

SYMMETRIC PRODUCTS OF MIXED HODGE MODULES

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ABSTRACT. Generalizing a theorem of Macdonald, we show a formula for the mixed Hodge structure on the cohomology of the symmetric products of bounded complexes of mixed Hodge modules by showing the existence of the canonical action of the symmetric group on the multiple external self-products of complexes of mixed Hodge modules. We also generalize a theorem of Hirzebruch and Zagier on the signature of the symmetric products of manifolds to the case of the symmetric products of symmetric pairings on bounded complexes with constructible cohomology sheaves where the pairing is not assumed to be non-degenerate.

Introduction

For a complex algebraic variety X , let $\mathcal{S}^n X$ denote the n -fold symmetric product of X . This is by definition the quotient of the n -fold self-product X^n by the action of the symmetric group \mathfrak{S}_n . We assume X is quasi-projective so that $\mathcal{S}^n X$ is an algebraic variety. Let $\pi : X^n \rightarrow \mathcal{S}^n X$ denote the canonical projection. We have a canonical isomorphism of sheaves of \mathbf{Q} -vector spaces

$$\mathbf{Q}_{\mathcal{S}^n X} = (\pi_* \mathbf{Q}_{X^n})^{\mathfrak{S}_n},$$

where the right-hand side is the \mathfrak{S}_n -invariant part. This implies canonical isomorphisms

$$H^\bullet(\mathcal{S}^n X, \mathbf{Q}) = H^\bullet(X^n, \mathbf{Q})^{\mathfrak{S}_n} = (\otimes^n H^\bullet(X, \mathbf{Q}))^{\mathfrak{S}_n},$$

where the last isomorphism follows from the multiple Künneth formula. So we get an isomorphism of bigraded vector spaces

$$H^\bullet(\mathcal{S}^n X, \mathbf{Q}) \left(:= \bigoplus_{n \geq 0} H^\bullet(\mathcal{S}^n X, \mathbf{Q}) \right) = \mathrm{Sym}^\bullet H_{\mathrm{even}}^\bullet(X, \mathbf{Q}) \otimes \bigwedge^\bullet H_{\mathrm{odd}}^\bullet(X, \mathbf{Q}),$$

where $H_{\mathrm{even}}^\bullet(X, \mathbf{Q}) := \bigoplus_{j: \mathrm{even}} H^j(X, \mathbf{Q})$ (similarly for H_{odd}^\bullet), and $\mathrm{Sym}^\bullet V^\bullet$ on the right-hand side denotes the direct sum of the usual symmetric tensor products $\bigoplus_{n \geq 0} \mathrm{Sym}^n V^\bullet$ for a graded vector space V^\bullet . The last formula, which was implicit in [MS], is pointed out to us by S. Kimura, and is closely related to the theory of finite dimensional motives [Ki] (see also [dBa]). We show that the above isomorphisms are compatible with mixed Hodge structures, and extend to the case of intersection cohomology (with local system coefficients). These imply formulas in [MS] for the Hodge numbers and Hirzebruch's χ_y -genus [Hi] of the (intersection) cohomology of symmetric products, generalizing earlier work by [Ma], [Ch] and others.

In the case when X is smooth, $\mathcal{S}^n X$ is a complex V -manifold, hence is a \mathbf{Q} -homology manifold, so the intersection cohomology of $\mathcal{S}^n X$ coincides with its usual cohomology (see [GM]). If furthermore X is projective (or, more generally, compact

with a Kähler desingularization), then the pure Hodge structure on the cohomology of X defined in [St] (which is reproduced in [PS], Section 2.5) coincides with the mixed Hodge structure in [D3] and also with the pure Hodge structure on the intersection cohomology in [Sa1] (which is obtained by applying the decomposition theorem to the desingularization in the non-projective case), see Proposition (2.8). Here we can prove only a weaker version of [DB], Th. 5.3 showing that if an algebraic complex V -manifold X is globally embeddable into a smooth variety (e.g. if X is quasi-projective), then the filtered Steenbrink complex $(\tilde{\Omega}_X^\bullet, F)$ is canonically isomorphic to the filtered Du Bois complex $(\underline{\Omega}_X^\bullet, F)$ as filtered differential complexes on the *ambient* variety. It is, however, unclear whether the isomorphism holds as filtered differential complexes on the original variety X as noted in loc. cit., see also a remark after (2.7.2) below.

In this paper we extend the above assertions on the symmetric products to the case of arbitrary bounded complexes of mixed Hodge modules $\mathcal{M} \in D^b\text{MHM}(X)$, where $\text{MHM}(X)$ is the abelian category of (algebraic) mixed Hodge modules on X , and $D^b\text{MHM}(X)$ is the derived category of bounded complexes of $\text{MHM}(X)$. There are, however, certain technical difficulties associated to mixed Hodge modules.

For instance, it is not clear a priori if there is a canonical action of the symmetric group \mathfrak{S}_n on the n -fold external self-product $\boxtimes^n \mathcal{M}$ in a compatible way with the natural action on the underlying \mathbf{Q} -complexes, since the difference in the t -structures of the underlying \mathcal{D} -modules and \mathbf{Q} -complexes gives certain differences of signs. In this paper we solve this problem by showing a cancellation of signs appearing in the morphisms of a commutative diagram, see Prop. (1.5) and Th. (1.9) below. It is rather surprising that the sign coming from certain changes of orders of the multiple external products of complexes cancels out with the sign coming from the anti-commutativity of the exterior algebra $\bigwedge^\bullet \Theta$ where Θ is the sheaf of vector fields.

We also prove the multiple Künneth formula for the n -fold external products of bounded complexes of mixed Hodge modules in a compatible way with the action of the symmetric group \mathfrak{S}_n , see (1.12). For the compatibility with the action of \mathfrak{S}_n , we have to construct a canonical multiple Künneth isomorphism. Once a canonical morphism is constructed, the assertion is reduced to the formula for the underlying \mathbf{Q} -complexes, which is well known.

As a consequence of these considerations, we get the following assertion, which is used in [MS]:

Theorem 1. *For any bounded complex of mixed Hodge modules \mathcal{M} on a complex quasi-projective variety X , the symmetric product can be defined by*

$$\mathcal{S}^n \mathcal{M} = (\pi_* \boxtimes^n \mathcal{M})^{\mathfrak{S}_n} \in D^b\text{MHM}(\mathcal{S}^n X),$$

and we have canonical isomorphisms of graded mixed Hodge structures

$$H^\bullet(\mathcal{S}^n X, \mathcal{S}^n \mathcal{M}) = H^\bullet(X^n, \boxtimes^n \mathcal{M})^{\mathfrak{S}_n} = \left(\bigotimes^n H^\bullet(X, \mathcal{M}) \right)^{\mathfrak{S}_n},$$

in a compatible way with the corresponding isomorphisms of the underlying \mathbf{Q} -complexes.

Forgetting mixed Hodge structures, the last assertion of Theorem 1 also holds for any bounded A -complexes with constructible cohomology sheaves $K \in D_c^b(X, A)$ on a topological stratified space X as in (3.1) with $\dim_A H^\bullet(X, K) < \infty$ where A is a field of characteristic 0, see (3.8) below.

In Theorem 1 we use the splitting of an idempotent in $D^b\text{MHM}(\mathcal{S}^n X)$ (see [BS] and also [LC] for a simpler argument) together with the complete reductivity of \mathfrak{S}_n to show the existence of the \mathfrak{S}_n -invariant part in $D^b\text{MHM}(\mathcal{S}^n X)$. The complete reductivity is also used to show the commutativity of the \mathfrak{S}_n -invariant part with the direct image by $X^n \rightarrow pt$. Note that a certain amount of representation theory is needed to justify the definition of the projector $\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma$ defining the invariant part in case of bounded complexes of mixed Hodge modules, see Remark (2.6) below.

Corollary 1. *With the above notation, there is a canonical isomorphism of bi-graded mixed Hodge structures*

$$H^\bullet(\mathcal{S}^\bullet X, \mathcal{S}^\bullet \mathcal{M}) = \text{Sym}^\bullet H_{\text{even}}^\bullet(X, \mathcal{M}) \otimes \wedge^\bullet H_{\text{odd}}^\bullet(X, \mathcal{M}),$$

where $H^\bullet(\mathcal{S}^\bullet X, \mathcal{S}^\bullet \mathcal{M})$ has the bigrading $\bigoplus_{j,n \geq 0} H^j(\mathcal{S}^n X, \mathcal{S}^n \mathcal{M})$, and Sym^\bullet on the right-hand side denotes the direct sum of the usual symmetric tensor products of the graded mixed Hodge structure $H_{\text{even}}^\bullet(X, \mathcal{M})$ consisting of the even degree part.

For this we use the fact that the \mathfrak{S}_n -invariant part coincides with the maximal quotient on which the action of \mathfrak{S}_n is trivial (this follows from the complete reductivity of \mathfrak{S}_n). Corollary 1 implies Th. 1.1 in [MS] for any bounded complexes of mixed Hodge module \mathcal{M} . More precisely, the Hodge numbers of $\mathcal{M} \in D^b\text{MHM}(X)$ are defined for $p, q, k \in \mathbf{Z}$ by

$$h^{p,q,k}(\mathcal{M}) := h^{p,q}(H^k(X, \mathcal{M})) := \dim_{\mathbf{C}}(\text{Gr}_F^p \text{Gr}_{p+q}^W H^k(X, \mathcal{M})_{\mathbf{C}}),$$

where $H^k(X, \mathcal{M})_{\mathbf{C}}$ denotes the underlying \mathbf{C} -vector space of a mixed Hodge structure. Taking the alternating sums over k , we get the E -polynomial in $\mathbf{Z}[y^{\pm 1}, x^{\pm 1}]$:

$$e(\mathcal{M})(y, x) := \sum_{p,q} e^{p,q}(\mathcal{M}) y^p x^q \quad \text{with} \quad e^{p,q}(\mathcal{M}) := \sum_k (-1)^k h^{p,q,k}(\mathcal{M}).$$

For the generating series of the above numbers and polynomials, Corollary 1 implies the following assertion in [MS], Th. 1.1:

Corollary 2. *For any bounded complex of mixed Hodge modules \mathcal{M} on a complex quasi-projective variety X , we have the following identities:*

$$\begin{aligned} \sum_{n \geq 0} \left(\sum_{p,q,k} h^{p,q,k}(\mathcal{S}^n \mathcal{M}) y^p x^q (-z)^k \right) t^n &= \prod_{p,q,k} \left(\frac{1}{1 - y^p x^q z^k t} \right)^{(-1)^k h^{p,q,k}(\mathcal{M})}, \\ \sum_{n \geq 0} e(\mathcal{S}^n \mathcal{M})(y, x) t^n &= \prod_{p,q} \left(\frac{1}{1 - y^p x^q t} \right)^{e^{p,q}(\mathcal{M})} = \exp \left(\sum_{r \geq 1} e(\mathcal{M})(y^r, x^r) \frac{t^r}{r} \right). \end{aligned}$$

Indeed, the isomorphism in Corollary 1 holds after passing to the bigraded quotients $\text{Gr}_F^\bullet \text{Gr}_\bullet^W$, and it implies that the direct sum decomposition $V^\bullet = V_{\text{even}}^\bullet \oplus V_{\text{odd}}^\bullet$ with $V^\bullet := H^\bullet(X, \mathcal{M})$ gives a multiplicative decomposition of the left-hand side in

a compatible way with the one on the right-hand side defined by the denominators and the numerators. So the first equality is reduced to the case $V^\bullet = V_{\text{even}}^\bullet$ or $V^\bullet = V_{\text{odd}}^\bullet$, and easily follows (using a basis of $\text{Gr}_F^\bullet \text{Gr}_\bullet^W V^\bullet$ if necessary). Here the degree of z corresponds to the degree of the graded vector space V^\bullet . The second equality follows from the first by substituting $z = 1$, and the last equality uses the identity $-\log(1-t) = \sum_{i \geq 1} t^i/i$.

Note that Corollary 2 in the constant coefficient case with X singular is stated in [Ch], Prop. 1.1 without any comments about the compatibility with the mixed Hodge structures as in Proposition (2.2) below (although [D3] should be used there). Note also that the formalism in Appendix A of [MS] does not directly apply to mixed Hodge modules since the external product \boxtimes for mixed Hodge modules are not defined by using smooth pull-backs pr_i^* and tensor product \otimes (both have certain shifts of degrees in case of mixed Hodge modules, which cause the problem of sign of the action of \mathfrak{S}_n). It seems also rather nontrivial whether the isomorphism in the multiple Künneth formula is independent of the order of $\{1, \dots, n\}$ if one proves the formula by induction on n reducing to the case $n = 2$ without assuming the existence of a canonical isomorphism for multiple products, even though this independence is crucial to the proof of the compatibility with the action of \mathfrak{S}_n . To avoid this problem, we use multiple tensor product in Section 1.

We define the χ_y -genus of $\mathcal{M} \in D^b\text{MHM}(X)$ in $\mathbf{Z}[y^{\pm 1}]$ by

$$\chi_{-y}(\mathcal{M}) := \sum_p f^p(\mathcal{M}) y^p \quad \text{with} \quad f^p(\mathcal{M}) := \sum_k (-1)^k \dim_{\mathbf{C}} \text{Gr}_F^p H^k(X, \mathcal{M})_{\mathbf{C}}.$$

Only this χ_y -genus has the corresponding characteristic class version, see [BSY], [Sc2]. A generating series formula for these characteristic classes of symmetric products is discussed in [CMSSY]. Since

$$f^p(\mathcal{M}) = \sum_q e^{p,q}(\mathcal{M}), \quad \chi_{-y}(\mathcal{M}) = e(\mathcal{M})(y, 1),$$

Corollary 2 implies the following assertion in [MS], Cor. 1.2:

Corollary 3. *For any bounded complex of mixed Hodge modules \mathcal{M} on a complex quasi-projective variety X , we have the following equalities:*

$$\sum_{n \geq 0} \chi_{-y}(\mathcal{S}^n \mathcal{M}) t^n = \prod_p \left(\frac{1}{1 - y^p t} \right)^{f^p(\mathcal{M})} = \exp \left(\sum_{r \geq 1} \chi_{-y^r}(\mathcal{M}) \frac{t^r}{r} \right).$$

Replacing cohomology by cohomology with compact supports, we get $h_c^{p,q,k}(\mathcal{M})$, $e_c(\mathcal{M})(y, x)$, etc. instead of $h^{p,q,k}(\mathcal{M})$, $e(\mathcal{M})(y, x)$, etc. and the assertions also hold for those numbers and polynomials as is stated in [MS]. Indeed, we can replace X with a compactification \overline{X} and \mathcal{M} with the zero extension $j_! \mathcal{M}$ where $j : X \rightarrow \overline{X}$ is the inclusion.

We can apply the above formulas, for instance, to the cases where $\mathcal{M} = (a_X)^* \mathbf{Q}$ with $a_X : X \rightarrow pt$ the canonical morphism or $\mathcal{M} = (\text{IC}_X \mathcal{L})[-\dim X]$ with X irreducible and \mathcal{L} a polarizable variation of Hodge structure defined on a smooth Zariski-open subset of X , see also [MS] for more examples and applications.

Setting $y = 1$ in the formula of Corollary 3, we get a formula for the underlying \mathbf{Q} -complex K . This formula also holds in case X is a topological stratified space as in (3.1) and $K \in D_c^b(X, A)$ with $\dim_A H^\bullet(X, K) < \infty$ where A is a field of characteristic 0, see also [MS], Th. 1.4(a). For such X and K with $A = \mathbf{R}$, we also have a generalization of [MS], Th. 1.4(c) on the signature of the induced pairing on the symmetric products as follows:

Assume there is a pairing

$$\phi : K \otimes_{\mathbf{R}} K \rightarrow \mathbf{D}_X := a_X^! \mathbf{R},$$

where $a_X : X \rightarrow pt$ is the natural morphism. This induces a pairing

$$\pi_* \boxtimes^n \phi : \pi_* \boxtimes^n K \otimes \pi_* \boxtimes^n K \rightarrow \pi_* \boxtimes^n \mathbf{D}_X = \pi_* \mathbf{D}_{X^n} \xrightarrow{\text{Tr}} \mathbf{D}_{\mathcal{S}^n X},$$

where the last morphism is the trace morphism $\text{Tr} : \pi_! \pi^! \mathbf{D}_{\mathcal{S}^n X} \rightarrow \mathbf{D}_{\mathcal{S}^n X}$ associated with the adjunction for $\pi_!, \pi^!$. Restricting this self-pairing to $\mathcal{S}^n K$, we get the induced pairing

$$\mathcal{S}^n \phi : \mathcal{S}^n K \otimes \mathcal{S}^n K \rightarrow \mathbf{D}_{\mathcal{S}^n X}.$$

Assume $\dim_{\mathbf{R}} H_c^\bullet(X, K) < \infty$, and moreover ϕ induces a graded-symmetric self-pairing ϕ_X of the graded vector space $H_c^\bullet(X, K)$ (in particular, its restriction to the odd degree part is anti-symmetric). The last condition is satisfied if $\phi : K \otimes_{\mathbf{R}} K \rightarrow \mathbf{D}_X$ is symmetric. We do not assume, however, that ϕ_X on $H_c^\bullet(X, K)$ is non-degenerate. Let $\sigma(\phi)$ be the signature of ϕ_X on $H_c^0(X, K)$. Let ρ_i be the rank of the induced pairing between $H_c^i(X, K)$ and $H_c^{-i}(X, K)$ ($i \in \mathbf{Z}$). Set $\chi(\phi) = \sum_i (-1)^i \rho_i$. This coincides with the Euler characteristic $\chi_c(X, K)$ if ϕ_X is non-degenerate. Let $\sigma(\mathcal{S}^n \phi)$ be the signature of the induced pairing on $H_c^0(\mathcal{S}^n X, \mathcal{S}^n K)$.

Theorem 2. *With the above notation and assumption, we have the identity:*

$$\sum_{n \geq 0} \sigma(\mathcal{S}^n \phi) t^n = \frac{(1+t)^{\frac{\sigma(\phi) - \chi(\phi)}{2}}}{(1-t)^{\frac{\sigma(\phi) + \chi(\phi)}{2}}}$$

This generalizes a result of Hirzebruch and Zagier [Za] which is closely related to the Hirzebruch signature theorem, see also [MS], Th. 1.4(c). If X is an even-dimensional complex analytic space and if X is a \mathbf{Q} -homology manifold, then we consider $\text{IC}_X \mathbf{R} = \mathbf{R}_X[\dim X]$ instead of \mathbf{R}_X , where the dualizing complex \mathbf{D}_X is given by $\mathbf{R}_X[2 \dim X]$ (here the Tate twist is omitted). However, this does not cause any problem of sign since the complex is shifted by an even degree.

Finally, note that Corollary 3 implies Theorem 2 in the case when X is projective, K underlies a pure \mathbf{R} -Hodge module \mathcal{M} of even weight with strict support, and ϕ gives a polarization of \mathcal{M} , see (3.7) below. Note also that Theorem 1 and its corollaries hold also for mixed \mathbf{R} -Hodge modules which is defined in the same way as in [Sa1] [Sa2] using induction on the dimension of the support with \mathbf{Q} -Hodge structure replaced by \mathbf{R} -Hodge structure in the zero-dimensional case. Here we assume the local monodromies are quasi-unipotent so that the V -filtrations are indexed by \mathbf{Q} . This is different from [Sa3], 1.11. Note that the proof of (0.10) (i.e. Th. 2.2) in loc. cit. is still incomplete even now (the problem is very difficult), and we have to wait until the detailed version of a paper quoted there will be published.

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In Section 1 we show the canonical action of the symmetric group together with the multiple Künneth formula for the multiple external products of bounded complexes of mixed Hodge modules. In Section 2 we prove Theorem 1 after showing Proposition (2.2) which is related to a certain λ -structure. We also prove Proposition (2.8) in case of complex V -manifolds. In Section 3 we prove Theorem 2 treating only bounded \mathbf{R} -complexes with constructible cohomology sheaves.

1. Multiple external products and the symmetric group

1.1. Multiple external products of complexes. Let K_i^\bullet be bounded complexes of A -modules for $i \in [1, n]$, where A is a field. We have the n -fold tensor complex $\bigotimes_{i=1}^n K_i^\bullet$ such that the j -th component is given by

$$\bigoplus_{|p|=j} \left(\bigotimes_{i=1}^n K_i^{p_i} \right),$$

with $|p| := \sum_{i=1}^n p_i$, and the restriction of the differential to $\bigotimes_{i=1}^n K_i^{p_i}$ is given by

$$(1.1.1) \quad \sum_{i=1}^n (-1)^{p_1 + \dots + p_{i-1}} d_i,$$

where d_i denotes also the morphism induced by the differential d_i of K_i^\bullet .

Let K_i^\bullet be bounded complexes of sheaves of A -modules on topological spaces X_i for $i \in [1, n]$. We have the n -fold external product $\boxtimes_{i=1}^n K_i^\bullet$ on $\prod_{i=1}^n X_i$ such that the j -th component is given by

$$\bigoplus_{|p|=j} \left(\boxtimes_{i=1}^n K_i^{p_i} \right),$$

and the differential is given as in (1.1.1).

Note that the n -fold external product of sheaves of A -modules E_i is defined by

$$\boxtimes_{i=1}^n E_i = E_1 \boxtimes \dots \boxtimes E_n := pr_1^{-1} E_1 \otimes_A \dots \otimes_A pr_n^{-1} E_n,$$

with pr_i the projection to the i -th factor. In the case $X = pt$, the n -fold tensor product $\bigotimes_{i=1}^n E_i$ is defined by using the universality for multilinear maps $E_1 \times \dots \times E_n \rightarrow E'$.

We will write ${}_A \boxtimes$ instead of \boxtimes when we have to specify A .

1.2. External products of \mathcal{D} -modules. Let $M_i \in M(\mathcal{D}_{X_i})$ for $i \in [1, n]$, where X_i is a complex manifold and $M(\mathcal{D}_{X_i})$ denotes the category of \mathcal{D}_{X_i} -modules. Set $\mathcal{X} = \prod_{i=1}^n X_i$. We have the n -fold external product

$${}_o \boxtimes_{i=1}^n M_i = M_1 \boxtimes \dots \boxtimes M_n \in M(\mathcal{D}_{\mathcal{X}}),$$

which is defined by the scalar extension of

$${}_c \boxtimes_{i=1}^n M_i := pr_1^{-1} M_1 \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} pr_n^{-1} M_n$$

by the inclusion

$$(1.2.1) \quad \mathcal{O}'_{\mathcal{X}} := pr_1^{-1} \mathcal{O}_{X_1} \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} pr_n^{-1} \mathcal{O}_{X_n} \hookrightarrow \mathcal{O}_{\mathcal{X}}.$$

For bounded complexes of \mathcal{D}_{X_i} -modules $M_i^\bullet \in C^b(\mathcal{D}_{X_i})$, we can then define the n -fold external product

$$\circ \boxtimes_{i=1}^n M_i^\bullet = M_1^\bullet \boxtimes \cdots \boxtimes M_n^\bullet \in C^b(\mathcal{D}_{\mathcal{X}}),$$

where the differential is given as in (1.1.1). This induces the n -fold external product in $D^b(\mathcal{D}_{\mathcal{X}})$ for $M_i^\bullet \in D^b(\mathcal{D}_{X_i})$.

Note that the above definition also applies to the case of \mathcal{O}_X -modules.

1.3. Action of the symmetric group \mathfrak{S}_n . With the notation of (1.2), assume $X_i = X$ so that $\mathcal{X} = X^n$. Let E_i be sheaves of A -modules on X . For $\sigma \in \mathfrak{S}_n$, we have a natural isomorphism (without any sign)

$$(1.3.1) \quad \sigma^\# : \boxtimes_{i=1}^n E_i \xrightarrow{\sim} \sigma_* (\boxtimes_{i=1}^n E_{\sigma(i)}).$$

Here σ_* , σ^* denote the sheaf-theoretic direct image and pull-back associated to the action of σ so that $\sigma_* = (\sigma^{-1})^*$, and hence $\sigma_* pr_i^* = (\sigma^{-1})^* pr_i^* = pr_{\sigma(i)}^*$. We have moreover

$$(1.3.2) \quad \sigma_* \tau^\# \circ \sigma^\# = (\sigma\tau)^\#,$$

as morphisms

$$\boxtimes_{i=1}^n E_i \rightarrow \sigma_* \tau_* (\boxtimes_{i=1}^n E_{\sigma\tau(i)}) = (\sigma\tau)_* (\boxtimes_{i=1}^n E_{\sigma\tau(i)}).$$

(Set $E'_i = E_{\sigma(i)}$ so that $E'_{\tau(i)} = E_{\sigma\tau(i)}$.) Thus we get a contravariant action. If one prefers a covariant action, it can be defined by $\sigma_\# = (\sigma^\#)^{-1}$.

Let K_i^\bullet be bounded complexes of sheaves of A -modules on X . We have a canonical isomorphism (see [D2])

$$(1.3.3) \quad \sigma^\# : \boxtimes_{i=1}^n K_i^\bullet \xrightarrow{\sim} \sigma_* (\boxtimes_{i=1}^n K_{\sigma(i)}^\bullet),$$

which is defined for $m_i \in K_i^{p_i}$ by

$$(1.3.4) \quad \begin{aligned} \boxtimes_{i=1}^n m_i &\mapsto (-1)^{\nu(\sigma,p)} \sigma_* (\boxtimes_{i=1}^n m_{\sigma(i)}), \\ \text{with } \nu(\sigma,p) &:= \sum_{i < j, \sigma(j) < \sigma(i)} p_i p_j. \end{aligned}$$

Here $(-1)^{\nu(\sigma,p)}$ coincides with the sign of the permutation of $\{i \in [1, n] \mid p_i : \text{odd}\}$ ignoring the i with p_i even. More precisely, let $1 \leq i_1 < \cdots < i_{n'} \leq n$ be the integers with p_{i_k} odd ($k \in [1, n']$), and $1 \leq j_1 < \cdots < j_{n'} \leq n$ be the integers such that $\{\sigma(i_k)\} = \{j_k\}$. There is $\sigma' \in \mathfrak{S}_{n'}$ such that $j_{\sigma'(k)} = \sigma(i_k)$, and $(-1)^{\nu(\sigma,p)}$ is the sign of σ' .

For $M_i^\bullet \in C^b(\mathcal{D}_{X_i})$, we have similarly a canonical isomorphism in $C^b(\mathcal{D}_{X^n})$

$$(1.3.5) \quad \sigma^\# : \circ \boxtimes_{i=1}^n M_i^\bullet \xrightarrow{\sim} \sigma_* (\circ \boxtimes_{i=1}^n M_{\sigma(i)}^\bullet),$$

inducing an isomorphism in $D^b(\mathcal{D}_{X^n})$ for $M_i^\bullet \in D^b(\mathcal{D}_{X_i})$. Here σ_* may be viewed as the direct image of \mathcal{D} -modules since the action of σ is an isomorphism.

1.4. Compatibility with the de Rham functor. With the notation of (1.2), the external product \boxtimes is compatible with the de Rham functor DR . This means that there is a canonical morphism for $M_i^\bullet \in C^b(\mathcal{D}_{X_i})$:

$$(1.4.1) \quad \mathbb{C} \boxtimes_{i=1}^n \text{DR}_{X_i}(M_i^\bullet) \rightarrow \text{DR}_{\mathcal{X}}(\circ \boxtimes_{i=1}^n M_i^\bullet),$$

where the left-hand side is defined for $K_i^\bullet = \mathrm{DR}_{X_i} M_i^\bullet$ as in (1.1) with $A = \mathbb{C}$. Here we use right \mathcal{D} -modules so that the j -th component of $\mathrm{DR}_X(M)$ for a right \mathcal{D}_X -module M on a complex manifold X is given by $M \otimes_{\mathcal{O}} \bigwedge^{-j} \Theta_X$ where Θ_X is the sheaf of holomorphic vector fields, and the complex is locally identified with the Koszul complex associated to the differential operators $\partial/\partial x_i$ if one chooses a local coordinate system (x_1, \dots, x_d) so that the $\bigwedge^{-j} \Theta_X$ are locally trivialized, see [Sa1]. Note that (1.4.1) is a quasi-isomorphism in the holonomic case (i.e. if the $\mathcal{H}^k M_i^\bullet$ are holonomic \mathcal{D} -modules).

For $m_i \in M_i^{p_i}$ and $\eta_j \in \bigwedge^{-q_j} \Theta_{X_j}$, the morphism (1.4.1) is then given by

$$(1.4.2) \quad \begin{aligned} \boxtimes_{i=1}^n (m_i \otimes \eta_i) &\mapsto (-1)^{\nu(p,q)} \left(\boxtimes_{i=1}^n m_i \right) \otimes \left(\bigwedge_{i=1}^n pr_i^* \eta_i \right), \\ \text{with } \nu(p,q) &:= \sum_{i>j} p_i q_j, \end{aligned}$$

where $\bigwedge_{i=1}^n pr_i^* \eta_i := pr_1^* \eta_1 \wedge \dots \wedge pr_n^* \eta_n \in \bigwedge^{-\sum_i q_i} \Theta_X$, and the sign comes from the same reason as (1.3.4).

Proposition 1.5. *With the notation of (1.3), the action of the symmetric group \mathfrak{S}_n is compatible with the de Rham functor DR , i.e. for $M_i^\bullet \in C^b(\mathcal{D}_X)$, there is a commutative diagram*

$$\begin{array}{ccc} \mathbb{C} \boxtimes_{i=1}^n \mathrm{DR}_X(M_i^\bullet) & \xrightarrow{\sigma^\#} & \sigma_* \left(\mathbb{C} \boxtimes_{i=1}^n \mathrm{DR}_X(M_{\sigma(i)}^\bullet) \right) \\ & & \downarrow \\ & & \sigma_* \left(\mathrm{DR}_{X^n} \left(\mathcal{O} \boxtimes_{i=1}^n M_{\sigma(i)}^\bullet \right) \right) \\ & & \downarrow \\ \mathrm{DR}_{X^n} \left(\mathcal{O} \boxtimes_{i=1}^n M_i^\bullet \right) & \xrightarrow{\mathrm{DR}(\sigma^\#)} & \mathrm{DR}_{X^n} \left(\sigma_* \left(\mathcal{O} \boxtimes_{i=1}^n M_{\sigma(i)}^\bullet \right) \right) \end{array}$$

where the horizontal morphisms are induced by (1.3.3) and DR_{X^n} of (1.3.5), and the vertical morphisms are induced by (1.4.1) together with the commutativity of DR_{X^n} and σ_* .

Proof. We may assume $\sigma = (k, k+1)$ for some $k \in [1, n-1]$, i.e. $\sigma(i) = i$ for $i \notin \{k, k+1\}$ and $\sigma \neq \mathrm{id}$. Indeed, \mathfrak{S}_n is generated by such elements and the action is compatible with the group law by (1.3.2). (The last property cannot be used to define the action of $\sigma \in \mathfrak{S}_n$ without showing the independence of factorizations of σ .)

Let $m_i \in M_i^{p_i}$, $\eta_j \in \bigwedge^{-q_j} \Theta_X$. Consider the image of $\boxtimes_{i=1}^n (m_i \otimes \eta_i)$ in each term of the diagram. These are given up to sign by

$$\begin{array}{ccc} \boxtimes_{i=1}^n (m_i \otimes \eta_i) & \rightarrow & \sigma_* \left(\boxtimes_{i=1}^n (m_{\sigma(i)} \otimes \eta_{\sigma(i)}) \right) \\ & & \downarrow \\ & & \sigma_* \left(\boxtimes_{i=1}^n m_{\sigma(i)} \right) \otimes \sigma_* \left(\bigwedge_{i=1}^n pr_i^* \eta_{\sigma(i)} \right) \\ & & \downarrow \\ \left(\boxtimes_{i=1}^n m_i \right) \otimes \left(\bigwedge_{i=1}^n pr_i^* \eta_i \right) & \rightarrow & \sigma_* \left(\boxtimes_{i=1}^n m_{\sigma(i)} \right) \otimes \left(\bigwedge_{i=1}^n pr_i^* \eta_i \right) \end{array}$$

We have to show that the signs associated to the morphisms of the diagram cancel out. Since $\sigma(i) = i$ for $i \notin \{k, k+1\}$, certain signs associated to the two vertical morphisms cancel out. Indeed, these are associated to the sum of $p_j q_j$ over $i < j$

with $(i, j) \neq (k, k + 1)$ in (1.4.2). So the assertion is reduced to the case $n = 2$ and $\sigma \neq id$. Then the signs coming from the horizontal and vertical morphisms are given by

$$(-1)^{(p_1+q_1)(p_2+q_2)}, \quad (-1)^{p_1p_2} \quad \text{and} \quad (-1)^{p_2q_1}, \quad (-1)^{p_1q_2}, \quad (-1)^{q_1q_2},$$

where the last sign comes from the anti-commutativity in $\bigwedge^{-q_1-q_2}\Theta_{X^2}$, and the other signs come from (1.3.4) and (1.4.2). So the assertion follows.

Remarks 1.6. (i) In the above argument it is also possible to use left \mathcal{D} -modules instead of right \mathcal{D} -modules if we replace the de Rham functor DR_X with $DR_X[-\dim X]$ and cancel the effect of the shift of complexes by using the twist of the character $\varepsilon^{\dim X}$. Here

$$(1.6.1) \quad \varepsilon : \mathfrak{S}_n \rightarrow \{-1, 1\}$$

is a character such that $\varepsilon(\sigma)$ is the sign of a permutation σ . In this case $\bigwedge^\bullet \Theta_X$ is replaced by $\Omega_X^\bullet = \bigwedge^\bullet \Omega_X^1$. Note that the shift of complex $[k]$ in general corresponds to the twist of the action by the character ε^k .

In the case $M = \mathcal{O}_X$, we have to twist the action by $\varepsilon^{\dim X}$ since the de Rham functor DR_X is shifted by $\dim X$ so that $DR_X(\mathcal{O}_X) = \mathbf{C}_X[\dim X]$.

(ii) In Proposition (1.5), we considered the de Rham complexes in the category of \mathbf{C} -complexes. However, Proposition (1.5) holds also if we use the filtered de Rham functor which is defined for bounded complexes of filtered \mathcal{D} -modules (M^\bullet, F) and whose value is in the category $C^bF(\mathcal{O}_X, \text{Diff})$ of bounded filtered differential complexes in [Sa1]. This filtered differential complex version is needed in [CMSSY]. Here the multiple external products $\mathbf{c}\boxtimes$ in the first row of the diagram in Proposition (1.5) is replaced by $\mathcal{O}\boxtimes$, since the multiple external product for bounded filtered differential complexes is defined by using the scalar extension by (1.2.1). However, we have to define first $\mathbf{c}\boxtimes$ for bounded filtered differential complexes before applying the scalar extension by (1.2.1). So we need the category $C^bF(\mathcal{O}'_{\mathcal{X}}, \text{Diff})$ consisting of bounded filtered differential complexes of $\mathcal{O}'_{\mathcal{X}}$ -modules, see (1.2.1) for $\mathcal{O}'_{\mathcal{X}}$. This category is defined in the same way as in [Sa1] using

$$\mathcal{D}'_{\mathcal{X}} := \mathbf{c}\boxtimes_{i=1}^n \mathcal{D}_{X_i} = \mathcal{O}'_{\mathcal{X}}\langle \partial_1, \dots, \partial_{d'} \rangle,$$

instead of $\mathcal{D}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}\langle \partial_1, \dots, \partial_{d'} \rangle$ where $\partial_i := \partial/\partial x_i$ for local coordinates $x_1, \dots, x_{d'}$ of \mathcal{X} with $d' = \dim \mathcal{X}$.

To prove the above variant of Proposition (1.5), we first prove the commutativity of the original diagram in Proposition (1.5) by the same argument as before where the multiple external product $\mathbf{c}\boxtimes$ in the first row is defined in $C^bF(\mathcal{O}'_{\mathcal{X}}, \text{Diff})$. Then we can take the scalar extension of the two terms in the first row by the morphism (1.2.1) since the other terms are filtered differential complexes of $\mathcal{O}_{\mathcal{X}}$ -modules.

1.7. Action of \mathfrak{S}_n on mixed Hodge modules. Let $\text{MHM}(X)$ be the category of mixed Hodge modules [Sa2], and $D^b\text{MHM}(X)$ be the derived category of bounded complexes of $\text{MHM}(X)$. Here we assume X is a complex manifold or a smooth complex algebraic variety. (Since X is assumed quasi-projective in Theorem 1, we may replace X with a smooth variety containing it.) In the algebraic case, we use

analytic \mathcal{D} -modules (assuming the stratifications are algebraic) to simplify the de Rham functor.

For $\mathcal{M}_i \in \text{MHM}(X)$, there is the n -fold external product

$$\boxtimes_{i=1}^n \mathcal{M}_i := \mathcal{M}_1 \boxtimes \cdots \boxtimes \mathcal{M}_n \in \text{MHM}(X^n),$$

and there is a natural action of \mathfrak{S}_n on it by Proposition (1.5). This implies a natural action of \mathfrak{S}_n on

$$\boxtimes_{i=1}^n \mathcal{M}_i \in C^b\text{MHM}(X^n) \quad \text{for} \quad \mathcal{M}_i \in C^b\text{MHM}(X),$$

and then on $\boxtimes_{i=1}^n \mathcal{M}_i \in D^b\text{MHM}(X^n)$ for $\mathcal{M}_i \in D^b\text{MHM}(X)$. By Proposition (1.5) this action is compatible with the natural action on the underlying \mathbf{Q} -complexes using the faithfulness of the functor

$$\otimes_{\mathbf{Q}} \mathbf{C} : D_c^b(X, \mathbf{Q}) \rightarrow D_c^b(X, \mathbf{C}).$$

Here the faithfulness follows from the well-known formula

$$\text{Hom}(K, K') = H^0(X, \mathbf{R}\mathcal{H}om(K, K')) \quad \text{for} \quad K, K' \in D_c^b(X, A).$$

Note also that the composition of the functor rat associating the underlying \mathbf{Q} -complex and the above functor $\otimes_{\mathbf{Q}} \mathbf{C}$ is canonically isomorphic to the de Rham functor

$$\text{DR}_X : D^b\text{MHM}(X) \rightarrow D_c^b(X, \mathbf{C}).$$

This follows from the construction of the realization functor in [BBD].

1.8. Mixed Hodge modules on singular varieties. Let X be a complex algebraic variety or a complex analytic space (assumed Hausdorff). A filtered \mathcal{D} -module (M, F) on X is a collection of filtered \mathcal{D}_Z -modules $(M_{U \hookrightarrow Z}, F)$ for any closed embeddings $U \hookrightarrow Z$ where U is an open subvariety (or an open subset) of X , Z is smooth, and $(M_{U \hookrightarrow Z}, F) \in MF(\mathcal{D}_Z)_U$. Here $MF(\mathcal{D}_Z)_U \subset MF(\mathcal{D}_Z)$ is defined by the condition that the $\text{Gr}_p^F M_{U \hookrightarrow Z}$ are \mathcal{O}_U -modules (in particular $M_{U \hookrightarrow Z}$ is supported on U), and it is satisfied by mixed Hodge modules supported on U , see [Sa1], Lemma 3.2.6. They satisfy some compatibility conditions, see [Sa1], 2.1.20 and also [Sa4], 1.5. For instance, if there are two closed embeddings $U_a \hookrightarrow Z_a$ ($a = 1, 2$), set

$$U_{1,2} := U_1 \cap U_2 \hookrightarrow Z_{1,2} := Z_1 \times Z_2.$$

Then we have isomorphisms for $a = 1, 2$

$$(1.8.1) \quad (M_{U_a \hookrightarrow Z_a}, F)|_{Z_a \setminus (U_a \setminus U_{a'})} \cong (pr_{Z_a})_*(M_{U_{1,2} \hookrightarrow Z_{1,2}}, F),$$

where $a' := 3 - a$ and $(pr_{Z_a})_*$ is the direct image as a filtered \mathcal{D} -module under the projection to Z_a . Here we use the following:

Let $f : X' \rightarrow Y'$ be a morphism of smooth varieties or complex manifolds inducing an isomorphism $X \xrightarrow{\sim} Y$ where $X \subset X'$, $Y \subset Y'$ are closed subvarieties or closed subspaces. Then the direct image of \mathcal{D} -modules induces an equivalence of categories

$$(1.8.2) \quad f_* : MF(\mathcal{D}_{X'})_X \xrightarrow{\sim} MF(\mathcal{D}_{Y'})_Y.$$

This assertion is local, and is reduced to the case where $X \hookrightarrow X'$ is a minimal embedding at $x \in X$, i.e. the Zariski tangent space of X at x has the same dimension as X' .

Mixed Hodge modules on singular varieties can be defined in the same way as above. In the above notation, we use an equivalence of categories

$$(1.8.3) \quad f_* : \mathrm{MHM}(X')_X \xrightarrow{\sim} \mathrm{MHM}(Y')_Y,$$

where $\mathrm{MHM}(X')_X \subset \mathrm{MHM}(X)$ is the full subcategory consisting of objects supported on X . Note that (1.8.3) holds also for complexes by replacing $\mathrm{MHM}(X')_X$ with $C^b\mathrm{MHM}(X')_X$ and $\mathrm{MHM}(Y')_Y$ with $C^b\mathrm{MHM}(Y')_Y$ since

$$(1.8.4) \quad f_*\mathcal{M} = H^0 f_*\mathcal{M}, \quad H^j f_*\mathcal{M} = 0 \ (j \neq 0) \quad \text{for } \mathcal{M} \in \mathrm{MHM}(X')_X.$$

Theorem 1.9. *Let X be complex algebraic variety or a complex analytic space. Assume X is globally embeddable into a smooth variety or space. Let $\mathcal{M}_i \in C^b\mathrm{MHM}(X)$. For $\sigma \in \mathfrak{S}_n$ we have a contravariant action*

$$\sigma^\# : \boxtimes_{i=1}^n \mathcal{M}_i \xrightarrow{\sim} \sigma_*(\boxtimes_{i=1}^n \mathcal{M}_{\sigma(i)}) \quad \text{in } C^b\mathrm{MHM}(X^n),$$

satisfying $\sigma_*\tau^\# \circ \sigma^\# = (\sigma\tau)^\#$ for any $\sigma, \tau \in \mathfrak{S}_n$. The induced isomorphism in $D^b\mathrm{MHM}(X^n)$ for $\mathcal{M}_i \in D^b\mathrm{MHM}(X)$ is compatible with the canonical action on the underlying \mathbf{Q} -complexes via the functor rat associating the underlying \mathbf{Q} -complex.

Proof. Note first that the assertion is proved in (1.7) if X is smooth. By hypothesis there is a closed embedding $X \hookrightarrow X'$ with X' smooth. Then we have an equivalence of categories

$$\mathrm{MHM}(X) \xrightarrow{\sim} \mathrm{MHM}(X')_X,$$

where $\mathrm{MHM}(X')_X \subset \mathrm{MHM}(X')$ denotes the full subcategory consisting of objects supported on X . This follows from the definition of \mathcal{D} -modules on complex varieties or complex analytic spaces, see (1.8). So any $\mathcal{M} \in C^b\mathrm{MHM}(X)$ is canonically represented by

$$\mathcal{M}' \in C^b\mathrm{MHM}(X')_X := C^b(\mathrm{MHM}(X')_X).$$

Here \mathcal{M}' is a bounded complex of mixed Hodge module defined by using filtered $\mathcal{D}_{X'}$ -modules in the usual sense. Then Theorem (1.9) in the smooth case can be applied to \mathcal{M}' , and we get the canonical action of \mathfrak{S}_n in $C^b\mathrm{MHM}((X')^n)_{X^n}$. This action is independent of the choice of X' . Indeed, if there are two closed embeddings $X \hookrightarrow X'_a$ ($a = 1, 2$), set $X'_3 = X'_1 \times X'_2$. By (1.8.3) we have an equivalence of categories

$$(pr_a)_* : C^b\mathrm{MHM}(X'_3)_X \xrightarrow{\sim} C^b\mathrm{MHM}(X'_a)_X,$$

induced by the direct image under the projection $pr_a : X'_3 \rightarrow X'_a$ ($a = 1, 2$), and similarly for the projection between their multiple fiber products where $\mathrm{MHM}(X'_a)_X$ is replaced by $\mathrm{MHM}((X'_a)^n)_{X^n}$ for $a = 1, 2, 3$. Since (1.8.4) holds for $f = pr_a$, we have the commutativity of the multiple external product with the direct image as in (1.12.2) below also in the analytic case. So the independence of the embedding follows. This finishes the proof of Theorem (1.9).

Remark 1.10. In the case $\mathcal{M}_i = \mathcal{M}(\forall i)$ and $\mathcal{M} = a_X^* \mathbf{Q}$ for a variety X or $\mathcal{M} = \mathrm{IC}_X \mathcal{L}$ with \mathcal{L} a variation of Hodge structure on a smooth dense Zariski-open

subset of an irreducible variety X , it is not difficult to show Theorem (1.9) using the following property:

(P) There is a sufficiently small smooth open subset U of X^n which is stable by the action of \mathfrak{S}_n and such that the restriction induces an isomorphism

$$\mathrm{End}(\boxtimes^n \mathcal{M}) \xrightarrow{\sim} \mathrm{End}((\boxtimes^n \mathcal{M})|_U).$$

Note that

$$\boxtimes^n a_X^* \mathbf{Q} = a_{X^n}^* \mathbf{Q}, \quad \boxtimes^n (\mathrm{IC}_X \mathcal{L}) = \mathrm{IC}_{X^n}(\boxtimes^n \mathcal{L}).$$

1.11. Direct image of mixed Hodge modules. The definition of the direct image of mixed Hodge modules under a morphism of complex algebraic varieties $f : X \rightarrow Y$ is as follows (see the proof of [Sa2], Th. 4.3): Take an affine open covering U_j of X , and let $U_J = \bigcap_{j \in J} U_j$ with f_J the restriction of f to U_J . Here X may be singular, and we have closed embeddings $U_j \hookrightarrow Z_j$ with Z_j smooth since U_j are affine, see (1.8) for mixed Hodge modules on singular varieties. We may also assume that there is an affine open covering $\{U'_j\}$ of Y together with closed embeddings $U'_j \hookrightarrow Z'_j$ such that Z'_j is smooth, $f(U_j) \subset U'_j$ and $f|_{U_j}$ is extended to $f'_j : Z_j \rightarrow Z'_j$, see also [Sa1], 2.3.9. Then, for any bounded complex of mixed Hodge modules \mathcal{M} on X , there is a quasi-isomorphism $\mathcal{N} \rightarrow \mathcal{M}$ such that

$$H^k(f_J)_*(\mathcal{N}^p|_{U_J}) = 0 \quad \text{for any } p \in \mathbf{Z}, k \neq 0 \text{ and } J.$$

(This is essentially the same argument as in [Be].) Here $(f_J)_*$ can be defined by using $f'_j := \prod_{j \in J} f'_j$. Combining this construction with the Čech complex for U_J , we get a double complex such that the associated single complex \mathcal{M}' is a representative of \mathcal{M} satisfying the condition:

$$(1.11.1) \quad H^k f_*(\mathcal{M}'^p) = 0 \quad \text{for any } p \in \mathbf{Z} \text{ and } k \neq 0.$$

Then the direct image $f_* \mathcal{M}$ is defined by $H^0 f_*(\mathcal{M}'^\bullet)$. This is independent of the choice of the above \mathcal{N} by the standard argument in the theory of derived categories since there is a quasi-isomorphism $\mathcal{N} \rightarrow \mathcal{M}$ for any \mathcal{M} . It is also independent of the choice of U_j by using a refinement of two affine coverings.

In case X is projective, we can take U_i to be the complement of a hyperplane section, and the resolution $\mathcal{N} \rightarrow \mathcal{M}$ can be constructed by using the dual of the Čech complex (using the direct images with proper supports) which is associated to an open covering defined by the complements of sufficiently general hyperplane sections, see [Be].

We can similarly define the direct image with compact support $f_!$ by the dual argument where the Čech complex associated to the U_i is replaced by the dual of the Čech complex using the direct images with proper supports and the directions of the morphisms are all reversed, e.g. we have $\mathcal{M} \rightarrow \mathcal{N}$ instead of $\mathcal{N} \rightarrow \mathcal{M}$.

1.12. Multiple Künneth formula for mixed Hodge modules. For morphisms of complex algebraic varieties $f_i : X_i \rightarrow Y_i$, set $\mathcal{X} = \prod_{i=1}^n X_i$, $\mathcal{Y} = \prod_{i=1}^n Y_i$, and $f = \prod_{i=1}^n f_i : \mathcal{X} \rightarrow \mathcal{Y}$. Let $\mathcal{M}_i \in D^b\mathrm{MHM}(X_i)$. We first show that the direct

image commutes with the multiple external products, i.e. there is a canonical isomorphism

$$(1.12.1) \quad \boxtimes_{i=1}^n (f_i)_* \mathcal{M}_i = f_* (\boxtimes_{i=1}^n \mathcal{M}_i) \quad \text{in } D^b\text{MHM}(\mathcal{Y}),$$

and this also holds with direct images $(f_i)_*$, f_* replaced by direct images with proper supports $(f_i)!$, $f!$. Moreover the isomorphism is compatible with the action of \mathfrak{S}_n in case $X_i = X$ and $Y_i = Y$ for any i . These assertions follow from the definition of the direct image of bounded complexes of mixed Hodge modules as is explained in (1.11). We note a proof for the usual direct images. The argument is similar for the direct images with proper supports.

Take a representative \mathcal{M}'_i in (1.11) for each \mathcal{M}_i so that (1.11.1) is satisfied. Then the isomorphism (1.12.1) follows, since there are natural isomorphisms of mixed Hodge modules

$$(1.12.2) \quad \boxtimes_{i=1}^n H^0(f_i)_* \mathcal{M}'_i{}^{p_i} = H^0 f_* (\boxtimes_{i=1}^n \mathcal{M}'_i{}^{p_i}).$$

By this argument, (1.12.1) is compatible with the action of \mathfrak{S}_n in case $X_i = X$ for any i . Moreover, in case $Y_i = pt$ for any i , (1.12.1) is compatible with the corresponding isomorphism (3.8.1) below for the underlying \mathbf{R} -complexes K_i of \mathcal{M}_i . This follows from the definition of the realization functor in [BBD] using Prop. 3.1.8 in loc. cit.

In case $Y_i = pt$ for any i , we also show that the above proof of (1.12.1) implies the multiple Künneth isomorphism of graded mixed Hodge structures

$$(1.12.3) \quad \bigotimes_{i=1}^n H^\bullet(X_i, \mathcal{M}_i) = H^\bullet(\mathcal{X}, \boxtimes_{i=1}^n \mathcal{M}_i),$$

and this also holds for cohomology with compact supports. Moreover, the isomorphism is compatible with the action of \mathfrak{S}_n in case X_i and \mathcal{M}_i are independent of i . Here the category of graded-polarizable mixed \mathbf{Q} -Hodge structures MHS is naturally identified with $\text{MHM}(pt)$ as in [Sa2] (where ‘graded-polarizable’ means that the graded quotients of the weight filtration are polarizable, see [D1]). The obtained mixed Hodge structure in the constant coefficient case in [Sa2] coincides with that in [D3], see [Sa5]. (Note that Ext^i in MHS vanishes for $i > 1$ by [Ca] although this is not used in our argument).

By the above argument, the proof of (1.12.3) is reduced to the multiple Künneth formula for n -fold tensor products of bounded complexes of mixed Hodge structures

$$(1.12.4) \quad \bigotimes_{i=1}^n H^\bullet \mathcal{N}_i \xrightarrow{\sim} H^\bullet(\bigotimes_{i=1}^n \mathcal{N}_i),$$

where $\mathcal{N}_i^p := H^0(f_i)_* \mathcal{M}'_i{}^p$. For the proof of (1.12.4), we have a canonical morphism induced by the natural inclusions

$$\bigotimes_{i=1}^n \text{Ker}(d : \mathcal{N}_i^{p_i} \rightarrow \mathcal{N}_i^{p_i+1}) \hookrightarrow \text{Ker}(d : \mathcal{N}^p \rightarrow \mathcal{N}^{p+1}),$$

where $\mathcal{N} = \bigotimes_{i=1}^n \mathcal{N}_i$ and $p = \sum_{i=1}^n p_i$. This is compatible with the multiple Künneth formula for the n -fold tensor products of the underlying complexes of \mathbf{Q} -vector spaces, and the assertion is well known in the latter case. So (1.12.4) follows.

Remark 1.13. It is also possible to prove the multiple Künneth formula for mixed Hodge modules by induction on n (reducing to the case $n = 2$). In this case it is not easy to show that the obtained isomorphism is independent of the choice of the order of $\{1, \dots, n\}$ although this is essential for the proof of the compatibility with the action of \mathfrak{S}_n . In the case $Y = pt$, however, there is a canonical isomorphism for the underlying \mathbf{Q} -vector spaces (see (3.8) below), and we can use this canonical isomorphism by showing its compatibility with the mixed Hodge structure.

2. Symmetric products

2.1. Representations of \mathfrak{S}_n . Since \mathfrak{S}_n is a finite group, it is completely reductive, and every finite dimensional representation of \mathfrak{S}_n over \mathbf{Q} is semisimple. (In fact, this is easily shown by taking a positive definite symmetric pairing $\langle u, v \rangle$ for any finite dimensional representation on a \mathbf{Q} -vector space V and replacing it with $\sum_{\sigma \in \mathfrak{S}_n} \langle \sigma(u), \sigma(v) \rangle$ so that $\langle \sigma(u), \sigma(v) \rangle = \langle u, v \rangle$ for any $u, v \in V$ and $\sigma \in \mathfrak{S}_n$.)

Applying this to the group ring $\mathbf{Q}[\mathfrak{S}_n]$ viewed as a left $\mathbf{Q}[\mathfrak{S}_n]$ -module, we get the decomposition by irreducible characters

$$\mathbf{Q}[\mathfrak{S}_n] = \bigoplus_{\chi} \mathbf{Q}[\mathfrak{S}_n]_{\chi},$$

where χ runs over the irreducible characters of \mathfrak{S}_n , and $\mathbf{Q}[\mathfrak{S}_n]_{\chi}$ is the sum of simple left $\mathbf{Q}[\mathfrak{S}_n]$ -submodules of $\mathbf{Q}[\mathfrak{S}_n]$ with character χ . Since this decomposition is compatible with the right action of $\mathbf{Q}[\mathfrak{S}_n]$, the direct factors $\mathbf{Q}[\mathfrak{S}_n]_{\chi}$ are two-sided ideals of $\mathbf{Q}[\mathfrak{S}_n]$. Hence

$$\mathbf{Q}[\mathfrak{S}_n]_{\chi} \mathbf{Q}[\mathfrak{S}_n]_{\chi'} \subset \mathbf{Q}[\mathfrak{S}_n]_{\chi} \cap \mathbf{Q}[\mathfrak{S}_n]_{\chi'} = 0 \quad \text{if } \chi \neq \chi'.$$

By the above decomposition, there are unique elements

$$e_{\chi} \in \mathbf{Q}[\mathfrak{S}_n]_{\chi} \quad \text{with } \sum_{\chi} e_{\chi} = 1 \in \mathbf{Q}[\mathfrak{S}_n].$$

The above property implies that the e_{χ} are mutually orthogonal idempotents and e_{χ} is the identity of $\mathbf{Q}[\mathfrak{S}_n]_{\chi}$.

Let V_{χ} be an irreducible representation over \mathbf{Q} with character χ . In the case of symmetric groups, it is known that any irreducible representation over \mathbf{C} is defined over \mathbf{Q} . So the multiplicity of V_{χ} in $\mathbf{Q}[\mathfrak{S}_n]$ as a left $\mathbf{Q}[\mathfrak{S}_n]$ -module coincides with $\dim V_{\chi}$ by the well-known orthonormal relation between the irreducible characters χ . This implies that $\dim \mathbf{Q}[\mathfrak{S}_n]_{\chi} = (\dim V_{\chi})^2$, and hence $\mathbf{Q}[\mathfrak{S}_n]_{\chi}$ is isomorphic to the full endomorphism algebra $\text{End}_{\mathbf{Q}}(V_{\chi})$. The last assertion is used in [D6].

As a corollary of the semisimplicity, the \mathfrak{S}_n -invariant part of a finite dimensional representation on a \mathbf{Q} -vector space is identified with the \mathfrak{S}_n -coinvariant part, which is by definition the maximal quotient on which the action of \mathfrak{S}_n is trivial.

By the theory of Young diagrams, the irreducible characters χ correspond to the Young diagrams, i.e. the partitions $\lambda = \{\lambda_1, \lambda_2, \dots\}$ of n such that $\lambda_1 \geq \lambda_2 \geq \dots$ and $\sum_i \lambda_i = n$. The corresponding representation can be constructed by using the Young symmetrizer. The trivial character 1 corresponds to the trivial partition $\{n\}$, and the corresponding idempotent is given by

$$(2.1.1). \quad e_1 = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \in \mathbf{Q}[\mathfrak{S}_n]$$

The sign character ε in (1.6.1) corresponds to $\{1, 1, \dots\} = {}^t\{n\}$, and

$$(2.1.2) \quad e_\varepsilon = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \sigma \in \mathbf{Q}[\mathfrak{S}_n].$$

It is not difficult to prove (2.1.1–2) by using the above definition of e_χ via the left action of $\mathbf{Q}[\mathfrak{S}_n]$ on itself by considering the condition: $\tau(\sum_\sigma a_\sigma \sigma) = \sum_\sigma a_\sigma \sigma$ (or $\varepsilon(\tau) \sum_\sigma a_\sigma \sigma$) for any $\tau \in \mathfrak{S}_n$, where $a_\sigma \in \mathbf{Q}$.

In general the relation between e_χ and the Young symmetrizer is not so trivial. For the proof of Theorem 1, the explicit form of the idempotents is not needed. We will need rather the coincidence of the \mathfrak{S}_n -invariant and coinvariant part as explained above.

The following is known to the specialists in a more general situation (i.e. for any decompositions $V^\bullet = V'^\bullet \oplus V''^\bullet$), see e.g. [D6], Sect. 1, [He], 4.2. It is closely related to a λ -ring structure of the Grothendieck group of a graded vector space although a canonical isomorphism as graded vector spaces is finer than an equality in the Grothendieck group.

Proposition 2.2. *Let V^\bullet be a finite dimensional graded vector space. We have the decomposition $V^\bullet = V_{\text{even}}^\bullet \oplus V_{\text{odd}}^\bullet$ by the parity of the degree. Then we have a canonical isomorphism of bigraded vector spaces*

$$(2.2.1) \quad \begin{aligned} \bigoplus_{n \geq 0} (\bigotimes^n V^\bullet)^{\mathfrak{S}_n} &= \text{Sym}^\bullet V_{\text{even}}^\bullet \otimes \bigwedge^\bullet V_{\text{odd}}^\bullet, \\ \text{i.e. } (\bigotimes^n V^\bullet)^{\mathfrak{S}_n} &= \bigoplus_{n'+n''=n} (\text{Sym}^{n'} V_{\text{even}}^\bullet \otimes \bigwedge^{n''} V_{\text{odd}}^\bullet), \end{aligned}$$

where the action of \mathfrak{S}_n is defined by identifying the graded vector space V^\bullet with a complex with zero differential so that the sign appears as in (1.3.4) with $X = pt$. If V^\bullet is graded mixed Hodge structure, then (2.2.1) is an isomorphism of bigraded mixed Hodge structures.

Proof. In general, the usual symmetric tensor product $\text{Sym}^n V'$ of a finite dimensional vector space V' can be identified with a maximal quotient of $\bigotimes^n V'$ on which the action of \mathfrak{S}_n is trivial, see (2.1). We have a similar assertion for $\bigwedge^n V'$ using the action of \mathfrak{S}_n twisted by the character ε in (1.6.1). Using these, we get a canonical surjection

$$(2.2.2) \quad \bigotimes^n V^\bullet \rightarrow \bigoplus_{n'+n''=n} (\text{Sym}^{n'} V_{\text{even}}^\bullet \otimes \bigwedge^{n''} V_{\text{odd}}^\bullet)$$

Indeed, for $v := v_1 \otimes \dots \otimes v_n \in \bigotimes^n V^\bullet$ with $v_i \in V^{k_i}$, its image is defined by the image of

$$(v_{p_1} \otimes \dots \otimes v_{p_{n'}}) \otimes (v_{q_1} \otimes \dots \otimes v_{q_{n''}})$$

in the right-hand side, where p_i and q_j are strictly increasing sequences such that

$$k_{p_i} \text{ is even, } \quad k_{q_j} \text{ is odd, } \quad \{p_1, \dots, p_{n'}\} \amalg \{q_1, \dots, q_{n''}\} = \{1, \dots, n\},$$

where $n' + n'' = n$. Then the morphism respects the action of \mathfrak{S}_n where the action is trivial on the target. We have moreover a canonical morphism from the right-hand side to the maximal quotient of $\bigotimes^n V^\bullet$ divided by the subspace generated by $\sigma v - v$ for any $\sigma \in \mathfrak{S}_n$ and $v \in \bigotimes^n V^\bullet$. So the first assertion follows.

The compatibility with the mixed Hodge structures follows from the property that any morphism of mixed Hodge structures is strictly compatible with the mixed

Hodge structures [D1] since (2.2.2) is a morphism of mixed Hodge structures. This finishes the proof of Proposition (2.2).

2.3. Proof of Theorem 1. Since X is assumed quasi-projective, we can apply Theorem (1.9), and get the canonical action of \mathfrak{S}_n on $\pi_* \boxtimes^n \mathcal{M} \in D^b\text{MHM}(\mathcal{S}^n X)$ which is compatible with the one on the underlying \mathbf{Q} -complexes. By the splitting of an idempotent in $D^b\text{MHM}(\mathcal{S}^n X)$ (see [BS] and also [LC] for a simpler argument) which is applied to e_1 in (2.1.1), we get a direct factor $\mathcal{S}^n \mathcal{M}$ of $\pi_* \boxtimes^n \mathcal{M}$ endowed with two morphisms in $D^b\text{MHM}(\mathcal{S}^n X)$

$$\mathcal{S}^n \mathcal{M} \rightarrow \pi_* \boxtimes^n \mathcal{M} \rightarrow \mathcal{S}^n \mathcal{M},$$

whose composition is the identity. Note that $\mathcal{S}^n \mathcal{M}$ is unique up to a canonical isomorphism using the above two morphisms together with the forgetful functor associating the underlying \mathbf{Q} -complexes. So the first assertion follows.

Using the decomposition of $\pi_* \boxtimes^n \mathcal{M}$ under the irreducible characters χ in (2.1) together with the compatibility of the direct image functor with the composition of $X^n \rightarrow \mathcal{S}^n X \rightarrow pt$, we get the canonical isomorphism

$$H^\bullet(\mathcal{S}^n X, \mathcal{S}^n \mathcal{M}) = H^\bullet(X^n, \boxtimes^n \mathcal{M})^{\mathfrak{S}_n}.$$

By the multiple Künneth formula (1.12) we get the second canonical isomorphism

$$H^\bullet(X^n, \boxtimes^n \mathcal{M}) = \bigotimes^n H^\bullet(X, \mathcal{M}).$$

These are compatible with the corresponding isomorphisms for the underlying \mathbf{Q} -complexes. Then the second isomorphism is compatible with the action of \mathfrak{S}_n using the action on the underlying \mathbf{Q} -complexes. Thus the remaining assertions are proved. This finishes the proof of Theorem 1.

Remarks 2.4. (i) We have

$$(2.4.1) \quad \mathcal{S}^n \mathcal{M} = a_{\mathcal{S}^n X}^* \mathbf{Q} \in D^b\text{MHM}(\mathcal{S}^n X) \text{ if } \mathcal{M} = a_X^* \mathbf{Q} \in D^b\text{MHM}(X),$$

where $a_X : X \rightarrow pt$, etc. denote the natural morphisms. This immediately follows from the characterization of $a_X^* \mathbf{Q}$ in [Sa2], (4.4.2), i.e. it is uniquely characterized by the conditions that $\text{rat}(\mathcal{M}) = \mathbf{Q}_X$ and $\mathcal{H}^0(X, \mathcal{M}) = \mathbf{Q}$ as a mixed Hodge structure.

(ii) We have

$$(2.4.2) \quad \mathcal{S}^n \mathcal{M} = \text{IC}_{\mathcal{S}^n X} \mathcal{S}^n \mathcal{L} \in \text{MHM}(\mathcal{S}^n X) \text{ if } \mathcal{M} = \text{IC}_X \mathcal{L} \in \text{MHM}(X),$$

where \mathcal{L} is a polarizable variation of Hodge structure on a smooth open subvariety U of an irreducible variety X , and $\mathcal{S}^n \mathcal{L}$ is defined on the smooth part of $\mathcal{S}^n U$. This follows from the fact that the intersection complexes are stable by multiple external product, direct factor, and also by the direct image π_* by a finite morphism π . Indeed, the intersection complexes are defined by using the intermediate direct image, and the latter commutes with the direct image π_* by a finite morphism π (since π_* is an exact functor of mixed Hodge modules).

2.5. Proof of Corollary 1. The assertion follows from Theorem 1 and Proposition (2.2).

Remark 2.6. Let $\mathcal{M} \in D^b\text{MHM}(X)$ with K its underlying \mathbf{Q} -complex. Set

$$\begin{aligned} I &:= \text{Im}(\mathbf{Q}[\mathfrak{S}_n] \rightarrow \text{End}(\pi_* \boxtimes^n \mathcal{M})), \\ I' &:= \text{Im}(\mathbf{Q}[\mathfrak{S}_n] \rightarrow \text{End}(\pi_* \boxtimes^n K)), \\ I'' &:= \text{Im}(\mathbf{Q}[\mathfrak{S}_n] \rightarrow \text{End}(\mathbf{Q}^n)), \end{aligned}$$

where the last representation is given by permutation matrices. In the notation of (2.1) there are sets of irreducible characters $\Lambda \supset \Lambda' \supset \Lambda''$ such that

$$I = \bigoplus_{\chi \in \Lambda} \mathbf{Q}[\mathfrak{S}_n]_{\chi}, \quad I' = \bigoplus_{\chi \in \Lambda'} \mathbf{Q}[\mathfrak{S}_n]_{\chi}, \quad I'' = \bigoplus_{\chi \in \Lambda''} \mathbf{Q}[\mathfrak{S}_n]_{\chi},$$

since there are surjections $I \rightarrow I' \rightarrow I''$. The last morphism can be defined by restricting to the fiber of π over a certain good point of the support of $\pi_* \boxtimes^n K$.

In certain cases (e.g. in the constant coefficient case), we have the equality $I = I' = I''$, and the \mathfrak{S}_n -invariant part is clearly given by the projector e_1 in (2.1.1) looking at the action of \mathfrak{S}_n on the fiber over a general point of $\mathcal{S}^n X$. In this case we would not need the representation theory as is explained in (2.1). For a general bounded complex of mixed Hodge modules \mathcal{M} , however, it is unclear whether the above three coincide, and we need some argument as in (2.1).

2.7. Hodge theory on compact complex V -manifolds [St]. Let X be a complex V -manifold with $j : X' \hookrightarrow X$ the inclusion of the smooth part X' of X . Following [St], set

$$(2.7.1) \quad \tilde{\Omega}_X^p := j_* \Omega_{X'}^p.$$

By definition, a complex V -manifold X is locally a quotient of a smooth complex manifold Y by an action of a finite group G . Let $\pi : Y \rightarrow X$ denote this quotient morphism (locally defined on X). Since π is finite and $X \setminus X'$ has at least codimension 2, we have locally a canonical isomorphism

$$(2.7.2) \quad \tilde{\Omega}_X^p = (\pi_* \Omega_Y^p)^G,$$

Indeed, this holds on X' and we can apply the Hartogs extension theorem on Y since the pull-back of $X \setminus X'$ in Y has at least codimension 2. (Note that $j_* \Omega_{X'}^\bullet \neq \mathbf{R}j_* \Omega_{X'}^\bullet$, even in the algebraic case by taking the global section functor and applying [Gr], Th. 1', see the proof of [DB], Th. 5.3. We can prove only a weaker version of loc. cit. by Proposition (2.8) below.)

By Steenbrink [St], there is Hodge theory for compact complex V -manifolds with a Kähler desingularization. (This is reproduced in [PS], Section 2.5.) As for the relation with the theory of Hodge modules, we first note the following:

The filtered complex $(\tilde{\Omega}_X^\bullet, F)$, with F^p defined by the truncation $\sigma_{\geq p}$ in [D1], is a filtered differential complex in the sense of [DB], and hence also in the sense of [Sa1], i.e. it belongs to $C^b F(\mathcal{O}_X, \text{Diff})$ in loc. cit. This assertion follows from (2.7.2). Note that the derived category of filtered differential complexes in [DB] is canonically equivalent to the one in [Sa1] if the variety is *smooth*, see [Fi].

There are canonical isomorphisms (see [BBD], [GM]):

$$(2.7.3) \quad \mathbf{Q}_X[\dim X] \xrightarrow{\sim} \text{IC}_X \mathbf{Q}, \quad H^\bullet(X, \mathbf{Q}) \xrightarrow{\sim} \text{IH}^\bullet(X, \mathbf{Q}),$$

since a complex V -manifold is a \mathbf{Q} -homology manifold, see Remark (2.10) below. Moreover, these isomorphisms are lifted to $\mathrm{MHM}(X)$ and MHS in the algebraic case, see [Sa2].

For a bounded filtered differential complex $(L^\bullet, F) \in C^bF(\mathcal{O}_X, \mathrm{Diff})$ on a complex manifold, we have the associated bounded complex of filtered right \mathcal{D} -modules $\mathrm{DR}_X^{-1}(L^\bullet, F)$, see [Sa1], 2.2.5. This is naturally extended to the case of singular spaces and also to the algebraic case.

Let (M, F) be the underlying filtered right \mathcal{D} -module of the polarizable Hodge module \mathcal{M} corresponding to the intersection complex $\mathrm{IC}_X \mathbf{Q}$. This is represented by filtered right \mathcal{D} -modules (M_Z, F) for closed embeddings $U \hookrightarrow Z$ where U is an open subset of X and Z is smooth, see [Sa1], 2.1.20. Note that $\mathcal{H}^i \mathrm{Gr}_p^F \mathrm{DR}_Z(M_Z, F)$ is an \mathcal{O}_U -module by [Sa1], Lemma 3.2, and is independent of Z . So it is globally well-defined on X , and is denoted by $\mathcal{H}^i \mathrm{Gr}_p^F \mathrm{DR}_X(M, F)$.

Proposition 2.8. *Let X be a compact complex V -manifold with a Kähler desingularization. Then the pure Hodge structure on $H^\bullet(X, \mathbf{Q})$ in [St] coincides with the one on the intersection cohomology $\mathrm{IH}^\bullet(X, \mathbf{Q})$ which is obtained by using the decomposition theorem in [Sa1], [Sa3] for the desingularization. Moreover, there is a canonical filtered quasi-isomorphism of complexes of filtered \mathcal{D} -modules on X :*

$$(2.8.1) \quad \mathrm{DR}_X^{-1}(\tilde{\Omega}_X^\bullet, F)[\dim X] \xrightarrow{\sim} (M, F),$$

where (M, F) is as above, and we have

$$(2.8.2) \quad \mathcal{H}^i \mathrm{Gr}_F^p \mathrm{DR}_X(M, F) = \tilde{\Omega}_X^p \text{ if } i = p - \dim X, \text{ and } 0 \text{ otherwise.}$$

In case X is algebraic, the pure Hodge structure on $H^\bullet(X, \mathbf{Q})$ in [St] also coincides with the mixed Hodge structure on $H^\bullet(X, \mathbf{Q})$ in [D3], [Sa2]. If furthermore X is a closed subvariety of a smooth complex algebraic variety Z , then there is an isomorphism in the derived category of filtered differential complexes on Z in the sense of [DB] or [Sa1]:

$$(2.8.3) \quad (\tilde{\Omega}_X^\bullet, F) = (\underline{\Omega}_X^\bullet, F),$$

where $(\underline{\Omega}_X^\bullet, F)$ is the filtered Du Bois complex in [DB].

Proof. We first show the analytic case. Using the decomposition theorem ([Sa3], Th. 0.5) for a Kähler desingularization $\rho : \tilde{X} \rightarrow X$, we can show that (M, F) is a direct factor of $\rho_*(\omega_{\tilde{X}}, F)$ (where [Sa1] is enough in case ρ is projective), and this is compatible with the \mathbf{Q} -structure using Deligne's canonical choice of the decomposition [D5]. This implies a pure Hodge structure on the intersection cohomology. By the definition of the direct image of filtered \mathcal{D} -modules in the analytic case in [Sa2], 2.13 (applied to $a_X : X \rightarrow pt$), it is then enough to show (2.8.1).

Since the assertion is local on X , we may assume that X is a quotient of Y as above, and moreover, X is a closed analytic subset of a complex manifold Z so that (M, F) is represented by a filtered \mathcal{D}_Z -module (M_Z, F) . Thus the assertion is reduced to showing the canonical filtered quasi-isomorphism of complexes of filtered

\mathcal{D}_Z -modules

$$(2.8.4) \quad \mathrm{DR}_Z^{-1}(\tilde{\Omega}_X^\bullet, F)[\dim X] \xrightarrow{\sim} (M_Z, F).$$

Let $\pi' : Y \rightarrow Z$ be the composition of $\pi : Y \rightarrow X$ and the inclusion $X \hookrightarrow Z$. Since π' is finite, the direct image as a filtered right \mathcal{D} -module $\pi'_*(\omega_Y, F)$ is a filtered \mathcal{D}_Z -module and underlies a pure Hodge module corresponding to an intersection complex with local system coefficients and with strict support X . So we get

$$(2.8.5) \quad (M_Z, F) = (\pi'_*(\omega_Y, F))^G.$$

Indeed, the assertion is clear on a Zariski-open subset X' of X over which π is étale, and (M_Z, F) is uniquely determined by its restriction to the complement of $X \setminus X'$ (using [Sa1], Prop. 3.2.2).

The direct image of filtered differential complexes are defined by the sheaf-theoretic direct image, and the direct image commutes with the de Rham functor, see [Sa1], Lemma 2.3.6. Since Y is smooth, we have

$$\mathrm{DR}_Y(\omega_Y, F) = (\Omega_Y^\bullet, F)[\dim Y].$$

So we get a canonical filtered quasi-isomorphism of complexes of filtered \mathcal{D}_Z -modules

$$\mathrm{DR}_Z^{-1}(\pi'_*(\Omega_Y^\bullet, F))[\dim Y] \xrightarrow{\sim} \pi'_*(\omega_Y, F).$$

This is equivalent to an isomorphism in the derived category since $\pi'_*(\omega_Y, F)$ is a filtered \mathcal{D}_Z -module and $\pi'_*\Omega_Y^p = 0$ for $p > \dim Y$. So the isomorphism is compatible with the action of G since it is clear on the complement of $X \setminus X'$ where X' is as in the proof of (2.8.5). By definition we have

$$\mathrm{DR}_Z^{-1}(\pi'_*\Omega_Y^p) := \pi'_*\Omega_Y^p \otimes_{\mathcal{O}_Z} \mathcal{D}_Z,$$

and the action of G is induced by that on $\pi'_*\Omega_Y^p$ (i.e. it is the identity on \mathcal{D}_Z). So (2.8.4) follows by taking the G -invariant part.

In the algebraic case the mixed Hodge structures on the cohomology of X defined in [D3] and [Sa2] coincide (see [Sa5]) and the canonical isomorphisms in (2.7.3) are lifted to $\mathrm{MHM}(X)$ and MHS . So it remains to show (2.8.3). By the same argument as above, we have the algebraic version of (2.8.1). In the case X is globally a closed subvariety of a smooth variety Z , we have a canonical isomorphism in the derived category of filtered \mathcal{D}_Z -modules

$$\mathrm{DR}_Z^{-1}(\tilde{\Omega}_X^\bullet, F)[\dim X] = (M_Z, F).$$

By [Sa5], Th. 0.2, we have

$$(M_Z, F) = \mathrm{DR}_Z^{-1}(\underline{\Omega}_X^\bullet, F)[\dim X].$$

We get thus

$$\mathrm{DR}_Z^{-1}(\tilde{\Omega}_X^\bullet, F) = \mathrm{DR}_Z^{-1}(\underline{\Omega}_X^\bullet, F).$$

So (2.8.3) follows by using [Sa1], Prop. 2.2.10 and applying the functor DR_Z which is denoted by $\widehat{\mathrm{DR}}_Z$ in loc. cit. This finishes the proof of Proposition (2.8).

Remarks 2.9. (i) For a complex algebraic V -manifold X , Proposition (2.8) implies the canonical isomorphisms of coherent \mathcal{O}_X -modules

$$(2.9.1) \quad \tilde{\Omega}_X^p = \mathrm{Gr}_{F \underline{\Omega}_X^\bullet}^p[p] \quad (p \in \mathbf{Z}),$$

since the assertion is local. It might be possible to prove [DB], Th. 5.3 by extending the isomorphisms in (2.9.1) if we have the following vanishing of the negative extensions in the derived category of filtered differential complexes in loc. cit.:

$$\mathrm{Ext}^{p-q+1}((\tilde{\Omega}_X^p, F), (\tilde{\Omega}_X^q, F)) = 0 \quad \text{if } q > p + 1.$$

There are, however, no truncations $\tau_{\leq k}$ for filtered differential complexes. The usual definition in [D1] does not work for filtered differential complexes even in the filtered acyclic case.

(ii) In case X is singular it is unclear whether for any filtered differential complex (K, F) , there is a *filtered injective resolution*, i.e. a quasi-isomorphism $(K, F) \xrightarrow{\sim} (I, F)$ such that the $\mathrm{Gr}_F^p I^i$ are injective \mathcal{O}_X -modules. If it always exists, then the extension group can be calculated by using an injective resolution, and a version of Th. 5.3 in [DB] can be proved where the isomorphism is considered in the derived category of filtered differential complexes in [Sa1]. (For the derived category in [DB], the definition of homotopy in loc. cit. is not compatible with the calculation using an injective resolution.)

(iii) Let $i : X \hookrightarrow Y$ be a closed embedding of algebraic varieties. In case X is singular (even if Y is smooth), it is unclear whether the following direct image functor is fully faithful:

$$i_* : D^b F(\mathcal{O}_X, \mathrm{Diff}) \rightarrow D^b F(\mathcal{O}_Y, \mathrm{Diff}).$$

For \mathcal{O}_Y -modules M , we have the functor $i_{\mathcal{O}}^!$ defined by

$$i_{\mathcal{O}}^! M := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, M) = \{m \in M \mid \mathcal{I}_X m = 0\} \subset M,$$

where \mathcal{I}_X is the sheaf of ideals of $X \subset Y$. This derives $i^! : D_{\mathrm{coh}}^b(\mathcal{O}_Y) \rightarrow D_{\mathrm{coh}}^b(\mathcal{O}_X)$ by using injective resolutions. However, differential operators do not necessarily preserve $i_{\mathcal{O}}^! M \subset M$ in general, e.g. if $X = \{0\} \subset Y = \mathrm{Spec} \mathbf{C}[t]$ with $M = \mathbf{C}[t, t^{-1}]/\mathbf{C}[t]$ and $i_{\mathcal{O}}^! M = \mathbf{C}$.

Remark 2.10. We say that a complex analytic space X is a \mathbf{Q} -homology manifold if the local cohomology $H_{\{x\}}^i \mathbf{Q}_X$ for any $x \in X$ is isomorphic to \mathbf{Q} if $i = 2 \dim X$, and vanishes otherwise. If a finite group G acts on a complex analytic space X and if X is a \mathbf{Q} -homology manifold, then the quotient complex analytic space X/G is also a \mathbf{Q} -homology manifold, since the action of the stabilizer G_x of x on the local cohomology $H_{\{x\}}^{2 \dim X} \mathbf{Q}_X$ is trivial. In particular, a complex V -manifold is a \mathbf{Q} -homology manifold.

3. Signature on the symmetric products

3.1. Induced pairings. Let X be a topological stratified space, i.e. X is a Hausdorff topological space with a stratification given by an increasing sequence

of closed subspaces X_k ($k \geq -1$) with $X_{-1} = \emptyset$, $X_d = X$ ($d \gg 0$), and for any $x \in X_d \setminus X_{d-1}$ with $d \geq 0$, there is an open neighborhood U_x of x in X together with a compact topological space L_x having an increasing sequence of closed subspaces $(L_x)_k$ ($k \geq -1$) with $(L_x)_{-1} = \emptyset$ and such that there is a homeomorphism $U_x \cong \mathbf{R}^d \times C(L_x)$ inducing $X_k \cap U_x \cong \mathbf{R}^d \times C((L_x)_{k-d-1})$ for any $k \geq d$ (see also [GM] and [Sc1], Def. 4.2.1). Here $C(Z)$ for a topological space Z denotes the open cone of Z if $Z \neq \emptyset$, and $C(Z) = pt$ if $Z = \emptyset$. (We do not assume X equidimensional.) It is known that the multiple Künneth formula holds for bounded complexes on such spaces having constructible cohomology sheaves and finite dimensional global cohomology groups (with compact supports), see (3.8) below.

Let $K \in D_c^b(X, \mathbf{R})$ endowed with a pairing

$$\phi : K \otimes_{\mathbf{R}} K \rightarrow \mathbf{D}_X,$$

where $\mathbf{D}_X = a_X^! \mathbf{R}$ (see [Ve]) with $a_X : X \rightarrow pt$ the canonical morphism. We say that ϕ is *symmetric* if the composition of the involution of $K \otimes_{\mathbf{R}} K$ with ϕ coincides with ϕ .

We have the induced pairing $\pi_* \boxtimes^n \phi$ which is the composition of

$$(3.1.1) \quad \begin{aligned} \pi_* \boxtimes^n K \otimes \pi_* \boxtimes^n K &\rightarrow \pi_* (\boxtimes^n K \otimes \boxtimes^n K) \xrightarrow{\gamma} \pi_* \boxtimes^n (K \otimes K) \\ &\xrightarrow{\phi} \pi_* \boxtimes^n \mathbf{D}_X = \pi_* \mathbf{D}_{X^n} \xrightarrow{\text{Tr}} \mathbf{D}_{\mathcal{S}^n X}, \end{aligned}$$

where the last morphism is given by the trace morphism Tr associated with the adjunction for π_1, π^1 . Note that we have a certain sign for the isomorphism γ as in (1.3.4).

Restricting this self-pairing to $\mathcal{S}^n K$, we get the induced pairing

$$\mathcal{S}^n \phi : \mathcal{S}^n K \otimes \mathcal{S}^n K \rightarrow \mathbf{D}_{\mathcal{S}^n X}.$$

Note that the subcomplex $\mathcal{S}^n K \hookrightarrow \pi_* \boxtimes^n K$ is defined by using the symmetrizer e_1 in (2.1.1).

The above construction is compatible with the global section functor with compact supports. Here we assume the nonzero $H_c^i(X, K)$ are bounded and finite dimensional. We have the induced self-pairing

$$\phi_X : V^\bullet \otimes V^\bullet \rightarrow \mathbf{R} \quad \text{with} \quad V^\bullet := H_c^\bullet(X, K).$$

This is graded-symmetric if ϕ is symmetric. It induces further

$$\otimes^n \phi_X : \otimes^n (V^\bullet \otimes V^\bullet) \rightarrow \mathbf{R}.$$

Then the induced pairing $\mathcal{S}^n \phi_X$ on $H_c^\bullet(\mathcal{S}^n X, \mathcal{S}^n K)$ coincides with the restriction of the composition

$$(3.1.2) \quad \phi_X^n := \otimes^n \phi_X \circ \gamma' : (\otimes^n V^\bullet) \otimes (\otimes^n V^\bullet) \xrightarrow{\gamma'} \otimes^n (V^\bullet \otimes V^\bullet) \rightarrow \mathbf{R},$$

to the \mathfrak{S}_n -invariant part

$$(3.1.3) \quad H_c^\bullet(\mathcal{S}^n X, \mathcal{S}^n K) = (\otimes^n V^\bullet)^{\mathfrak{S}_n} \hookrightarrow \otimes^n V^\bullet.$$

Here the last inclusion is defined by using the symmetrizer e_1 in (2.1.1). Note that we have a certain sign for γ' as in (1.3.4).

3.2. Good bases of the cohomology groups. With the notation and the assumption of (3.1), set $r_i := \dim V^i$. We have bases $v_{i,1}, \dots, v_{i,r_i}$ of V^i for $i \in \mathbf{Z}$ satisfying

$$(3.2.1) \quad \phi_X(v_{i,j}, v_{i',j'}) \neq 0 \iff i + i' = 0, \quad j = j' \leq \rho_i,$$

where the ρ_i is the rank of ϕ_X on $H_c^i(X, K) \otimes H_c^{-i}(X, K)$. Note that $\rho_i = \rho_{-i}$ for any i . Set

$$J = \{(i, j) \in \mathbf{Z} \times \mathbf{N} \mid j \leq r_i (\forall i)\}.$$

Let $\Lambda_n \subset \mathbf{N}^J$ consisting of $\mu = (\mu_{i,j})_{(i,j) \in J} \in \mathbf{N}^J$ satisfying the condition:

$$(3.2.2) \quad \sum_{i,j} \mu_{i,j} = n, \quad \sum_{i,j} i \mu_{i,j} = 0, \quad \mu_{i,j} \in \{0, 1\} \text{ for } i \text{ odd}.$$

Then we have a basis of $(\mathcal{S}^n V^\bullet)^0$ defined by the images of

$$v^\mu := \bigotimes_i \left(\bigotimes_j v_{i,j}^{\mu_{i,j}} \right) \quad (\mu \in \Lambda_n),$$

where \bigotimes_i and \bigotimes_j are the ordered tensor products as in (1.1) which are applied successively. Here we identify

$$\mathcal{S}^n V^\bullet := \left(\bigotimes^n V^\bullet \right)^{\mathfrak{S}_n}$$

with the maximal quotient of $\bigotimes^n V^\bullet$ on which the action of \mathfrak{S}_n is trivial. We have to apply the symmetrizer e_1 in (2.1.1) to get an element in the \mathfrak{S}_n -invariant subspace.

By the above argument, the $v_{i,j}$ for $j > \rho_i$ do not contribute to the signature of $\mathcal{S}^n \phi_X$. Then, replacing V^\bullet with the subspace generated by $v_{i,j}$ with $j \leq \rho_i$, the proof of Theorem 2 is reduced to the case where the self-pairing ϕ_X on V^\bullet is non-degenerate.

3.3. Proof of Theorem 2. By the above argument we may assume

$$(3.3.1) \quad \rho_i = r_i \quad (\forall i \in \mathbf{Z}).$$

With the notation of (3.2), let ι be an involution of Λ_n defined by

$$\iota(\mu) = \mu' \quad \text{with} \quad \mu'_{i,j} := \mu_{-i,j} \quad (\forall i, j).$$

Then

$$\phi_X^n(v^\mu, v^{\mu'}) \neq 0 \iff \mu' = \iota(\mu).$$

This gives an orthogonal decomposition of $\mathcal{S}^n V^\bullet$ into the direct factors of the form

$$V^\mu := \mathbf{R}v^\mu + \mathbf{R}v^{\iota(\mu)},$$

which has dimension 1 or 2 depending on whether $\iota(\mu) = \mu$ or not. If $\dim V^\mu = 2$, then this orthogonal direct factor is hyperbolic, and hence the signature is zero. So this can be neglected. Thus it is enough to consider only the V^μ with $\iota(\mu) = \mu$ (and hence $\dim V^\mu = 1$). Set

$$\Lambda'_n = \{\mu \mid \iota(\mu) = \mu\} \subset \Lambda_n.$$

We have an additive structure on $\Lambda' := \coprod_n \Lambda'_n$ defined by

$$(\mu + \nu)_{i,j} = \begin{cases} \mu_{i,j} + \nu_{i,j} & \text{if } \mu_{i,j} + \nu_{i,j} \leq 1 \text{ for any } (i, j) \text{ with } i \text{ odd,} \\ 0 & \text{if } \mu_{i,j} + \nu_{i,j} > 1 \text{ for some } (i, j) \text{ with } i \text{ odd.} \end{cases}$$

This additive structure is compatible with an orthogonal direct sum decomposition

$$V^\bullet = V_1^\bullet \oplus V_2^\bullet,$$

if the latter is compatible with the basis $v_{i,j}$ (i.e. if it corresponds to a partition of the basis $v_{i,j}$).

The right-hand side of the formula in Theorem 2 is compatible with the above direct sum decomposition since σ and χ are additive. So we first calculate the right-hand side of the formula in the primitive cases of 2 or 1-dimensional vector subspaces of the form:

$$(3.3.2) \quad V'^\bullet = \mathbf{R}v_{i,j} + \mathbf{R}v_{-i,j} \quad (i \neq 0) \quad \text{or} \quad V'^\bullet = \mathbf{R}v_{0,j}.$$

Note that $\sum_{\mu \in \Lambda'} \mathbf{R}v^\mu \subset \mathcal{S}^n V^\bullet$ is generated by the images of the multiple tensor products of vector subspaces of the form $(\bigotimes^{\dim V'} V'^\bullet)^0$ where V'^\bullet is as in (3.3.2).

In the first case of (3.3.2) we have $\sigma_\phi = 0$ and $\chi_\phi = \pm 2$, depending on the parity of the degree i . So the right-hand side of the formula is given in this case by

$$(1 - t^2)^{-1} = 1 + t^2 + t^4 + \dots \quad \text{or} \quad 1 - t^2,$$

depending on the parity of the degree i .

In the second case of (3.3.2) we have $\chi_\phi = 1$ and $\sigma_\phi = \pm 1$, depending on the signature of ϕ_X . So the right-hand side of the formula is given in this case by

$$(1 - t)^{-1} = 1 + t + t^2 + \dots \quad \text{or} \quad (1 + t)^{-1} = 1 - t + t^2 - t^3 \pm \dots,$$

depending on the signature of ϕ_X .

The compatibility with the above direct sum decomposition is rather nontrivial for the left-hand side for the odd degree part since there is a problem of sign associated to γ' in (3.1.2). This is trivial for the even degree elements since they commute with any elements (even with any odd degree elements) without any signs. In the above primitive case with even degrees, we can calculate the left-hand side of the formula, and verify the formula in these cases. So the assertion is proved in the case V^\bullet has only the even degree part using the above compatibility with direct sum decompositions. Then, by Proposition (2.2) together with the commutativity of even degree elements with any elements (without any signs), it now remains to calculate the left-hand side of the formula in the case V^\bullet has only the odd degree part.

So the proof of Theorem 2 is reduced to the calculation in the next subsection.

3.4. The odd degree case. With the notation and the assumption of (3.3), assume further $V^\bullet = V_{\text{odd}}^\bullet$. Take any $\mu \in \Lambda'_n$ where $\mu_{i,j} = 0$ for i even by the above hypothesis. We have to calculate the sign of

$$\phi_X^n(e_1(v^\mu), e_1(v^\mu)),$$

see (2.1.1) for e_1 . Here we replace v^μ with

$$u := u_1 \otimes u'_1 \otimes \dots \otimes u_r \otimes u'_r,$$

where $u_k = v_{i_k, j_k}$, $u'_k = v_{-i_k, j_k}$, and $n = 2r$. Then $e_1(u)$ coincides with $e_1(v^\mu)$ up to the sign $\varepsilon(\tau)$ of $\tau \in \mathfrak{S}_n$ such that $\tau(u) = \varepsilon(\tau)v^\mu$. Note that the action of \mathfrak{S}_n

on $\bigotimes^n V_{\text{odd}}^\bullet$ is twisted by the sign character ε in (1.6.1). Since the two signs cancel out, we get

$$\phi_X^n(e_1(v^\mu), e_1(v^\mu)) = \phi_X^n(e_1(u), e_1(u)).$$

We then replace the second u in $\phi_X^n(e_1(u), e_1(u))$ by

$$u' := u'_1 \otimes u_1 \otimes \cdots \otimes u'_r \otimes u_r.$$

Here we get the first sign $(-1)^r$. This is the sign of τ' such that $\tau'(u) = (-1)^r u'$. Hence

$$e_1(u) = (-1)^r e_1(u').$$

By (2.1.1), we have to calculate the sign of

$$(3.4.1) \quad \sum_{\sigma, \sigma' \in \mathfrak{S}_n} \phi_X^n(\sigma u, \sigma' u') = \sum_{\sigma \in \mathfrak{S}_n} \phi_X^n(\sigma u, \sigma u') = \phi_X^n(u, u') n!.$$

Here the middle equality follows from the vanishing of $\phi_X^n(\sigma u, \sigma' u')$ for $\sigma \neq \sigma'$, since u' coincides with $v^{(\mu)}$ up to a sign. For the last equality, we need

$$(3.4.2) \quad \phi_X^n(\sigma v, \sigma v') = \phi_X^n(v, v') \quad \text{for any } v, v' \in \bigotimes^n V_{\text{odd}}^\bullet.$$

This follows from the definition of ϕ_X^n in (3.1.2). Indeed, we have $n = 2r$, and the sign of γ' in (3.1.2) is given in this case by

$$(3.4.3) \quad (-1)^{n(n-1)/2} = (-1)^r.$$

So (3.4.2) and hence (3.4.1) are shown. Thus the sign of $\phi_X^n(e_1(v^\mu), e_1(v^\mu))$ coincides with that of $\phi_X^n(u, u')$ up to the sign $(-1)^r$.

We also get the second sign $(-1)^r$ from γ' in the definition of ϕ_X^n in (3.1.2) as is shown in (3.4.3). We then get the third sign $(-1)^r$ from the products

$$\phi_X(u_k, u'_k) \phi_X(u'_k, u_k) \quad (k \in [1, r]),$$

since ϕ_X is anti-symmetric on V_{odd}^\bullet and $\phi_X(u_k, u'_k) \in \mathbf{R}$.

Thus we get the sign $(-1)^r$ in total (since we got it three times). This sign coincides with that of the corresponding term on the right-hand side, which is the sign of the coefficient of t^{2r} in the polynomial $(1 - t^2)^m$ where $m := \sum_{i \in \mathbf{N}} \rho_{2i+1}$. The absolute value of the coefficient is $\binom{m}{r}$, and this also coincides with that for the left-hand side in this case. So Theorem 2 is proved.

Remark 3.5. The above calculation in the even degree case is closely related to [MG2]. In the odd degree case, however, we get an anti-symmetric pairing, and this is different from [MG1].

3.6. Abstract Hodge index theorem. Let $(V^\bullet; l, \phi)$ be a graded \mathbf{R} -Hodge structure of Lefschetz type of weight w in [Sa1], Sect. 4 with the precise signs (see [D4]). This means that V^k is a pure \mathbf{R} -Hodge structure of weight $w + k$ endowed with a morphism of Hodge structures $l : V^\bullet \rightarrow V^{\bullet+2}(1)$ and a graded-symmetric pairing of vector space $\phi : V^\bullet \otimes V^\bullet \rightarrow \mathbf{R}$ such that ϕ induces a self-pairing of graded \mathbf{R} -Hodge structures with value in $\mathbf{R}(-w)$, we have $\phi(lu, v) = \phi(u, lv)$ for any u, v , and $(-1)^{k(k-1)/2} \phi(id \otimes l^k)$ gives a polarization of Hodge structure on the primitive part $V_{\text{prim}}^{-k} := \text{Ker } l^{k+1} \subset V^{-k}$ ($k \in \mathbf{N}$). These conditions imply

$$i^{q-p} \phi(l^k v, \bar{v}) > 0 \quad \text{for any } v \in V_{\text{prim}, \mathbf{C}}^{p, q} \setminus \{0\},$$

where $k := w - p - q \in \mathbf{N}$. Here we use the Hodge decomposition

$$V_{\text{prim}, \mathbf{C}}^{-k} = \bigoplus_{p+q=w-k} V_{\text{prim}, \mathbf{C}}^{p,q}.$$

In some references, i^{p-q} is used instead of i^{q-p} . However, this does not cause a problem if $p + q$ is even. Set

$$\chi_y(V^\bullet) = \sum_{p,q} (-1)^{q-w} h^{p,q}(V^\bullet) y^p \quad \text{with} \quad h^{p,q}(V^\bullet) = \dim_{\mathbf{C}} \text{Gr}_F^p V_{\mathbf{C}}^{p+q-w}.$$

Since $\chi_{-y}(V^\bullet) = \sum_{p,q} (-1)^{p+q-w} h^{p,q}(V^\bullet) y^p$, this agrees with the previous definition.

Assume w is *even* so that $(-1)^w = 1$. Let $\sigma(\phi|V^0)$ denote the signature of the restriction of the graded-symmetric self-pairing ϕ to V^0 . Then

$$(3.6.1) \quad \sigma(\phi|V^0) = \chi_1(V^\bullet) = \sum_{p,q} (-1)^q h^{p,q}(V^\bullet).$$

This follows from the same calculation as in [Hi], p. 125, Thm. 15.8.2, using the above conditions together with the primitive decomposition. Since w is even, we also have

$$(3.6.2) \quad \chi(V^\bullet) = \chi_{-1}(V^\bullet) = \sum_{p,q} (-1)^{p+q} h^{p,q}(V^\bullet).$$

3.7. Relation between Corollary 3 and Theorem 2. Using the above properties, we can show that Corollary 3 implies Theorem 2 in the case when X is a projective variety, the complex K is an intersection complex underlying a polarizable Hodge module \mathcal{M} of even weight, and ϕ is a polarization of \mathcal{M} . In this case $H^\bullet(X, \mathcal{M})$ is a graded Hodge structure of Lefschetz type of weight w , see [Sa1], Th. 5.3.1. Then Theorem 2 is shown by splitting the summation inside the exponential in the last term of the formula in Corollary 3 in two parts according to the parity of the index of summation. (For a similar formulation in case X is a smooth projective variety, see [Mo], p. 173, Cor. 2.13, which is based on the Hodge index theorem for global projective complex V -manifolds in loc. cit., p. 171, Cor. 2.11.)

3.8. Multiple Künneth formula for A -complexes. In case of bounded A -complexes with constructible cohomology sheaves on topological stratified spaces as in (3.1) (or [Ve]), the multiple Künneth formula holds by assuming the finiteness of the global cohomology, where A a field of characteristic 0. Indeed, let $K_i \in D_c^+(X_i, A)$ with $\dim H^\bullet(X_i, K_i) < \infty$ for $i \in [1, n]$. There is a *canonical morphism* of complexes

$$(3.8.1) \quad \bigotimes_{i=1}^n \mathbf{R}\Gamma(X_i, K_i) \rightarrow \mathbf{R}\Gamma\left(\prod_i X_i, \boxtimes_{i=1}^n K_i\right),$$

which is defined by taking a flasque resolution

$$(3.8.2) \quad \boxtimes_{i=1}^n K_i \xrightarrow{\sim} \tilde{\mathcal{K}}.$$

Here we may assume that each K_i is flasque by replacing it with a flasque resolution if necessary. It is shown that the canonical morphism (3.8.1) is a quasi-isomorphism as follows.

Let pr_n denote the projection to the n -th factor X_n with fiber $\mathcal{X}' := \prod_{i=1}^{n-1} X_i$ where $n \geq 2$. Note that (3.8.1) holds for $\mathcal{K}' := \boxtimes_{i=1}^{n-1} K_i$ by inductive assumption if $n-1 > 1$ (and it is trivial if $n-1 = 1$). We first show the quasi-isomorphism

$$(3.8.3) \quad \mathbf{R}\Gamma(\mathcal{X}', \mathcal{K}') \otimes K_n \xrightarrow{\sim} \mathbf{R}(pr_n)_*(\boxtimes_{i=1}^n K_i),$$

where $\mathbf{R}\Gamma(\mathcal{X}', \mathcal{K}')$ on the left-hand side is identified with a constant sheaf complex on X_n . The morphism in (3.8.3) is defined by using the flasque resolution (3.8.2) (together with the inductive hypothesis for \mathcal{K}'). To show that (3.8.3) is a quasi-isomorphism, we have to determine the stalk of the right-hand side at each $x_n \in X_n$. For this we take $U_{x_n} \subset X_n$ in (3.1) and prove the following canonical quasi-isomorphism induced by the restriction morphism for the inclusion $\mathcal{X}' \times \{x_n\} \hookrightarrow \mathcal{X}' \times U_{x_n}$:

$$(3.8.4) \quad \mathbf{R}\Gamma(\mathcal{X}' \times U_{x_n}, \mathcal{K}' \boxtimes (K_n|_{U_{x_n}})) \xrightarrow{\sim} \mathbf{R}\Gamma(\mathcal{X}', \mathcal{K}' \otimes K_{n,x_n}).$$

Note that the right-hand side is canonically isomorphic to $\mathbf{R}\Gamma(\mathcal{X}', \mathcal{K}') \otimes K_{n,x_n}$ using the hypotheses on the finiteness, and the left-hand side is independent of the size of U_{x_n} under the restriction morphisms using the cone structure in (3.1) since K_n is cohomologically constructible with respect to the stratification in (3.1). For the proof of (3.8.4), consider the direct image by the projection pr' to \mathcal{X}' with fiber U_{x_n} . We get a canonical quasi-isomorphism induced by the restriction under the inclusion $\mathcal{X}' \times \{x_n\} \hookrightarrow \mathcal{X}' \times U_{x_n}$:

$$(3.8.5) \quad \mathbf{R}pr'_*(\mathcal{K}' \boxtimes (K_n|_{U_{x_n}})) \xrightarrow{\sim} \mathcal{K}' \otimes K_{n,x_n}.$$

This is proved by restricting it over $\prod_{i=1}^{n-1} U_{x_i}$ and reducing to the fact that the resolution (3.8.2) induces a quasi-isomorphism at each stalk. Then (3.8.4) and hence (3.8.3) follow by applying the global section functor over \mathcal{X}' to (3.8.5). We now apply the global section functor over X_n to (3.8.3), and conclude that (3.8.1) is a quasi-isomorphism by increasing induction on n . (The argument seems to work also for $K_i \in D_c^b(X_i, A)$ in the sense of [Ve] assuming $\dim H^\bullet(X_i, K_i) < \infty$.)

The above quasi-isomorphism (3.8.1) implies the multiple Künneth isomorphism and also a remark after Theorem 1. Note that similar assertions hold for cohomology with compact supports where U_{x_i} is replaced by its closure in X_i after shrinking it. (See [Sc1] for an argument using the base change theorem.)

REFERENCES

- [BS] Balmer, P. and Schlichting, M., Idempotent completion of triangulated categories, *Journal of Alg.* 236 (2001), 819–834.
- [Be] Beilinson, A., On the derived category of perverse sheaves, *Lecture Notes in Math.*, 1289, Springer 1987, pp. 27–41.
- [BBD] Beilinson, A., Bernstein, J. and Deligne, P., *Faisceaux pervers*, Astérisque 100, Soc. Math. France, Paris, 1982.
- [BSY] Brasselet, J.-P., Schürsmann, J. and Yokura, S., Hirzebruch classes and motivic Chern classes of singular spaces, *Journal of Topology and Analysis* 2, (2010), 1–55.
- [CMSSY] Cappell, S. E., Maxim, L., Schürsmann, J., Shaneson, J. L., Yokura, S., Characteristic classes of symmetric products of complex quasi-projective varieties, arXiv:1008.4299.

- [Ca] Carlson, J., Extensions of mixed Hodge structures, in Journées de Géométrie Algébrique d'Angers 1979, Sijthoff-Noordhoff Alphen a/d Rijn, 1980, pp. 107–128.
- [Ch] Cheah, J., On the cohomology of Hilbert schemes of points, *J. Alg. Geom.* 5 (1996), 479–511.
- [dBa] del Baño, S., On the Chow motive of some moduli spaces, *J. reine angew. Math.* 532 (2001), 105–132.
- [D1] Deligne, P., Théorie de Hodge, II, *Publ. Math. IHES* 40 (1971), 5–57.
- [D2] Deligne, P., Cohomologie à supports propres, SGA4 XVII, *Lect. Notes in Math.* 305, Springer, Berlin, 1973, pp. 250–461.
- [D3] Deligne, P., Théorie de Hodge, III, *Publ. Math. IHES* 44 (1974), 5–77.
- [D4] Deligne, P., Positivité: signe, I (manuscrit, 16-2-84), II (manuscrit, 6-11-85).
- [D5] Deligne, P., Décompositions dans la catégorie dérivée, in *Motives* (Seattle, WA, 1991), *Proc. Sympos. Pure Math.*, 55, Part 1, A.M.S. Providence, RI, 1994, pp. 115–128.
- [D6] Deligne, P., Catégories tensorielles, *Mosc. Math. J.* 2 (2002), 227–248.
- [DB] Du Bois, Ph., Complexe de De Rham filtré d'une variété singulière, *Bull. Soc. Math. France* 109 (1981) 41–81.
- [Fi] Fiorot, L., On derived categories of differential complexes, *J. Algebra* 312 (2007), 362–376.
- [GM] Goresky, M. and MacPherson, R., Intersection homology theory, *Topology* 19 (1980), 135–162.
- [Gr] Grothendieck, A., On the de Rham cohomology of algebraic varieties, *Publ. Math. IHES* 29 (1966), 95–103.
- [He] Heinloth F., A note on functional equations for zeta functions with values in Chow motives, *Ann. Inst. Fourier* 57 (2007), 1927–1945.
- [Hi] Hirzebruch, F., *Topological method in algebraic geometry*, Springer-Verlag, Berlin, 1966.
- [Ki] Kimura, S.-i., Chow groups are finite-dimensional, in some sense, *Math. Ann.* 331 (2005), 173–201.
- [LC] Le, J. and Chen, X.-W, Karoubianness of a triangulated category, *Journal of Algebra* 310 (2007), 452–457.
- [Ma] Macdonald, I. G., The Poincaré polynomial of a symmetric product, *Proc. Cambridge Philos. Soc.* 58 (1962), 563–568.
- [Mo] Moonen, B., *Das Lefschetz-Riemann-Roch-Theorem für singuläre Varietäten*, *Bonner Mathematische Schriften* 106 (1978).
- [MG1] McGarraghy, S., Exterior powers of symmetric bilinear forms, *Algebra Colloq.* 9 (2002), 197–218.
- [MG2] McGarraghy, S., Symmetric powers of symmetric bilinear forms, *Algebra Colloq.* 12 (2005), 41–57.
- [MS] Maxim, L. and Schürmann, J., Twisted genera of symmetric products, arXiv:0906.1264v1.
- [PS] Peters, C.A. and Steenbrink, J.H.M., *Mixed Hodge structures*, Springer, Berlin, 2008.
- [Sa1] Saito, M., Modules de Hodge polarisables, *Publ. RIMS, Kyoto Univ.* 24 (1988), 849–995.
- [Sa2] Saito, M., Mixed Hodge modules, *Publ. RIMS, Kyoto Univ.* 26 (1990), 221–333.
- [Sa3] Saito, M., Decomposition theorem for proper Kähler morphisms, *Tohoku Math. J.* 42 (1990), 127–147.
- [Sa4] Saito, M., \mathcal{D} -modules on analytic spaces, *Publ. RIMS, Kyoto Univ.* 27 (1991), 291–332.
- [Sa5] Saito, M., Mixed Hodge complexes on algebraic varieties, *Math. Ann.* 316 (2000), 283–331.
- [Sc1] Schürmann, J., *Topology of singular spaces and constructible sheaves*, *Monografie Matematyczne* 63 (New Series), Birkhäuser, Basel, 2003.

- [Sc2] Schürsmann, J., Characteristic classes of mixed Hodge modules, arXiv:0907.0584, to appear in the Proceedings of the 2008 MSRI Workshop “Topology of Stratified Spaces”.
- [St] Steenbrink, J.H.M., Mixed Hodge structure on the vanishing cohomology, in Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976) Stijhoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 525–563.
- [Ve] Verdier, J.-L., Dualité dans la cohomologie des espaces localement compacts, Séminaire Bourbaki, no 300, Benjamin, 1966.
- [Za] Zagier, D., Equivariant Pontrjagin classes and applications to orbit spaces, Lect. Notes in Math. 290, Springer-Verlag, Berlin, 1972.

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