On the geometry and topology of aspherical compact Kähler manifolds and related questions

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(based on joint work with D. Arapura, Y. Liu and B. Wang)

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Hopf Conjecture

♣ Curvature conditions restrict the topology of a smooth manifold.

Sample Theorem (Berger, Klingenberg, Brendle-Schoen)

If a simply-connected Riemannian manifold has sectional curvature satisfying $1/4 < sec(X) \leq 1$, then it is homeomorphic to a sphere. Moreover, it is diffeomorphic to the standard sphere.

Conjecture (Hopf, 1931)

Let X be a closed Riemannian manifold of real dimension $2n$, with sectional curvature $sec(X)$. Then:

- $sec(X) \leq 0 \Longrightarrow (-1)^n \cdot \chi(X) \geq 0$
- $sec(X) < 0 \Longrightarrow (-1)^n \cdot \chi(X) > 0$
- $sec(X) \geq 0 \Longrightarrow \chi(X) \geq 0$
- $sec(X) > 0 \Longrightarrow \chi(X) > 0$

Singer Conjecture

Definition

A connected CW complex X is called asphererical if its universal cover $\widetilde{X}$ is contractible. (E.g., abelian varieties, ball quotients, etc.)

Remark

(Cartan-Hadamard) If X is a closed Riemannian manifold with $sec(X) \leq 0$, then X is aspherical.

Conjecture (Singer)

If X is a closed aspherical topological manifold of real dimension $2n$, then all L^2-Betti numbers of $\widetilde{X}$ vanish except (possibly) in degree n, hence $(-1)^n \cdot \chi(X) \geq 0$.

What's known?

♣ The Singer and Hopf ($sec \leq 0$) conjectures are true for:

- $n = 1$: by the uniformization theorem for Riemann surfaces
- $n = 2$ & $sec \leq 0$, via Gauss-Bonnet (Chern & Milnor)
- aspherical compact complex surfaces, via Enriques–Kodaira classification (Johnson-Kotschick)
- if X compact Kähler:
  - hyperbolic manifolds, e.g., $sec < 0$ (Gromov)
  - nonelliptic manifolds, e.g., $sec \leq 0$ (Cox-Xavier, Jost-Zuo)
  - if X carries a holomorphic 1-form with finitely many zeros (Llosa Isenrich-Py)
- $\cdots$

♣ new perspectives in recent works of Liu-M.-Wang, Arapura-Wang, M., Albanese-Di Cerbo-Lombardi, ...

Relation to (semi)positivity

♣ In the Kähler context, curvature constraints are related to (semi)positivity of the complex (co)tangent bundle.

Definition

A vector bundle E is *ample* (resp. *nef*) if the line bundle $\mathcal{O}(1)$ on $\mathbf{P}(E)$ is ample (resp. nef).

Example

If E is globally generated, then E is nef.

Proposition (Demailly-Peternell-Schneider)

If X is a compact Kähler manifold, then:

- $sec(X) < 0 \Longrightarrow T^*X$ *is ample*
- $sec(X) \leq 0 \Longrightarrow T^*X$ *is nef (K_X is nef, X is minimal)*
- $sec(X) > 0 \Longrightarrow TX$ *is ample ($X \cong \mathbb{P}^n$ by Mori, Siu-Yau)*
- $sec(X) \geq 0 \Longrightarrow TX$ *is nef (classified by Campana-Peternell Conjecture).*

Theorem (Fulton-Lazarsfeld, Demailly-Peternell-Schneider)

Let E be a rank r nef (resp., ample) vector bundle on a Kähler manifold X. For any r-dim. conic subvariety C of E, one has

$$C \cdot Z_E \geq 0 \quad (\text{resp., } > 0),$$

where Z_E is the zero section of E, and $C \cdot Z_E$ is the intersection number of cycles in E.

♣ In the Kähler context, the Singer-Hopf conjecture can be approached in relation to the (semi)positivity of T^*X.
♣ If X is a compact Kähler manifold of cx. dim. n,

$$(-1)^n \cdot \chi(X) = \int_X c_n(T^*X) = T^*_X X \cdot T^*_X X,$$

with $T^*_X X$ the zero section of T^*X. So, if

- T^*X is nef, e.g., $sec(X) \leq 0$: (DPS) $\Longrightarrow (-1)^n \cdot \chi(X) \geq 0$
- T^*X is ample, e.g., $sec(X) < 0$: (FL) $\Longrightarrow (-1)^n \cdot \chi(X) > 0$

♣ So the Singer-Hopf conjecture in the Kähler setting reduces to:

Conjecture (Liu-M.-Wang)

If X is an aspherical compact Kähler manifold then T^*X is nef.

Example

The class of Kähler manifolds whose cotangent bundles are nef is closed under taking products, finite unramified covers, and subvarieties (e.g. smooth subvarieties of abelian varieties).

Remark

$(-1)^n \cdot \chi(X) = \chi(X, \mathbb{C}_X[n]) = T^*_X X \cdot T^*_X X$
$\mathbb{C}_X[n] \in Perv(X)$ *with* $CC(\mathbb{C}_X[n]) = T^*_X X$.

More generally,

Theorem (Liu-M.-Wang)

If X is a compact Kähler manifold with T^*X nef, and $P \in Perv(X)$, then $\chi(X, P) \geq 0$.

♣ proof uses 2 ingredients:

- Kashiwara index theorem: $\chi(X, P) = CC(P) \cdot T^*_X X$, where CC is the characteristic cycle.
- Demailly-Peternell-Schneider theorem, using that $CC(P)$ is an *effective* cycle for $P \in Perv(X)$.

Conjecture (Liu-M.-Wang, Arapura-Wang)

If X is an aspherical compact Kähler manifold and $P \in Perv(X)$, then $\chi(X,P) \geq 0$.

Remark

The theorem and conjecture apply to any bounded constructible complex on X with an effective characteristic cycle, e.g., the IC-complex, a characteristic complex, or the DT-sheaf for any closed irreducible subvariety $Z \subseteq X$.

Corollary

If T^*X is nef, and $Z \subseteq X$ is a closed irreducible subvariety, then:

- $(-1)^{\dim Z} \cdot \chi^{IH}(Z) \geq 0$.
- $(-1)^{\dim Z} \cdot \chi(Z, Eu_Z) \geq 0$, *where Eu_Z is the local Euler obstruction function of MacPherson.*
- $\chi_{vir}(Z) := \chi(Z, \nu_Z) \geq 0$, *where ν_Z is Behrend's function and $\chi_{vir}(Z)$ is the DT invariant.*

Remark

If X is an aspherical complex projective manifold, the Shafarevich conjecture implies that the universal cover $\widetilde{X}$ is Stein.

Theorem (Kratz)

If X is a complex projective manifold whose universal cover $\widetilde{X}$ is a bounded domain in a Stein manifold, then T^*X is nef.

♣ It is not true that if $\widetilde{X}$ is a Stein manifold then T^*X is nef (Y. Wang, 2022). However, the known such examples are not aspherical (Di Cerbo-Pardini, 2023).

Conjecture (M.)

If X is a complex projective manifold with universal cover $\widetilde{X}$ a Stein manifold, then X admits a finite morphism $f : X \to Y$ to a complex projective manifold Y with T^*Y nef.

The conjecture is motivated by the following result, and it reduces Singer-Hopf in the projective context to the Shafarevich conjecture:

Theorem (M.)

Let X be a complex projective manifold and let $F^{\bullet}$ be a constructible complex on X with effective characteristic cycle. Assume X admits a finite morphism $f : X \to Y$ to a complex projective manifold Y with nef cotangent bundle (e.g., Y has non-positive sectional curvature). Then $\chi(X, F^{\bullet}) \geq 0$. Moreover, the inequality is strict if T^*Y is ample (e.g., Y has negative sectional curvature).

Hodge refinements of the Singer-Hopf Conjecture

♣ If X is a compact complex manifold , then

$$\chi(X) = \sum_{p \geq 0} (-1)^p \cdot \chi^p(X),$$

with $\chi^p(X) := \chi(X, \Omega_X^p)$, and $\chi^p(X) = (-1)^n \chi^{n-p}(X)$ by Serre duality.

Conjecture (Arapura-M.-Wang)

If X is a compact Kähler manifold of dimension n which is asphérical or has a nef cotangent bundle, then for any $0 \leq p \leq n$ one has:

$$(-1)^{n-p} \cdot \chi^p(X) \geq 0.$$

♣ If $p = n$, Kollár's conjecture: $\chi(X, K_X) \geq 0$ (for $\pi_1(X)$ generically large)

Known cases of the conjecture

♣ The conjecture is true in the following cases:

- Kähler hyperbolic manifolds, e.g., $sec < 0$ (Gromov)
- Kähler nonelliptic manifolds, e.g., $sec \leq 0$ (Jost-Zuo)
- If $alb_X : X \to Alb(X)$ is semi-small, by a Nakano-type generic vanishing theorem for Ω^p_X's (Popa-Schnell)
- if T^*X is globally generated (since alb_X is an immersion)
- aspherical compact complex surfaces, via Enriques–Kodaira classification, Winkelnkemper's inequality and Riemann-Roch (Arapura-M.-Wang, Albanese-Di Cerbo-Lombardi)
- smooth projective curves and surfaces with T^*X nef, via Riemann-Roch, positivity for T^*X, and BMY inequality (Arapura-M.-Wang).

Theorem (Arapura-M.-Wang, Albanese-Di Cerbo-Lombardi)

Let X be an aspherical smooth complex compact surface. Then $\chi(X, \Omega_X^2) = \chi(X, \mathcal{O}_X) \geq 0$ *and* $\chi(X, \Omega_X^1) \leq 0$

♣ Ingredients:

- with $\chi = \chi(X), \sigma = \sigma(X)$, one gets by Riemann-Roch

$$\chi(X, \Omega_X^2) = \chi(X, \mathcal{O}_X) = \frac{1}{4}(\chi + \sigma), \quad \chi(X, \Omega_X^1) = \frac{1}{2}(\sigma - \chi)$$

- Johnson-Kotschick: Winkelnkemper's inequality $\chi \geq |\sigma|$ holds, unless X is a ruled surface over a curve of genus ≥ 2 (none of these is aspherical).

Theorem (Arapura-M.-Wang)

Let X be a complex projective manifold of dimension $n \leq 4$ with T^*X nef. Then

$$\chi(X, K_X) = (-1)^n \cdot \chi(X, \mathcal{O}_X) \geq 0.$$

♣ Ingredients:

- Riemann-Roch
- Demailly-Peternell-Schneider for T^*X
- BMY inequality

Hopf conjecture for $sec \geq 0$

Conjecture (Hopf, 1931)

If X is an even-dimensional closed Riemannian manifold with $sec(X) \geq 0$, then $\chi(X) \geq 0$. If $sec(X) > 0$, then $\chi(X) > 0$.

- proved in dim. 2 (via Gauss-Bonnett) and 4 (via Bonnet-Myers)
- very few examples of spaces with $sec > 0$

(Semi)positivity for the tangent bundle

♣ Recall that if X is a compact Kähler manifold with $sec(X) \geq 0$, then TX is nef.

Proposition (Demailly-Peternell-Schneider)

If X is a compact Kähler manifold with TX nef, then $\chi(X) \geq 0$.

♣ Recall that

$$\chi(X) = \sum_{p \geq 0} (-1)^p \cdot \chi^p(X),$$

with $\chi^p(X) := \chi(X, \Omega_X^p)$.

Conjecture (Arapura-M.-Wang)

If X is a compact Kähler manifold of dimension n with TX nef, then for any $0 \leq p \leq n$ one has

$$(-1)^p \cdot \chi^p(X) \geq 0$$

♣ Demailly-Peternell-Schneider: it suffices to assume X is a Fano manifold (i.e., a complex projective manifold with K_X^{-1} ample).

Lemma

If the complex projective manifold X has a cellular decomposition (e.g., X is rational homogenous), then $(-1)^p \cdot \chi^p(X) > 0$ for all $p \leq \dim X$.

Example

If TX is globally generated, the conjecture holds. Indeed, a smooth projective variety with TX globally generated is a homogeneous variety, so it has a cellular decomposition.

More generally, the conjecture can be reduced to:

Conjecture (Campana-Peternell)

Any Fano manifold with nef tangent bundle is rational homogeneous (i.e., G/P, with G a semi-simple Lie group and P a parabolic subgroup).

Theorem (Arapura-M.-Wang)

If X is a compact Kähler manifold of dimension n with non-negative bisectional curvature (e.g., non-negative sectional curvature), then for $0 \leq p \leq n$ one has

$$(-1)^p \cdot \chi^p(X) \geq 0.$$

♣ If X has non-negative bisectional curvature then TX is nef, so the theorem proves a special case of the conjecture.

♣ Ingredients of the proof:

- X Fano $\Longrightarrow$ X is simply-connected
- Mok's classification of compact Kähler manifolds with non-negative bisectional curvature implies that X admits a cellular decomposition.

Other open questions

The conjecture of Arapura-M.-Wang can also be deduced from the following weaker conjectures:

Conjecture (Arapura-M.-Wang)

If X is a Fano manifold with TX nef, then $h^{p,q}(X) = 0$, $\forall p \neq q$.

Conjecture (Arapura-M.-Wang)

If X is a Fano manifold with TX nef, then X admits a holomorphic vector field with only isolated zeros.

♣ True if TX is globally generated (by Bott's residue formula and Bertini's theorem).

Thank you !