Toric varieties, characteristic classes and lattice point counting

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Motivation: Counting lattice points in Polytopes

- Lattice Polytope:

$$P := conv(S) \subset M_{\mathbb{R}} \cong \mathbb{R}^n,$$

  for $S \subset M \cong \mathbb{Z}^n$ a finite set in a lattice M.
- assume P is full-dimensional, i.e., $\dim P = n$.
- *Problem*: Calculate $\#(\ell P \cap M)$, for $\ell \in \mathbb{Z}_{>0}$.
- *Geometric Approach*:
  - consider the associated (possibly singular) projective *toric variety* X_P with Cartier *divisor* D_P.
  - Ehrhart polynomial of P:

$$\mathrm{Ehr}_P(\ell) := \#(\ell P \cap M) = \sum_{k \geq 0} a_k \ell^k,$$

    with

$$a_k = \frac{1}{k!} \int_{X_P} [D_P]^k \cap td_k(X_P).$$

- So computing $\#(\ell P \cap M)$ amounts to the computation of the (homology) *Todd classes* $td_*(X_P)$.

Cones

- $M \simeq \mathbb{Z}^n$ n-dimensional lattice in $\mathbb{R}^n$.
- $N = \mathrm{Hom}(M, \mathbb{Z})$ the dual lattice.
- natural pairing: $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$.
- A *cone* $\sigma \subset N_{\mathbb{R}} := N \otimes \mathbb{R}$ is a subset

$$\sigma = cone(S) := \{\sum_{u \in S} \lambda_u \cdot u \mid \lambda_u \geq 0\},$$

  for some finite set $S \subset N$.
- For each *ray* (i.e., 1-dimensional face) ρ of σ, let u_ρ be the unique generator of the semigroup $\rho \cap N$.
- The $\{u_\rho\}_{\rho \in \sigma(1)}$ are the *generators* of σ, with $\sigma(1)$ the collection of rays of σ.
- A cone σ is called
  - *smooth* if it is generated by a subset of a $\mathbb{Z}$-basis of N.
  - *simplicial* if its generators are linearly independent over $\mathbb{R}$.

Cones and Affine Toric Varieties: $\sigma \rightsquigarrow U_\sigma$

- *dual cone* of σ is: $\check{\sigma} = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0, \forall u \in \sigma\}$.
- cone $\sigma \rightsquigarrow$ affine toric variety $U_\sigma := Spec\big(\mathbb{C}[M \cap \check{\sigma}]\big)$.
- **Example**: If $0 \leq r \leq n$ and $\sigma = cone(e_1, \cdots, e_r) \subset \mathbb{R}^n$, then:

$$\check{\sigma} = cone(e_1^*, \cdots, e_r^*, \pm e_{r+1}^*, \cdots, \pm e_n^*)$$

$$U_\sigma = Spec\big(\mathbb{C}[x_1, \cdots, x_r, x_{r+1}^{\pm 1}, \cdots x_n^{\pm 1}]\big) \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}.$$

In particular, $U_{\{0\}} = (\mathbb{C}^*)^n$.

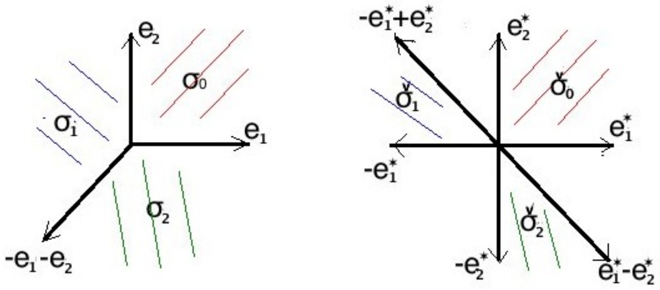
Fans and Toric Varieties: $\Sigma \rightsquigarrow X_\Sigma$

- A *fan* Σ in $N_\mathbb{R}$ is a finite collection of cones so that:
  - if $\sigma \in \Sigma$, then any face of σ is also in Σ.
  - if $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of each.
- fan $\Sigma \rightsquigarrow$ toric variety X_Σ defined as:

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma,$$

with U_{σ_1} and U_{σ_2} glued along $U_{\sigma_1 \cap \sigma_2}$.

Example



- $\begin{cases} U_{\sigma_0} = Spec(\mathbb{C}[x,y]) = \mathbb{C}^2_{(x,y)} \\ U_{\sigma_1} = Spec(\mathbb{C}[x^{-1}, x^{-1}y]) = \mathbb{C}^2_{(x^{-1},x^{-1}y)} \\ U_{\sigma_2} = Spec(\mathbb{C}[y^{-1}, xy^{-1}]) = \mathbb{C}^2_{(y^{-1},xy^{-1})} \end{cases}$
- $X_\Sigma = \mathbb{P}^2_{\mathbb{C}}$ of coordinates $(z_0 : z_1 : z_2)$, with $x = \frac{z_1}{z_0}$, $y = \frac{z_2}{z_0}$.

Toric Varieties

- $U_{\{0\}} = (\mathbb{C}^*)^n$ is an affine open subset of X_Σ.
- action of $T_N = (\mathbb{C}^*)^n$ on itself extends to an algebraic action of T_N on X_Σ with finitely many orbits, one for each $\sigma \in \Sigma$.
- cone $\sigma \rightsquigarrow$ orbit $O_\sigma \rightsquigarrow$ orbit closure $V_\sigma := \bar{O}_\sigma$.
- ray $\rho \in \Sigma(1) \rightsquigarrow$ irreducible T_N-invariant divisor $V_\rho \subset X_\Sigma$.
- geometry of X_Σ depends on the properties of the fan Σ, e.g.:
  - X_Σ is *smooth* iff all cones of Σ are *smooth*.
  - X_Σ is *simplicial* (i.e., orbifold) iff all cones of Σ are *simplicial*.
  - X_Σ is *compact* iff Σ is a *complete fan*, i.e., $\bigcup_{\sigma \in \Sigma} \sigma = N_\mathbb{R}$.

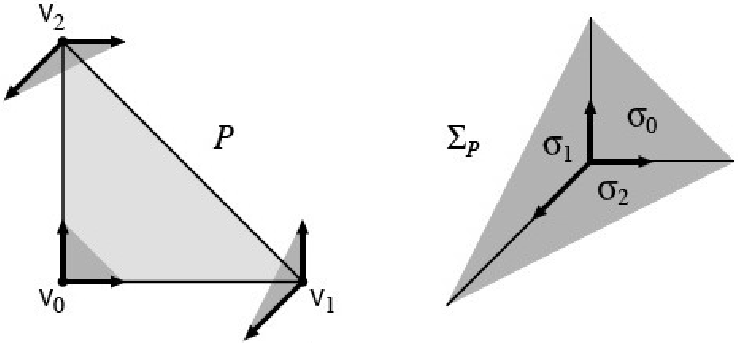
Lattice polytopes, Fans and Toric Varieties: $P \rightsquigarrow \Sigma_P \rightsquigarrow X_P$

- $P \subset M_{\mathbb{R}} = \mathbb{R}^n$ full-dimensional lattice polytope.
- each *facet* F (codimension one face of P) has a unique *inward-pointing facet normal* with ray generator u_F.
- $P \rightsquigarrow$ *inner normal fan* $\Sigma_P := \{\sigma_Q \mid Q \text{ face of } P\}$, with

$$Q \rightsquigarrow \sigma_Q := cone\,(u_F \mid F \supseteq Q)$$

- toric variety $X_P := X_{\Sigma_P}$.

Example



$P = k\Delta_2 \subseteq \mathbb{R}^2$ and its normal fan Σ_P

$$X_P = \mathbb{P}^2_{\mathbb{C}}$$

- *facet* $F \rightsquigarrow$ *ray* $\rho_F = cone(u_F) \rightsquigarrow$ *divisor* $V_F := \bar{O}_{\rho_F}$.
- P has a unique *facet presentation*:

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F, \forall F \text{ facet of } P\}$$

- $D_P := \sum_F a_F V_F$ is *ample Cartier divisor*, so X_P is projective.
- if P is *simple* (each vertex is the intersection of $\dim(P)$ facets) then X_P is simplicial.

Counting lattice points. Pick's formula.

- Danilov:

$$\#(P \cap M) = \chi(X_P, \mathcal{O}(D_P)) \overset{(RR)}{=} \int_{X_P} ch(\mathcal{O}(D_P)) \cap td_*(X_P)$$
$$= \sum_{k \geq 0} \frac{1}{k!} \int_{X_P} [D_P]^k \cap td_k(X_P).$$

- if $n = 2$:

$$td_*(X_P) = [X_P] + \frac{1}{2} \sum_F [V_F] + [pt],$$

so by evaluating $\int_{X_P}$, get *Pick's formula*:

$$\#(P \cap M) = Area(P) + \frac{\#(\partial P \cap M)}{2} + 1.$$

Characteristic classes of toric varieties

homology *Hirzebruch classes* of X (Brasselet-Schürmann-Yokura):

$$T_{y*}, \widehat{T}_{y*} : K_0(var/X) \to H_{2*}^{BM}(X) \otimes \mathbb{Q}[y],$$

so that

$$T_{y*}(X) := T_{y*}([id_X]) \ , \ \widehat{T}_{y*}(X) := \widehat{T}_{y*}([id_X]),$$

and if X compact and $* = 0$:

$$\chi_y(X) := \sum_{j,p} (-1)^j \dim_{\mathbb{C}} Gr_F^p H^j(X, \mathbb{C}) \cdot (-y)^p$$
$$= \int_X T_{y*}(X) = \int_X \widehat{T}_{y*}(X).$$

- if X is smooth and $\{\alpha\}$ are the Chern roots of TX, then:

$$T_{y*}(X) = \prod_{\alpha} Q_y(\alpha) \cap [X], \qquad Q_y(\alpha) = \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}}.$$

$$\widehat{T}_{y*}(X) = \prod_{\alpha} \widehat{Q}_y(\alpha) \cap [X], \quad \widehat{Q}_y(\alpha) = \frac{\alpha(1+ye^{-\alpha(1+y)})}{1-e^{-\alpha(1+y)}}.$$

- $$\widehat{Q}_y(\alpha) = \begin{cases} 1+\alpha & y=-1 \\ \frac{\alpha}{1-e^{-\alpha}} & y=0 \\ \frac{\alpha}{\tanh \alpha} & y=1. \end{cases}$$

- So, if X is smooth:

$$\widehat{T}_{y*}(X) = \begin{cases} c^*(X) \cap [X] & y=-1 \\ td^*(X) \cap [X] & y=0 \\ L^*(X) \cap [X] & y=1. \end{cases}$$

If X is a singular toric variety, then:

$\widehat{T}_{y*}(X)$		
$y = -1$	$y = 0$	$y = 1$ X proj. simplicial
$c_*(X) \otimes \mathbb{Q}$ $c_*(X) := c_*(1_X)$ $c_* : \mathrm{Cons(X)} \to H_*(X)$	$td_*(X)$ $td_*(X) := td_*([\mathcal{O}_X])$ $td_* : K_0(\mathrm{Coh}(X)) \to H_*(X)$	$L_*(X)$ $L_*(X) := L_*([IC_X])$
MacPherson	Baum-Fulton-MacPherson	Milnor-Thom

Problem: Compute these classes for a given toric variety $X = X_\Sigma$.

Proposition (M-Schürmann)

If $X := X_\Sigma$ is a toric variety, then:

$$T_{y*}(X) = \sum_{p=0}^{\dim(X)} td_*([\widetilde{\Omega}_X^p]) \cdot y^p,$$

where $\widetilde{\Omega}_X^p$ is the sheaf of Zariski p-forms (e.g., $\widetilde{\Omega}_X^{\dim(X)} \cong \omega_X$ is the canonical sheaf of X, while $\widetilde{\Omega}_X^0 \cong \mathcal{O}_X$ is the structure sheaf).

Additivity

Theorem A (M-Schürmann)

If $X = X_\Sigma$ is a toric variety, and for $\sigma \in \Sigma$ we let $V_\sigma := \bar{O}_\sigma$, then:

$$T_{y*}(X) = \sum_{\sigma \in \Sigma} (1+y)^{\dim(O_\sigma)} \cdot td_*([\omega_{V_\sigma}]).$$

$$td_*([\omega_X]) = \sum_{\sigma \in \Sigma} (-1)^{\mathrm{codim}(O_\sigma)} \cdot td_*(V_\sigma).$$

$$\widehat{T}_{y*}(X) = \sum_{\sigma, i} (-1-y)^{\dim(O_\sigma)-i} td_i(V_\sigma).$$

Note: $td_k([\omega_X]) = (-1)^{\dim(X)-k} td_k(X)$.

Corollary

(*a*) (*Ehler's formula*) *The (rational) MacPherson-Chern class* $c_*(X)$ *of a toric variety* $X = X_\Sigma$ *is computed by:*

$$c_*(X) = \sum_{\sigma \in \Sigma} [V_\sigma].$$

(*b*) *The Todd class* $td_*(X)$ *of a toric variety satisfies:*

$$td_*(X) = \sum_{\sigma \in \Sigma} td_*([\omega_{V_\sigma}]) = \sum_{\sigma, i} (-1)^{\dim(O_\sigma) - i} td_i(V_\sigma).$$

(*c*) *The L-classes of a projective simplicial toric variety is given by:*

$$L_*(X) = \sum_{\sigma, i} (-2)^{\dim(O_\sigma) - i} \cdot td_i(V_\sigma).$$

Corollary

If $X = X_\Sigma$ is a compact toric variety, then:
(a) *The Hirzebruch polynomial $\chi_y(X)$ is computed by:*

$$\chi_y(X) = \sum_{\sigma \in \Sigma} (-1-y)^{\dim(O_\sigma)}.$$

(b) *The Euler characteristic $e(X)$ is computed by:*

$$e(X) = \text{number of maximal cones of } \Sigma.$$

(c) *The signature of a projective simplicial toric variety is given by:*

$$sign(X) = \sum_{\sigma \in \Sigma} (-2)^{\dim(O_\sigma)}.$$

Weighted lattice point counting

Corollary (*Generalized Pick formula*)

If $X = X_P$ is the projective toric variety associated to a full-dimensional lattice polytope $P \subset M_{\mathbb{R}} \cong \mathbb{R}^n$, and $\ell \in \mathbb{Z}_{>0}$ then:

$$\sum_{Q \preceq P} (1+y)^{\dim(Q)} \cdot \#\big(\mathrm{Relint}(\ell Q) \cap M\big) = \int_{X_P} e^{\ell [D_P]} \cap T_{y*}(X_P)$$

$$\overset{n=2}{=} (1+y)^2 \cdot Area(P) + \frac{1-y^2}{2} \#(\partial P \cap M) + \chi_y(P).$$

Remark ($y = 0$)

$$\#(\ell P \cap M) = \sum_{Q \preceq P} \#(\mathrm{Relint}(\ell Q) \cap M) = \int_{X_P} e^{\ell [D_P]} \cap td_*(X_P).$$

- Our results above hold for any *polytopal subcomplex* $P' \subset P$ (e.g., $P' = \partial P$), and the associated torus-invariant closed algebraic subset $X := X_{P'}$ of X_P.
- **Theorem:** The Ehrhart polynomial of the polytopal subcomplex $P' \subset P$ is computed by:

$$\mathrm{Ehr}_{P'}(\ell) := \#(\ell P' \cap M) = \sum_{k \geq 0} a_k \ell^k,$$

with

$$a_k = \frac{1}{k!} \int_X [D_P|_X]^k \cap td_k(X).$$

- **Corollary:** $(\ell = 0)$

$$\chi(P') := \sum_{Q \preceq P'} (-1)^{\dim Q} = \chi_0(X) = \int_X td_*(X) = \chi(X, \mathcal{O}_X).$$

Mock Hirzebruch classes of simplicial toric varieties

Definition

The *mock* (naive) *Hirzebruch class* of a *simplicial* toric variety $X = X_\Sigma$ is defined as:

$$\widehat{T}^{(m)}_{y*}(X) = \left(\prod_{\rho \in \Sigma(1)} \widehat{Q}_y([V_\rho]) \right) \cap [X],$$

where $\widehat{Q}_y(\alpha) = \frac{\alpha(1+ye^{-\alpha(1+y)})}{1-e^{-\alpha(1+y)}}$.

Remark

If X is smooth, then: $\widehat{T}_{y*}(X) = \widehat{T}^{(m)}_{y*}(X)$.

Theorem B (M-Schürmann)

Let $X = X_\Sigma$ be a simplicial toric variety. Then:

$$\widehat{T}_{y*}(X) = \widehat{T}^{(m)}_{y*}(X) + \sum_{\sigma \in \Sigma_{\text{sing}}} \mathcal{A}_y(\sigma) \cap \widehat{T}^{(m)}_{y*}(V_\sigma)$$

with

$$\mathcal{A}_y(\sigma) := \frac{1}{m_\sigma} \cdot \sum_{g \in P^\circ_\sigma \cap N} \prod_{\rho \in \sigma(1)} \frac{1 + y \cdot a_\rho(g) \cdot e^{-[V_\rho](1+y)}}{1 - a_\rho(g) \cdot e^{-[V_\rho](1+y)}},$$

where, for a cone $\sigma = cone(\rho_1, \cdots, \rho_k)$ with generators $u_1, \cdots, u_k$,

$$P_\sigma := \{\sum_{i=1}^k \lambda_i u_i \mid 0 \le \lambda_i < 1\}, \quad P^\circ_\sigma := \{\sum_{i=1}^k \lambda_i u_i \mid 0 < \lambda_i < 1\},$$

$m_\sigma := \#(P_\sigma \cap N)$, and $a_\rho(g) \neq 1$ are roots of unity of order m_σ.

Corollary

Since X_Σ is smooth in codimension one,

$$\widehat{T}_{y*}(X) = [X] + \frac{1-y}{2} \sum_{\rho \in \Sigma(1)} [V_\rho] + l.o.t.$$

If Σ is complete, then the degree-zero term is $\chi_y(X) \cdot [pt]$.

Remark

- *The proof of thm uses the Cox geometric quotient realization $X = W/G$ of a simplicial toric variety, together with the Lefschetz-Riemann-Roch theorem.*
- *The mock classes $\widehat{T}_{y*}^{(m)}(V_\sigma)$ can be regarded as tangential data for the fixed point sets of the action, while the coefficient $\mathcal{A}_y(\sigma)$ encodes normal data information.*

Remark

- *Can get a closed expression for the Ehrhart polynomial of a lattice n-simplex in $\mathbb{R}^n$.*
- *Indeed, the above formula becomes very concrete in this case, since the associated toric variety is a (fake) weighted projective space, whose cohomology ring has only one generator.*

Example (Weighted Projective Spaces)

If $r > 1$ and $n > 1$, the n-dimensional weighted projective space $X = \mathbb{P}(1, \cdots, 1, r)$ has a unique isolated singularity.

$$\widehat{T}_{y*}(X) = \widehat{T}^{(m)}_{y*}(X) + \frac{1}{r} \sum_{\lambda^r=1, \lambda \neq 1} \left(\frac{1+\lambda y}{1-\lambda} \right)^n \cdot [pt]$$

$$sign(X) = \int_X L^{(m)}_{y*}(X) + \underbrace{\frac{1}{r} \sum_{\lambda^r=1, \lambda \neq 1} \left(\frac{1+\lambda}{1-\lambda} \right)^n}_{\neq 0 \text{ for } \mathbb{P}(1,1,3)}$$

$$\chi_a(X) = \int_X td^{(m)}_{y*}(X) + \underbrace{\frac{1}{r} \sum_{\lambda^r=1, \lambda \neq 1} \left(\frac{1}{1-\lambda} \right)^n}_{\neq 0 \text{ for } \mathbb{P}(1,1,2)}$$

THANK YOU !!!