

PROBLEMS IN ALGEBRAIC TOPOLOGY

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1. MANIFOLDS AND POINCARÉ DUALITY

- (1) Show that if a connected manifold M is the boundary of a compact manifold, then the Euler characteristic of M is even.
- (2) Show that $\mathbb{R}\mathbb{P}^{2n}$, $\mathbb{C}\mathbb{P}^{2n}$, $\mathbb{H}\mathbb{P}^{2n}$ cannot be boundaries.
- (3) Show that if M^{4n} is a connected manifold which is the boundary of a compact oriented $(4n + 1)$ -dimensional manifold V , then the signature of M is zero.
- (4) Show that $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ cannot be the boundary of an orientable 5-manifold. However, it is the boundary of a non-orientable manifold.
- (5) Show that if M^n is connected, non-compact manifold, then $H_i(M; R) = 0$ for $i \geq n$.
- (6) The Euler characteristic of a closed manifold of odd dimension is zero.
- (7) True or False: Any orientable manifold is a 2-fold covering of a non-orientable manifold.
- (8) Show that the Euler characteristic of a closed, oriented, $(4n+2)$ -dimensional manifold is even.
- (9) Let M be a closed, oriented $4n$ -dimensional manifold. Show that the signature $\sigma(M)$ is congruent mod 2 to the Euler characteristic $\chi(M)$.
- (10) If M is a compact, closed, oriented manifold of dimension n , show that the torsion subgroups of $H^i(M)$ and $H^{n-i+1}(M)$ are isomorphic.
- (11) Let M be a closed, connected, n -dimensional manifold. Show that:

$$\text{Tor}(H_{n-1}(M; \mathbb{Z})) = \begin{cases} 0, & \text{if } M \text{ is orientable} \\ \mathbb{Z}_2, & \text{if } M \text{ is non-orientable} \end{cases}$$

- (12) Show that $\mathbb{R}\mathbb{P}^{2n}$ cannot be embedded in S^{2n+1} .
- (13) Show that $\mathbb{R}\mathbb{P}^{2n}$, $\mathbb{C}\mathbb{P}^{2n}$ are fixed-point free spaces.

2. HOMOTOPY THEORY

- (1) Find spaces with the same homotopy (homology) groups, but having different homotopy type.
- (2) Describe the cell structure on lens spaces.
- (3) Show that $V_n(\mathbb{R}^\infty)$, $V_n(\mathbb{C}^\infty)$, $V_n(\mathbb{H}^\infty)$ are contractible.

3. COHOMOLOGY RING

- (1) Find the cohomology ring of real/complex projective spaces and of lens spaces.
- (2) Calculate $H^*(\mathbb{R}\mathbb{P}^\infty; \mathbb{Z})$, $H^*(\mathbb{R}\mathbb{P}^{2n}; \mathbb{Z})$, $H^*(\mathbb{R}\mathbb{P}^{2n+1}; \mathbb{Z})$.
- (3) Prove Borsuk-Ulam theorem.

- (4) If there exists a division algebra on \mathbb{R}^n , then $\mathbb{R}\mathbb{P}^{n-1}$ is orientable, hence $n = 2^r$, for some r .
- (5) Find the ring structure on $H^*(SU(n))$, $H^*(U(n))$, $H^*(V_k(\mathbb{C}^n))$.
- (6) Find the cohomology ring of $\Omega\mathbb{C}\mathbb{P}^n$, $\Omega\mathbb{R}\mathbb{P}^n$. (Hint: use a spectral sequence argument)

4. SPECTRAL SEQUENCES AND APPLICATIONS

- (1) Find the ring structure on $H^*(SU(n))$, $H^*(U(n))$, $H^*(V_k(\mathbb{C}^n))$.
- (2) Prove the Suspension Theorem for homotopy groups.
- (3) Calculate $H_*(\Omega S^n)$ and the ring structure on $H^*(\Omega S^n)$.
- (4) Find the cohomology ring of $\Omega\mathbb{C}\mathbb{P}^n$, $\Omega\mathbb{R}\mathbb{P}^n$. (Hint: use a spectral sequence argument)
- (5) Find the cohomology ring of Lens Spaces.
- (6) Calculate $H^*(K(\mathbb{Z}, 3))$.
- (7) Find $H^*(K(\mathbb{Z}, n); \mathbb{Q})$.
- (8) Calculate $\pi_4(S^3)$, $\pi_5(S^3)$.
- (9) Show that the p -torsion in $\pi_i(S^3)$ appears first for $i = 2p$ and it is \mathbb{Z}_p .
- (10) Where does the 7-torsion appear first in the homotopy groups of S^n ?
- (11) Prove Serre's theorem: (a) The homotopy groups of odd spheres S^n are torsion except in dimension n ; (b) The homotopy groups of even spheres S^n are torsion except in dimension n and $2n - 1$.
- (12) Calculate the cohomology of the space of maps from $S^1 \rightarrow S^3$, and similarly for the space of maps $S^1 \rightarrow S^2$ and $S^1 \rightarrow \mathbb{C}\mathbb{P}^n$.
- (13) Find the cohomology ring of $BU(n)$, $BO(n)$.

5. FIBRE BUNDLES

- (1) Classify the S^1 -bundles over S^2 .
- (2) Show that any vector bundle over a simply-connected base space is orientable.
- (3) Show that if an oriented vector bundle has a non-zero section, then its Euler class is zero.
- (4) Construct an orientable sphere bundle with zero Euler class, but no section.

6. CHARACTERISTIC CLASSES

- (1) Calculate $w(\mathbb{R}\mathbb{P}^n)$, $c(\mathbb{C}\mathbb{P}^n)$, $p(\mathbb{C}\mathbb{P}^n)$.
- (2) Show that a manifold M is orientable if and only if its first Stiefel-Whitney class vanishes.
- (3) Study possible immersions (embeddings) of $\mathbb{R}\mathbb{P}^n$ into \mathbb{R}^{n+k} . (Hint: use S-W classes)
- (4) Show that the only real projective spaces which can be parallelizable are \mathbb{P}^1 , \mathbb{P}^3 and \mathbb{P}^7 .
- (5) Show that if n is a power of 2, then $\mathbb{R}\mathbb{P}^n$ cannot be smoothly embedded in \mathbb{R}^{2n-1} .
- (6) Show that $\mathbb{R}\mathbb{P}^{2k+1}$ cannot be a boundary.
- (7) Show that $\mathbb{C}\mathbb{P}^4$ cannot be smoothly embedded in \mathbb{R}^n with $n \leq 11$. (Hint: use Pontrjagin classes)
- (8) Find the smallest k such that $\mathbb{C}\mathbb{P}^n$ can be smoothly embedded in \mathbb{R}^{2n+k} .

- (9) Show that $\mathbb{C}\mathbb{P}^{2n}$ is not an oriented boundary. (Hint: use the Pontrjagin number)
- (10) Find obstructions to the existence of a complex structure on an even dimensional manifold.
- (11) Find the cohomology ring of $BU(n)$, $BO(n)$.
- (12) Show that if M is an oriented boundary, then all its Pontrjagin numbers are zero.
- (13) $\mathbb{C}\mathbb{P}^n$ cannot be expressed non-trivially as a product of complex manifolds.