X_P} e^{\ell [D_P]} \cap T_{y*}(X_P)$$

$$\overset{n=2, \ell=1}{=} (1+y)^2 \cdot Area(P) + \frac{1-y^2}{2} \#(\partial P \cap M) + \chi_y(P).$$

Remark ($y=0$, *Danilov*)

$$\#(\ell P \cap M) = \sum_{Q \preceq P} \#(\mathrm{Relint}(\ell Q) \cap M) = \int_{X_P} e^{\ell [D_P]} \cap td_*(X_P).$$

♣ Equivariant versions (for a torus action) of Hodge-Chern and Hirzebruch classes have been recently developed, and used e.g., for proving weighted Euler-Maclaurin type formulae for lattice polytopes (Cappell-M.-Schürmann-Shaneson, 2023).

Singularities through the lens of characteristic classes

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Lecture 2.

Characteristic classes of hypersurfaces via specialization

Motivation. Overview

Virtual tangent bundle of a hypersurface

♣ Let $X \stackrel{i}{\hookrightarrow} Y$ be a complex algebraic *hypersurface* (or *lci*) in a complex algebraic manifold Y, with *normal bundle* $N_X Y$.

♣ The *virtual tangent bundle* of X is:

$$T_X^{\mathrm{vir}} := [T_Y|_X] - [N_X Y] \in K^0(X)$$

♣ T_X^{vir} is *independent of the embedding* in Y, so it is a well-defined element in $K^0(X)$, the Grothendieck group of algebraic vector bundles on X.

♣ If X is *smooth*: $T_X^{\mathrm{vir}} = [T_X] \in K^0(X)$.

Characteristic classes

♣ Let R be a commutative ring with unit, and

$$cl^* : \left(K^0(X), \oplus\right) \to (H^*(X) \otimes R, \cup)$$

a *multiplicative characteristic class theory* of complex algebraic vector bundles on X, with $H^*(X) = H^{2*}(X;\mathbb{Z})$.

♣ Associate to a hypersurface (or lci) X an *intrinsic* homology class (i.e., independent of the embedding $X \hookrightarrow Y$):

$$cl_*^{\mathrm{vir}}(X) := cl^*(T_X^{\mathrm{vir}}) \cap [X] \in H_*(X) \otimes R,$$

with $[X] \in H_*(X)$ the fundamental class of X in a suitable homology theory $H_*(X)$ (e.g., $H_{2*}^{BM}(X;\mathbb{Z})$).

♣ Assume $cl_*(-)$ is a *homology characteristic class theory* for complex algebraic varieties, so that if X smooth:

$$cl_*(X) = cl^*(T_X) \cap [X] \quad (\text{normalization})$$

Example

(a) *Chern classes*
$cl^* = c^* =$ *Chern class*, and
$c_* : K_0(D^b_c(X)) \to CF(X) \to H_*(X)$ the functorial *Chern class transformation of MacPherson*, with $c_*(X) := c_*([\mathbb{Q}_X])$.
(Here $CF(X)$ is the group of *constructible functions* on X.)

(b) *Todd classes*
$cl^* = td^* =$ *Todd class*, and
$td_* : K_0(X) \to H_*(X) \otimes \mathbb{Q}$ the Baum-Fulton-MacPherson *Todd class transformation*, with $td_*(X) := td_*([\mathcal{O}_X])$.

(c) *Hirzebruch classes*
$cl^* = \widehat{T}^*_y =$ *Hirzebruch class*, and
$\widehat{T}_{y*} : K_0(\mathrm{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}]$ the normalized homology Hirzebruch class transformation,
with $\widehat{T}_{y*}(X) := \widehat{T}_{y*}([\mathbb{Q}^H_X])$.

♣ If X is *smooth*:

$$cl_*^{\mathrm{vir}}(X) \overset{\mathrm{def}}{:=} cl^*(T_X^{\mathrm{vir}}) \cap [X] \overset{\mathrm{smooth}}{=} cl^*(T_X) \cap [X] \overset{norm}{=} cl_*(X) \,.$$

♣ If X is *singular*, the difference

$$\mathcal{M}cl_*(X) := cl_*^{\mathrm{vir}}(X) - cl_*(X)$$

depends in general on the singularities of X.

♣ If $i : X_{\mathrm{sing}} \hookrightarrow X$, get:

$$\mathcal{M}cl_*(X) \in \mathrm{Image}(i_*)$$

so $\mathcal{M}cl_*(X)$ *measures the complexity of singularities of* X.

♣ **Corollary**: $cl_k^{\mathrm{vir}}(X) = cl_k(X) \in H_k(X) \otimes R$, for $k > \dim X_{\mathrm{sing}}$.

Problem: *Describe $\mathcal{M}cl_*(X)$ in terms of the geometry of the singular locus $X_{\rm sing}$ of X.*

Upshot: Compute the (very) *complicated* "actual" homology class $cl_*(X)$ in terms of the simpler (cohomological) virtual class and invariants of the singularities of X.

Byproduct: Same method applies to optimization for the study, e.g., of the *Euclidean distance degree defect*.

Globally defined hypersurfaces: nearby & vanishing cycles, Verdier specialization

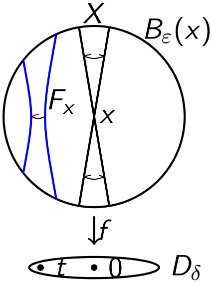
Milnor fibration, nearby/vanishing cycles

♣ $X = f^{-1}(0)$, $f\colon Y \to \mathbb{C}$ regular, Y smooth, $\dim Y = n+1$

♣ For $x \in X_{\rm sing}$ and $0 < \delta \ll \epsilon$, there is a *Milnor fibration*:

$$B_\epsilon(x) \cap f^{-1}(D^*_\delta) \xrightarrow{f} D^*_\delta,$$

whose *Milnor fiber* F_x is a local smoothing of X near x.



♣ If $x \in X_{\rm sing}$ is isolated, then $F_x \simeq \bigvee_{\mu_x} S^n$, with μ_x the *Milnor number* of f at x; the S^n's are called *vanishing cycles* at x

♣ Deligne: nearby & vanishing cycle functors $\psi_f, \varphi_f : D^b_c(Y) \to D^b_c(X)$, with a monodromy action T, s.t.

$$\mathcal{H}^k(\psi_f \mathbb{Q}_Y)_x \simeq H^k(F_x;\mathbb{Q}) \ , \ \ \mathcal{H}^k(\varphi_f \mathbb{Q}_Y)_x \simeq \widetilde{H}^k(F_x;\mathbb{Q})$$

♣ If $x \in X_{\rm reg}$, then F_x is contractible, so $\mathrm{Supp}(\varphi_f \mathbb{Q}_Y) \subseteq X_{\rm sing}$.

Verdier specialization for globally defined hypersurfaces

♣ Let Y be a smooth complex algebraic variety, and $f : Y \to \mathbb{C}$ an algebraic function, with $X := \{f = 0\}$ of codimension one.

♣ Let $X \overset{i}{\hookrightarrow} Y$, so $N_X Y$ is a trivial line bundle.

♣ Let $\psi_f, \varphi_f : D^b_c(Y) \to D^b_c(X)$ be Deligne's *nearby* and resp. *vanishing cycle* functors.

Theorem (Verdier)

(a)

$$td_* \circ i^!_K = i^! \circ td_* : K_0(Y) \to H_{*-1}(X) \otimes \mathbb{Q}$$

with $i^!_K : K_0(Y) \to K_0(X)$ *induced from* Li^**, and* $i^! : H_*(Y) \to H_{*-1}(X)$ *the corresponding Gysin morphisms.*

(b)

$$c_* \circ \psi_f = i^! \circ c_* : K_0(D^b_c(Y)) \to H_{*-1}(X)$$

Corollary (of Verdier specialization)

(a) $\mathcal{M}td_*(X) := td_*^{\mathrm{vir}}(X) - td_*(X) = 0$

(b) $c_*^{\mathrm{vir}}(X) = c_*(\psi_f(\mathbb{Q}_Y))$ *hence the Milnor class*

$$\mathcal{M}_*(X) := \mathcal{M}c_*(X) := c_*^{\mathrm{vir}}(X) - c_*(X)$$

is given by

$$\mathcal{M}_*(X) = c_*(\varphi_f(\mathbb{Q}_Y))$$

with $c_*(\varphi_f(\mathbb{Q}_Y)) \in H_*(X_{\mathrm{sing}})$.

Example (Reason for terminology)

If $X = f^{-1}(0)$ has only *isolated* singularities, then

$$\mathcal{M}_*(X) = \sum_{x \in X_{\mathrm{sing}}} (-1)^n \mu_x,$$

for μ_x the *Milnor number* of the IHS $(X, x) \subset (\mathbb{C}^{n+1}, 0)$.

Hirzebruch-Milnor classes via specialization

Specialization of Hodge-Chern and Hirzebruch classes

♣ ψ_f, φ_f admit lifts $\psi_f^H[1], \varphi_f^H[1]$ to Saito's mixed Hodge modules.

♣ Schürmann: proved the counterpart of Verdier's specialization for DR_y and $\widehat{T}_{y*}$.

Motivating example

♣ Let $i : X := \{f = 0\} \hookrightarrow Y$ be a *smooth* hypersurface inclusion, and L a *good variation of MHS* on Y.

♣ *Atiyah-Meyer*: $\mathrm{DR}_y([L^H]) = \mathrm{DR}^y(L) \cap (\Lambda_y(T^*_Y) \cap [\mathcal{O}_Y])$

♣ Using the multiplicativity of $\Lambda_y(-)$ and triviality of $N^*_X Y$, get

$$\begin{aligned} i^! \mathrm{DR}_y([L^H]) &= i^* \left(\mathrm{DR}^y(L) \cup \Lambda_y(T^*_Y)\right) \cap i^!([\mathcal{O}_Y]) \\ &= \left(\mathrm{DR}^y(i^* L) \cup \Lambda_y(i^* T^*_Y)\right) \cap [\mathcal{O}_X] \\ &= \Lambda_y(N^*_X Y) \cap \mathrm{DR}_y([i^* L^H]) \\ &= (1+y) \cdot \mathrm{DR}_y(i^*[L^H]) \\ &= -(1+y) \cdot \mathrm{DR}_y([\psi^H_f(L^H)]) \end{aligned}$$

♣ This identity holds for a singular hypersurface X and any $M \in \mathrm{MHM}(Y)$!

Specialization of Hirzebruch classes

Theorem (Schürmann)

Let Y be a smooth complex algebraic variety, and $f : Y \to \mathbb{C}$ an algebraic function, with $X := \{f = 0\} \overset{i}{\hookrightarrow} Y$ of codimension one. Then

(a)
$$-(1+y)\cdot DR_y(\ \psi_f^H(-)\ ) = i^! DR_y(-)$$

as transformations $K_0(MHM(Y)) \to K_0(X)[y^{\pm 1}]$.

(b)
$$-\widehat{T}_{y*}(\psi_f^H(-)) = i^! \, \widehat{T}_{y*}(-)$$

as transformations $K_0(MHM(Y)) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}]$.

Corollary (Cappell–M.–Schürmann–Shaneson)

1. $\widehat{T}^{\mathrm{vir}}_{y*}(X) := \widehat{T}^*_y(T^{\mathrm{vir}}_X) \cap [X] = -\widehat{T}_{y*}(\psi^H_f([\mathbb{Q}^H_Y]))$
2. $\mathcal{M}\widehat{T}_{y*}(X) := \widehat{T}^{\mathrm{vir}}_{y*}(X) - \widehat{T}_{y*}(X) = -\widehat{T}_{y*}(\varphi^H_f([\mathbb{Q}^H_Y]))$

Definition (Cappell-M.-Schürmann-Shaneson)

The class

$$\mathcal{M}\widehat{T}_{y*}(X) := \widehat{T}^{\mathrm{vir}}_{y*}(X) - \widehat{T}_{y*}(X)$$

is called the (normalized) *Hirzebruch-Milnor class* of X.

♣ Degree: If $X = \{f = 0\}$, with f proper, then:

$$deg\left(\mathcal{M}\widehat{T}_{y*}(X)\right) = \chi_y(X_t) - \chi_y(X),$$

for X_t the generic (smooth) fiber of f.

Example (Isolated singularities)

If the n-dimensional hypersurface X has *only isolated singularities*, then

$$\mathcal{M}\widehat{T}_{y*}(X) = \sum_{x \in X_{\rm sing}} (-1)^n \chi_y([\widetilde{H}^n(F_x;\mathbb{Q})])$$
$$= (-1)^n \sum_{x \in X_{\rm sing}} \sum_p \dim_{\mathbb{C}} Gr_F^p \widetilde{H}^n(F_x;\mathbb{C}) \cdot (-y)^p,$$

where F_x is the Milnor fiber of the IHS (X, x).

Example (Smooth singular locus)

Assume X has a connected *smooth singular locus* $\Sigma = X_{\mathrm{sing}}$, with $r = \dim_{\mathbb{C}} \Sigma < n$, and s.t. $X \supset \Sigma$ is a Whitney stratification of X. Let F_x be the Milnor fiber at $x \in \Sigma$. Then, in $K_0(\mathrm{MHM}(X))$:

$$\varphi_f^H[1]([\mathbb{Q}_Y^H]) = (-1)^{n-r} \cdot [L_\Sigma^H],$$

for L_Σ the variation of $\mathbb{Q}$-MHS (on Σ) with $(L_\Sigma)_x \simeq H^{n-r}(F_x; \mathbb{Q})$. So:

$$\mathcal{M}\widehat{T}_{y*}(X) = (-1)^{n-r} \cdot \widehat{T}_{y*}(\Sigma; L_\Sigma),$$

with $\widehat{T}_{y*}(\Sigma; L_\Sigma) := \widehat{T}_{y*}([L_\Sigma^H])$, for L_Σ^H the MHM defined by L_Σ.

Remark

The (twisted) Hirzebruch class $\widehat{T}_{y*}(\Sigma; L_\Sigma)$ is computed by the Atiyah-Meyer type formula. In particular, if $\pi_1(\Sigma) = 0$, get:

$$\mathcal{M}\widehat{T}_{y*}(X) = (-1)^{n-r} \cdot \chi_y([\widetilde{H}^{n-r}(F_x; \mathbb{Q})]) \cdot \widehat{T}_{y*}(\Sigma).$$

Example

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial function so that

1. f depends only on the first $n - r + 1$ coordinates $x_1, \cdots, x_{n-r+1}$ of $\mathbb{C}^{n+1}$.
2. f has an isolated singularity at $0 \in \mathbb{C}^{n-r+1}$ when regarded as defined on $\mathbb{C}^{n-r+1}$.

Set $X := f^{-1}(0) \subset \mathbb{C}^{n+1}$, hence $\Sigma = X_{\mathrm{sing}} = \mathbb{C}^r$ and $X \supset \Sigma$ is a Whitney stratification. Then:

$$\mathcal{M}\widehat{T}_{y*}(X) = (-1)^{n-r}\chi_y([\widetilde{H}^{n-r}(F_0;\mathbb{Q})]) \cdot [\mathbb{C}^r],$$

with F_0 the Milnor fiber of $f : \mathbb{C}^{n-r+1} \to \mathbb{C}$ at 0.

Theorem (Cappell-M.-Schürmann-Shaneson)

If $\Sigma = X_{\mathrm{sing}}$ has dimension r, then:

$$\mathcal{M}\widehat{T}_{y*}(X) = (-1)^{n-r}\chi_y([\widetilde{H}^{n-r}(F_{N,x};\mathbb{Q})]) \cdot [\Sigma] + \ell.o.t$$

where $F_{N,x}$ is the transversal Milnor fiber at $x \in \Sigma_{\mathrm{reg}}$, i.e., the Milnor fiber of the isolated singularity germ $(X \cap N, x)$ defined (locally in the analytic topology) by restricting f to a normal slice N at a regular point $x \in \Sigma_{\mathrm{reg}}$.

Theorem (Cappell-M.-S.-Shaneson)

Let $X = \{f = 0\} \subset Y$, for $f : Y \to \mathbb{C}$ an algebraic function on a complex algebraic manifold Y. Let $\mathcal{S}_0$ be a partition of the singular locus X_{sing} into disjoint locally closed algebraic submanifolds S, such that the restrictions $\varphi_f(\mathbb{Q}_Y)|_S$ have constant cohomology sheaves (e.g., these are locally constant sheaves on each S, and the pieces S are simply-connected). For each $S \in \mathcal{S}_0$, let F_s be the Milnor fiber of a point $s \in S$. Then:

$$\mathcal{M}\widehat{T}_{y*}(X) = \sum_{S \in \mathcal{S}_0} \underbrace{\left(\widehat{T}_{y*}(\bar{S}) - \widehat{T}_{y*}(\bar{S} \setminus S)\right)}_{\text{horizontal info}} \cdot \underbrace{\chi_y([\widetilde{H}^*(F_s;\mathbb{Q})])}_{\text{vertical info}}$$

Example

In particular, the theorem applies to the Hilbert scheme

$$(\mathbb{C}^3)^{[4]} = \{df_4 = 0\} \subset Y_4,$$

which has an "adapted" partition with all strata simply-connected. The vanishing cycle module corresponding to $f_4 : Y_4 \to \mathbb{C}$ and its Hodge polynomial were computed by Dimca-Szendröi.

The case $y = -1$

♣ $\mathcal{M}\widehat{T}_{y*}(X)|_{y=-1} = \mathcal{M}_*(X) \otimes \mathbb{Q} = c_*(\varphi_f(1_Y))$. with $\varphi_f : CF(Y) \to CF(X)$ the *motivic vanishing cycle functor*.

♣ Hence, for $y = -1$, the previous theorem holds without any monodromy assumptions along strata.

♣ Recall: $(\mathbb{C}^3)^{[m]} = \{df_m = 0\}$, with $f_m : Y_m \to \mathbb{C}$.
Here, $\varphi_{f_m}(1_{Y_m})$ is the Behrend function, whose Euler characteristic over $(\mathbb{C}^3)^{[m]}$ computes the corresponding Donaldson-Thomas invariant.

The case $y = 0$

♣ If X has only Du Bois (e.g., rational) singularities, then:

1. $\widehat{T}_{y*}(X)|_{y=0} = T_{y*}(X)|_{y=0} = td_*(X)$.
2. Hence: $\mathcal{M}\widehat{T}_{y*}(X)|_{y=0} = \mathcal{M}td_*(X) = 0$, a class version of Dolgachev-Steenbrink cohomological insignificance.

Theorem (M.-Saito-Schürmann)

Assume X_{sing} is projective. Then:

$$\mathcal{M}\widehat{T}_{y*}(X)|_{y=0} = 0 \implies X \text{ has only Du Bois singularities.}$$

Corollary (Ishii)

If X has only isolated singularities, then:

$$X \text{ is Du Bois} \iff \dim_{\mathbb{C}} Gr_F^0 \widetilde{H}^n(F_x;\mathbb{C}) = 0 \text{ for all } x \in X_{\mathrm{sing}}.$$

Saito, Schwede: X is Du Bois $\iff$ X is log canonical ($lct(f) = 1$)

Hirzebruch-Milnor classes of very ample divisors

♣ Let Y be a complex projective manifold and X a (possibly singular) hypersurface on Y which is a very ample divisor.

♣ Let X' be a general hyperplane section of Y in the linear system $|X|$.

♣ Define a one-parameter family: $\mathcal{X} := \bigsqcup_{t \in \mathbb{CP}^1} X_t$, with $X_0 = X$ and $X_\infty = X'$, and let $\pi : \mathcal{X} \to \mathbb{CP}^1$ be the projection.

♣ By definition, X and X' are defined by sections s and s' of the same line bundle.

♣ Let $f := s/s' : Y \setminus X' \subset \mathcal{X} \to \mathbb{C}$, with $f^{-1}(0) = X \setminus X'$.

♣ Note: $f = \pi^* t$, with t the affine coordinate of $\mathbb{C} \subset \mathbb{CP}^1$.

♣ Key point: $(\varphi_{\pi^* t} \mathbb{Q}_{\mathcal{X}})|_{X \cap X'} = 0$.

♣ Adapt the previous results to $f : Y \setminus X' \to \mathbb{C}$, and get a description of $\mathcal{M}\widehat{T}_{y*}(X)$ in terms of the vanishing cycles restricted to the complement of the generic hyperplane section X'.

Parusinski–Pragacz Euler characteristic formula

Corollary (Parusinski-Pragacz, M.-Saito-Schürmann)

Let L be a very ample line bundle over the complex projective manifold Y. Assume that the hypersurface X in Y is the zero set of a holomorphic section $s \in H^0(Y; L)$, and let $s' \in H^0(Y; L)$ be a section of L so that its zero set X' is nonsingular and transverse to a Whitney stratification $\mathcal{S}$ of X. Then

$$\chi(X) = \chi(X') - \sum_{S \in \mathcal{S}} \chi(S \setminus X') \cdot \mu_S,$$

where

$$\mu_S := \chi\left(\widetilde{H}^*(F_{x_S}; \mathbb{Q})\right)$$

is the Euler characteristic of the reduced cohomology of the Milnor fiber F_{x_S} of X at some (any) point $x_S \in S$.

Singularities through the lens of characteristic classes

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Lecture 3.

Spectral classes. Applications to rational and du Bois singularities

(Higher) rational and du Bois singularities

Two generalizations of the De Rham complex

♣ For X a reduced complex algebraic variety, there are two generalizations of the classical De Rham complex:

- De Rham complex $(\Omega^\bullet_X, F)$ of Kähler differentials;
- Du Bois complex $(\underline{\Omega}^\bullet_X, F)$.

♣ There is a natural morphism of filtered complexes

$$(\Omega^\bullet_X, F) \to (\underline{\Omega}^\bullet_X, F),$$

which is a filtered quasi-isomorphism if X is smooth.

Higher Du Bois singularities

- For $p \geq 0$, set

$$\underline{\underline{\Omega}}_X^p := gr_F^p(\underline{\Omega}_X^\bullet)[p] \in D^b_{\mathrm{coh}}(X)$$

- E.g., if X is smooth, then $\underline{\underline{\Omega}}_X^p \simeq \Omega_X^p$.
- E.g., if X has only quotient or toroidal singularities, then $\underline{\underline{\Omega}}_X^p \simeq \widehat{\Omega}_X^p := j_*\Omega^p_{X_{reg}}$, the p-th Zariski sheaf (for $j\colon X_{reg} \hookrightarrow X$).

Definition (Jung-Kim-Saito-Yoon, Mustață-Olano-Popa-Witaszek)

For $k \geq 0$, X has k-Du Bois singularities if the induced morphism

$$\Omega_X^p \to \underline{\underline{\Omega}}_X^p$$

is an isomorphism in $D^b_{\mathrm{coh}}(X)$ for all $0 \leq p \leq k$.

Remark

When $k = 0$, this recovers the usual notion of Du Bois singularities.

Higher rational singularities

♣ Higher versions of rational singularities were introduced by Friedman-Laza:

Definition (Friedman-Laza)

(1) Assume X is irreducible, with $\mu : (\widetilde{X}, D) \to (X, X_{\mathrm{sing}})$ a log resolution of singularities. Say that X has *k-rational singularities* if the natural morphism

$$\Omega^p_X \to R\mu_*\Omega^p_{\widetilde{X}}(\log D)$$

is an isomorphism for all $0 \leq p \leq k$.
(2) An arbitrary variety X has *k-rational singularities* if all its connected components are irreducible, with k-rational singularities.

Remark

When $k = 0$, this recovers the usual notion of rational singularities.

Relations between higher Du Bois and higher rational

Theorem

Assume X is a hypersurface in a smooth variety Y. Then:

(a) *(Mustață-Popa, Friedman-Laza)*

$$X \text{ is } k\text{-rational} \Longrightarrow X \text{ is } k\text{-Du Bois}$$

(b) *(Mustață-Popa)*

$$X \text{ is } k\text{-Du Bois} \Longrightarrow X \text{ is } (k-1)\text{-rational}$$

Remark

This result is also of consequence of what I will talk about today. (The theorem is also true for X a locally complete intersection.)

Remark

I focus here on work with Yang on globally defined hypersurfaces $X = f^{-1}(0)$ in a smooth variety Y, with $f : Y \to \mathbb{C}$ non-constant. Many of our results hold for arbitrary hypersurfaces (Saito).

Spectral characteristic classes

Spectral Hirzebruch classes

- X complex algebraic variety
- MHM(X)= mixed Hodge modules on X
- $H_i(X)$ is either $H_{2i}^{BM}(X;\mathbb{Q})$ or $CH_i(X)_{\mathbb{Q}}$
- $K_0(X) := K_0(Coh(X))$
- *Spectral Hirzebruch class transformation* (M.-Saito-Schürmann):

$$T^{sp}_{t*} : K_0^{mon}(\mathrm{MHM}(X)) \to \bigcup_{e \geq 1} H_*(X)[t^{\pm \frac{1}{e}}],$$

where $K_0^{mon}(\mathrm{MHM}(X))$ is the Grothendieck group of mixed Hodge modules on X with a finite order automorphism.

- This is a characteristic class version of the Hodge spectrum (Varchenko, Steenbrink, Saito).

♣ If X is *smooth*, and $(M, T_s) \in \mathrm{MHM}(X)$ with $T_s^e = id$ and underlying filtered (left) $\mathcal{D}_X$-module $(\mathcal{M}, F)$, have

$$(\mathcal{M}, F) = \sum_{\lambda^e=1} (\mathcal{M}_\lambda, F),$$

s.t. $T_s = \lambda \cdot Id$ on $\mathcal{M}_\lambda$. Set

$$T_{t*}^{sp}[M, T_s] := \sum_{p,\lambda} td_* \left([Gr_{-p}^F \mathrm{DR}(\mathcal{M}_\lambda)] \right) \cdot t^{p+\ell(\lambda)} \in H_*(X)[t^{\pm 1/e}]$$

where $td_* : K_0(X) \to H_*(X)$ is the Todd class transformation, $\mathrm{DR}(\mathcal{M}_\lambda)$ is the De Rham complex with its induced filtration (e.g., $\mathrm{DR}(\mathcal{O}_X) = \Omega_X^\bullet[\dim X]$ with the stupid filtration), and $\ell(\lambda) \in [0, 1)$ is s.t. $\exp 2\pi i \ell(\lambda) = \lambda$.

♣ If X *singular*, the same definition applies using local embeddings into smooth varieties, and extend to complexes $M^\bullet \in D^b\mathrm{MHM}(X)$ by applying the above to each $H^i(M^\bullet)$.

Vanishing cycles

♣ $X = f^{-1}(0)$, $f\colon Y \to \mathbb{C}$ regular, Y smooth, $\dim X = n$
♣ Deligne: vanishing cycle functor $\varphi_f : D^b_c(Y) \to D^b_c(X)$, endowed with a monodromy action T, s.t.

$$\mathcal{H}^k(\varphi_f \mathbb{Q}_Y)_x \simeq \widetilde{H}^k(F_x; \mathbb{Q}),$$

with F_x the Milnor fiber of f at $x \in X_{\rm sing}$.
♣ φ_f lifts (up to a shift) to mixed Hodge modules

$$\varphi^H_f := \varphi_f[-1] : \mathrm{MHM}(Y) \to \mathrm{MHM}(X)$$

with $\varphi^H_f = \varphi^H_{f,1} \oplus \varphi^H_{f,\neq 1}$.

Spectral Hirzebruch-Milnor classes of hypersurfaces

♣ $X = f^{-1}(0)$, $f\colon Y \to \mathbb{C}$ regular, Y smooth, $\dim X = n$

Definition (M.-Saito-Schürmann)

The spectral Hirzebruch-Milnor class of X is defined as:

$$\mathcal{M}T^{sp}_{t*}(X) := T^{sp}_{t*}([\varphi_f \mathbb{Q}^H_Y, T_s]) \in H_*(X_{\mathrm{sing}})[t^{1/ord(T_s)}]$$

with T_s be the semisimple part of the monodromy action.

Remark

$\mathcal{M}T^{sp}_{t*}(X)$ detects Du Bois/rational singularities, as well as the jumping coefficients of the multiplier ideals $\mathcal{J}(\alpha X)$ of X.

Let $\mathcal{M}T^{sp}_{t*}(X)|_{t^\alpha} \in H_*(X_{\mathrm{sing}})$ be the *coefficient of t^α* in $\mathcal{M}T^{sp}_{t*}(X)$. So

$$\mathcal{M}T_{y*}(X)|_{y=0} = \bigoplus_{\alpha \in \mathbb{Q}\cap(0,1)} \mathcal{M}T^{sp}_{t*}(X)|_{t^\alpha} \in H_*(X_{\mathrm{sing}})$$

Theorem (M.-Saito-Schürmann)

X has Du Bois singularities $\Longrightarrow \mathcal{M}T^{sp}_{t*}(X)|_{t^\alpha} = 0$ for $\alpha < 1$.

The converse holds if X_{sing} is projective.

Corollary (Ishii)

If X has only isolated singularities, then:

$$X \text{ is Du Bois} \iff \dim Gr^0_F H^{n-1}(F_x;\mathbb{C}) = 0 \text{ for all } x \in X_{\mathrm{sing}}.$$

Saito, Schwede: X is Du Bois $\iff$ X is log canonical $(lct(f) = 1)$

Higher Du Bois/rational singularities via characteristic classes

♣ $X = f^{-1}(0)$, $f\colon Y \to \mathbb{C}$ regular, Y smooth, $\dim Y = n$

Theorem (M.-Yang, Saito)

X is k-Du Bois $\Longrightarrow \mathcal{M}T^{sp}_{t*}(X)|_{t^\alpha} = 0$ for $\alpha < k+1$.

X is k-rational $\Longrightarrow \mathcal{M}T^{sp}_{t*}(X)|_{t^\alpha} = 0$ for $\alpha \leq k+1$.

The converse implications are true if $X_{\rm sing}$ is projective.

This is a consequence of describing k-Du Bois/k-rational singularities via the Hodge filtration on $\varphi^H_f \mathbb{Q}^H_Y[n] \in \mathrm{MHM}(X)$.

Theorem (M.-Yang)

$$X \text{ is } k\text{-rational} \iff Gr_p^F \varphi_f^H \mathbb{Q}_Y^H[n] = 0, \quad p \leq k+1.$$

$$X \text{ is } k\text{-Du Bois} \iff \begin{cases} Gr_p^F \varphi_{f,\neq 1}^H \mathbb{Q}_Y^H[n] = 0, & p \leq k+1, \\ Gr_p^F \varphi_{f,1}^H \mathbb{Q}_Y^H[n] = 0, & p \leq k. \end{cases}$$

Corollary

X is k-rational $\Longrightarrow$ *X is k-Du Bois* $\Longrightarrow$ *X is $(k-1)$-rational*

Together with the Thom-Sebastiani theorem for vanishing cycles (Saito, M-Saito-Schürmann), this gives:

Corollary

Let Y_a be smooth, $f_a : Y_a \to \mathbb{C}$ with $X_a = f_a^{-1}(0)$, $a = 1, 2$.
Set $Y = Y_1 \times Y_2$, $X = f^{-1}(0) \subseteq Y$, with $f = f_1 + f_2$ on Y. Then
X_1 is k_1-Du Bois, X_2 is k_2-Du Bois $\Longrightarrow$ X is $(k_1 + k_2 + 1)$-Du Bois
X_1 is k_1-rational, X_2 is k_2-Du Bois $\Longrightarrow$ X is $(k_1 + k_2 + 1)$-rational.

Bernstein-Sato polynomial, minimal exponent

- $X = f^{-1}(0)$, $f\colon Y \to \mathbb{C}$ regular, Y smooth, $\dim Y = n$
- Bernstein-Sato polynomial of f: minimal degree polyn. $b_f(s)$ s.t.

$$b_f(s) \cdot f^s = P(x, \partial_x, s) \cdot f^{s+1}$$

for some $P(x, \partial_x, s) \in D_Y[s]$.

- $b_f(s) = s + 1 \iff X$ is smooth.
- Kashiwara: All roots of b_f are in $\mathbb{Q}_{<0}$.
- Lichtin, Kollár: largest root of b_f is $-lct(f)$.
- minimal exponent $\tilde{\alpha}_f$ of f is the smallest root of $\frac{b_f(-s)}{-s+1}$.
- $lct(f) = \min\{1, \tilde{\alpha}_f\}$, hence: X is Du Bois $\iff \tilde{\alpha}_f \geq 1$
- Saito: X has rational singularities $\iff \tilde{\alpha}_f > 1$
- $\tilde{\alpha}_f \leq \frac{\dim Y}{2}$

Example

Let $f = x_1^{a_1} + \cdots + x_n^{a_n}$, with integer exponents $a_i \geq 2$ for $i = 1, \ldots, n$. Then $\tilde{\alpha}_f = \sum_{i=1}^n \frac{1}{a_i}$.

Example

If X is singular and has quotient or toroidal singularities, then $1 < \tilde{\alpha}_f \leq 2$. Here, the lower bound is due to the fact that these are rational singularities, and the upper bound is sharp (e.g., the minimal exponent for the toric hypersurface $x_1x_2 - x_3x_4 = 0$ in $\mathbb{C}^4$ equals 2).

Sketch of proof

Theorem (Jung-Kim-Saito-Yoon, Friedman-Laza)

- X *has* *k-Du Bois* *singularities* $\iff \tilde{\alpha}_f \geq k+1$.
- X *has* *k-rational* *singularities* $\iff \tilde{\alpha}_f > k+1$.

♣ Let $(\mathcal{B}_f, F) = \mathcal{O}_Y \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ with $Gr_k^F \mathcal{B}_f = \mathcal{O}_Y \otimes \partial_t^k$, $\forall\ k \in \mathbb{N}$.
♣ Kashiwara-Malgrange: the (decreasing) V-filtration $\{V^\alpha\}_{\alpha \in \mathbb{Q}}$ along f satisfies

$$\varphi^H_{f,\neq 1}\mathbb{Q}^H_Y[\dim Y] = \bigoplus_{0<\alpha<1} Gr_V^\alpha \mathcal{B}_f, \quad \varphi^H_{f,1}\mathbb{Q}^H_Y[\dim Y] = Gr_V^0 \mathcal{B}_f,$$

$$F_k\varphi^H_{f,\neq 1}\mathbb{Q}^H_Y[\dim Y] = \bigoplus_{0<\alpha<1} F_{k-1} Gr_V^\alpha \mathcal{B}_f, \;\; F_k\varphi^H_{f,1}\mathbb{Q}^H_Y[\dim Y] = F_k Gr_V^0 \mathcal{B}_f.$$

Theorem (Saito, Schnell-Yang, M.-Yang)

$$\tilde{\alpha}_f \geq k+1 \iff Gr_p^F Gr_V^\alpha \mathcal{B}_f = 0 \text{ for } 0 \leq p \leq k,\ 0 \leq \alpha < 1$$

$$\tilde{\alpha}_f > k+1 \iff Gr_p^F Gr_V^\alpha \mathcal{B}_f = 0 \text{ for } 0 \leq p \leq k,\ 0 \leq \alpha < 1, \text{ and } Gr_{k+1}^F Gr_V^0 \mathcal{B}_f = 0.$$

Higher multiplier ideals and their jumping coefficients

♣ $X := f^{-1}(0)$ reduced hypersurface in a smooth variety Y.

Definition

For $k \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$, the *k*-th multiplier ideal of X is given by

$$\mathcal{J}_k(\alpha X) \otimes \partial_t^k := Gr_k^F V^{>\alpha}\mathcal{B}_f \subseteq Gr_k^F \mathcal{B}_f = \mathcal{O}_Y \otimes \partial_t^k.$$

♣ The higher multiplier ideals $\mathcal{J}_k(\alpha X) \subset \mathcal{O}_Y$ satisfy:

(i) Budur-Saito: $\mathcal{J}_0(\alpha X) = \mathcal{J}(\alpha X)$, the (classical) multiplier ideal of X

(ii) $\mathcal{J}_k(\alpha X)$ is decreasing and right-continuos in α,

(iii) $\mathcal{J}_k(-kX) = \mathcal{O}_Y$.

Definition

For any $k \in \mathbb{N}$, $\alpha \in \mathbb{Q}$ is a jumping coefficient of the k-th multiplier ideal $\mathcal{J}_k$ of X if

$$\mathcal{J}_k((\alpha - \epsilon)X)/\mathcal{J}_k(\alpha X) \cong Gr_k^F Gr_V^\alpha \mathcal{B}_f \neq 0.$$

Theorem (M.-Yang)

X is k-Du Bois $\iff$ $\mathcal{J}_\ell$ has no jumping coefficients in $[0,1) \cap \mathbb{Q}$ for all $0 \leq \ell \leq k$.
X is k-rational $\iff$ $\mathcal{J}_\ell$ has no jumping coefficients in $[0,1) \cap \mathbb{Q}$ for all $0 \leq \ell \leq k$, and $\alpha = 0$ is not a jumping coefficient of $\mathcal{J}_{k+1}$.

This is a consequence of the Hodge theoretic interpretation of jumping coefficients:

Theorem (M.-Yang)

$\alpha \in (0,1) \cap \mathbb{Q}$ *is not a jumping number of* $\mathcal{J}_\ell$ *for all* $0 \leq \ell \leq k$

$$\iff Gr^F_\ell \varphi^H_{f,e^{-2\pi i \alpha}} \mathbb{Q}^H_Y[\dim Y] = 0 \text{ for all } 0 \leq \ell \leq k+1.$$

$\alpha = 0$ *is not a jumping number of* $\mathcal{J}_\ell$ *for all* $0 \leq \ell \leq k$ $\iff$

$$Gr^F_\ell \varphi^H_{f,1} \mathbb{Q}^H_Y[\dim Y] = 0 \text{ for all } 0 \leq \ell \leq k.$$

Spectral classes detect jumping coefficients

Jumping coefficients of higher multiplier ideals are also detected by the spectral Hirzebruch-Milnor classes:

Corollary (M.-Yang)

If $\alpha \in [0,1) \cap \mathbb{Q}$ is not a jumping number of the ℓ-th multiplier ideal $\mathcal{J}_\ell$ of X for all $0 \le \ell \le k$, then

$$\mathcal{M}T^{sp}_{t*}(X)|_{t^{\ell+\alpha}} = 0, \text{ for all } 0 \le \ell \le k.$$

The converse holds if $X_{\rm sing}$ is projective.

♣ The case $k = 0$ (i.e., for the classical multiplier ideal of X) was proved earlier by M.-Saito-Schürmann.

Arbitrary hypersurfaces

♣ Spectral characteristic classes can be defined for arbitrary hypersurfaces in smooth complex varieties (by glueing the previous definition for locally defining equations), and all characteristic class results hold in this more general setup (M. Saito).

On the projectivity of the singular locus

♣ The converse assertions involving characteristic classes rely on the *positivity of Todd class transformation* in the projective case:

Proposition (M.-Saito-Schürmann)

If X is a complex projective variety and $\mathcal{F} \in Coh(X)$ so that $\dim_{\mathbb{C}} \mathrm{Supp}(\mathcal{F}) = k$, then $td_*[\mathcal{F}]_{2k} \in H_{2k}(X;\mathbb{Q})$ does not vanish.

♣ By the functoriality of td_*, this can be reduced to the case of the projective space, where it follows from the positivity of the degrees of subvarieties.

Remark

Cannot replace projectivity by compactness in the assumption of the converse assertions (Hironaka's example).
Cannot omit the compactness assumption, since the Chow groups of affine varieties are usually small.

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THANK YOU !!!