# A DECOMPOSITION THEOREM FOR THE PERIPHERAL COMPLEX ASSOCIATED WITH HYPERSURFACES

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ABSTRACT. We give necessary and sufficient conditions for a decomposition (in the category of perverse sheaves) of the Cappell-Shaneson peripheral complex associated with a complex affine hypersurface. We also find a general class of hypersurfaces for which such a decomposition exists.

## 1. INTRODUCTION

The deepest result in the theory of perverse sheaves on algebraic varieties is the famous *Decomposition theorem* of Beilinson, Bernstein, Deligne and Gabber ([2]), which asserts that: (a) the category of perverse sheaves on an algebraic variety $Y$ is an artinian abelian category, whose simple objects are the middle perversity intersection homology complexes $IC_{\bar{m}}(V; L)$, associated to irreducible subvarieties $V$ of $Y$ and irreducible local system $L$ on $V$; (b) if $f : X \to Y$ is a proper algebraic map, then $Rf_* IC_{\bar{m}}(X)$ is a sum of (possibly shifted) intersection homology complexes of subvarieties of $Y$. A nice exposition of the Decomposition theorem is contained in [24], §12, and a new geometric proof is given in [7]. The result has many remarkable consequences for the topology of algebraic maps, some of which are presented by Goresky-MacPherson in [16] and [23]: the degeneration of the spectral sequence for $f$ in case $f$ is a topological fibration, generalized invariant cycle theorems, realization of intersection homology groups of algebraic varieties as direct summand of the homology of their resolutions etc. The decomposition theorem is one of the most powerful techniques for calculating intersection homology.

In contrast to the algebraic case, Cappell and Shaneson ([6]) have proven a general decomposition theorem for self-dual complexes of sheaves (under an appropriate cobordism relation) for arbitrary stratified maps. Their theorem can be interpreted in part as the statement that, up to cobordism, the BBD decomposition theorem holds in the topological category. For many important topological invariants, a decomposition theorem up to cobordism is sufficient to provide exact formulae: e.g., if $f : X \to Y$ is any stratified map, the $L$-classes (hence signatures as well) of $X$ and $Y$ can be related via $f$.

In the attempt of understanding the relation between the local and global topological structure of stratified spaces, Cappell and Shaneson ([5]) also investigated the invariants associated with a stratified pseudomanifold $X$, PL-embedded in codimension two in a manifold $Y$ (e.g., $X$ might be a hypersurface in a smooth complex algebraic variety). They exhibit a duality formula (called 'superduality') which holds for 'superdual' perversities and over certain rings, an analogue of the Poincaré duality for dual perversities which only holds over a field. In describing the $L$-classes of the subspace $X$, they use as a main tool *the peripheral complex* $\mathcal{R}^\bullet$, a torsion, self-dual, perverse sheaf, supported on $X$. Using a general splitting theorem, up

to cobordism, for an arbitrary self-dual perverse torsion sheaf, they give a decomposition theorem (up to cobordism) of $\mathcal{R}^\bullet$. This seems to be the correct analogue (in the topological category) to their decomposition theorem for stratified maps, suited to the study of codimension two sub-pseudomanifolds.

As the singular spaces that arise in applications are usually complex algebraic varieties, it is natural to ask if there exists a genuine decomposition of the peripheral complex associated with complex hypersurfaces (usually hypersurfaces in a projective or affine space). In the category of algebraic varieties, such a decomposition should be interpreted as the analogue of the BBD decomposition theorem. It would also provide a parallel between the study of singular spaces in topology and algebraic geometry.

The aim of this paper is to find necessary and sufficient conditions for a decomposition of the peripheral complex associated with a complex affine hypersurface $X$, and to provide examples when these conditions are satisfied. Our main result (Theorem 3.3) asserts that the peripheral complex can be decomposed in simple parts if and only if the natural maps between middle- and logarithmic-perversity lower-middle intersection homology groups of links of strata of $X$ in the pair $(\mathbb{C}^{n+1}, X)$, are isomorphisms of $\Gamma$-modules. (Here $\Gamma = \mathbb{Q}[t, t^{-1}]$ stands for the stalk of local coefficient systems defined on the hypersurface complement and on the link complement by the linking number with $X$ and the link, respectively). The key ingredient in proving the theorem is a splitting criterion in the category of perverse, self-dual, torsion sheaves (Lemma 3.1, adapted from [7]). Among examples for which the necessary and sufficient conditions for a splitting of $\mathcal{R}^\bullet$ are satisfied, we mention the following result (Theorem 3.11): if the links of singular strata of $X$ in $(\mathbb{C}^{n+1}, X)$ are filtered by rational homology spheres, and have simply-connected singular strata, then the peripheral complex splits. In the case of hypersurfaces with one or two singular strata, these conditions can be realized geometrically, by imposing restrictions on the monodromy operators of the Milnor fibrations associated with the link pairs of singular strata of the hypersurface (see Proposition 3.7).

## 2. Preliminaries

2.1. **Homological Algebra.** The underlying space $X$ will be complex analytic or algebraic. The base ring $R$ will be a Dedekind domain. All sheaves on $X$ are sheaves of $R$-modules. For a more detailed exposition, the reader is advised to consult [3], [9] or [25].

A map $\phi^\bullet : A^\bullet \to B^\bullet$ of complexes of sheaves is a *quasi-isomorphism* provided that the induced sheaf maps $\mathcal{H}^p(\phi^\bullet) : \mathcal{H}^p(A^\bullet) \to \mathcal{H}^p(B^\bullet)$ are isomorphisms for all $p$. If $A^\bullet$ and $B^\bullet$ are quasi-isomorphic, they become isomorphic in the *derived category* and we write $A^\bullet \cong B^\bullet$ in $D(X)$. If $\mathcal{H}^i(A^\bullet) = 0$ in degrees $i \neq t$, for some $t \in \mathbb{Z}$, then $A^\bullet$ is quasi-isomorphic to the complex $\mathcal{H}^t(A^\bullet)[-t]$.

The *derived category* $D^b(X)$ of bounded complexes of sheaves is the (triangulated) category whose objects consist of bounded differential complexes, and where the morphisms are obtained by 'inverting' the quasi-isomorphisms so that they become isomorphisms in the derived category. The cone construction for a morphism of complexes $\phi : A^\bullet \to B^\bullet$ gives rise, in a non-unique way, to a diagram of

morphisms of complexes:

$$A^\bullet \xrightarrow{\phi} B^\bullet \to M^\bullet(\phi) \xrightarrow{[1]} A^\bullet[1]$$

where $M^\bullet(\phi)$ is the algebraic mapping cone of $\phi$. A triangle in $D^b(X)$ is called *distinguished* if it is isomorphic in $D^b(X)$ to a diagram arising from a cone. A morphism $\phi : A^\bullet \to B^\bullet$ in $D^b(X)$ can be completed to a distinguished triangle:

$$A^\bullet \xrightarrow{\phi} B^\bullet \to C^\bullet \xrightarrow{[1]} A^\bullet[1]$$

If $\phi = 0$, then $C^\bullet \cong A^\bullet[1] \oplus B^\bullet$ and we say that *the triangle splits*.

A distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ determines long exact sequences on cohomology and hypercohomology:

$$\cdots \to \mathcal{H}^p(A^\bullet) \to \mathcal{H}^p(B^\bullet) \to \mathcal{H}^p(C^\bullet) \to \mathcal{H}^{p+1}(A^\bullet) \to \cdots$$

$$\cdots \to \mathbb{H}^p(X; A^\bullet) \to \mathbb{H}^p(X; B^\bullet) \to \mathbb{H}^p(X; C^\bullet) \to \mathbb{H}^{p+1}(X; A^\bullet) \to \cdots$$

A complex $A^\bullet$ is *constructible* with respect to a complex (analytic) stratification $\mathcal{S} = \{S_\alpha\}$ of $X$ provided that, for all $\alpha$ and $p$, the cohomology sheaves $\mathcal{H}^p(A^\bullet|_{S_\alpha})$ are locally constant and have finitely-generated stalks. If $A^\bullet$ is bounded and constructible with respect to some stratification $\mathcal{S}$, we write $A^\bullet \in D^b_c(X)$.

For any $A^\bullet \in D^b_c(X)$, there is the hypercohomology spectral sequence:

$$E_2^{p,q} = H^p(X; \mathcal{H}^q(A^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X; A^\bullet)$$

Let $A^\bullet \in D^b_c(X)$. The dual of $A^\bullet$, $\mathcal{D}_X(A^\bullet)$, is well-defined up to quasi-isomorphism by : for any open $U \subseteq X$, there is a natural split exact sequence (recall that $R$ is a Dedekind domain):

$$0 \to Ext(\mathbb{H}^{q+1}_c(U, A^\bullet), R) \to \mathbb{H}^{-q}(U, \mathcal{D}_X A^\bullet) \to Hom(\mathbb{H}^q_c(U, A^\bullet), R) \to 0$$

If $j : Y \hookrightarrow X$ is the inclusion of a closed subspace and $i : U \hookrightarrow X$ the inclusion of the open complement, then for all $A^\bullet \in D^b_c(X)$, there exist distinguished triangles:

$$Ri_! i^! A^\bullet \to A^\bullet \to Rj_* j^* A^\bullet \xrightarrow{[1]}$$

and

$$Rj_! j^! A^\bullet \to A^\bullet \to Ri_* i^* A^\bullet \xrightarrow{[1]}$$

where the second triangle can be obtained from the first by dualizing. Note that $Ri_! = i_!$, $Rj_* = j_* = j_! = Rj_!$, and $i^! = i^*$.

We will state for future use the following simple lemma: let $\mathcal{C}$ be an abelian category and we let $C(\mathcal{C})$ be the category whose objects are complexes of objects of $\mathcal{C}$. We denote by $\tau_{\leq}$ and $\tau_{\geq}$ the natural truncation functors on $C(\mathcal{C})$.

**Lemma 2.1.** *Let $t \in \mathbb{Z}$, $A^\bullet, B^\bullet \in C(\mathcal{C})$ such that $A^\bullet \cong \tau_{\leq t} A^\bullet$ and $B^\bullet \cong \tau_{\geq t} B^\bullet$. Then the natural map:*

$$Hom_{D(\mathcal{C})}(A^\bullet, B^\bullet) \to Hom_{\mathcal{C}}(\mathcal{H}^t(A^\bullet), \mathcal{H}^t(B^\bullet))$$

*is an isomorphism of abelian groups.*

2.2. **Perverse Sheaves.** We will use the notations and conventions of [2]. Let $Y$ be a complex algebraic variety and $D_c^b(Y)$ the derived category of bounded, constructible complexes of sheaves of $R$-modules on $Y$. Consider the t-structure $({}^p D^{\leq 0}(Y), {}^p D^{\geq 0}(Y))$ on $D_c^b(Y)$ associated with the middle perversity. The associated *heart* is $Perv(Y)$, a full abelian sub-category of $D_c^b(Y)$. Its objects are called *perverse sheaves*.

We have the following structure: two full subcategories ${}^p D^{\leq 0}(Y)$ and ${}^p D^{\geq 0}(Y)$, and

$$ {}^p D^{\leq m}(Y) := {}^p D^{\leq 0}(Y)[-m] $$
$$ {}^p D^{\geq m}(Y) := {}^p D^{\geq 0}(Y)[-m] $$

and, if $\mathcal{X}$ is a stratification with respect to which $F^\bullet$ is constructible, and $\alpha_l : S_l \hookrightarrow Y$ is the embedding of a stratum of complex dimension $l$, the following hold:

- *condition of support*:

    $F^\bullet \in Ob({}^p D^{\leq 0}(Y))$ iff $\mathcal{H}^j(\alpha_l^* F^\bullet) = 0$ for any l and j, with $j > -l$

- *condition of cosupport*:

    $F^\bullet \in Ob({}^p D^{\geq 0}(Y))$ iff $\mathcal{H}^j(\alpha_l^! F^\bullet) = 0$ for any l and j, with $j < -l$.

If $F^\bullet \in Ob({}^p D^{\leq m}(Y))$ and $G^\bullet \in Ob({}^p D^{\geq m+t}(Y))$, $t > 0$, then:

$$ Hom_{D_c^b(Y)}(F^\bullet, G^\bullet) = 0 $$

The abelian category of perverse sheaves is defined by:

$$ Perv(Y) := {}^p D^{\leq 0}(Y) \cap {}^p D^{\geq 0}(Y). $$

Recall that if $Y$ is non-singular, then a perverse sheaf $F^\bullet$ on $Y$ (constructible with respect to the obvious stratification) is just a local system, i.e., a locally constant sheaf with finitely generated stalks. More precisely:

$$ F^\bullet \cong \mathcal{H}^{-dimY}(F^\bullet)[dimY]. $$

We note the following fact for future reference (see [7]):

**Lemma 2.2.** *Let*

$$ Y = Y_n \supset Y_{n-1} \supset Y_{n-2} \supset \cdots \supset Y_0 \supset Y_{-1} = \emptyset $$

*be the filtration associated to a Whitney stratification of $Y$, with $dim_{\mathbb{C}} Y_j = j$, and set $S_j := Y_j \setminus Y_{j-1}$, $U_j := Y \setminus Y_{j-1}$. If $F^\bullet$ is constructible with respect to the given stratification, perverse, and supported on a closed s-dimensional stratum $S_s$, then $F^\bullet \cong \mathcal{H}^{-s}(F^\bullet)[s]$.*

2.3. **Intersection Homology Complexes.** Let $Y$ be an algebraic variety of pure complex dimension $n$ and let $\mathcal{L}$ be a local system defined on an open subvariety of the regular part of $Y$. We assume that $\mathcal{L}$ has coefficients in finitely generated $R$-modules, where $R$ is a Dedekind ring. Fix a perversity $\bar{p}$ associated with a Whitney stratification of $Y$ inducing the filtration:

$$ Y = Y_n \supset Y_{n-1} \supset Y_{n-2} \supset \cdots \supset Y_0 \supset Y_{-1} = \emptyset, $$

where $dim_{\mathbb{C}} Y_j = j$. Define the pure strata $S_j := Y_j \setminus Y_{j-1}$, the Zariski-dense open sets $U_j := Y \setminus Y_{j-1}$, therefore: $U_j = U_{j+1} \cup S_j$.

*The intersection homology complex $IC_{\bar{p}}^\bullet(Y, \mathcal{L})$ is defined up to quasi-isomorphism* by the following set of axioms (using the conventions of [2]):

- $\mathcal{H}^j(IC_{\bar{p}}^\bullet(Y, \mathcal{L})) = 0$ if $j < -n$

- $\mathcal{H}^{-n}(IC_{\bar{p}}^{\bullet}(Y,\mathcal{L})) \simeq \mathcal{L}$
- $\mathcal{H}^{j}(\alpha_{l}^{*}IC_{\bar{p}}^{\bullet}(Y,\mathcal{L})) = 0$ for any $l$ and $j > -n$ s.t. $j > \bar{p}(2n-2l)-n$
- $\mathcal{H}^{j}(\alpha_{l}^{!}IC_{\bar{p}}^{\bullet}(Y,\mathcal{L})) = 0$ for any $l$ and $j > -n$ s.t. $j \leq \bar{p}(2n-2l)-n+1$

where $\alpha_{l} : S_{l} \to Y$ is the embedding of a stratum $S_{l}$ of pure complex dimension $l$.

This is a special case of [2], (2.1.9): $IC_{\bar{p}}^{\bullet}(Y,\mathcal{L}) \simeq j_{!*}\mathcal{L}[n]$, where $j$ is the inclusion of the dense open stratum, and $j_{!*}\mathcal{L}[n]$ is the *intermediate extension* of the perverse sheaf $\mathcal{L}[n]$ from the top stratum to $Y$, for the function $p : \{strata\ of\ Y\} \to \mathbb{Z}$, $p(S) = 0$ for $dim(S) = dim(Y)$, and $p(S) = \bar{p}(2k)+1$ for $dim_{\mathbb{C}}(S) = n-k,\ k \geq 1$.

The *perversity $\bar{p}$ intersection homology groups* of $Y$ with twisted coefficients are defined as:
$$^{\Phi}IH_{k}^{\bar{p}}(Y,\mathcal{L}) := \mathbb{H}_{\Phi}^{n-k}(Y; IC_{\bar{p}}^{\bullet}(Y,\mathcal{L})),$$
where $\Phi$ stands for the family of supports (in our case, either closed or compact).

The local calculation on stalks at points $x \in S_{l}$ gives (in the notations of [2]):
$$\mathcal{H}^{j}(IC_{\bar{p}}^{\bullet}(Y,\mathcal{L}))_{x} \cong \begin{cases} IH_{n-2l-j-1}^{\bar{p}}(L_{x};\mathcal{L}_{|L_{x}}), & j \leq \bar{p}(2n-2l)-n \\ 0, & j > \bar{p}(2n-2l)-n. \end{cases}$$

where $L_{x}$ is the link in $Y$ of the component of $S_{l}$ containing $x$. This formula holds for classical perversities (see [3], V.3.15). For super-perversities (i.e., perversities satisfying $\bar{p}(2) = 1$), a similar formula holds since it can be derived only from the axiomatic definition of the intersection complex.

Note that the complexes $IC_{\bar{m}}^{\bullet}(Y,\mathcal{L})$ and $IC_{\bar{l}}^{\bullet}(Y,\mathcal{L})$ are perverse by definition.

The middle-perversity intersection homology complex is also denoted by $IC_{Y}(\mathcal{L})$. Given a closed subvariety $i : Y' \to Y$ and a complex of type $IC_{Y'}(\mathcal{L}') \in Perv(Y')$, we denote $i_{*}IC_{Y'}(\mathcal{L}')$ simply by $IC_{Y'}(\mathcal{L}')$. It is an object of $Perv(Y)$ since $i_{*}$ is a t-exact functor.

## 3. THE PERIPHERAL COMPLEX ASSOCIATED TO A COMPLEX HYPERSURFACE

In this section we give necessary and sufficient condition for a splitting (in the category of perverse sheaves) of the peripheral complex associated to a complex hypersurface. Then we present examples when such a decomposition holds.

3.1. **The decomposition theorem.** Let $\bar{m}$ and $\bar{l}$ be the middle and logarithmic perversities, respectively (i.e., $\bar{m}(s) = [(s-1)/2]$ and $\bar{l}(s) = [(s+1)/2]$). Let $X$ be a reduced hypersurface in a smooth complex algebraic variety $Y$ of pure dimension $n$. We will assume that $Y$ is the affine space $\mathbb{C}^{n}$ or, more generally, that the fundamental class of $X$ maps trivially to the homology of $Y$. Fix a Whitney stratification of $Y$ made of $Y \setminus X$ and a Whitney stratification $\mathcal{X}$ of $X$. Consider a local system $\mathcal{L}$ defined on $Y \setminus X$, with stalk $\Gamma := \mathbb{Q}[t,t^{-1}]$ and action by an element $\alpha \in \pi_{1}(Y \setminus X)$ determined by multiplication by $t^{\ell_{X}(\alpha)}$, where $\ell_{X}(\alpha)$ denotes the linking number of $\alpha$ with $X$ (for a definition of the linking numbers in the affine space, see [1], §1A). Then $X \subset Y$ is of finite local type, i.e., the link of any component of the stratification of the pair $(Y,X)$ is of finite type (see [5], §2). Under these assumptions and notations, the *Cappell-Shaneson superduality isomorphism* holds, i.e., one has ([5], Theorem 3.3; recall we are using the indexing conventions of [2]):
$$IC_{\bar{m}}^{\bullet}(Y,\mathcal{L})^{op} \cong \mathcal{D}(IC_{\bar{l}}^{\bullet}(Y,\mathcal{L}))$$
(here $IC_{\bar{m}}^{\bullet}(Y,\mathcal{L})^{op}$ is obtained by composing all module structures with the involution $t \mapsto t^{-1}$).

The *peripheral complex* $\mathcal{R}_Y^\bullet(\mathcal{L})$ is defined by the distinguished triangle:

$$IC_{\bar{m}}^\bullet(Y, \mathcal{L}) \to IC_{\bar{l}}^\bullet(Y, \mathcal{L}) \to \mathcal{R}_Y^\bullet(\mathcal{L}) \overset{[1]}{\to}$$

Note that the superduality induces a canonical isomorphism:

$$\mathcal{R}_Y^\bullet(\mathcal{L}) \cong \mathcal{D}(\mathcal{R}_Y^\bullet(\mathcal{L}))[1]^{op}.$$

Since $X \subset Y$ is of finite local type, the stalks of the cohomology sheaves of the peripheral complex are torsion $\Gamma$-modules. Also, by definition, $\mathcal{R}_Y^\bullet(\mathcal{L})$ is supported on $X = \Sigma_Y$, the union of the non-top-dimensional strata of $Y$. Moreover, $\mathcal{R}_Y^\bullet(\mathcal{L})$ is a perverse sheaf since the category of perverse sheaves is stable by extensions, and $IC_{\bar{l}}^\bullet$ and $IC_{\bar{m}}[1]$ are perverse by definition. Therefore, $\mathcal{R}_Y^\bullet(\mathcal{L})$ is a perverse, torsion, self-dual sheaf in the sense of [5].

Let $S_l$ be the union of the $l$-dimensional components of strata of $Y$. Let $U_s = U_{s+1} \sqcup S_s = U \sqcup S$, where $U_s = Y \setminus Y_{s-1}$ and $S_s = Y_s \setminus Y_{s-1}$. Therefore, $U = \sqcup_{l>s} S_l$ and $S = S_s$ is a closed stratum in $U_s$. Let $\alpha_s = \alpha$ and $\beta_s = \beta$ be the corresponding closed and open embeddings. For simplicity, we will use in the sequel the notation $\mathcal{R}^\bullet = \mathcal{R}_Y^\bullet(\mathcal{L})_{|U_s}$. Note that $\mathcal{R}^\bullet$ is a perverse sheaf on $U_s$, for if $j$ is the open inclusion of $U_s$ in $Y$, then the functor $j^* = j^!$ is t-exact. Also, $\mathcal{R}^\bullet$ is self-dual since:

$$\begin{aligned}
\mathcal{R}^\bullet &= j^* \mathcal{R}_Y^\bullet(\mathcal{L}) \cong j^* \mathcal{D}(\mathcal{R}_Y^\bullet(\mathcal{L}))[1]^{op} \\
&\cong \mathcal{D}(j^! \mathcal{R}_Y^\bullet(\mathcal{L}))[1]^{op} \cong \mathcal{D}(j^* \mathcal{R}_Y^\bullet(\mathcal{L}))[1]^{op} \\
&= \mathcal{D}(\mathcal{R}^\bullet)[1]^{op}.
\end{aligned}$$

Using the support condition for perverse sheaves we obtain:

$$\mathcal{H}^j(\alpha^* \mathcal{R}^\bullet) = 0 \text{ if } j > -dim(S) = -s,$$

and for any $l > s$:

$$\mathcal{H}^j(\mathcal{R}^\bullet|_{S_l}) = 0 \text{ if } j > -l,$$

in particular for $j > -s$. So the natural maps:

$$\tau_{\leq -s} \mathcal{R}^\bullet \to \mathcal{R}^\bullet$$

and

$$\tau_{\leq -s-1} \beta^* \mathcal{R}^\bullet \to \beta^* \mathcal{R}^\bullet$$

are isomorphisms (in the derived category). Then $\mathcal{R}^\bullet \in D^{\leq -s}$ (where $(D^{\leq 0}, D^{\geq 0})$ is the natural t-structure on $D_c^b(U_s)$), and there is an isomorphism:

$$Hom_{D_c^b(U_s)}(\mathcal{R}^\bullet, \mathcal{P}^\bullet) \cong Hom_{D^{\leq -s}}(\mathcal{R}^\bullet, \tau_{\leq -s} \mathcal{P}^\bullet)$$

for any $\mathcal{P}^\bullet \in D_c^b(U_s)$. Therefore, by setting $\mathcal{P}^\bullet = \beta_* \beta^* \mathcal{R}^\bullet$, the natural adjunction map:

$$\mathcal{R}^\bullet \to \beta_* \beta^* \mathcal{R}^\bullet$$

admits a natural lifting:

$$t : \mathcal{R}^\bullet \to \tau_{\leq -s} \beta_* \beta^* \mathcal{R}^\bullet.$$

On the other hand, we have a natural map:

$$c : \mathcal{R}^\bullet \to \tau_{\geq -s} \mathcal{R}^\bullet \cong \mathcal{H}^{-s}(\mathcal{R}^\bullet)[s].$$

The support condition $\mathcal{H}^{-s}(\mathcal{R}^\bullet|_{S_l}) = 0$ if $l > s$ leads to:

$$supp\ \mathcal{H}^{-s}(\mathcal{R}^\bullet) \subset S.$$

Also, the constructibility of $\mathcal{R}^\bullet$ implies that the sheaves $\mathcal{H}^j(\mathcal{R}^\bullet)$ are local systems on $S$. In particular, the complex $\mathcal{H}^{-s}(\mathcal{R}^\bullet)[s]$ is perverse.

Now we will state and prove a key-ingredient for finding a decomposition of the peripheral complex (this lemma is adapted from [7], and the proof is almost identical; we include the proof for completeness):

**Lemma 3.1.** Splitting criterion
*Let everything be as above and suppose that the local systems $\mathcal{H}^{-s}(\alpha_!\alpha^!\mathcal{R}^\bullet)$ and $\mathcal{H}^{-s}(\mathcal{R}^\bullet)$ on $S$ have the same rank as (finite dimensional) rational vector spaces. Then the following are equivalent:*

(1) *$\mathcal{H}^{-s}(\alpha_!\alpha^!\mathcal{R}^\bullet) \to \mathcal{H}^{-s}(\mathcal{R}^\bullet)$ is an isomorphism of sheaves of $\Gamma$-modules;*
(2) *the map $t : \mathcal{R}^\bullet \to \tau_{\leq -s}\beta_*\beta^*\mathcal{R}^\bullet$ has a unique lifting $\bar{t} : \mathcal{R}^\bullet \to \tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet$ and the map:*

$$(\bar{t}, c) : \mathcal{R}^\bullet \to \tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet \oplus \mathcal{H}^{-s}(\mathcal{R}^\bullet)[s]$$

*is an isomorphism, i.e.,*

$$\mathcal{R}_Y^\bullet(\mathcal{L})_{|U_s} \cong \tau_{\leq -s-1}\beta_{s*}(\mathcal{R}_Y^\bullet(\mathcal{L})_{|U_{s+1}}) \oplus \mathcal{H}^{-s}(\mathcal{R}_Y^\bullet(\mathcal{L})_{|U_s})[s].$$

**Remark 3.2.** As we will see in the proof of the decomposition theorem below and using [5], Proposition 2.4, and the superduality isomorphism for algebraic links ([5], Corollary 3.4; [10], Theorem 5.1), the equal-rank condition required above is always satisfied. However, the lemma applies to any perverse, self-dual, torsion sheaf.

*Proof.* (adapted from [7])
Consider the distinguished triangle of cohomologically constructible complexes:

$$\tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet \to \tau_{\leq -s}\beta_*\beta^*\mathcal{R}^\bullet \to \mathcal{H}^{-s}(\beta_*\beta^*\mathcal{R}^\bullet)[s] \xrightarrow{[1]}$$

Apply the cohomological functor $Hom_{D_c^b(U_s)}(\mathcal{R}^\bullet, -)$ to the triangle and look at the corresponding long exact sequence:

$$\cdots \to Hom_{D_c^b(U_s)}^{-1}(\mathcal{R}^\bullet, \mathcal{H}^{-s}(\beta_*\beta^*\mathcal{R}^\bullet)[s]) \to Hom_{D_c^b(U_s)}(\mathcal{R}^\bullet, \tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet) \to$$

$$\to Hom_{D_c^b(U_s)}(\mathcal{R}^\bullet, \tau_{\leq -s}\beta_*\beta^*\mathcal{R}^\bullet) \to Hom_{D_c^b(U_s)}(\mathcal{R}^\bullet, \mathcal{H}^{-s}(\beta_*\beta^*\mathcal{R}^\bullet)[s]) \to \cdots$$

By perversity, we have:

$$Hom_{D_c^b(U_s)}^{-1}(\mathcal{R}^\bullet, \mathcal{H}^{-s}(\beta_*\beta^*\mathcal{R}^\bullet)[s]) = 0.$$

So, if a lifting $\bar{t}$ of $t$ exists, it must be unique. Such a lifting exists if and only if the image of $t$ in $Hom_{D_c^b(U_s)}(\mathcal{R}^\bullet, \mathcal{H}^{-s}(\beta_*\beta^*\mathcal{R}^\bullet)[s])$ is zero, which is equivalent to the fact that the natural map:

$$\mathcal{H}^{-s}(\mathcal{R}^\bullet) \xrightarrow{b} \mathcal{H}^{-s}(\beta_*\beta^*\mathcal{R}^\bullet)$$

is zero (see Lemma 2.1).
Consider the relevant piece of the long exact sequence associated to the attaching triangle:

$$\alpha_!\alpha^!\mathcal{R}^\bullet \to \mathcal{R}^\bullet \to \beta_*\beta^*\mathcal{R}^\bullet \xrightarrow{[1]}$$

namely,

$$\to \mathcal{H}^{-s}(\alpha_!\alpha^!\mathcal{R}^\bullet) \xrightarrow{a} \mathcal{H}^{-s}(\mathcal{R}^\bullet) \xrightarrow{b} \mathcal{H}^{-s}(\beta_*\beta^*\mathcal{R}^\bullet) \to$$

The map $b$ is trivial, i.e., the lifting exists, if and only if $a$ is surjective.

If a splitting as above exists, then $\bar{t}$ exists, hence $a$ is surjective, hence isomorphism because of the equal-rank condition.

Conversely, if $a$ is isomorphism, then we have a lifting $\bar{t}$. Use $\bar{t}$ and $c$ to define the map $\bar{t} \oplus c$ and to check that it is an isomorphism. The map $\bar{t}$ gives rise to an exact sequence in the abelian category $Perv(U_s)$:

$$0 \to \mathcal{K} \to \mathcal{R}^\bullet \xrightarrow{\bar{t}} \tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet \to \mathcal{Q} \to 0$$

Note that

$$\tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet \cong \beta_{!*}\beta^*\mathcal{R}^\bullet$$

is the intermediate extension of $\beta^*\mathcal{R}^\bullet$, hence it is a perverse sheaf on $U_s$ and has no non-trivial sub-object or quotient supported on a closed subvariety of $U_s \setminus U$ (see [2], 1.4.25, 2.1.11). Since $\mathcal{R}^\bullet$ and $\beta_{!*}\beta^*\mathcal{R}^\bullet$ are isomorphic on $U$, $\mathcal{Q}$ must have support contained in $U_s \setminus U$, hence it must be trivial, as a quotient of $\beta_{!*}\beta^*\mathcal{R}^\bullet$. The restriction of $\bar{t}$ to $U$ produces the map:

$$\beta^*\mathcal{R}^\bullet \to \beta^*\tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet \cong \tau_{\leq -s-1}\beta^*\beta_*\beta^*\mathcal{R}^\bullet \cong \tau_{\leq -s-1}\beta^*\mathcal{R}^\bullet.$$

Hence, by $\tau_{\leq -s-1}\beta^*\mathcal{R}^\bullet \cong \beta^*\mathcal{R}^\bullet$, $\mathcal{K}$ must be supported on $S$. We obtain a short exact sequence in $Perv(U_s)$, hence a distinguished triangle in $D_c^b(U_s)$ :

$$\mathcal{K} \to \mathcal{R}^\bullet \xrightarrow{\bar{t}} \tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet \xrightarrow{[1]}$$

so $\mathcal{K}$ must be cohomologically constructible (since the other two vertices of the triangle are so).

Since $\mathcal{K}$ is perverse and supported on the closed $s$-dimensional stratum $S$, by Lemma 2.2 we have that:

$$\mathcal{K} \cong \mathcal{H}^{-s}(\mathcal{K})[s].$$

By applying the cohomology functor to the above distinguished triangle we obtain the exact sequence:

$$0 \to \mathcal{H}^{-s-1}(\mathcal{R}^\bullet) \xrightarrow{d} \mathcal{H}^{-s-1}(\beta_*\beta^*\mathcal{R}^\bullet) \xrightarrow{e} \mathcal{H}^{-s}(\mathcal{K}) \to \mathcal{H}^{-s}(\mathcal{R}^\bullet) \to 0$$

Since $a$ is injective, and $\bar{t}$ is a lifting of $t$, $d$ is surjective (this follows from the cohomology long exact sequence of the attaching triangle), so $e = 0$. Therefore,

$$\mathcal{K} \cong \mathcal{H}^{-s}(\mathcal{K})[s] \cong \mathcal{H}^{-s}(\mathcal{R}^\bullet)[s].$$

Since (by using Lemma 2.1)

$$Hom_{D_c^b(U_s)}(\tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet, \mathcal{K}[1]) \cong Hom_{Sh(U_s)}(\mathcal{H}^{-s-1}(\beta_*\beta^*\mathcal{R}^\bullet), \mathcal{H}^{-s}(\mathcal{K}))$$

and $e = 0$, we see that the distinguished triangle

$$\mathcal{K} \to \mathcal{R}^\bullet \xrightarrow{\bar{t}} \tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet \xrightarrow{[1]}$$

splits, i.e., there is some isomorphism:

$$\mathcal{R}^\bullet \cong \tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet \oplus \mathcal{K} \cong \tau_{\leq -s-1}\beta_*\beta^*\mathcal{R}^\bullet \oplus \mathcal{H}^{-s}(\mathcal{R}^\bullet)[s].$$

So $\bar{t} \oplus c$ is an isomorphism.

$\square$

Using inductively the above lemma, by adding one stratum at a time, we obtain the following result on the decomposition of the peripheral complex $\mathcal{R}_Y^\bullet(\mathcal{L})$ in the category $Perv(Y)$:

**Theorem 3.3.**

$$\mathcal{R}_Y^\bullet(\mathcal{L}) \cong \sum_{l=0}^{n-1} IC_{\bar{S}_l}(\mathcal{H}^{-l}(\alpha_l^* \mathcal{R}_Y^\bullet(\mathcal{L}))) \cong \sum_{\mathcal{V}} IC_{\bar{V}}(\mathcal{H}^{-dim_\mathbb{C} V}(j_V^* \mathcal{R}_Y^\bullet(\mathcal{L})))$$

*if and only if, for all l, the natural maps:*

$$IH_{n-l-1}^{\bar{m}}(L_y, \Gamma) \to IH_{n-l-1}^{\bar{l}}(L_y, \Gamma)$$

*are isomorphisms of $\Gamma$-modules, where $L_y \cong S^{2n-2l-1}$ is the link of a point $y$ of a connected component $V$ of a stratum $S_l$, $\mathcal{V}$ is the set of components of singular strata of $Y$ (i.e., strata of $X$), $j_V$ is the inclusion map of $V$ in $Y$, and $\Gamma$ also denotes the local system defined by the linking number homomorphism on the link complement (cf. [5], [10]).*

*Proof.* Using the notations of the previous lemma, we need to translate the first condition of the lemma in more concrete terms.

Let $y$ be a point in $S$. We calculate the stalks of $\mathcal{H}^{-s}(\alpha_! \alpha^! \mathcal{R}^\bullet)$ and $\mathcal{H}^{-s}(\mathcal{R}^\bullet)$ at $y$.

$$\mathcal{H}^{-s}(\mathcal{R}^\bullet)_y \overset{(1)}{\cong} \mathcal{H}^{-s}(IC_{\bar{l}}^\bullet(Y, \mathcal{L}))_y \overset{(2)}{\cong} IH_{n-s-1}^{\bar{l}}(L_y, \Gamma),$$

where the isomorphism (1) follows from the defining triangle and the stalk calculation for $IC_{\bar{m}}^\bullet(Y, \mathcal{L})$: $\mathcal{H}^j(IC_{\bar{m}}^\bullet)_y = 0$ for $j > -s-1$; (2) is just the local computation of the stalk of the intersection homology complex $IC_{\bar{l}}^\bullet(Y, \mathcal{L})$ (see [3], (3.15)).

Since $\alpha$ is a closed embedding, we have: $\alpha_! \cong \alpha_*$. Also, if we denote by $i_y$ and $j_y$ the inclusions of the point $y$ in $S$ and $U_s$, respectively, then $\alpha \circ i_y = j_y$, $\alpha^* \alpha_* \cong id$ and $i_y^! \cong i_y^*[-2s]$. Using all of these, we have:

$$
\begin{aligned}
\mathcal{H}^{-s}(\alpha_! \alpha^! \mathcal{R}^\bullet)_y &\cong \mathcal{H}^{-s}(\alpha^* \alpha_! \alpha^! \mathcal{R}^\bullet)_y \\
&\cong \mathcal{H}^{-s}(\alpha^* \alpha_* \alpha^! \mathcal{R}^\bullet)_y \\
&\cong \mathcal{H}^{-s}(\alpha^! \mathcal{R}^\bullet)_y \\
&\cong H^{-s}(i_y^* \alpha^! \mathcal{R}^\bullet) \\
&\overset{(1)}{\cong} H^{-s}(i_y^* \alpha^! \mathcal{D}_{U_s} \mathcal{R}^\bullet[1]^{op}) \\
&\cong H^{-s+1}(\mathcal{D}_{\{y\}} i_y^! \alpha^* \mathcal{R}^{\bullet op}) \\
&\cong Ext(H^s(i_y^! \alpha^* \mathcal{R}^{\bullet op}); \Gamma) \oplus Hom(H^{s-1}(i_y^! \alpha^* \mathcal{R}^{\bullet op}); \Gamma) \\
&\cong Ext(H^s(i_y^* \alpha^* \mathcal{R}^\bullet[-2s]^{op}); \Gamma) \oplus Hom(H^{s-1}(i_y^* \alpha^* \mathcal{R}^\bullet[-2s]^{op}); \Gamma) \\
&\cong Ext(H^{-s}(j_y^* \mathcal{R}^{\bullet op}); \Gamma) \oplus Hom(H^{-s-1}(j_y^* \mathcal{R}^{\bullet op}); \Gamma) \\
&\overset{(2)}{\cong} Hom(\mathcal{H}^{-s}(\mathcal{R}^{\bullet op})_y, \mathbb{Q}(t)/\Gamma) \oplus Hom(\mathcal{H}^{-s-1}(\mathcal{R}^{\bullet op})_y, \Gamma) \\
&\overset{(3)}{\cong} Hom(\mathcal{H}^{-s}(\mathcal{R}^{\bullet op})_y, \mathbb{Q}(t)/\Gamma) \\
&\overset{(4)}{\cong} Hom(IH_{n-s-1}^{\bar{l}}(L_y, \Gamma)^{op}, \mathbb{Q}(t)/\Gamma) \\
&\overset{(5)}{\cong} IH_{n-s-1}^{\bar{m}}(L_y, \Gamma),
\end{aligned}
$$

where (1) follows from the self-duality of $\mathcal{R}^\bullet$, (2) and (3) follow from the fact that the stalks of $\mathcal{R}^\bullet$ are torsion $\Gamma$-modules, (4) is just the local calculation, and (5) is the superduality isomorphism (see [5], Theorem 3.2 and Corollary 3.4).

Since all the above isomorphism are natural, the first condition of the lemma, on stalks at points $y$ in $S$ becomes: the natural map

$$IH^{\bar{m}}_{n-s-1}(L_y, \Gamma) \to IH^{\bar{l}}_{n-s-1}(L_y, \Gamma)$$

is an isomorphism.

To end the proof of the theorem, use induction on strata and the Deligne's (axiomatic) characterization of the intersection cohomology complex ([12], [3]). $\square$

**Remark 3.4.** (1) An equivalent formulation of the above condition is that the superduality isomorphism on the links of singular strata induces a non-degenerate bilinear pairing:

$$IH^{\bar{m}}_{n-l-1}(L_y, \Gamma) \otimes IH^{\bar{m}}_{n-l-1}(L_y, \Gamma)^{op} \xrightarrow{\mathcal{B}_y} \mathbb{Q}(t)/\Gamma$$

where $L_y$ is the link of a point $y$ of a connected component of the stratum $S_l$.

(2) In the proof of the splitting criterion and the decomposition theorem above we only need the superduality isomorphism, which in turn is implied by the assumption that $X$ is a sub-pseudomanifold of finite local type of $Y$. This will then imply the self-duality of the peripheral complex and the fact that its stalks are torsion $\Gamma$-modules. Therefore, our decomposition theorem is true in the more general setting of [5]. Note the similarity between our theorem and Theorem 4.2 of [5]: under the assumption that the maps on the lower-middle intersection homology groups of the links are isomorphisms, the latter gives the same decomposition of the peripheral complex, only up to algebraic cobordism.

It remains to prove that the statement of Theorem 3.3 doesn't depend on the choice of the Whitney stratification $\mathcal{X}$ of $X$, i.e., it is invariant under refinements of the stratification of $X$.

For this, fix a Whitney stratification $\mathcal{X}$ and let $\mathcal{Y}$ be a refinement of $\mathcal{X}$. Then the link $L'$ of a stratum of $\mathcal{Y}$ has the form $L' \cong S^k * L$, where $L$ is the link of a stratum in the original stratification $\mathcal{X}$. (Here $A * B$ is the join of $A$ and $B$.) Only the case $k \geq 0$ is of interest and, if this is the case, $L'$ is a suspension of a compact pseudomanifold. We will make use of the formula for the intersection homology of a suspension (see [20] or [17]; the proof for local coefficients is the same):

$$IH^{\bar{p}}_i(\Sigma L; \Gamma) = \begin{cases} IH^{\bar{p}}_{i-1}(L; \Gamma), & i > l - \bar{p}(l) - 1 \\ 0, & i = l - \bar{p}(l) - 1 \\ IH^{\bar{p}}_i(L; \Gamma), & i < l - \bar{p}(l) - 1. \end{cases}$$

where $L$ is a compact pseudomanifold of (real) dimension $l$.

Now let $V^s$ be a component of a stratum of complex dimension $s$ of $\mathcal{Y}$ which is not a stratum in the original stratification, and say $W^r$ is a component of a stratum of $\mathcal{X}$ containing $V^s$. Hence, by our previous assumption, we will consider the case $r \geq s + 1$. The link $L'$ in $Y$ of a point $x \in V$ is $L'^{2n-2s-1} \cong S^{2r-2s-1} * L^{2n-2r-1}$, where $L^{2n-2r-1}$ is the link of $x$ regarded in $W$. By the above formula we obtain:

$$IH^{\bar{m}}_{n-s-1}(L', \Gamma) = 0.$$

(For a different proof of this fact, one can use [14], Proposition 3). On the other hand, by superduality on links, we have:

$$IH^{\bar{m}}_{n-s-1}(L', \Gamma)^{op} \cong IH^{\bar{l}}_{n-s-1}(L', \Gamma),$$

as torsion $\Gamma$-modules. Hence, we also have: $IH_{n-s-1}^{\bar{l}}(L', \Gamma) = 0$ and therefore, the map required to be an isomorphism in the theorem (applied for $\mathcal{Y}$), namely $IH_{n-s-1}^{\bar{m}}(L', \Gamma) \to IH_{n-s-1}^{\bar{l}}(L', \Gamma)$, is the map $0 \to 0$, so it doesn't bring any new condition to be satisfied for the decomposition of the peripheral complex to hold.

We also need to check that the new terms that appear in the decomposition of $\mathcal{R}_Y^\bullet(\mathcal{L})$ corresponding to $\mathcal{Y}$ are zero. More precisely, we will show that the complexes (on $Y$):
$\mathcal{A} := IC_{\bar{W}}(\mathcal{H}^{-r}(\mathcal{R}_Y^\bullet(\mathcal{L})|_W))$ (corresponding to $\mathcal{X}$) and
$\mathcal{B} := IC_{\overline{W \setminus V}}(\mathcal{H}^{-r}(\mathcal{R}_Y^\bullet(\mathcal{L})|_{W \setminus V})) \oplus IC_{\bar{V}}(\mathcal{H}^{-s}(\mathcal{R}_Y^\bullet(\mathcal{L})|_V))$ (corresponding to $\mathcal{Y}$)
are quasi-isomorphic. By [3], V.4.17, it follows that $\mathcal{A}$ and the first summand of $\mathcal{B}$ are quasi-isomorphic. On the other hand, the second summand of $\mathcal{B}$ is quasi-isomorphic to 0. Indeed, the stalk of $\mathcal{H}^{-s}(\mathcal{R}_Y^\bullet(\mathcal{L})|_V)$ at $x$ is $IH_{n-s-1}^{\bar{l}}(L_x'^{2n-2s-1}; \Gamma)$, which is zero as we have seen before.

3.2. **When does the decomposition hold? Examples.** In what follows, we will give examples of affine hypersurfaces for which the decomposition of the peripheral complex holds. As a convention, whenever we consider the intersection cohomology complexes associated to odd-dimensional spaces (e.g. links of complex algebraic varieties), we will adopt the indexing conventions from [12], according to which the restriction of the intersection homology complex to the dense open stratum is just the local system shifted by the *real* dimension of the space. For complex algebraic varieties we will keep using the indexing from [2], i.e., the restriction to the dense open stratum is the local system shifted by the *complex* dimension of the variety. We will consider as ambient space the affine space $\mathbb{C}^{n+1}$ (note the dimension change from $n$ to $n+1$).

Let $X$ be a hypersurface in $\mathbb{C}^{n+1}(= Y)$, a local coefficient system $\mathcal{L}$ defined on the complement of $X$ by the linking number with the hypersurface, and fix a Whitney stratification of $\mathbb{C}^{n+1}$ made of a Whitney stratification of $X$ and $\mathbb{C}^{n+1} \setminus X$. We will always work with the coarsest such Whitney stratification because, as noted before, the decomposition of the peripheral complex is invariant under refinements. Note that the local system $\mathcal{L}$ induces local coefficient systems (denoted by the symbol $\Gamma$ and defined by a linking number homomorphism as well) on complements of links of strata of $X$ in the pair $(\mathbb{C}^{n+1}, X)$.

(1) If $X$ is a *nonsingular hypersurface*, then we may consider the trivial Whitney stratification on $\mathbb{C}^{n+1}$, made of $X$ and its complement. The above decomposition of the peripheral complex $\mathcal{R}_Y^\bullet(\mathcal{L})$ holds trivially in this case. Indeed, the link of the stratum $X$ in $\mathbb{C}^{n+1}$ is a circle $S^1$, and $\mathcal{L}$ carries a generator of $\pi_1(S^1)$ to $t$; moreover, $IH_0^{\bar{m}}(S^1; \Gamma) = IH_0^{\bar{l}}(S^1; \Gamma) = H_0(S^1; \Gamma) = \mathbb{Q}$. In this case we obtain for the peripheral complex:

$$\mathcal{R}_Y^\bullet(\mathcal{L}) \cong IC_X(\mathbb{Q}) \cong \mathbb{Q}[n].$$

(2) If $X$ has a *manifold singularity*, let $s := \dim_{\mathbb{C}}(\mathrm{Sing}(X))$, and let $X^s$ be the singular locus of $X$, which is assumed to be a manifold. We assume that the filtration:

$$X^s \subset X \subset \mathbb{C}^{n+1}$$

is induced by a Whitney stratification of $\mathbb{C}^{n+1}$ with two singular strata: $S_n = X \setminus X^s$ and $S_s = X^s$ (this is the case if, for example, $X$ has only isolated singularities, i.e., $s = 0$).

The link pair of a component of $S_n$ is $(S^1, \emptyset)$ and the condition of the theorem is trivially satisfied. The link pair $(L_y, L_y \cap X)$ of a point $y$ in $S_s$ is homeomorphic to $(S^{2n-2s+1}, S^{2n-2s+1} \cap X = K)$, where $S^{2n-2s+1}$ is a small sphere centered at $y$ in a submanifold of $\mathbb{C}^{n+1}$ which is tranversal to $S_s$ and intersects $X^s$ only at $y$. This is a smooth algebraic knot, the link pair of an isolated singularity of a hypersurface in $\mathbb{C}^{n-s+1}$, therefore we have the following isomorphisms ([5], [10]):

$$IH_{n-s}^{\bar{m}}(L_y, \Gamma) \cong H_{n-s}(L_y \setminus (L_y \cap X); \Gamma) \cong H_{n-s}(C; \Gamma),$$

where $C$ is the complement of the interior of a tubular neighborhood of $K$ in $S^{2n-2s+1}$; and by superduality ([5]):

$$IH_{n-s}^{\bar{l}}(L_y, \Gamma) \cong H_{n-s}(C, \partial C; \Gamma).$$

Since, in general, the natural map $H_{n-s}(C; \Gamma) \to H_{n-s}(C, \partial C; \Gamma)$ is not an isomorphism of $\Gamma$-modules (even though, by superduality, the two modules have the same rank as rational vector spaces), the first *examples where the decomposition of the peripheral complex does not hold* appear for the case of a hypersurface with a manifold singularity.

We claim that, if $n - s \geq 2$ (hence $\pi_0(K) = 0$, by [27]) and $K$ is a rational homology sphere, then the above map is an isomorphism and the decomposition holds. In fact, the above map of $\Gamma$-modules will be surjective and this is enough for being an isomorphism (by the equal-rank condition). By [27], $K$ is a rational homology sphere if and only if the monodromy operator of the Milnor fibration associated to the isolated hypersurface singularity $X^n \cap \mathbb{C}^{n-s+1} \subset \mathbb{C}^{n-s+1}$ has no trivial eigenvalues. Here, and in the sequel, the eigenvalue 1 is referred to as the *trivial* eigenvalue.

Let $IC_{\bar{m}}^{\bullet}$ and $IC_{\bar{l}}^{\bullet}$ stand for $IC_{\bar{m}}^{\bullet}(S^{2n-2s+1}, \Gamma)$ and $IC_{\bar{l}}^{\bullet}(S^{2n-2s+1}, \Gamma)$, respectively. Also denote by $\mathcal{R}$ the associated peripheral complex. The decomposition of the peripheral complex $\mathcal{R}$ associated to the link of $S_s$ holds since this is a smooth link pair and proceed as in the previous case. More precisely, using the indexing notations of [12], we have:

$$\mathcal{R} \cong j_*^K IC_{\bar{m}}(K, \mathcal{H}^{-2n}(\mathcal{R}|_K))[1] \cong j_*^K IC_{\bar{m}}(K, \mathbb{Q})[1],$$

where $j^K$ is the inclusion map of the manifold $K$ in $S^{2n-2s+1}$. Therefore:

$$
\begin{aligned}
\mathbb{H}^{-n+s}(S^{2n-2s+1}; \mathcal{R}) &\cong \mathbb{H}^{-n+s+1}(K; IC_{\bar{m}}(K; \mathbb{Q})) \\
&\cong H^{-n+s+1}(K; \mathbb{Q}[2n-2s-1]) \\
&= H^{n-s}(K; \mathbb{Q}) \\
&\cong 0,
\end{aligned}
$$

the last isomorphism following from the assumption that $K$ is a rational homology sphere and $n - s \geq 2$. By applying the long exact hypercohomology sequence to the triangle defining the peripheral complex $\mathcal{R}$, we obtain that the map $IH_{n-s}^{\bar{m}}(L_y, \Gamma) \to IH_{n-s}^{\bar{l}}(L_y, \Gamma)$ is onto, as desired.

(3)  The next case to study is that of a hypersurface $X$ with *two singular strata*. Suppose that $X$ has an s-dimensional singular locus, $X^s$, and assume that the filtration

$$\{p\} \subset X^s \subset X \subset \mathbb{C}^{n+1}$$

corresponds to a Whitney stratification of the ambient space. As in the previous cases, the link pairs of components of the stratum $S_n$ satisfy the condition stated in the theorem. Moreover, the same is true for the stratum $S_s$ if, for example, $n - s \geq 2$ and the monodromy operator of the Milnor fibration associated to the isolated hypersurface singularity $X^n \cap \mathbb{C}^{n-s+1} \subset \mathbb{C}^{n-s+1}$ has no trivial eigenvalues. Let's assume this in what follows.

The link pair of $S_0 = \{p\}$ is homeomorphic to a singular knot $(S^{2n+1}, S^{2n+1} \cap X)$ where $S^{2n+1} = L_{\{p\}}$ is a small sphere centered at $p$ in $\mathbb{C}^{n+1}$. If $K := S^{2n+1} \cap X$ and $\Sigma := S^{2n+1} \cap X^s$, then $L_{\{p\}}$ has a stratification as a topological pseudomanifold induced by the filtration:

$$\Sigma \subset K \subset S^{2n+1},$$

and $\dim_{\mathbb{R}} K = 2n - 1$, $\dim_{\mathbb{R}} \Sigma = 2s - 1$.

For simplicity, we assume that $p$ is the origin in $\mathbb{C}^{n+1}$ and we will assume that the above stratification satisfies the following conditions:

- the monodromy operators of the Milnor fibrations associated to the link pairs of components of $S_s = X^s - \{0\}$ have no trivial eigenvalues ($n - s \geq 2$).
- 
  $h_q - I : H_q(F; \mathbb{Q}) \to H_q(F; \mathbb{Q})$ is an isomorphism for $n - r \leq q \leq n$,

  where $h$ and $F$ are the monodromy and respectively the Milnor fiber of the Milnor fibration at the origin, and we assume that $F$ is $(n-r-1)$-connected, for some $r \leq s$.
- $X^s$ is a complete intersection at the origin ($s \geq 2$) and $\delta(1) \neq 0$, where

  $$\delta(t) := det(\bar{h}_* - tI : H_s(\bar{F}; \mathbb{Q}) \to H_s(\bar{F}; \mathbb{Q}))$$

  and $\bar{h}$ and $\bar{F}$ are the monodromy and respectively the Milnor fiber associated to the isolated singularity of the complete intersection $X^s$ at the origin.

**Claim 3.5.** *Under the above assumptions, the map:*

$$IH_n^{\bar{m}}(S^{2n+1}, \Gamma) \to IH_n^{\bar{l}}(S^{2n+1}, \Gamma)$$

*is an isomorphism of $\Gamma$-modules.*

In order to be able to formulate a more general result, we first interpret the above conditions:

Since the monodromy operator of the Milnor fibration associated to the smooth link pair of a point in a component of $S_s$ has no trivial eigenvalues, the associated smooth knot $X \cap S^{2n-2s+1}$ is a rational homology sphere by [27].

The link pair of the origin $(L_{\{0\}} \cong S_\epsilon^{2n+1}, K)$ is singular and admits a filtration:

$$\Sigma = \Sigma^{2s-1} \subset K = K^{2n-1} \subset S^{2n+1}.$$

If the Milnor fiber $F$ at the origin is $(n - r - 1)$-connected ($r \leq s$), then by the Wang sequence of the Milnor fibration and by Alexander duality we get ([29]):

$$H^q(K) \cong 0, \ q > n + r.$$

Since $K$ is $(n - 2)$-connected ([27]), we see that failure of Poincaré duality for $K$ is measured by the homomorphisms $h_q - Id$. In particular we have, as a generalization of [27], Theorem 8.5, the following (see [29]):

**Proposition 3.6.** *For $n \neq 2$ ($n = 2$) the simplicial complex $K$ is homotopy ($\mathbb{Z}$-homology) equivalent to $S^{2n-1}$ if and only if*

$$h_q - Id : H_q(F) \to H_q(F) \text{ is an isomorphism for } n - r \leq q \leq n.$$

Therefore, by the Poincaré conjecture in high dimensions, if $n \geq 3$, $K$ is homeomorphic to $S^{2n-1}$ if and only if:

$$h_q - Id : H_q(F) \to H_q(F) \text{ is an isomorphism for } n - r \leq q \leq n.$$

Also, for $K$ to be a rational homology sphere is enough to require $n \geq 2$ and that the characteristic polynomials of the monodromy operators in the above range do not vanish at 1 (i.e., the monodromy operators have no trivial eigenvalues).

Now assume that $X^s$ is a *complete intersection*. Since we work in a neighborhood of the origin, it suffices to consider $X^s$ a complete intersection at the origin. The origin is at most an isolated singularity for $X^s$, so $\Sigma$ is a manifold. By [21], Corollary 3.2.1, $\Sigma$ is $(s-2)$-connected. We can choose holomorphic functions $f_1, \ldots, f_p$ generating the ideal of $X^s$ at the origin such that $p = n + 1 - s$ and such that $f_1, \ldots, f_{p-1}$ define a complete intersection $X^* \subset \mathbb{C}^{n+1}$ at the origin, having at most an isolated singularity at the origin (see [21], 1.4). Let $\Sigma^* := X^* \cap S_\epsilon^{2n+1}$. Then the map:

$$\frac{f_p}{|f_p|} : \Sigma^* \setminus \Sigma \xrightarrow{\phi} S^1$$

is a smooth locally trivial fibration ([21], Theorem 2.1.1).
Since $X^s$ and $X^*$ are complete intersections at the origin and $X^s \setminus \{0\}$ and $X^* \setminus \{0\}$ are regular, the fiber $\bar{F}$ of $\phi$ has the homotopy type of a bouquet of $s$-spheres ([21], Corollary 4.1.8). Let $\bar{h} : \bar{F} \to \bar{F}$ be the monodromy of the above fibration. The Wang exact sequence of the fibration gives the exact sequence (with $\mathbb{Z}$-coefficients):

$$0 \to H_{s+1}(\Sigma^* \setminus \Sigma) \to H_s(\bar{F}) \xrightarrow{\bar{h}_* - I} H_s(\bar{F}) \to H_s(\Sigma^* \setminus \Sigma) \to 0.$$

The homology long exact sequence of the pair $(\Sigma^*, \Sigma^* \setminus \Sigma)$, the Alexander duality, Poincaré duality and the fact that $\Sigma^*$ is $(s-1)$-connected and $\Sigma$ is $(s-2)$-connected, yield the following exact sequence:

$$0 \to \tilde{H}^{s-1}(\Sigma) \to H_{s+1}(\Sigma^* \setminus \Sigma) \to H_{s+1}(\Sigma^*) \to H^s(\Sigma) \to H_s(\Sigma^* \setminus \Sigma) \to H_s(\Sigma^*) \to 0$$

If $\delta(t) := det(\bar{h}_* - tI)$, then $\delta(1) = \pm 1$ if and only if $\Sigma^*$ and $\Sigma$ are $\mathbb{Z}$-homology spheres. Therefore, for $s \geq 3$, $\Sigma^*$ and $\Sigma$ are topological spheres if and only if $\delta(1) = \pm 1$. Also, if $s \geq 2$ and $\delta(1) \neq 0$ then $\Sigma$ is a rational homology sphere.

*Therefore, under our assumptions (with $n - s \geq 2$ and $s \geq 2$), the link pairs of strata of $(\mathbb{C}^{n+1}, X)$ are filtered by rational homology spheres.*

*Proof of the Claim.*
Since we will be working with intersection homology complexes of stratified spaces with odd dimensional strata, it is more convenient to use from now on the indexing conventions from [12].
By the above assumptions, we work with the stratification of $L_{\{0\}} \cong S^{2n+1}$ corresponding to the filtration:

$$\Sigma = \Sigma^{2s-1} \subset K = K^{2n-1} \subset S^{2n+1},$$

and recall that $K$ and $\Sigma$ are rational homology spheres. Because of the superduality isomorphism, the $\Gamma$-modules $IH_n^{\bar{m}}(L_{\{0\}}, \Gamma)$ and $IH_n^{\bar{l}}(L_{\{0\}}, \Gamma)$ have the same rank as $\mathbb{Q}$-vector spaces. Therefore it suffices to show that the map in question is onto.

Let $IC_{\bar{m}}^{\bullet}$ and $IC_{\bar{l}}^{\bullet}$ stand for $IC_{\bar{m}}^{\bullet}(S^{2n+1}, \Gamma)$ and $IC_{\bar{l}}^{\bullet}(S^{2n+1}, \Gamma)$ respectively. Also denote by $\mathcal{R}$ the associated peripheral complex. The decomposition of the peripheral complex associated to the link pair of the origin holds since the link pairs of strata of $S^{2n+1}$ are links of the singular strata of the hypersurface for which our condition is already satisfied (by the assumptions we made). More precisely, we have:

$$\mathcal{R} \cong j_*^K IC_{\bar{m}}(K, \mathcal{H}^{-2n}(\mathcal{R}|_{K \setminus \Sigma}))[1] \oplus j_*^{\Sigma} IC_{\bar{m}}(\Sigma, \mathcal{H}^{-n-s}(\mathcal{R}|_{\Sigma}))[n - s + 1],$$

where $j^K$ and $j^{\Sigma}$ are the inclusion maps of $K$ and $\Sigma$ in $S^{2n+1}$. By applying the long exact hypercohomology sequence to the triangle defining the peripheral complex, we obtain:

$$\cdots \to \mathbb{H}^{-n}(S^{2n+1}; IC_{\bar{m}}^{\bullet}) \to \mathbb{H}^{-n}(S^{2n+1}; IC_{\bar{l}}^{\bullet}) \to \mathbb{H}^{-n}(S^{2n+1}; \mathcal{R}) \to$$
$$\to \mathbb{H}^{-n+1}(S^{2n+1}; IC_{\bar{m}}^{\bullet}) \to \cdots$$

Note that the map: $\mathbb{H}^{-n}(S^{2n+1}; \mathcal{R}) \to \mathbb{H}^{-n+1}(S^{2n+1}; IC_{\bar{m}}^{\bullet})$ is the zero map. This follows from the fact that $\mathbb{H}^{-n}(S^{2n+1}; \mathcal{R})$ is the abutment of a spectral sequence whose $E_2$ term is $E_2^{p,q} = H^p(S^{2n+1}; \mathcal{H}^q(\mathcal{R}))$, and similarly, $\mathbb{H}^{-n+1}(S^{2n+1}; IC_{\bar{m}}^{\bullet}) = \mathbb{H}^{-n}(S^{2n+1}; IC_{\bar{m}}^{\bullet}[1])$ is the abutment of the spectral sequence with $E_2$ term defined by $E_2'^{p,q} = H^p(S^{2n+1}; \mathcal{H}^q(IC_{\bar{m}}^{\bullet}[1])) = H^p(S^{2n+1}; \mathcal{H}^{q+1}(IC_{\bar{m}}^{\bullet}))$. Moreover, the map in discussion is induced by the map on coefficients $\mathcal{H}^q(\mathcal{R}) \to \mathcal{H}^{q+1}(IC_{\bar{m}}^{\bullet})$. But the latter is the zero map since the peripheral complex is supported on $K$, while $IC_{\bar{m}}^{\bullet}$ is only supported on $S^{2n+1} \setminus K$.

In order to prove the last claim, we calculate the stalk cohomology of the complex $IC_{\bar{m}}^{\bullet}$. For $x \in \Sigma$ we have

$$\mathcal{H}^q(IC_{\bar{m}}^{\bullet}(S^{2n+1}, \Gamma))_x \cong \begin{cases} IH_{-q-2s}^{\bar{m}}(L_x; \Gamma), & q \leq -n - s - 1 \\ 0, & q > -n - s - 1, \end{cases}$$

where $L_x$ is the link of the stratum $\Sigma$ in $S^{2n+1}$. Since $L_x$ is homeomorphic to the link of $X^s$ in $\mathbb{C}^{n+1}$, the link pair of $x$ is a smooth algebraic knot $(L_x, K_x)$, therefore for all $j$:

$$IH_j^{\bar{m}}(L_x; \Gamma) \cong H_j(L_x \setminus K_x; \Gamma).$$

On the other hand, using a Milnor fibration argument for $L_x \cong S^{2n-2s+1}$ (see [27]), we see that

$$H_j(L_x \setminus K_x; \Gamma) = 0 \quad \text{for} \quad j > n - s.$$

Altogether, $\mathcal{H}^q(IC_{\bar{m}}^{\bullet}|\Sigma) = 0$ for all $q$.

The link pair of the stratum $K \setminus \Sigma$ in $S^{2n+1}$ is $(S^1, \emptyset)$ and the local calculation on stalks yields:

$$IC_{\bar{m}|K \setminus \Sigma}^{\bullet} \cong 0.$$

Thus,

$$IC_{\bar{m}|K}^{\bullet} \cong 0$$

and $IC_{\bar{m}}^{\bullet}$ is supported on $S^{2n+1} \setminus K$.

Therefore, from the long exact sequence of hypercohomology, the natural homomorphism $IH_n^{\bar{m}}(S^{2n+1}, \Gamma) \to IH_n^{\bar{l}}(S^{2n+1}, \Gamma)$ is an isomorphism of $\Gamma$-modules if and

only if $\mathbb{H}^{-n}(S^{2n+1}; \mathcal{R})$ is the zero module, which from the above decomposition is equivalent to asking that:

$$IH_{n-1}^{\bar{m}}(K; \mathbb{Q}) = 0$$

(here, $\mathbb{Q} \cong \Gamma/(t-1) \cong IH_0^{\bar{l}}(S^1; \Gamma)$ is the trivial local system defined on $K \setminus \Sigma$) and

$$IH_{s-1}^{\bar{m}}(\Sigma; \mathcal{IH}_{n-s}^{\bar{l}}(S^{2n-2s+1}; \Gamma)) = 0,$$

where $\mathcal{IH}_{n-s}^{\bar{l}}(S^{2n-2s+1}; \Gamma)$ is a local coefficient system on $\Sigma$, whose stalk at $x \in \Sigma$ is the torsion $\Gamma$-module $IH_{n-s}^{\bar{l}}(S_x^{2n-2s+1}; \Gamma)$, for $S_x^{2n-2s+1}$ the link of $x \in \Sigma$ in $S^{2n+1}$, i.e., the link of the component of $x \in X^s$ in $\mathbb{C}^{n+1}$.

Assume $s > 2$. Then $\Sigma$ is a manifold and a simply-connected rational homology sphere, and the second condition above is trivially satisfied. Moreover, we have:

$$IH_i^{\bar{m}}(\Sigma; \mathcal{IH}_{n-s}^{\bar{l}}(S^{2n-2s+1}; \Gamma)) \cong \begin{cases} 0, & 0 < i < 2s-1 \\ IH_{n-s}^{\bar{l}}(S^{2n-2s+1}; \Gamma), & i = 2s-1. \end{cases}$$

In order to verify the first condition, we note that $K$ is a (simply-connected) rational homology sphere and it is known that the rational homology and intersection homology of $K$ agree. In our settings, we can prove the last assertion as follows. By the long exact sequence of compactly supported hypercohomology for the decomposition $K = (K \setminus \Sigma) \sqcup \Sigma$ we obtain the following (where we use $IC$ for $IC_{\bar{m}}^\bullet(K; \mathbb{Q})$):

$$\cdots \to \mathbb{H}^{-n}(\Sigma; IC) \to \mathbb{H}_c^{-n+1}(K \setminus \Sigma; IC) \to \mathbb{H}^{-n+1}(K; IC) \to$$
$$\to \mathbb{H}^{-n+1}(\Sigma; IC) \to \mathbb{H}_c^{-n+2}(K \setminus \Sigma; IC) \to \cdots$$

Moreover, if we consider the long exact sequence of compactly supported cohomology for the covering $K = (K \setminus \Sigma) \sqcup \Sigma$, we obtain:

$$\cdots \to H^{n-1}(\Sigma; \mathbb{Q}) \to H_c^n(K \setminus \Sigma; \mathbb{Q}) \to H^n(K; \mathbb{Q}) \to$$
$$\to H^n(\Sigma; \mathbb{Q}) \to H_c^{n+1}(K \setminus \Sigma; \mathbb{Q}) \to \cdots$$

There is a natural map between the two exact sequences above, induced by the sheaf map $\mathbb{Q}[2n-1] \to IC$. Therefore we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^{n-1}(\Sigma; \mathbb{Q}) & \longrightarrow & H_c^n(K \setminus \Sigma; \mathbb{Q}) & \longrightarrow & H^n(K; \mathbb{Q}) & \longrightarrow \\
& \downarrow{\scriptstyle(1)} & & \downarrow{\scriptstyle(2)} & & \downarrow{\scriptstyle(3)} & \\
\cdots \longrightarrow & \mathbb{H}^{-n}(\Sigma; IC) & \longrightarrow & \mathbb{H}_c^{-n+1}(K \setminus \Sigma; IC) & \longrightarrow & \mathbb{H}^{-n+1}(K; IC) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^n(\Sigma; \mathbb{Q}) & \longrightarrow & H_c^{n+1}(K \setminus \Sigma; \mathbb{Q}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle(4)} & & \downarrow{\scriptstyle(5)} & \\
\longrightarrow & \mathbb{H}^{-n+1}(\Sigma; IC) & \longrightarrow & \mathbb{H}_c^{-n+2}(K \setminus \Sigma; IC) & \longrightarrow \cdots
\end{array}$$

The stalk of the intersection complex $IC_{\bar{m}}^\bullet(K; \mathbb{Q})$ at a point $x \in \Sigma$ is:

$$\mathcal{H}^q(IC)_x \cong \begin{cases} IH_{-q-2s}^{\bar{m}}(L_x; \mathbb{Q}), & q \leq -n-s \\ 0, & q > -n-s, \end{cases}$$

where $L_x$ is the link of the stratum $\Sigma$ in $K$, i.e., the intersection of $K$ with the link of $\Sigma$ in $S^{2n+1}$; and this is a manifold rational homology sphere by our assumptions, therefore $IH^{\bar{m}}_{-q-2s}(L_x;\mathbb{Q}) = H_{-q-2s}(L_x;\mathbb{Q})$, which is zero, unless $-q - 2s = 2n - 2s - 1$, i.e., $q = -2n+1$, in which case it is $\mathbb{Q}$ (we don't consider the case $-q - 2s = 0$ since we already work under $q \leq -n-s$). Therefore we obtain a quasi-isomorphism:

$$IC^{\bullet}_{\bar{m}}(K;\mathbb{Q})|_{\Sigma} \cong \mathbb{Q}_{\Sigma}[2n-1].$$

Thus, by the hypercohomology spectral sequence, the maps (1) and (4) are isomorphisms. On the other hand, by the definition of the intersection homology complex, $IC^{\bullet}_{\bar{m}}(K;\mathbb{Q})|_{K\setminus\Sigma} \cong \mathbb{Q}[2n-1]$. Hence, the maps (2) and (5) are isomorphisms. By the five-lemma, we obtain the isomorphism:

$$0 \cong H^n(K;\mathbb{Q}) \cong \mathbb{H}^{-n+1}(K;IC) = IH^{\bar{m}}_{n-1}(K;\mathbb{Q}),$$

thus proving our claim.□

Recall that by our assumptions $K$ is a rational homology sphere of dimension $2n - 1$ and $n > s > 2$ (thus $K$ is also simply-connected). From the above local computation, we obtain the following quasi-isomorphism:

$$IC^{\bullet}_{\bar{m}}(K;\mathbb{Q}) \cong \mathbb{Q}[2n-1].$$

Hence, by applying the hypercohomology functor, we have:

$$IH^{\bar{m}}_i(K;\mathbb{Q}) := \mathbb{H}^{-i}(K;IC^{\bullet}_{\bar{m}}(K;\mathbb{Q})) \cong \mathbb{H}^{-i}(K;\mathbb{Q}[2n-1]) =$$
$$= H^{2n-1-i}(K;\mathbb{Q}) \cong H_i(K;\mathbb{Q}) = \begin{cases} 0, & 0 < i < 2n - 1 \\ \mathbb{Q}, & i = 2n - 1. \end{cases}$$

Therefore we have proved the following:

**Proposition 3.7.** *Assume that $X$ is a hypersurface with two singular strata, and let $\{0\} \subset X^s \subset X^n = X \subset \mathbb{C}^{n+1}$ be the filtration associated to the corresponding Whitney stratification of the pair $(\mathbb{C}^{n+1}, X)$. Suppose that the link of the origin has the following induced filtration: $\Sigma^{2s-1} \subset K^{2n-1} \subset S^{2n+1}$. If $n - s \geq 2$ and $s > 2$, the decomposition of the peripheral complex associated to the embedding $X \subset \mathbb{C}^{n+1}$ holds (as in Theorem 3.3) provided that:*

- *the monodromy operators of the Milnor fibrations associated to the link pairs of components of $X^s - \{0\}$ have no trivial eigenvalues.*
- 

  $h_q - I : H_q(F;\mathbb{Q}) \to H_q(F;\mathbb{Q})$ *is an isomorphism for $n - r \leq q \leq n$,*

  *where $h$ and $F$ are the monodromy and respectively the Milnor fiber of the Milnor fibration at the origin and we assume that $F$ is $(n-r-1)$-connected for some $r \leq s$.*
- *$X^s$ is a complete intersection at the origin and $\delta(1) \neq 0$, where*

  $$\delta(t) := det(\bar{h}_* - tI : H_s(\bar{F};\mathbb{Q}) \to H_s(\bar{F};\mathbb{Q}))$$

  *and $\bar{h}$ and $\bar{F}$ are the monodromy and respectively the Milnor fiber associated to the isolated singularity of the complete intersection $X^s$ at the origin.*

**Remark 3.8.** By taking transversal intersections, a similar result can be stated for any affine hypersurface with two singular strata.

(4)  **More generally**, let $X$ be a complex hypersurface in $\mathbb{C}^{n+1}$ and suppose that

$$\mathbb{C}^{n+1} \supset X = Y_n \supset Y_{n-1} \supset \cdots \supset Y_1 \supset Y_0 \supset Y_{-1} = \emptyset$$

is the filtration associated to a Whitney stratification of $\mathbb{C}^{n+1}$, made of the hypersurface complement and a Whitney stratification of $X$. Assume $\dim_{\mathbb{C}} Y_l = l$ and denote the pure strata by $S_l := Y_l \setminus Y_{l-1}$ (which are either $\emptyset$ or locally closed algebraic subsets of pure dimension $l$; the components of $S_l$ are a finite number of nonsingular algebraic varieties). The local system of coefficients $\mathcal{L}$ on $\mathbb{C}^{n+1} \setminus X$ is defined as before by the linking number with $X$.

We are looking for sufficient conditions such that the condition stated in the decomposition theorem on link pairs of singular strata is satisfied. It is trivially satisfied on the links of components of the stratum $S_n$. We will use an induction on the dimension of the links of singular strata as follows. Suppose that we find sufficient conditions such that the map on lower-middle intersection homology groups of the link of $S_j$, $j > k$, is an isomorphism of $\Gamma$-modules and we want to do the same for the link $L_y \cong S^{2n-2k+1}$ of a point $y \in S_k$. This link is stratified by the transversal intersections with the strata of $X$. Moreover, after intersecting transversally with a generic linear subspace $\mathbb{C}^{n-k+1}$, we may assume that $k = 0$ and $S_k = S_0 = \{y\}$. Since the link pairs of a link of a stratum are also link pairs in the original pseudomanifold, the link pair of $y$ will be of finite local type and local type in the sense of [5], hence the superduality isomorphism for the intersection complexes $IC_{\bar{m}}^{\bullet}(L_y; \Gamma)$ and $IC_{\bar{l}}^{\bullet}(L_y; \Gamma)$ holds. The splitting criterion can be used again to give, in the notations of [12], the following decomposition theorem for the peripheral complex $\mathcal{R}$ associated with the link pair $(L_y, L_y \cap X)$:

**Proposition 3.9.**

$$\mathcal{R} \cong \sum_{V \in \mathcal{V}} j_* IC_{\bar{m}}^{\bullet}(\bar{V}, \mathcal{H}^{c(V)-m}(j_V^* \mathcal{R}))[c(V)]$$

*where $\mathcal{V}$ is the set of components of singular strata of $L_y$, $m = 2n + 1 = dim(L_y)$, $c(V) = \frac{1}{2} codim(V)$, and $j$, $j_V$ are the inclusion maps of $\bar{V}$ and $V$ respectively.*

**Remark 3.10.** Using the inductive hypothesis it is easy to check that, for $V \in \mathcal{V}$, the complex:

$$j_* IC_{\bar{m}}^{\bullet}(\bar{V}, \mathcal{H}^{c(V)-m}(j_V^* \mathcal{R}))[c(V)]$$

is a perverse, torsion, self-dual sheaf on $L_y$ (in the sense of [5]). Moreover, the condition on the links of $L_y$ stated in the decomposition theorem will always be satisfied by the inductive hypothesis.

Since by superduality the groups $IH_n^{\bar{m}}(L_y, \Gamma)$ and $IH_n^{\bar{l}}(L_y, \Gamma)$ have the same rank as rational vector spaces, in order to show that the natural map $IH_n^{\bar{m}}(L_y, \Gamma) \to IH_n^{\bar{l}}(L_y, \Gamma)$ is an isomorphism of $\Gamma$-modules, it suffices to show that it is an epimorphism. From the defining triangle of the peripheral complex $\mathcal{R}$ for the pair $(L_y, L_y \cap X)$, this will be the case if and only if $\mathbb{H}^{-n}(L_y; \mathcal{R}) = 0$ (this follows as before by studying the map $\mathcal{H}^q(\mathcal{R}) \to \mathcal{H}^{q+1}(IC_{\bar{m}}^{\bullet}(L_y; \Gamma))$), which yields that the morphism $\mathbb{H}^{-n}(L_y; \mathcal{R}) \to \mathbb{H}^{-n+1}(L_y; IC_{\bar{m}}^{\bullet}(L_y; \Gamma))$ is the zero map; see [26], Lemma 3.2 for a calculation of the support of the middle-perversity intersection homology complex of an algebraic link pair). By applying the hypercohomology functor in the decomposition given by the above proposition, this is equivalent to:

$$IH_{r-1}^{\bar{m}}(\Sigma^{2r-1}; \mathcal{IH}_{n-r}^{\bar{l}}(L_x, \Gamma)) = 0 \ \text{ for } \ 1 \le r \le n,$$

where $\Sigma^{2r-1}$ is the closure of a $(2r-1)$-dimensional singular stratum of $L_y \cong S^{2n+1}$, and $L_x \cong S^{2n-2r+1}$ is the link of $\Sigma^{2r-1}$ in $L_y$, i.e., the link of $Y_r$ in $\mathbb{C}^{n+1}$. Here $\mathcal{IH}^{\bar{l}}_{n-r}(L_x, \Gamma)$ is a local system of coefficients defined on the top (dense) stratum of $\Sigma^{2r-1}$, whose stalk at a point $x$ is $IH^{\bar{l}}_{n-r}(L_x, \Gamma)$.

The main theorem of this section asserts that under 'reasonable' assumptions, the peripheral complex splits:

**Theorem 3.11.** *Let $X$ be a reduced hypersurface in $\mathbb{C}^{n+1}$ and suppose that all the links of singular strata of $X$ in a stratification of the pair $(\mathbb{C}^{n+1}, X)$ are filtered by (simply-connected) rational homology spheres, and have simply-connected singular strata. Then the peripheral complex $\mathcal{R}^\bullet(\mathcal{L})$ associated with $X$ splits.*

**Remark 3.12.** Note that we require that $X$ is a rational homology manifold.

*Proof.* Under our assumptions, it follows that the local system defined on the top stratum of $\Sigma^{2r-1}$ is a constant sheaf $\mathcal{G}$, whose stalk is the finite dimensional rational vector space ([5]) $G := IH^{\bar{l}}_{n-r}(L_x, \Gamma)$. We will show inductively that:

$$(*) \qquad IC^\bullet_{\bar{m}}(\Sigma^{2r-1}; \mathcal{G}) \cong \mathcal{G}[2r-1]$$

where the above isomorphism holds in the derived category of bounded, constructible sheaves on $\Sigma^{2r-1}$, and will thus prove the following:

$$
\begin{aligned}
IH^{\bar{m}}_i(\Sigma^{2r-1}; \mathcal{G}) &\cong \mathbb{H}^{-i}(\Sigma^{2r-1}; IC^\bullet_{\bar{m}}(\Sigma^{2r-1}; \mathcal{G})) \\
&\cong \mathbb{H}^{-i}(\Sigma^{2r-1}; \mathcal{G}[2r-1]) \\
&\cong \mathbb{H}^{2r-1-i}(\Sigma^{2r-1}; \mathcal{G}) \\
&\cong H^{2r-1-i}(\Sigma^{2r-1}; G) \\
&\cong \begin{cases} 0, & 0 < i < 2r-1 \\ G = IH^{\bar{l}}_{n-r}(L_x, \Gamma), & i = 2r-1. \end{cases}
\end{aligned}
$$

The inductive hypothesis will be the following: the links of dimension $\leq 2n-1$ (i.e., the links of strata $S_j$, $j > 0$) are filtered by rational homology spheres which satisfy the condition $(*)$. As the first step of induction, see the case of hypersurfaces with manifold singularities described above.

As the link $L_y \cong S^{2n+1}$ is stratified by the transversal intersections with the strata of $X$, $\Sigma^{2r-1}$ is a union of strata of $L_y$, and inherits a pseudomanifold stratification. Let $\Sigma^{2s-1} \supset \Sigma^{2l-1}$, $1 \leq s < r$ be two consecutive terms in the filtration of $\Sigma^{2r-1}$. The stalk cohomology of the intersection complex $IC^\bullet_{\bar{m}}(\Sigma^{2r-1}; \mathcal{G})$ at a point $z \in \Sigma^{2s-1} \setminus \Sigma^{2l-1}$ is given by the local calculation formula:

$$
\mathcal{H}^q(IC^\bullet_{\bar{m}}(\Sigma^{2r-1}; \mathcal{G}))_z \cong \begin{cases} 0, & q > -r-s \\ IH^{\bar{m}}_{-q-2s}(L_z; \mathcal{G}_{|L_z}), & q \leq -r-s, \end{cases}
$$

where $L_z$ is the link of $\Sigma^{2s-1}$ in $\Sigma^{2r-1}$, i.e., $L_z \cong Y_r \cap S^{2n-2s+1}_z$, for $S^{2n-2s+1}_z$ the link of $Y_s$ in $\mathbb{C}^{n+1}$. Since $2n - 2s + 1 \leq 2n - 1$, by the inductive hypothesis, $L_z$ is a rational homology sphere of dimension $2r - 2s - 1$, which satisfies the condition $(*)$. Therefore, $IH^{\bar{m}}_{-q-2s}(L_z; \mathcal{G}_{|L_z}) = 0$ unless $-q - 2s = 2r - 2s - 1$, i.e., $q = -2r+1$, in which case it is $G$. Therefore,

$$
IC^\bullet_{\bar{m}}(\Sigma^{2r-1}; \mathcal{G})|_{\Sigma^{2s-1} \setminus \Sigma^{2l-1}} \cong G[2r-1].
$$

The same reasoning applies to all the strata of $\Sigma^{2r-1}$, thus obtaining the condition
$(*)$ for $IC_{\bar{m}}^{\bullet}(\Sigma^{2r-1}; \mathcal{G})$.

$\square$

## 4. ALEXANDER MODULES OF HYPERSURFACE COMPLEMENTS. A CONJECTURE.

In [26], we discuss the Alexander modules of hypersurface complements. We recall here the main constructions and results.

Let $V$ be a degree $d$ reduced hypersurface in $\mathbb{CP}^{n+1}$, which is transversal to the hyperplane at infinity, $H$. Let $\mathcal{S}$ be a Whitney stratification of $V$. This yields a Whitney stratification of the pair $(\mathbb{CP}^{n+1}, V)$. By the transversality condition, we also obtain a stratification of $(\mathbb{CP}^{n+1}, V \cup H)$. Let $\mathcal{U}$ denote the (affine) hypersurface complement $\mathcal{U} := \mathbb{CP}^{n+1} - (V \cup H)$. Define a local system $\mathcal{L}_H$ on $\mathcal{U}$, with stalk $\Gamma := \mathbb{Q}[t, t^{-1}]$ and action by an element $\alpha \in \pi_1(\mathcal{U})$ determined by multiplication by $t^{\mathrm{lk}(V \cup -dH, \alpha)}$, where $\mathrm{lk}(V \cup -dH, \alpha)$ is the linking number of $\alpha$ with the divisor $V \cup -dH$ of $\mathbb{CP}^{n+1}$. Then, for any (super-)perversity $\bar{p}$, the intersection homology complex $IC_{\bar{p}}^{\bullet}(\mathbb{CP}^{n+1}, \mathcal{L}_H)$ is defined by using Deligne's axiomatic construction ([3], [12]). With all of these, the following hold:

(1) There is an isomorphism of $\Gamma$-modules ([26], Corollary 3.4):
$$IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H) \cong H_i(\mathcal{U}; \mathcal{L}_H) \cong H_i(\mathcal{U}^c; \mathbb{Q})$$
where $\mathcal{U}^c$ is the infinite cyclic cover of $\mathcal{U} = \mathbb{CP}^{n+1} - V \cup H$ corresponding to the kernel of the total linking number isomorphism, i.e., $lk : \pi_1(\mathcal{U}) \to \mathbb{Z}$, $\alpha \mapsto \mathrm{lk}(\alpha, V \cup -dH)$. The module structure on the Alexander module $H_i(\mathcal{U}^c; \mathbb{Q})$ is induced by the action of a generating covering transformation.

(2) For any $i \leq n$, the module $IH_i^{\bar{m}}(\mathbb{CP}^{n+1}; \mathcal{L}_H)$ is a torsion $\Gamma$-module (see [26], Theorem 3.6). Denote its order by $\delta_i(t)$.

(3) For any $i \leq n$, the zeros of $\delta_i(t)$ are roots of unity of order $d$ ([26], Theorem 4.1).

(4) (Divisibility Theorem, [26], Theorem 4.2)
Fix an irreducible component of $V$, say $V_1$. Then for a fixed integer $i$ $(1 \leq i \leq n)$, the prime factors of the global Alexander polynomial $\delta_i(t)$ of $V$ are among the prime factors of local polynomials $\xi_j^s(t)$ associated to the local Alexander modules $H_j(S^{2n-2s+1} - K^{2n-2s-1}; \Gamma)$ of link pairs $(S^{2n-2s+1}, K^{2n-2s-1})$ of components of strata $S \in \mathcal{S}$ such that: $S \subset V_1$, $n - i \leq s = \dim S \leq n$, and $j$ is in the range $2n - 2s - i \leq j \leq n - s$.

Since the peripheral complex plays an important role in proving the above theorems, it is natural to ask what simplifications a decomposition of $\mathcal{R}^{\bullet}|_{\mathbb{C}^{n+1}}$ can bring. We propose the following conjecture which can be easily verified in the case of projective hypersurfaces with at most two singular strata:

**Conjecture 4.1.** *Suppose that $\mathcal{R}^{\bullet}|_{\mathbb{C}^{n+1}}$ splits. Then for fixed $i \leq n$, a prime element $\gamma \in \Gamma$ divides $\delta_i(t)$ only if $\gamma$ divides (up to a power of $t - 1$) one of the polynomials of the local Alexander modules $H_{n-\dim_{\mathbb{C}}S}(S^{2n-2\dim_{\mathbb{C}}S+1} - K; \Gamma)$, associated with links of singular strata $S \in \mathcal{S}$ of $(\mathbb{CP}^{n+1}, V)$ (contained in some fixed irreducible component), provided that $\dim_{\mathbb{C}}S \geq n - i$.*

It is conceivable that a decomposition of the peripheral complex associated to a complex hypersurface would have implications in the study of other topological invariants of the hypersurface. We expect to obtain applications similar to those

obtained by Goresky-MacPherson ([16]) for the BBD decomposition theorem. This will make the object of a future work.

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