Perverse Sheaves on Semi-abelian Varieties: Properties and Applications

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Cohomology jump loci

Let X be a smooth connected complex quasi-projective variety with $b_1(X) > 0$.
The (identity component of the) moduli space of rank-one $\mathbb{C}$-local systems on X is defined as:

$$\mathrm{Char}(X) := \mathrm{Hom}(H_1(X, \mathbb{Z})/\mathrm{Torsion}, \mathbb{C}^*) \cong (\mathbb{C}^*)^{b_1(X)}$$

Definition

The *i-th cohomology jumping locus of X* is defined as:

$$\mathcal{V}^i(X) = \{\rho \in \mathrm{Char}(X) \mid H^i(X, L_\rho) \neq 0\},$$

where L_ρ is the rank-one $\mathbb{C}$-local system on X associated to the representation $\rho \in \mathrm{Char}(X)$.

$\mathcal{V}^i(X)$ are closed subvarieties of $\mathrm{Char}(X)$ and homotopy invariants of X.

Computational tools

The jump loci $\mathcal{V}^i(X)$ can be defined for any finite CW complex X.

(a) If X is a finite connected CW complex, then $\mathcal{V}^0(X) = \{\mathbb{C}_X\}$ is a point, the trivial rank-one local system.

(b) If X is a closed oriented manifold of real dimension n, Poincaré duality yields $H^i(X, L_\rho)^\vee \cong H^{n-i}(X, L_{\rho^{-1}})$, for all i.

(c) For all $\rho \in \mathrm{Char}(X)$ with associated rank-one local system L_ρ, one has $\chi(X) = \chi(X, L_\rho)$.

Sample computations

Example

(a) If $X = S^1$, Poincaré duality yields:

$$\mathcal{V}^i(X) = \begin{cases} \{\mathbb{C}_X\}, & \text{if } i = 0, 1 \\ \emptyset, & \text{otherwise.} \end{cases}$$

(b) If $X = (S^1)^n$, Künneth Theorem yields

$$\mathcal{V}^i(X) = \begin{cases} \{\mathbb{C}_X\}, & \text{if } 0 \leq i \leq n \\ \emptyset, & \text{otherwise.} \end{cases}$$

(c) If $X = \Sigma_g$ is a smooth complex projective curve of genus $g \geq 2$, then $\chi(X) = 2 - 2g \neq 0$, and Poincaré duality yields:

$$\mathcal{V}^i(X) = \begin{cases} \{\mathbb{C}_X\}, & \text{if } i = 0, 2, \\ \mathrm{Char}(X), & \text{if } i = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Semi-abelian varieties

A complex *abelian variety* of dimension g is a compact complex torus $\mathbb{C}^g/\mathbb{Z}^{2g}$ which is also a complex projective variety.
A *semi-abelian variety* G is an abelian complex algebraic group which is an extension

$$1 \to T \to G \to A \to 1,$$

where A is an abelian variety of dimension g and $T \cong (\mathbb{C}^*)^m$ is an algebraic affine torus of dimension m. In particular,

$$\pi_1(G) \cong \mathbb{Z}^{m+2g}, \text{ with } \dim G = m + g.$$

Definition

Let X be a smooth complex quasi-projective variety.
The *Albanese map* of X is a morphism $\mathrm{alb} : X \to \mathrm{Alb}(X)$ from X to a semi-abelian variety $\mathrm{Alb}(X)$ such that for any morphism $f : X \to G$ to a semi-abelian variety G, there exists a unique morphism $g : \mathrm{Alb}(X) \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{alb}} & \mathrm{Alb}(X) \\ & \searrow^{f} & \downarrow{\scriptstyle \exists!\ g} \\ & & G \end{array}$$

$\mathrm{Alb}(X)$ is called the *Albanese variety* associated to X.

If

$$1 \to (\mathbb{C}^*)^m \to \mathrm{Alb}(X) \to (S^1)^{2g} \to 1,$$

then:

- $g = \dim_{\mathbb{C}} H^{1,0}$, with $H^{1,0}$ the weight $(1,0)$ part of $H^1(X;\mathbb{C})$.
- $m = \dim_{\mathbb{C}} H^{1,1}$, with $H^{1,1}$ the weight $(1,1)$ part of $H^1(X;\mathbb{C})$.

The Albanese map induces an isomorphism on the free part of H_1:

$$H_1(X,\mathbb{Z})/\mathrm{Torsion} \xrightarrow{\cong} H_1(\mathrm{Alb}(X),\mathbb{Z}).$$

In particular,

$$\mathrm{Char}(X) \cong \mathrm{Char}(\mathrm{Alb}(X)).$$

Constructible complexes enter the scene

By the projection formula, for any $\rho \in \mathrm{Char}(X) \cong \mathrm{Char}(\mathsf{Alb}(X))$:

$$H^i(X, L_\rho) \cong H^i(X, \mathbb{C}_X \otimes L_\rho) \cong H^i\left(\mathsf{Alb}(X), (R\,\mathsf{alb}_* \mathbb{C}_X) \otimes L_\rho\right).$$

Hence,

$$\mathcal{V}^i(X) = \mathcal{V}^i(\mathsf{Alb}(X), R\,\mathsf{alb}_* \mathbb{C}_X).$$

If alb is proper (e.g., X is projective), the BBDG decomposition theorem yields that $R\,\mathsf{alb}_* \mathbb{C}_X$ is a direct sum of (shifted) *perverse sheaves*.
This motivates the study of *cohomology jumping loci of constructible complexes* (resp., *perverse sheaves*) *on semi-abelian varieties*.

Cohomology jump loci of constructible complexes

Definition

Let $\mathcal{F} \in D^b_c(G, \mathbb{C})$ be a bounded constructible complex of $\mathbb{C}$-sheaves on a semi-abelian variety G. The *i-th cohomology jumping locus of $\mathcal{F}$* is defined as:

$$\mathcal{V}^i(G, \mathcal{F}) := \{\rho \in \mathrm{Char}(G) \mid \mathbb{H}^i(G, \mathcal{F} \otimes_{\mathbb{C}} L_\rho) \neq 0\}.$$

Theorem (Budur-Wang)

Each $\mathcal{V}^i(G, \mathcal{F})$ is a finite union of translated subtori of $\mathrm{Char}(G)$.

Mellin transformation

$\mathrm{Char}(G) = \operatorname{Spec} \Gamma_G$, with $\Gamma_G := \mathbb{C}[\pi_1(G)] \cong \mathbb{C}[t_1^{\pm 1}, \cdots, t_{m+2g}^{\pm 1}]$. Let $\mathcal{L}_G$ be the (universal) rank 1 local system of Γ_G-modules on G, defined by mapping the generators of $\pi_1(G) \cong \mathbb{Z}^{m+2g}$ to the multiplication by the corresponding variables of Γ_G.

Definition

The *Mellin transformation* $\mathcal{M}_* : D^b_c(G, \mathbb{C}) \to D^b_{coh}(\Gamma_G)$ is given by

$$\mathcal{M}_*(\mathcal{F}) := Ra_*(\mathcal{L}_G \otimes_{\mathbb{C}} \mathcal{F}),$$

where $a : G \to pt$ is the constant map, and $D^b_{coh}(\Gamma_G)$ denotes the bounded coherent complexes of Γ_G-modules.

Theorem (Gabber-Loeser, Liu-M.-Wang)

If $G = T$ is a complex affine torus, then:

$$\mathcal{F} \in \mathrm{Perv}(T, \mathbb{C}) \iff H^i(\mathcal{M}_*(\mathcal{F})) = 0 \text{ for all } i \neq 0.$$

(By the projection formula) cohomology jump loci of $\mathcal{F}$ are determined by those of $\mathcal{M}_*(\mathcal{F})$, i.e.,

$$\mathcal{V}^i(G, \mathcal{F}) = \mathcal{V}^i(\mathcal{M}_*(\mathcal{F})),$$

where if R is a Noetherian domain and $E^\bullet$ is a bounded complex of R-modules with finitely generated cohomology, we set

$$\mathcal{V}^i(E^\bullet) := \{\chi \in \operatorname{Spec} R \mid H^i(F^\bullet \otimes_R R/\chi) \neq 0\},$$

with $F^\bullet$ a bounded above finitely generated *free* resolution of $E^\bullet$. So understanding $\mathcal{V}^i(G, \mathcal{F})$ becomes a commutative algebra problem!

Propagation package

Theorem (Liu-M.-Wang)

For any $\mathbb{C}$-perverse sheaf $\mathcal{P}$ on a semi-abelian variety G, the cohomology jump loci of $\mathcal{P}$ satisfy the following properties:

(i) *Propagation property:*

$$\mathcal{V}^{-m-g}(G,\mathcal{P}) \subseteq \cdots \subseteq \mathcal{V}^0(G,\mathcal{P}) \supseteq \mathcal{V}^1(G,\mathcal{P}) \supseteq \cdots \supseteq \mathcal{V}^g(G,\mathcal{P}).$$

Moreover, $\mathcal{V}^i(G,\mathcal{P}) = \emptyset$ *if* $i \notin [-m-g, g]$.

(ii) *Codimension lower bound: for any* $i \geq 0$,

$$\mathrm{codim}\mathcal{V}^i(G,\mathcal{P}) \geq i \quad \text{and} \quad \mathrm{codim}\mathcal{V}^{-i}(G,\mathcal{P}) \geq i.$$

Remark ((Hyper)cohomology concentration)

Let $\mathcal{P}$ be a $\mathbb{C}$-perverse sheaf so that not all $\mathbb{H}^j(G,\mathcal{P})$ are zero. Let

$$k_+ := \max\{j \mid \mathbb{H}^j(G,\mathcal{P}) \neq 0\} \text{ and } k_- := \min\{j \mid \mathbb{H}^j(G,\mathcal{P}) \neq 0\}.$$

The propagation property is equivalent to: $k_+ \geq 0$, $k_- \leq 0$ and

$$\mathbb{H}^j(G,\mathcal{P}) \neq 0 \iff k_- \leq j \leq k_+.$$

(If $G = A$ is an abelian variety, a similar result was proved by Weissauer.)

Application: Generic vanishing

Corollary (Kramer, Liu-M.-Wang, Franecki-Kapranov)

For any $\mathbb{C}$-perverse sheaf $\mathcal{P}$ on a semi-abelian variety G,

$$\mathbb{H}^i(G, \mathcal{P} \otimes_{\mathbb{C}} L_\rho) = 0$$

for any generic rank-one $\mathbb{C}$-local system L_ρ and all $i \neq 0$.
In particular, (for such generic ρ)

$$\chi(G, \mathcal{P}) = \chi(G, \mathcal{P} \otimes_{\mathbb{C}} L_\rho) = \dim_{\mathbb{C}} \mathbb{H}^0(G, \mathcal{P} \otimes_{\mathbb{C}} L_\rho) \geq 0.$$

Moreover, the equality holds if and only if $\mathcal{V}^0(G, \mathcal{P}) \neq \mathrm{Char}(G)$.

Corollary (Liu-M.-Wang)

Let X be a smooth quasi-projective variety of complex dimension n. Assume that $R\,\mathrm{alb}_*\,\mathbb{C}_X[n]$ is a perverse sheaf on $\mathrm{Alb}(X)$ (e.g., alb is proper and semi-small). Then:

(1) *Propagation property:*

$$\mathcal{V}^{2n}(X) \subseteq \cdots \mathcal{V}^{n+1}(X) \subseteq \mathcal{V}^{n}(X) \supseteq \mathcal{V}^{n-1}(X) \supseteq \cdots \supseteq \mathcal{V}^{0}(X) = \{\mathbb{C}_X\}$$

(2) *Codimension lower bound: for any $i \geq 0$,*

$$\mathrm{codim}\mathcal{V}^{n-i}(X) \geq i \;\text{ and }\; \mathrm{codim}\mathcal{V}^{n+i}(X) \geq i.$$

(3) *Generic vanishing: $H^i(X, L_\rho) = 0$ for generic $\rho \in \mathrm{Char}(X)$ and all $i \neq n$.*

(4) *Signed Euler characteristic property: $(-1)^n \cdot \chi(X) \geq 0$.*

(5) *Betti property: $b_i(X) > 0$ for any $i \in [0, n]$, and $b_1(X) \geq n$.*

$b_1(X) - \dim \mathcal{V}^0(X) =: \mathrm{codim}\mathcal{V}^0(X) \geq n$, so $b_1(X) \geq n$.

Topological characterization of semi-abelian varieties

Corollary (Liu-M.-Wang)

Let X be a smooth quasi-projective variety with proper Albanese map (e.g., X is projective), and assume that X is homotopy equivalent to a torus. Then X is isomorphic to a semi-abelian variety.

Hint: Use the BBDG decomposition theorem and the propagation package to show that $\mathrm{alb} : X \to \mathrm{Alb}(X)$ is an isomorphism.

Application: Homological duality

Definition (Denham-Suciu-Yuzvinsky)

A connected finite CW complex X, with $H := H_1(X, \mathbb{Z})$, is an *abelian duality space of dimension n* if:

(a) $H^i(X, \mathbb{Z}[H]) = 0$ for $i \neq n$,

(b) $H^n(X, \mathbb{Z}[H])$ is a (non-zero) torsion-free $\mathbb{Z}$-module.

In what follows we work with the *full* character variety

$$\mathrm{Char}(X) = \mathsf{Hom}(H, \mathbb{C}^*).$$

Using properties of the Mellin transformation, we get:

Theorem (Liu-M.-Wang)

Let X be an n-dimensional smooth complex quasi-projective variety, which is homotopy equivalent to an n-dimensional CW complex (e.g., X is affine). Suppose the Albanese map alb *is proper and semi-small (e.g., a closed embedding), or* alb *is quasi-finite. Then X is an abelian duality space of dimension n.*

Example (Very affine manifolds)

Let X be an n-dimensional *very affine manifold*, i.e., a smooth closed subvariety of a complex affine torus $T = (\mathbb{C}^*)^m$ (e.g., the complement of an essential hyperplane / toric arrangement). Then X is an abelian duality space of dimension n. (Here, alb is proper and semi-small.)

Theorem (Denham-Suciu-Yuzvinsky, Liu-M.-Wang)

Let X be an abelian duality space X of dimension n. Then:

(i) *Propagation property:* $\mathcal{V}^n(X) \supseteq \mathcal{V}^{n-1}(X) \supseteq \cdots \supseteq \mathcal{V}^0(X)$.

(ii) *Codimension lower bound: for any* $i \geq 0$,

$$\mathrm{codim}\mathcal{V}^{n-i}(X) = b_1(X) - \dim \mathcal{V}^{n-i}(X) \geq i.$$

(iii) *Generic vanishing:* $H^i(X, L_\rho) = 0$ *for* ρ *generic and all* $i \neq n$.

(iv) *Signed Euler characteristic property:*

$$(-1)^n \chi(X) \geq 0.$$

(v) *Betti property:* $b_i(X) > 0$, *for* $0 \leq i \leq n$, *and* $b_1(X) \geq n$.

The following provides a new topological characterization of compact complex tori and, resp., abelian varieties in terms of homological duality properties:

Theorem (Liu-M.-Wang)

Let X be a compact Kähler manifold. Then X is an abelian duality space if and only if X is a compact complex torus. In particular, abelian varieties are the only complex projective manifolds that are abelian duality spaces.

Characterization of perverse sheaves

Theorem (Schnell)

If A is an abelian variety and $\mathcal{F} \in D^b_c(A, \mathbb{C})$, then:

$$\mathcal{F} \in \mathsf{Perv}(A, \mathbb{C}) \iff \forall i \in \mathbb{Z} : \mathrm{codim}\mathcal{V}^i(A, \mathcal{P}) \geq |2i|.$$

If $1 \to T \to G \to A \to 1$ defines a semi-abelian variety G, and $\Gamma_G := \mathbb{C}[\pi_1(G)]$, $\Gamma_T = \mathbb{C}[\pi_1(T)]$ and $\Gamma_A = \mathbb{C}[\pi_1(A)]$, then $\operatorname{Spec}\Gamma_G$, $\operatorname{Spec}\Gamma_T$ and $\operatorname{Spec}\Gamma_A$ are affine tori fitting into a short exact sequence of linear algebraic groups

$$1 \to \operatorname{Spec}\Gamma_A \to \operatorname{Spec}\Gamma_G \xrightarrow{p} \operatorname{Spec}\Gamma_T \to 1.$$

Definition

Let V be an irreducible subvariety of $\operatorname{Spec}\Gamma_G$. Define:

torus dimension: $\dim_t V = \dim p(V)$,

abelian dimension: $\dim_a V = \frac{1}{2}\left(\dim V - \dim_t V\right)$,

semi-abelian dimension: $\dim_{sa} V = \dim_t V + \dim_a V$.

$$\operatorname{codim}_t V = m - \dim_t V, \quad \operatorname{codim}_a V = g - \dim_a V,$$
$$\operatorname{codim}_{sa} V = m + g - \dim_{sa} V.$$

Remark

1. *If $G = T$ is a complex affine torus:* $\dim_{sa}(V) = \dim(V)$, $\mathrm{codim}_{sa}(V) = \mathrm{codim}(V)$, $\dim_a(V) = \mathrm{codim}_a(V) = 0$.
2. *If $G = A$ is an abelian variety:* $\dim_{sa}(V) = \dim_a(V) = \frac{1}{2}\dim(V)$, $\mathrm{codim}_{sa}(V) = \mathrm{codim}_a(V) = \frac{1}{2}\mathrm{codim}(V)$.

Theorem (Liu-M.-Wang)

A constructible complex $\mathcal{F} \in D^b_c(G, \mathbb{C})$ is perverse on G $\iff$

(i) $\mathrm{codim}_a \mathcal{V}^i(G, \mathcal{F}) \geq i$ *for any* $i \geq 0$, *and*

(ii) $\mathrm{codim}_{sa} \mathcal{V}^i(G, \mathcal{F}) \geq -i$ *for any* $i \leq 0$.

Corollary

$\mathcal{F} \in D^b_c(T, \mathbb{C})$ is perverse on a complex affine torus T $\iff$

(i) *For any* $i > 0$: $\mathcal{V}^i(T, \mathcal{F}) = \emptyset$, *and*

(ii) *For any* $i \leq 0$: $\mathrm{codim} \mathcal{V}^i(T, \mathcal{F}) \geq -i$.

Thank you !