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Hodge-theoretic Atiyah-Meyer formulae and the stratified multiplicative property

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Dedicated to Lê Dũng Tráng on His 60th Birthday

Abstract. In this note we survey Hodge-theoretic formulae of Atiyah-Meyer type for genera and characteristic classes of complex algebraic varieties, and derive some new and interesting applications. We also present various extensions to the singular setting of the Chern-Hirzebruch-Serre signature formula.

1. Introduction

In the mid 1950's, Chern, Hirzebruch and Serre [**CHS**] showed that if $F \hookrightarrow E \xrightarrow{\pi} B$ is a fiber bundle of closed, coherently oriented, topological manifolds such that the fundamental group of the base B acts trivially on the cohomology of the fiber F, then the signatures of the spaces involved are related by a simple multiplicative relation:

$$\sigma(E) = \sigma(F) \cdot \sigma(B). \tag{1.1}$$

A decade later, Kodaira [**Ko**], Atiyah [**At**], and respectively Hirzebruch [**H69**] observed that without the assumption on the (monodromy) action of $\pi_1(B)$ the multiplicativity relation fails. In the case when π is a differentiable fiber bundle of compact oriented manifolds so that both B and F are even-dimensional, Atiyah obtained a formula for $\sigma(E)$ involving a contribution from the monodromy action. Let $k = \frac{1}{2}\dim_{\mathbb{R}} F$. Then the flat bundle $\mathcal{V}$ over B with fibers $H^k(F_x;\mathbb{R})$ $(x \in B)$ has a K-theory signature, $[\mathcal{V}]_K \in KO(B)$ for k even (resp. in $KU(B)$ for k odd), and the Atiyah signature theorem [**At**] asserts that

$$\sigma(E) = \langle ch^*_{(2)}([\mathcal{V}]_K) \cup L^*(B), [B] \rangle, \tag{1.2}$$

where $ch^*_{(2)}$ is a modified Chern character (obtained by precomposing with the second Adams operation), and $L^*(B)$ is the total Hirzebruch L-polynomial of B.

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Meyer [**Me**] extended Atiyah's formula to the case of *twisted signatures* of closed manifolds endowed with Poincaré local systems (that is, local systems with duality) not necessarily arising from a fibre bundle projection. If B is a closed, oriented, smooth manifold of even dimension, and $\mathcal{L}$ is a local system equipped with a nondegenerate (anti-)symmetric bilinear pairing $\mathcal{L} \otimes \mathcal{L} \to \mathbb{R}_B$, then the twisted signature $\sigma(B;\mathcal{L})$ is defined to be the signature of the nondegenerate form on the sheaf cohomology group $H^{dim(B)/2}(B;\mathcal{L})$, and can be computed by Meyer's signature formula:

$$\sigma(B;\mathcal{L}) = \langle ch^*_{(2)}([\mathcal{L}]_K) \cup L^*(B), [B]\rangle, \tag{1.3}$$

where $[\mathcal{L}]_K$ is the K-theory signature of $\mathcal{L}$ defined as follows. For k even (resp. for k odd) the nondegenerate pairing induces a splitting of the associated flat bundle $\mathcal{L} = \mathcal{L}^+ \oplus \mathcal{L}^-$ into a positive and negative definite part (resp. induces a complex structure on the associated flat bundle $\mathcal{L}$ with $\mathcal{L}^*$ the complex conjugate bundle). Then

$$[\mathcal{L}]_K := \begin{cases} \mathcal{L}^+ - \mathcal{L}^- \in KO(B) & \text{if } k \text{ is even,} \\ \mathcal{L}^* - \mathcal{L} \in KU(B) & \text{if } k \text{ is odd.} \end{cases}$$

Geometric mapping situations that involve singular spaces generally lead to Poincaré local systems that are only defined on the top stratum of a stratified space. For example, Cappell and Shaneson [**CS91**] proved that if $f : Y \to X$ is a stratified map of even relative dimension between oriented, compact, Whitney stratified spaces with only strata of even codimension, then:

$$\sigma(Y) = \sigma(X;\mathcal{L}^f_{X-\Sigma}) + \sum_{\text{pure strata } Z\subset X} \sigma(\bar{Z};\mathcal{L}^f_Z), \tag{1.4}$$

where $\Sigma \subset X$ is the singular set of f and, for an open stratum Z in X, $\mathcal{L}^f_Z$ is a certain Poincaré local system defined on it. In particular, if all strata $Z \subset X$ of f are simply-connected then, as an extension of the Chern-Hirzebruch-Serre formula (1.1) to the stratified case, we obtain from (1.4) that

$$\sigma(Y) = \sum_{\text{strata } Z\subset X} \sigma(\bar{Z}) \cdot \sigma(N_Z), \tag{1.5}$$

where for a pure stratum Z of real codimension at least two and with link L_Z in X,

$$N_Z := f^{-1}(\text{cone } L_Z) \cup_{f^{-1}(L_Z)} \text{cone } f^{-1}(L_Z)$$

is the topological completion of the preimage under f of the normal slice to Z in X; if Z is a component of the top stratum $X \setminus \Sigma$, then N_Z is the fiber of f over Z.

More generally, similar formulae hold for the push-forward of the Goresky-MacPherson L-classes $L_k(Y) \in H_k(Y;\mathbb{Q})$, $k \geq 0$. On a space X with only even-codimensional strata and singular set Σ, the twisted homology L-classes $L_k(X;\mathcal{L})$ and the twisted signature $\sigma(X;\mathcal{L})$ for a Poincaré local system $\mathcal{L}$ on $X - \Sigma$ can be defined by noting that the duality of the local system extends to a self-duality of the corresponding middle-perversity intersection chain sheaf $IC_X(\mathcal{L})$ on X (for complete details on the construction, the reader is advised to consult the book [**Ba**] and the references therein).

It is therefore natural at this point to ask for extensions of Meyer's signature formula to the singular setting. In [**BCS**], Banagl, Cappell and Shaneson proved the following. Suppose X is a closed oriented Whitney stratified normal Witt space (that is, a space on which the middle-perversity intersection chain sheaf IC_X is self-dual, cf. [**Si**]) of even dimension with singular set Σ, and let $\mathcal{L}$ be a Poincaré local system defined on $X - \Sigma$ such that $\mathcal{L}$ is *strongly transverse to* Σ. On normal spaces, this technical assumption is equivalent to saying that $\mathcal{L}$ has a unique extension as a Poincaré local system to all of X. Such a local system possesses a K-theory signature $[\mathcal{L}]_K$ in the K-theory of X (cf. [**BCS**], Corollary 2), and $IC_X(\mathcal{L})$ is again self-dual. Then the twisted L-classes are well-defined, and they can be computed by the formula (cf. [**BCS**], Theorems 1 and 3)

$$L_*(X;\mathcal{L}) = ch^*_{(2)}\left([\mathcal{L}]_K\right) \cap L_*(X) \tag{1.6}$$

(here L_* stands for the total homology L-classes respectively); in particular, the twisted signature is given by

$$\sigma(X;\mathcal{L}) = \langle ch^*_{(2)}\left([\mathcal{L}]_K\right), L_*(X)\rangle. \tag{1.7}$$

In this note, we survey Hodge theoretic Atiyah-Meyer type formulae for genera and characteristic classes of complex algebraic varieties. In fact, these are Hodge theoretic analogues of the above formulae (see [**CLMSa, CLMSb**]), and various extensions to the singular setting (see [**CMSS**]). We also present the main ideas and constructions that lead to the stratified multiplicative property for Hodge genera and the Hirzebruch characteristic classes of complex algebraic varieties; for more details on part of this work, see [**CMSa, CMSb**]. Some of the results in this note were announced in the present form in the paper [**CLMSb**].

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2. Hirzebruch characteristic classes.

In this section we first define the Hirzebruch class of a smooth complex projective algebraic variety, then, following [**BSY, SY**], we describe its recent generalization to the singular setting. The construction in the singular case yields characteristic classes in (Borel-Moore) homology, and makes use of Saito's theory of algebraic mixed Hodge modules. In this section, we only survey formal properties of this deep theory which will be needed in the sequel.

2.1. The non-singular case. If Z is a smooth projective complex algebraic variety, the signature and the L-classes of Z are special cases of more general Hodge theoretic invariants encoded by the Hirzebruch characteristic class (also called "the generalized Todd class") $T^*_y(T_Z)$ of the tangent bundle of Z (cf. [**H66**]). This is defined by the normalized power series

$$Q_y(\alpha) = \frac{\alpha(1+y)}{1-e^{-\alpha(1+y)}} - \alpha y \in \mathbb{Q}[y][[\alpha]], \tag{2.1}$$

that is,

$$T^*_y(T_Z) = \prod_{i=1}^{dim(Z)} Q_y(\alpha_i), \tag{2.2}$$

where $\{\alpha_i\}$ are the Chern roots of the tangent bundle T_Z. Note that $Q_y(\alpha)$ is equal to $1+\alpha$ for $y=-1$, to $\frac{\alpha}{1-e^{-\alpha}}$ for $y=0$, and it equals $\frac{\alpha}{\tanh \alpha}$ if $y=1$. Therefore, the Hirzebruch class $T_y^*(T_Z)$ coincides with the total Chern class $c^*(T_Z)$ if $y=-1$, with the total Todd class $td^*(T_Z)$ if $y=0$, and with the total Thom-Hirzebruch L-class $L^*(T_Z)$ if $y=1$.

The Hirzebruch class appears in the *generalized Hirzebruch-Riemann-Roch theorem* (cf. [**H66**], §21.3), which asserts that if Ξ is a holomorphic vector bundle on a smooth complex projective variety Z, then the χ_y-characteristic of Ξ, which is defined by
(2.3)

$$\chi_y(Z,\Xi) := \sum_{p\geq 0} \chi(Z, \Xi\otimes\Lambda^p T_Z^*)\cdot y^p = \sum_{p\geq 0}\left(\sum_{i\geq 0}(-1)^i \dim H^i(Z,\Omega(\Xi)\otimes\Lambda^p T_Z^*)\right)\cdot y^p,$$

with T_Z^* the holomorphic cotangent bundle of Z and $\Omega(\Xi)$ the coherent sheaf of germs of sections of Ξ, can in fact be expressed in terms of the Chern classes of Ξ and the tangent bundle of Z, or more precisely,

$$\chi_y(Z,\Xi) = \langle ch^*_{(1+y)}(\Xi)\cup T_y^*(T_Z), [Z]\rangle, \tag{2.4}$$

where $ch^*_{(1+y)}(\Xi) = \sum_{j=1}^{\mathrm{rk}(\Xi)} e^{\beta_j(1+y)}$, for $\{\beta_j\}_j$ the Chern roots of Ξ. In particular, if $\Xi = \mathcal{O}_Z$, the Hirzebruch genus $\chi_y(Z) := \chi_y(Z,\mathcal{O}_Z)$ can be computed by

$$\chi_y(Z) = \langle T_y^*(T_Z), [Z]\rangle. \tag{2.5}$$

2.2. Mixed Hodge modules. Before discussing extensions of the Hirzebruch class to the singular setting, we need to briefly recall some aspects of Saito's theory of algebraic mixed Hodge modules. Generic references for this theory are Saito's papers [**Sa88, Sa89, Sa90**].

To each complex algebraic variety Z, Saito associated an abelian category $\mathrm{MHM}(Z)$ of algebraic mixed Hodge modules on Z (cf. [**Sa88, Sa90**]). If Z is smooth, an object of this category consists of a bifiltered regular holonomic D-module (M,W,F) together with a filtered perverse sheaf (K,W) that corresponds, after tensoring with $\mathbb{C}$, to (M,W) under the Riemann-Hilbert correspondence. In general, for a singular variety Z one works with suitable local embeddings into manifolds and corresponding filtered D-modules supported on Z. In addition, these objects are required to satisfy a long list of complicated properties.

The forgetful functor from $\mathrm{MHM}(Z)$ to the category of perverse sheaves extends to a functor $rat : D^b\mathrm{MHM}(Z) \to D^b_c(Z)$ to the derived category of complexes of $\mathbb{Q}$-sheaves with constructible cohomology. The usual truncation $\tau_{\leq}$ on $D^b\mathrm{MHM}(Z)$ corresponds to the perverse truncation ${}^p\tau_{\leq}$ on $D^b_c(Z)$. Saito also constructed a t-structure $\tau'_{\leq}$ on $D^b\mathrm{MHM}(Z)$ which is compatible with the usual t-structure on $D^b_c(Z)$ ([**Sa90**], Remark 4.6(2)). There are functors f_*, $f_!$, f^*, $f^!$, $\otimes$, $\boxtimes$ on $D^b\mathrm{MHM}(Z)$ which are "lifts" via rat of the similar functors defined on $D^b_c(Z)$. If f is a proper algebraic morphism then $f_* = f_!$.

It follows from the definition that every $M\in\mathrm{MHM}(Z)$ has an increasing weight filtration W so that the functor $M\to Gr^W_k M$ is exact. We say that $M\in\mathrm{MHM}(Z)$

is pure of weight k if $Gr_i^W M = 0$ for all $i \neq k$. The weight filtration is extended to the derived category $D^b\mathrm{MHM}(Z)$ by requiring that a shift $M \mapsto M[1]$ increases the weights by one. So $M \in D^b\mathrm{MHM}(Z)$ is pure of weight k if $H^i(M)$ is pure of weight $i+k$ for all $i \in \mathbb{Z}$. If f is a map of algebraic varieties, then $f_!$ and f^* preserve weight $\leq k$, and f_* and $f^!$ preserve weight $\geq k$. In particular, if $M \in D^b\mathrm{MHM}(X)$ is pure and $f : X \to Y$ is proper, then $f_*M \in D^b\mathrm{MHM}(Y)$ is pure of the same weight as M.

We say that $M \in \mathrm{MHM}(Z)$ is supported on S if and only if $rat(M)$ is supported on S. There are abelian subcategories $\mathrm{MH}(Z,k)^p \subset \mathrm{MHM}(Z)$ of pure polarizable Hodge modules of weight k. For each $k \in \mathbb{Z}$, the abelian category $\mathrm{MH}(Z,k)^p$ is semi-simple, in the sense that every polarizable Hodge module on Z can be uniquely written as a direct sum of polarizable Hodge modules with strict support in irreducible closed subvarieties of Z. Let $\mathrm{MH}_S(Z,k)^p$ denote the subcategory of polarizable Hodge modules of weight k with strict support in S. Then every $M \in \mathrm{MH}_S(Z,k)^p$ is generically a polarizable variation of Hodge structures $\mathbb{V}_U$ on a Zariski dense open subset $U \subset S$, with quasi-unipotent monodromy at infinity. Conversely, every such polarizable variation of Hodge structures can be extended in an unique way to a pure Hodge module. Under this correspondence, for $M \in \mathrm{MH}_S(Z,k)^p$ we have that $rat(M) = IC_S(\mathbb{V})$, for $\mathbb{V}$ the corresponding variation of Hodge structures.

Saito showed that the category of mixed Hodge modules supported on a point, $\mathrm{MHM}(pt)$, coincides with the category mHs^p of (graded) polarizable rational mixed Hodge structures. Here one has to switch the increasing D-module filtration F_* of the mixed Hodge module to the decreasing Hodge filtration of the mixed Hodge structure by $F^* := F_{-*}$, so that $gr^p_F \simeq gr^F_{-p}$. In this case, the functor rat associates to a mixed Hodge structure the underlying rational vector space. There exists a unique object $\mathbb{Q}^H \in \mathrm{MHM}(pt)$ such that $rat(\mathbb{Q}^H) = \mathbb{Q}$ and $\mathbb{Q}^H$ is of type $(0,0)$. In fact, $\mathbb{Q}^H = ((\mathbb{C}, F), \mathbb{Q}, W)$, with $gr_i^F = 0 = gr_i^W$ for all $i \neq 0$. For a complex variety Z, define $\mathbb{Q}^H_Z := a_Z^*\mathbb{Q}^H \in D^b\mathrm{MHM}(Z)$, with $rat(\mathbb{Q}^H_Z) = \mathbb{Q}_Z$, for $a_Z : Z \to pt$ the map to a point. If Z is *smooth* of complex dimension n then $\mathbb{Q}_Z[n]$ is perverse on Z, and $\mathbb{Q}^H_Z[n] \in \mathrm{MHM}(Z)$ is a single mixed Hodge module (in degree 0), explicitly described by $\mathbb{Q}^H_Z[n] = ((\mathcal{O}_Z, F), \mathbb{Q}_Z[n], W)$, where F and W are trivial filtrations so that $gr_i^F = 0 = gr^W_{i+n}$ for all $i \neq 0$. So if Z is smooth of dimension n, then $\mathbb{Q}^H_Z[n]$ is a pure mixed Hodge module of weight n. Next, note that if $j : U \hookrightarrow Z$ is a Zariski-open dense subset in Z, then the intermediate extension $j_{!*}$ (cf. [**BBD**]) preserves the weights. This shows that if Z is a complex algebraic variety of pure dimension n and $j : U \hookrightarrow Z$ is the inclusion of a smooth Zariski-open dense subset then the intersection cohomology module $IC^H_Z := j_{!*}(\mathbb{Q}^H_U[n])$ is pure of weight n, with underlying perverse sheaf $rat(IC^H_Z) = IC_Z$.

If Z is smooth of dimension n, an object $M \in \mathrm{MHM}(Z)$ is called *smooth* if and only if $rat(M)[-n]$ is a local system on Z. Smooth mixed Hodge modules are (up to a shift) admissible (at infinity) variations of mixed Hodge structures (in the sense of Steenbrink-Zucker [**SZ**] and Kashiwara [**Ka**]). Conversely, an admissible variation of mixed Hodge structures $\mathcal{L}$ (e.g., a geometric variation, or a pure polarizable variation) on a smooth variety Z of pure dimension n gives rise to a smooth mixed Hodge module (cf. [**Sa90**]), i.e., to an element $\mathcal{L}^H[n] \in \mathrm{MHM}(Z)$

with $rat(\mathcal{L}^H[n]) = \mathcal{L}[n]$. A pure polarizable variation of weight k yields an element of $\mathrm{MH}(Z, k+n)^p$. By the stability by the intermediate extension functor it follows that if Z is an algebraic variety of pure dimension n and $\mathcal{L}$ is an admissible variation of (pure) Hodge structures (of weight k) on a smooth Zariski-open dense subset $U \subset Z$, then $IC_Z^H(\mathcal{L})$ is an algebraic mixed Hodge module (pure of weight $k+n$), so that $rat(IC_Z^H(\mathcal{L})|_U) = \mathcal{L}[n]$.

2.3. Grothendieck groups of algebraic mixed Hodge modules. In this section, we describe the functorial calculus of Grothendieck groups of algebraic mixed Hodge modules. Let Z be a complex algebraic variety. By associating to (the class of) a complex the alternating sum of (the classes of) its cohomology objects, we obtain the following identification (e.g. compare [[**KS**], p. 77], [[**Sc**], Lemma 3.3.1])

$$K_0(D^b\mathrm{MHM}(Z)) = K_0(\mathrm{MHM}(Z)). \tag{2.6}$$

In particular, if Z is a point, then

$$K_0(D^b\mathrm{MHM}(pt)) = K_0(mHs^p), \tag{2.7}$$

and the latter is a commutative ring with respect to the tensor product, with unit $[\mathbb{Q}^H_{pt}]$. Let $\tau_{\leq}$ be the natural truncation on $D^b\mathrm{MHM}(Z)$ with associated cohomology H^*. Then for any complex $M^\bullet \in D^b\mathrm{MHM}(Z)$ we have the identification

$$[M^\bullet] = \sum_{i\in\mathbb{Z}} (-1)^i [H^i(M^\bullet)] \in K_0(D^b\mathrm{MHM}(Z)) \cong K_0(\mathrm{MHM}(Z)). \tag{2.8}$$

In particular, if for any $M \in \mathrm{MHM}(Z)$ and $k \in \mathbb{Z}$ we regard $M[-k]$ as a complex concentrated in degree k, then

$$[M[-k]] = (-1)^k [M] \in K_0(\mathrm{MHM}(Z)). \tag{2.9}$$

All functors f_*, $f_!$, f^*, $f^!$, $\otimes$, $\boxtimes$ induce corresponding functors on $K_0(\mathrm{MHM}(\cdot))$. Moreover, $K_0(\mathrm{MHM}(Z))$ becomes a $K_0(\mathrm{MHM}(pt))$-module, with the multiplication induced by the exact exterior product

$$\boxtimes : \mathrm{MHM}(Z) \times \mathrm{MHM}(pt) \to \mathrm{MHM}(Z \times \{pt\}) \simeq \mathrm{MHM}(Z).$$

Also note that

$$M \otimes \mathbb{Q}^H_Z \simeq M \boxtimes \mathbb{Q}^H_{pt} \simeq M$$

for all $M \in \mathrm{MHM}(Z)$. Therefore, $K_0(\mathrm{MHM}(Z))$ is a unitary $K_0(\mathrm{MHM}(pt))$-module. The functors f_*, $f_!$, f^*, $f^!$ commute with exterior products (and f^* also commutes with the tensor product $\otimes$), so that the induced maps at the level of Grothendieck groups $K_0(\mathrm{MHM}(\cdot))$ are $K_0(\mathrm{MHM}(pt))$-linear. Moreover, by the functor

$$rat : K_0(\mathrm{MHM}(Z)) \to K_0(D^b_c(Z)) \simeq K_0(Perv(\mathbb{Q}_Z)),$$

these transformations lift the corresponding ones from the (topological) level of Grothendieck groups of constructible (or perverse) sheaves.

2.4. Hirzebruch classes in the singular setting. For any complex variety Z, and for any $p \in \mathbb{Z}$, Saito constructed a functor of triangulated categories

$$gr_p^F DR : D^b\mathrm{MHM}(Z) \to D^b_{coh}(Z) \tag{2.10}$$

commuting with proper push-down, with $gr_p^F DR(M) = 0$ for almost all p and M fixed, where $D^b_{coh}(Z)$ is the bounded derived category of sheaves of $\mathcal{O}_Z$-modules with coherent cohomology sheaves. If $\mathbb{Q}_Z^H \in D^b\mathrm{MHM}(Z)$ denotes the constant Hodge module on Z, and if Z is smooth and pure dimensional, then $gr_{-p}^F DR(\mathbb{Q}_Z^H) \simeq \Omega_Z^p[-p]$. The transformations $gr_p^F DR$ induce functors on the level of Grothendieck groups. Therefore, if $G_0(Z) \simeq K_0(D^b_{coh}(Z))$ denotes the Grothendieck group of coherent sheaves on Z, we get a group homomorphism (the *motivic Chern class transformation*)

$$\begin{gathered} MHC_* : K_0(\mathrm{MHM}(Z)) \to G_0(Z) \otimes \mathbb{Z}[y, y^{-1}] \;; \\ [M] \mapsto \sum_{i,p} (-1)^i [\mathcal{H}^i(gr_{-p}^F DR(M))] \cdot (-y)^p \;. \end{gathered} \tag{2.11}$$

We let $td_{(1+y)}$ be the natural transformation (cf. [**Y, BSY**])

$$\begin{gathered} td_{(1+y)} : G_0(Z) \otimes \mathbb{Z}[y, y^{-1}] \to H_{2*}^{BM}(Z) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}] \;; \\ [\mathcal{F}] \mapsto \sum_{k \geq 0} td_k([\mathcal{F}]) \cdot (1+y)^{-k} \;, \end{gathered} \tag{2.12}$$

where H_*^{BM} stands for Borel-Moore homology, and td_k is the degree k component (i.e., in $H_{2k}^{BM}(Z)$) of the Todd class transformation $td_* : G_0(Z) \to H_{2*}^{BM}(Z) \otimes \mathbb{Q}$ of Baum-Fulton-MacPherson [**BFM**], which is linearly extended over $\mathbb{Z}[y, y^{-1}]$.

Definition 2.1. The motivic Hirzebruch class transformation MHT_y is defined by the composition (cf. [**BSY**])

$$MHT_y := td_{(1+y)} \circ MHC_* : K_0(\mathrm{MHM}(Z)) \to H_{2*}^{BM}(Z) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}] \,. \tag{2.13}$$

The motivic Hirzebruch class $T_{y_*}(Z)$ of a complex algebraic variety Z is defined by

$$T_{y_*}(Z) := MHT_y([\mathbb{Q}_Z^H]). \tag{2.14}$$

Similarly, if Z is an n-dimensional complex algebraic manifold, and $\mathcal{L}$ is a local system on Z underlying an admissible variation of mixed Hodge structures, we define twisted Hirzebruch characteristic classes by

$$T_{y_*}(Z; \mathcal{L}) = MHT_y([\mathcal{L}^H]), \tag{2.15}$$

where $\mathcal{L}^H[n]$ is the smooth mixed Hodge module on Z with underlying perverse sheaf $\mathcal{L}[n]$.

Example 2.2. Let $\mathbb{V} = ((V_\mathbb{C}, F), V_\mathbb{Q}, K) \in \mathrm{MHM}(pt) = mHs^p$. Then:

$$MHT_y([\mathbb{V}]) = \sum_p td_0([gr_F^p V_\mathbb{C}]) \cdot (-y)^p = \sum_p \dim_\mathbb{C}(gr_F^p V_\mathbb{C}) \cdot (-y)^p = \chi_y([\mathbb{V}]), \tag{2.16}$$

so over a point the transformation MHT_y coincides with the χ_y-genus ring homomorphism $\chi_y : K_0(mHs^p) \to \mathbb{Z}[y, y^{-1}]$.

By definition, the transformations MHC_* and MHT_y commute with proper push-forward. The following *normalization* property holds (cf. [**BSY**]): If Z is smooth and pure dimensional, then

$$T_{y_*}(Z) = T_y^*(T_Z) \cap [Z] \,,$$

where $T_y^*(T_Z)$ is the cohomology Hirzebruch class of Z defined in §2.1. So, if Z is smooth and projective, then $T_1^*(T_Z)$ is the total Hirzebruch L-polynomial of Z and $\chi_1(Z) = \sigma(Z)$.

For a complete (possibly singular) variety Z with $k : Z \to pt$ the constant map to a point, the pushdown $k_* T_{y_*}(Z)$ is the Hodge genus

$$\chi_y(Z) := \chi_y([H^*(Z;\mathbb{Q})]) = \sum_{i,p} (-1)^i dim_{\mathbb{C}}(gr_F^p H^i(Z;\mathbb{C})) \cdot (-y)^p, \tag{2.17}$$

with $\chi_{-1}(Z) := \chi([H^*(Z;\mathbb{Q})])$ the topological Euler characteristic of Z. For Z smooth $k_* T_{y_*}(Z;\mathcal{L})$ is the twisted χ_y-genus $\chi_y(Z;\mathcal{L})$ defined in a similar manner ([**CLMSa**]) [1].

It was shown in [**BSY**] that for any variety Z the limits $T_{y*}(Z)$ for $y = -1, 0$ exist, with

$$T_{-1_*}(Z) = c_*(Z) \otimes \mathbb{Q}$$

the total (rational) Chern class of MacPherson (for a construction of the latter see [**M**]). Moreover, for a variety Z with at most Du Bois singularities (e.g., toric varieties), we have that

$$T_{0*}(Z) = td_*(Z) := td_*([\mathcal{O}_Z]) \,,$$

for td_* the Baum-Fulton-MacPherson transformation [**BFM**]. It is still a conjecture that for a rational homology manifold $T_{1*}(Z)$ coincides with the total Goresky-MacPherson homology L-class of Z (see [**BSY**], p.4 and Remark 5.4). As will be shown elsewhere, this conjecture is true at least for $Z = M/G$ the quotient of a complex projective manifold M by the algebraic action of a finite group.

The Hirzebruch class of Section 2.1 also admits another extension to the singular setting, which is defined by means of intersection homology. Let $IC_Z^H \in \mathrm{MHM}(Z)$ be the intersection homology (pure) Hodge module on a pure-dimensional variety Z, so $rat(IC_Z^H) = IC_Z$. Similarly, for an admissible variation of mixed Hodge structures $\mathcal{L}$ defined on a smooth Zariski dense open subset of Z, let $IC_Z^H(\mathcal{L})$ be the corresponding mixed Hodge module with underlying perverse sheaf $IC_Z(\mathcal{L})$. In order to simplify the notations in the following definition, we set

$$IC_Z'^H := IC_Z^H[-\dim_{\mathbb{C}} Z] \quad \text{and} \quad IC_Z'^H(\mathcal{L}) := IC_Z^H(\mathcal{L})[-\dim_{\mathbb{C}} Z].$$

Definition 2.3. We define intersection characteristic classes by

$$IT_{y_*}(Z) := MHT_y(\big[IC_Z'^H\big]) \in H_{2*}^{BM}(Z) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}], \tag{2.18}$$

and similarly,

$$IT_{y_*}(Z;\mathcal{L}) := MHT_y(\big[IC_Z'^H(\mathcal{L})\big]), \tag{2.19}$$

for $\mathcal{L}$ an admissible variation of mixed Hodge structures defined on a smooth Zariski dense open subset of Z.

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[1] Note that by Deligne's theory, if Z is smooth and projective then $\chi_y(Z)$ defined in (2.17) yields the same invariant as $\chi_y(Z;\mathcal{O}_Z)$ defined by the equation (2.3).

As we will see later on, the limit $IT_{y_*}(Z;\mathcal{L})$ for $y=-1$ always exists (as well as $IT_{y_*}(Z;\mathcal{L})$ for $y=0$, if $\mathcal{L}$ is of non-negative weight, e.g. $\mathcal{L}=\mathbb{Q}_Z$). If Z is complete, then by pushing $IT_{y_*}(Z)$ down to a point we recover the intersection homology χ_y-genus, $I\chi_y(Z)$, which is a polynomial in the Hodge numbers of $IH^*(Z;\mathbb{Q})$ defined by

$$I\chi_y(Z) := \chi_y([IH^*(Z;\mathbb{Q})])$$

Similarly, in the above notations and if Z is complete, one has that

$$I\chi_y(Z;\mathcal{L}) = k_* IT_{y_*}(Z;\mathcal{L})\ ,$$

for $k : Z \to pt$ the constant map. Note that $I\chi_{-1}(Z) = \chi([IH^*(Z;\mathbb{Q})]$ for Z complete is the intersection (co)homology Euler characteristic of Z, whereas for Z projective, $I\chi_1(Z)$ is the intersection (co)homology signature of Z due to Goresky-MacPherson. If Z is a $\mathbb{Q}$-homology manifold, then

$$\mathbb{Q}^H_Z \simeq IC'^H_Z \in D^b\mathrm{MHM}(Z)\ ,$$

so we get that $T_{y_*}(Z) = IT_{y_*}(Z)$. It is conjectured that for a compact variety Z, $IT_{1*}(Z)$ is the Goresky-MacPherson homology L-class $L_*(Z)$ ([**BSY**], Remark 5.4).

3. The stratified multiplicative property.

In this section we give a brief survey of the main ideas and results concerning the behavior of the singular Hirzebruch classes under proper algebraic morphisms. The main references are the papers [**CMSa, CMSb**]. Similar results were originally predicted by Cappell and Shaneson (cf. [**CS94, Sh**]), and were referred to as "the stratified multiplicative property for χ_y-genera and Hirzebruch characteristic classes". The results surveyed in this section are motivated by the attempt of adapting the Cappell-Shaneson formulae (1.4) and (1.5) for the (topological) signature and L-classes to the setting of complex algebraic (analytic) geometry.

Let Y be an irreducible complex algebraic variety endowed with a complex algebraic Whitney stratification $\mathcal{V}$ so that the intersection cohomology complexes

$$IC'_{\bar{W}} := IC_{\bar{W}}[-\dim_{\mathbb{C}}(W)]$$

are $\mathcal{V}$-constructible for all strata $W \in \mathcal{V}$. (All these complexes are regarded as complexes on all of Y.) Define a partial order on $\mathcal{V}$ by "$V \le W$ if and only if $V \subset \bar{W}$". Denote by S the top-dimensional stratum, so S is Zariski open and dense, and $V \le S$ for all $V \in \mathcal{V}$. Let us fix for each $W \in \mathcal{V}$ a point $w \in W$ with inclusion $i_w : \{w\} \hookrightarrow Y$. Then

$$i_w^*[IC'^H_{\bar{W}}] = [i_w^* IC'^H_{\bar{W}}] = [\mathbb{Q}^H_{pt}] \in K_0(\mathrm{MHM}(w)) = K_0(\mathrm{MHM}(pt)), \tag{3.1}$$

and $i_w^*[IC'^H_{\bar{V}}] \neq [0] \in K_0(\mathrm{MHM}(pt))$ only if $W \le V$. Moreover, for any $j \in \mathbb{Z}$, we have

$$\mathcal{H}^j(i_w^* IC'_{\bar{V}}) \simeq IH^j(c^\circ L_{W,V}), \tag{3.2}$$

with $c^\circ L_{W,V}$ the open cone on the link $L_{W,V}$ of W in $\bar{V}$ for $W \le V$ (cf. [**Bo**], p.30, Prop. 4.2). So

$$i_w^*[IC'^H_{\bar{V}}] = [IH^*(c^\circ L_{W,V})] \in K_0(\mathrm{MHM}(pt)),$$

with the mixed Hodge structures on the right hand side defined by the isomorphism (3.2).

The main technical result of this section is the following

Theorem 3.1. *([**CMSb**], Thm. 3.2) For each stratum $V \in \mathcal{V} \setminus \{S\}$ define inductively*

$$\widehat{IC^H}(\bar{V}) := [IC'^H_{\bar{V}}] - \sum_{W<V} \widehat{IC^H}(\bar{W}) \cdot i_w^*[IC'^H_{\bar{V}}] \in K_0(D^b MHM(Y)). \tag{3.3}$$

Assume $[M] \in K_0(D^b MHM(Y))$ is an element of the $K_0(MHM(pt))$-submodule $\langle [IC'^H_{\bar{V}}] \rangle$ of $K_0(D^b MHM(Y))$ generated by the elements $[IC'^H_{\bar{V}}]$, $V \in \mathcal{V}$. Then we have the following equality in $K_0(D^b MHM(Y))$:

$$[M] = [IC'^H_Y] \cdot i_s^*[M] + \sum_{V<S} \widehat{IC^H}(\bar{V}) \cdot \left(i_v^*[M] - i_s^*[M] \cdot i_v^*[IC'^H_Y] \right). \tag{3.4}$$

Before stating immediate consequences of the above theorem, let us recall from [**CMSb**] some cases when the technical hypothesis $[M] \in \langle [IC'^H_{\bar{V}}] \rangle$ is satisfied for a fixed $M \in D^b(\mathrm{MHM}(Y))$. Assume that all sheaf complexes $IC'_{\bar{V}}$, $V \in \mathcal{V}$, are not only $\mathcal{V}$-constructible, but satisfy the stronger property that they are "cohomologically $\mathcal{V}$-constant", i.e., all cohomology sheaves $\mathcal{H}^j(IC'_{\bar{V}})|_W$ $(j \in \mathbb{Z})$ are constant for all $V, W \in \mathcal{V}$ (e.g., this is the case if Y is a toric variety with its natural Whitney stratification by orbits, cf. [**BL**]). Moreover, assume that either

(1) $rat(M)$ is also cohomologically $\mathcal{V}$-constant, or
(2) all perverse cohomology sheaves $rat(H^j(M))$ $(j \in \mathbb{Z})$ are cohomologically $\mathcal{V}$-constant, e.g., each $H^j(M)$ is a pure Hodge module with the property that $\mathcal{H}^{-dim(V)}(rat(H^j(M))|_V)$ is constant for all $V \in \mathcal{V}$.

Then $[M] \in \langle [IC'^H_{\bar{V}}] \rangle$. In particular, if all strata $V \in \mathcal{V}$ are simply-connected, then we have that $[M] \in \langle [IC'^H_{\bar{V}}] \rangle$ for all $M \in D^b\mathrm{MHM}(X)$ so that $rat(M)$ is $\mathcal{V}$-constructible.

In the following, we specialize to the relative context of a proper algebraic map $f : X \to Y$ of complex algebraic varieties, with Y irreducible. For a given $M \in D^b\mathrm{MHM}(X)$, assume that $Rf_* rat(M)$ is constructible with respect to the given complex algebraic Whitney stratification $\mathcal{V}$ of Y, with open dense stratum S. By proper base change, we get

$$i_v^* f_*[M] = [H^*(\{f = v\}, rat(M))] \in K_0(\mathrm{MHM}(pt)).$$

So under the assumption $f_*[M] \in \langle [IC'^H_{\bar{V}}] \rangle$, Theorem 3.1 yields the following identity in $K_0(\mathrm{MHM}(Y))$:

Corollary 3.2.

$$\begin{aligned} f_*[M] &= [IC'^H_Y] \cdot [H^*(F; rat(M))] \\ &+ \sum_{V<S} \widehat{IC^H}(\bar{V}) \cdot \left([H^*(F_V; rat(M))] - [H^*(F; rat(M))] \cdot [IH^*(c^\circ L_{V,Y})] \right), \end{aligned}$$

where F is the (generic) fiber over the top-dimensional stratum S, and F_V is the fiber over a stratum $V \in \mathcal{V} \setminus \{S\}$.

Note that the corresponding classes $[H^*(F; rat(M))]$ and $[H^*(F_V; rat(M))]$ may depend on the choice of fibers of f, but the above formula holds for any such choice. If all strata $V \in \mathcal{V}$ are simply connected, then these classes are independent of the choices made. By pushing the identity in Corollary 3.2 down to a point via

k'_*, for $k' : Y \to pt$ the constant map, and using the fact that k'_* is $K_0(\mathrm{MHM}(pt))$-linear, an application of the χ_y-genus (ring) homomorphism yields the following:

PROPOSITION 3.3. *Under the above notations and assumptions, the following identity holds in* $\mathbb{Z}[y, y^{-1}]$*:*

$$\chi_y([H^*(X; rat(M)]) = I\chi(Y) \cdot \chi_y([H^*(F; rat(M))]) + \sum_{V<S} \widehat{I\chi_y}(\bar{V}) \cdot (\chi_y([H^*(F_V; rat(M))]) - \chi_y([H^*(F; rat(M))]) \cdot I\chi_y(c^\circ L_{V,Y})),$$

where for $V < S$, $\widehat{I\chi_y}(\bar{V})$ *is defined inductively by*

$$\widehat{I\chi_y}(\bar{V}) = I\chi_y(\bar{V}) - \sum_{W<V} \widehat{I\chi_y}(\bar{W}) \cdot I\chi_y(c^\circ L_{W,V}).$$

In particular, if in Proposition 3.3 we take $M = \mathbb{Q}^H_X$, we obtain the following [2]

THEOREM 3.4. *(*[**CMSb**]*, Thm. 2.5) Let* $f : X \to Y$ *be a proper algebraic map of complex algebraic varieties, with* Y *irreducible. Let* $\mathcal{V}$ *be the set of components of strata of* Y *in an algebraic stratification of* f*, and assume* $\pi_1(V) = 0$ *for all* $V \in \mathcal{V}$*. For each* $V \in \mathcal{V}$ *with* $\dim(V) < \dim(Y)$*, define inductively*

$$\widehat{I\chi_y}(\bar{V}) = I\chi_y(\bar{V}) - \sum_{W<V} \widehat{I\chi_y}(\bar{W}) \cdot I\chi_y(c^\circ L_{W,V}),$$

where $c^\circ L_{W,V}$ *denotes the open cone on the link of* W *in* $\bar{V}$*. Then:*

$$\chi_y(X) = I\chi_y(Y) \cdot \chi_y(F) + \sum_{V<S} \widehat{I\chi_y}(\bar{V}) \cdot (\chi_y(F_V) - \chi_y(F) \cdot I\chi_y(c^\circ L_{V,Y})), \tag{3.5}$$

where F *is the (generic) fiber over the top-dimensional stratum* S*, and* F_V *is the fiber of* f *above the stratum* $V \in \mathcal{V} \setminus \{S\}$*.*

REMARK 3.5. Formula (3.5) yields calculations of classical topological and algebraic invariants of the complex algebraic variety X, e.g. Euler characteristic, and if X is smooth and projective, signature and arithmetic genus, in terms of singularities of proper algebraic maps defined on X. In particular, if in Theorem 3.4 we take $f = id$, then formula (3.5) yields an interesting relationship between the χ_y- and respectively the $I\chi_y$-genus of an irreducible complex algebraic variety Y:

$$\chi_y(Y) = I\chi_y(Y) + \sum_{V<S} \widehat{I\chi_y}(\bar{V}) \cdot (1 - I\chi_y(c^\circ L_{V,Y})). \tag{3.6}$$

Similarly, for X pure dimensional, if we let $M = IC'^H_X$ then, in the above notations and assumptions on the monodromy along the strata, Proposition 3.3 yields the following formula (cf. [**CMSb**] for complete details):

$$I\chi_y(X) = I\chi_y(Y) \cdot I\chi_y(F) + \sum_{V<S} \widehat{I\chi_y}(\bar{V}) \cdot \left(I\chi_y(f^{-1}(c^\circ L_{V,Y})) - I\chi_y(F) \cdot I\chi_y(c^\circ L_{V,Y})\right). \tag{3.7}$$

By applying the transformation MHT_y to the identity of Corollary 3.2 for $M = \mathbb{Q}^H_X$, and resp. for $M = IC'^H_X$, and by using the fact that MHT_y commutes with the exterior product, we obtain the following result:

[2] Here we use the deep result due to Saito [**Sa00**] that Deligne's and Saito's mixed Hodge structures on cohomology groups coincide.

Theorem 3.6. *([**CMSb**], Thm. 4.7) Let $f : X \to Y$ be a proper morphism of complex algebraic varieties, with Y irreducible. Let $\mathcal{V}$ be the set of components of strata of Y in a stratification of f, with S the top-dimensional stratum (which is Zariski-open and dense in Y), and assume $\pi_1(V) = 0$ for all $V \in \mathcal{V}$. For each $V \in \mathcal{V} \setminus \{S\}$, define inductively*

$$\widehat{IT}_{y*}(\bar{V}) := IT_{y_*}(\bar{V}) - \sum_{W<V} \widehat{IT}_{y*}(\bar{W}) \cdot I\chi_y(c^\circ L_{W,V}),$$

where $c^\circ L_{W,V}$ denotes the open cone on the link of W in $\bar{V}$, and all homology characteristic classes are regarded in the Borel-Moore homology of the ambient variety Y (with coefficients in $\mathbb{Q}[y, y^{-1}, (1+y)^{-1}]$). Then:

(3.8)

$$f_*T_{y_*}(X) = IT_{y_*}(Y) \cdot \chi_y(F) + \sum_{V<S} \widehat{IT}_{y*}(\bar{V}) \cdot \left(\chi_y(F_V) - \chi_y(F) \cdot I\chi_y(c^\circ L_{V,Y})\right),$$

where F is the generic fiber of f, and F_V denotes the fiber over a stratum $V \in \mathcal{V} \setminus \{S\}$.

If, moreover, X is pure-dimensional, then:

$$f_*IT_{y_*}(X) = IT_{y_*}(Y) \cdot I\chi_y(F) + \sum_{V<S} \widehat{IT}_{y*}(\bar{V}) \cdot \left(I\chi_y(f^{-1}(c^\circ L_{V,Y})) - I\chi_y(F) \cdot I\chi_y(c^\circ L_{V,Y})\right). \tag{3.9}$$

These formulae can be viewed as, on the one hand, yielding powerful methods of inductively calculating (even parametrized families of) characteristic classes of algebraic varieties (e.g., by applying them to resolutions of singularities). On the other hand, they can be viewed as yielding topological and analytic constraints on the singularities of any proper algebraic morphism (e.g., even between smooth varieties), expressed in terms of (even parametrized families of) their characteristic classes.

Remark 3.7. For the value $y = -1$ of the parameter, i.e., in the case of (intersection (co)homology) Euler characteristics and MacPherson-Chern homology characteristic classes, all formulae in this section hold (even in the compact complex analytic case) without any assumption on the monodromy along the strata. This fact is a consequence of a formula similar to (3.4), which holds in the abelian group of $\mathcal{V}$-constructible functions on Y (see [**CMSa**], Theorem 3.1(2)).

It is interesting to see how the results of this section simplify in the following situation:

Proposition 3.8. *If $f : X \to Y$ is a proper algebraic map between irreducible n-dimensional complex algebraic varieties so that f is* homologically small of degree 1 *(in the sense of [**GM**], §6.2), then*

$$f_*IT_{y_*}(X) = IT_{y_*}(Y) \quad \text{and} \quad I\chi_y(X) = I\chi_y(Y). \tag{3.10}$$

In particular, if $f : X \to Y$ is a small resolution*, that is a resolution of singularities that is also small (in the sense of [**GM**]), then* [3]*:*

$$f_*T_{y_*}(X) = IT_{y_*}(Y) \quad \text{and} \quad \chi_y(X) = I\chi_y(Y). \tag{3.11}$$

[3]Finding numerical invariants of complex varieties, more precisely Chern numbers that are invariant under small resolutions, was Totaro's guiding principle in his paper [**To**].

PROOF. Indeed, for such a map we have that $f_*IC_X \in Perv(\mathbb{Q}_Y)$, more precisely there is a (canonical) isomorphism ([**GM**], Theorem 6.2):

$$f_*IC_X \simeq {}^p\mathcal{H}^0(f_*IC_X) \simeq IC_Y \in D^b_c(Y). \tag{3.12}$$

Moreover, as $rat : \mathrm{MHM}(Y) \to Perv(\mathbb{Q}_Y)$ is a faithful functor, this isomorphism can be lifted to the level of mixed Hodge modules. Then, since MHT_y commutes with proper push-downs and $[IC'^H_X] = (-1)^n[IC^H_X]$ in $K_0(\mathrm{MHM}(X))$, we obtain:

$$\begin{aligned} f_*IT_{y_*}(X) &= f_*MHT_y([IC'^H_X]) = (-1)^n MHT_y(f_*[IC^H_X]) \\ &= (-1)^n MHT_y([IC^H_Y]) = MHT_y([IC'^H_Y]) = IT_{y_*}(Y)\,. \end{aligned}$$

The claim about genera follows by noting that the isomorphism (3.12) (when regarded at the level of mixed Hodge modules) induces a (canonical) isomorphism of mixed Hodge structures $IH^*(X) \simeq IH^*(Y)$.

□

3.1. Lifts of characteristic classes to intersection homology. For a singular space Y, the usual characteristic class theories are natural transformations taking values in the (Borel-Moore) homology. If Y is a closed manifold, then by Poincaré Duality these homology characteristic classes are in the image of the cap product map

$$H^{dim_{\mathbb{R}}(Y)-*}(Y) \stackrel{\cap[Y]}{\to} H_*(Y),$$

so they lift to classes in cohomology. But the Poincaré Duality ceases to hold if the space Y has singularities. However, if Y is a topological pseudomanifold which for simplicity we assume to be compact, and for $\bar{p}$ a fixed perversity, the cap product map factors through the perversity $\bar{p}$ intersection homology groups:

$$H^{dim_{\mathbb{R}}(Y)-*}(Y) \to IH^{\bar{p}}_*(Y) \to H_*(Y).$$

It is therefore natural to ask what homology characteristic classes of Y admit lifts to intersection homology. In the case of the topological L_*-classes this is not obvious, and discussed in [**CS91**, (6.2)] based on their mapping theorem for these L_*-classes.

But for a complex algebraic variety Z, the MacPherson-Chern class transformation c_* and the Baum-Fulton-MacPherson Todd class transformation td_* factorize through the (rationalized) Chow group $CH_*(Z)_{\mathbb{Q}}$ of Z (cf. [**Ke, F**]). So the same applies to the Hirzebruch class transformation MHT_y (specialized at any value of y, compare [**BSY, SY**]). And by a deep result from [**BB, W**] (compare also with the more recent [**HS**]), the image of the fundamental class map:

$$cl : CH_i(Z)_{\mathbb{Q}} \to H_{2i}(Z;\mathbb{Q})$$

can be lifted (in general non-uniquely) to the middle intersection homology, i.e.,

$$im(cl : CH_i(Z)_{\mathbb{Q}} \to H_{2i}(Z;\mathbb{Q})) \subset im(IH^{\bar{m}}_{2i}(Z;\mathbb{Q}) \to H_{2i}(Z;\mathbb{Q}))\,.$$

As a corollary, we obtain the following result

THEOREM 3.9. *Let Z be a complete complex algebraic variety. Then for any rational value $y = a \in \mathbb{Q}$ of the parameter y the i-th piece of the Hirzebruch homology class $T_{a*}(Z)$, and for Z pure-dimensional also of the homology class $IT_{a*}(Z)$, is in the image of the natural map*

$$IH^{\bar{m}}_{2i}(Z;\mathbb{Q}) \to H_{2i}(Z;\mathbb{Q}).$$

REMARK 3.10. The conjectured equality $IT_{-1_*}(Z) = L_*(Z)$ would imply that the L-class $L_*(Z)$ of the pure-dimensional compact complex algebraic variety Z has a canonical lift to (rationalized) Chow groups $CH_*(Z)_{\mathbb{Q}}$, and therefore also (non-canonically) to middle intersection homology $IH^{\bar{m}}_{2*}(Z;\mathbb{Q})$.

4. The contribution of monodromy. Atiyah-Meyer type formulae.

If we drop the assumption of trivial monodromy along the strata in a stratification of a proper algebraic morphism, then the right hand side of the formulae in the previous section should be written in terms of twisted intersection homology genera and respectively twisted Hirzebruch characteristic classes. Indeed, for any complex algebraic variety Z we have the identification

$$K_0(\mathrm{MHM}(Z)) = K_0(\mathrm{MH}(Z)^p), \tag{4.1}$$

where $\mathrm{MH}(Z)^p$ denotes the abelian category of pure polarizable Hodge modules. And by the decomposition by strict support, it follows that $K_0(\mathrm{MH}(Z)^p)$ is generated by elements of the form $[IC^H_S(\mathcal{L})]$, for S an irreducible closed subvariety of Z and $\mathcal{L}$ a polarizable variation of Hodge structures (admissible at infinity) defined on a smooth Zariski open and dense subset of S. Thus the image of the natural transformation MHT_y is generated by twisted characteristic classes $IT_{y_*}(S;\mathcal{L})$, for S and $\mathcal{L}$ as before. It is therefore natural to look for Atiyah-Meyer type formulae for the twisted Hirzebruch classes.

The central result of this section is the following Meyer-type formula for twisted Hirzebruch classes of algebraic manifolds (see [**CLMSa**] for complete details), whose proof is included here for the sake of completeness:

THEOREM 4.1. *([**CLMSa**]) Let Z be a complex algebraic manifold of pure dimension n, and $\mathcal{L}$ an admissible variation of mixed Hodge structures on Z with associated flat bundle with Hodge filtration $(\mathcal{V},\mathcal{F}^\bullet)$. Then*

$$T_{y_*}(Z;\mathcal{L}) = \left(ch^*_{(1+y)}(\chi_y(\mathcal{V})) \cup T^*_y(T_Z)\right) \cap [Z] = ch^*_{(1+y)}(\chi_y(\mathcal{V})) \cap T_{y_*}(Z), \tag{4.2}$$

where

$$\chi_y(\mathcal{V}) := \sum_p \left[Gr^p_{\mathcal{F}}\mathcal{V}\right] \cdot (-y)^p \in K^0(Z)[y,y^{-1}]$$

is the K-theory χ_y-characteristic of $\mathcal{V}$ (with $K^0(Z)$ the Grothendieck group of algebraic vector bundles on Z), and $ch^*_{(1+y)}$ is the twisted Chern character defined in Section 2.1.

PROOF. Let $\mathcal{V} := \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_Z$ be the flat bundle with holomorphic connection $\bigtriangledown$, whose sheaf of horizontal sections is $\mathcal{L} \otimes \mathbb{C}$. The bundle $\mathcal{V}$ comes equipped with its Hodge (decreasing) filtration by holomorphic sub-bundles $\mathcal{F}^p$, and these are required to satisfy the Griffiths' transversality condition

$$\bigtriangledown(\mathcal{F}^p) \subset \Omega^1_Z \otimes \mathcal{F}^{p-1}.$$

The bundle $\mathcal{V}$ becomes a holonomic D-module bifiltered by

$$W_k\mathcal{V} := W_k\mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_Z,$$

$$F_p\mathcal{V} := \mathcal{F}^{-p}\mathcal{V}.$$

This data constitutes the smooth mixed Hodge module $\mathcal{L}^H[n]$. It follows from Saito's work that there is a filtered quasi-isomorphism between $(DR(\mathcal{L}^H), F_{-\bullet})$

and the usual filtered de Rham complex $(\Omega^\bullet_Z(\mathcal{V}), F^\bullet)$ with the filtration induced by Griffiths' transversality, that is,

$$F^p\Omega^\bullet_Z(\mathcal{V}) := \left[\mathcal{F}^p \overset{\nabla}{\to} \Omega^1_Z \otimes \mathcal{F}^{p-1} \overset{\nabla}{\to} \cdots \overset{\nabla}{\to} \Omega^i_Z \otimes \mathcal{F}^{p-i} \overset{\nabla}{\to} \cdots \right].$$

Therefore,

$$\begin{aligned} MHC_*([\mathcal{L}^H]) &= \sum_{p,i}(-1)^i[\mathcal{H}^i(gr^F_{-p}DR(\mathcal{L}^H))]\cdot(-y)^p \\ &= \sum_{p,i}(-1)^i[\mathcal{H}^i(gr^p_F\Omega^\bullet_Z(\mathcal{V}))]\cdot(-y)^p \\ &= \sum_{p,i}(-1)^i[\Omega^i_Z\otimes Gr^{p-i}_{\mathcal{F}}\mathcal{V}]\cdot(-y)^p \\ &= \chi_y(\mathcal{V})\otimes\lambda_y(T^*_Z)\in G_0(Z)\otimes\mathbb{Z}[y,y^{-1}], \end{aligned}$$

where $\lambda_y(T^*_Z) := \sum_p \Lambda^p T^*_Z \cdot y^p$ the total λ-class of Z. Since Z is an algebraic manifold, the Todd class transformation of the classical Grothendieck-Riemann-Roch theorem is explicitly described by [4]

$$td_*(\cdot) = ch^*(\cdot)td^*(Z)\cap[Z].$$

Therefore, by applying td_* (which is linearly extended over $\mathbb{Z}[y,y^{-1}]$) to the above equation, we have that

$$td_*\left(MHC_*([\mathcal{L}^H])\right) = \left(ch^*(\chi_y(\mathcal{V}))\cup\tilde{T}^*_y(T_Z)\right)\cap[Z], \tag{4.3}$$

where $\tilde{T}^*_y(T_Z) := ch^*(\lambda_y(T^*_Z))\cup td^*(Z)$ is the un-normalized Hirzebruch class (in cohomology). The claimed formula (4.2) follows now from the definition of $td_{(1+y)}$, by noting that the identities

$$ch^*_{(1+y)}(\cdot)_{2k} = (1+y)^k\cdot ch^*(\cdot)_{2k}, \quad \text{and} \quad T^k_y(T_Z) = (1+y)^{k-n}\cdot\tilde{T}^k_y(T_Z)$$

hold in $H^{2k}(Z)\otimes\mathbb{Q}[y]$. Indeed, we have in $H^{BM}_{2k}(Z)\otimes\mathbb{Q}[y,y^{-1}]$ the following sequence of equalities

$$\begin{aligned} td_k\left(MHC_*([\mathcal{L}^H])\right) &= \left(ch^*(\chi_y(\mathcal{V}))\cup\tilde{T}^*_y(T_Z)\right)^{2(n-k)}\cap[Z] \\ &= \left(\sum_{i+j=n-k} ch^*(\chi_y(\mathcal{V}))_{2i}\cup\tilde{T}^j_y(T_Z)\right)\cap[Z] \\ &= \left(\sum_{i+j=n-k}(1+y)^{-i}ch^*_{(1+y)}(\chi_y(\mathcal{V}))_{2i}\cup(1+y)^{n-j}T^j_y(T_Z)\right)\cap[Z] \\ &= (1+y)^k\left(ch^*_{(1+y)}(\chi_y(\mathcal{V}))\cup T^*_y(T_Z)\right)^{2(n-k)}\cap[Z]. \end{aligned}$$

□

[4] This formula is the counterpart of the Atiyah-Meyer formula in the coherent context of the Todd-class transformation of Baum-Fulton-MacPherson ([**BFM**]). More generally, the counterpart of the Banagl-Cappell-Shaneson formula (1.6) in the coherent context is $td_*(\mathcal{G}) = ch^*([\mathcal{G}])\cap td_*(Z)$, for a locally free coherent sheaf $\mathcal{G}$ on the singular algebraic variety Z.

Corollary 4.2. *If the variety Z in Theorem 4.1 is also* complete, *then by pushing down to a point, we obtain a Hodge theoretic Meyer-type formula for the twisted χ_y-genus:*

$$\chi_y(Z;\mathcal{L}) = \langle ch^*_{(1+y)}(\chi_y(\mathcal{V})) \cup T^*_y(T_Z), [Z]\rangle. \tag{4.4}$$

Remark 4.3. Assume that the local system $\mathcal{L}$ underlies a polarizable variation of pure Hodge structures of weight i on Z. Then the choice of such a polarization defines after identifying the Tate twists $\mathbb{Q}_Z(i) \simeq \mathbb{Q}_Z$ a suitable duality structure on $\mathcal{L}$, i.e. makes it a Poincaré local system. Then it is easy to see that the image of $\chi_1(\mathcal{V})$ under the natural map

$$can : K^0(Z) \to KU(Z) \to KU(Z)[1/2] \supset KO(Z)[1/2]$$

agrees with the K-theory signature $[\mathcal{L}]_K$ of this Poincaré local system. So this class

$$can(\chi_1(\mathcal{V})) \in KU(Z)[1/2]$$

does not depend on the choice of the polarization. In the same way one also gets for Z projective the equality

$$\chi_1(Z;\mathcal{L}) = \sigma(Z;\mathcal{L}) \ ,$$

so that in this case the formula (4.4) exactly specializes for $y = 1$ to Meyer's signature formula (1.3). Recall that $T^*_1(T_Z) = L^*(T_Z)$ for Z smooth.

Similarly, for any variation of mixed Hodge structures one gets by definition that

$$\chi_{-1}(\mathcal{V}) = [\mathcal{V}] \in K^0(Z) \quad \text{and} \quad ch^*_{(0)}(\chi_{-1}(\mathcal{V})) = \mathrm{rk}(\mathcal{V}) = \mathrm{rk}(\mathcal{L}) \in H^0(Z;\mathbb{Q}) \ .$$

So the formula (4.4) specializes for $y = -1$ to the well-known formula for the Euler characteristic of Z with coefficients in $\mathcal{L}$:

$$\chi(H^*(Z;\mathcal{L})) = \mathrm{rk}(\mathcal{L}) \cdot \chi(H^*(Z;\mathbb{Q})) = \mathrm{rk}(\mathcal{L}) \cdot \langle c^*(T_Z), [Z]\rangle \ .$$

Remark 4.4. Without the compactness assumption on Z, we can obtain directly a formula for $\chi_y(Z;\mathcal{L})$ by noting that the twisted logarithmic de Rham complex $\Omega^\bullet_X(\log D) \otimes \bar{\mathcal{V}}$ associated to the Deligne extension of $\mathcal{L}$ on a good compactification (X, D) of Z (with X smooth and compact, and D a simple normal crossing divisor), with its Hodge filtration induced by Griffiths' transversality, is part of a cohomological mixed Hodge complex that calculates $H^*(Z;\mathcal{L}\otimes\mathbb{C})$. In the above notation, we then obtain (cf. [**CLMSa**], Theorem 4.10):

$$\chi_y(Z;\mathcal{L}) = \langle ch^*(\chi_y(\bar{\mathcal{V}})) \cup ch^*\left(\lambda_y(\Omega^1_X(logD))\right) \cup td^*(X), [X]\rangle. \tag{4.5}$$

Here $\langle , \rangle$ denotes the Kronecker pairing on X, $td^*(X) := td^*(T_X)$ is the total Todd class of X (in cohomology),

$$\lambda_y\left(\Omega^1_X(\log D)\right) := \sum_i \Omega^i_X(\log D)\cdot y^i, \quad \text{and} \quad \chi_y(\bar{\mathcal{V}}) = \sum_p \left[Gr^p_{\bar{\mathcal{F}}}\bar{\mathcal{V}}\right]\cdot(-y)^p \ ,$$

with $(\bar{\mathcal{V}}, \bar{\mathcal{F}}^\bullet)$ the unique extension of $(\mathcal{V}, \mathcal{F}^\bullet)$ to X corresponding to the Deligne extension of $\mathcal{L}$ (cf. [**De**]).

For future reference, we mention here a different way of proving formula (4.5). Under the above notations and for $j : Z \hookrightarrow X$ the inclusion map, Saito's work implies that there is a filtered quasi-isomorphism between $(DR(j_*\mathcal{L}^H), F_{-\bullet})$ and the usual filtered logarithmic de Rham complex of $(\bar{\mathcal{V}}, \bar{\mathcal{F}}^\bullet)$. Then, as in the proof of Theorem 4.1, it follows that

$$MHC_*([j_*\mathcal{L}^H]) = \chi_y(\bar{\mathcal{V}}) \otimes \lambda_y \left(\Omega^1_X(\log D)\right) \tag{4.6}$$

(Note that all coherent sheaves appearing in the above formula are locally free). Therefore, by applying the transformation td_* (which is linearly extended over $\mathbb{Z}[y, y^{-1}]$) to the above equation, we have that

$$td_*\left(MHC_*([j_*\mathcal{L}^H])\right) = \left(ch^*(\chi_y(\bar{\mathcal{V}})) \cup ch^*\left(\lambda_y(\Omega^1_X(\log D))\right) \cup td^*(X)\right) \cap [X]. \tag{4.7}$$

Formula (4.5) can be now obtained by pushing (4.7) down to a point via the constant map $k : X \to pt$, and by using an argument similar to that of [[**CLMSa**], Proposition 5.4].

In the relative setting, as an application of Theorem 4.1 we obtain the following Atiyah-type result:

THEOREM 4.5. *([**CLMSa**]) Let $f : E \to B$ be a projective morphism of complex algebraic varieties, with B smooth and connected. Assume that the sheaves $R^s f_*\mathbb{Q}_E$, $s \in \mathbb{Z}$, are locally constant on B, e.g., f is a locally trivial topological fibration. Then*

$$f_* T_{y_*}(E) = ch^*_{(1+y)}\left(\chi_y(f)\right) \cap T_{y_*}(B), \tag{4.8}$$

where

$$\chi_y(f) := \sum_{i,p} (-1)^i \left[Gr^p_{\mathcal{F}}\mathcal{H}_i\right] \cdot (-y)^p \in K^0(B)[y]$$

is the K-theory χ_y-characteristic of f, for $\mathcal{H}_i$ the flat bundle with connection $\bigtriangledown_i : \mathcal{H}_i \to \mathcal{H}_i \otimes_{\mathcal{O}_B} \Omega^1_B$, whose sheaf of horizontal sections is $R^i f_*\mathbb{C}_E$.

If, moreover, B is complete, then by pushing down to a point, we obtain:

$$\chi_y(E) = \langle ch^*_{(1+y)}\left(\chi_y(f)\right) \cup T^*_y(T_B), [B]\rangle. \tag{4.9}$$

PROOF. If in (2.8) we let $M^\bullet = f_*\mathbb{Q}^H_E$, then by using (2.9) we obtain the following identity in $K_0(\mathrm{MHM}(B))$:

$$\left[f_*\mathbb{Q}^H_E\right] = \sum_{i \in \mathbb{Z}} (-1)^i [H^i(f_*\mathbb{Q}^H_E)] = \sum_{i \in \mathbb{Z}} (-1)^i \left[H^{i+\mathrm{dim}B}(f_*\mathbb{Q}^H_E)[-\mathrm{dim}B]\right]. \tag{4.10}$$

Note that $H^{i+\mathrm{dim}B}(f_*\mathbb{Q}^H_E) \in \mathrm{MHM}(B)$ is the smooth mixed Hodge module on B whose underlying rational complex is (recall that B is smooth)

$$rat(H^{i+\mathrm{dim}B}(f_*\mathbb{Q}^H_E)) = {}^p\mathcal{H}^{i+\mathrm{dim}B}(Rf_*\mathbb{Q}_E) = (R^i f_*\mathbb{Q}_E)[\mathrm{dim}B], \tag{4.11}$$

where ${}^p\mathcal{H}$ denotes the perverse cohomology functor. In this case, each of the local systems $\mathcal{L}_i := R^i f_*\mathbb{Q}_E$ underlies a geometric (hence admissible) variation of Hodge structures. By applying the natural transformation MHT_y to the equation (4.10), and using the fact that f is proper, we have that

$$f_* T_{y_*}(E) = \sum_i (-1)^i T_{y_*}(B; \mathcal{L}_i).$$

In view of Theorem 4.1 this yields the formula in equation (4.8).

□

REMARK 4.6. If the monodromy action of $\pi_1(B)$ on $H^*(F)$ is *trivial* (e.g., $\pi_1(B)=0$), i.e., if the local systems $R^i f_* \mathbb{Q}_E$ ($i \in \mathbb{Z}$) are constant on B, then by the "rigidity theorem" (e.g., see the discussion in the last paragraph of [**CMSb**], §3.1) the underlying variations of mixed Hodge structures are constant, so that

$$ch^*_{(1+y)}\left(\chi_y(f)\right) = \chi_y(F) \in H^0(B;\mathbb{Q}). \tag{4.12}$$

In this case, formula (4.9) yields the multiplicative relation

$$\chi_y(E) = \chi_y(F) \cdot \chi_y(B)\,,$$

thus extending the Chern-Hirzebruch-Serre theorem (in the context of complex algebraic varieties).

Theorem 4.1 can also be used for computing invariants arising from intersection homology (cf. Definition 2.3). In the above notations, we have the following

PROPOSITION 4.7. *([**CLMSb**]) Let $f: E \to B$ be a proper morphism of complex algebraic varieties, with E pure-dimensional and B smooth and connected. Assume that f is a locally trivial topological fibration with fiber F. Then*

$$f_* IT_{y_*}(E) = \sum_i (-1)^{dimF+i}\; T_{y_*}(B;\mathcal{L}_i), \tag{4.13}$$

where $\mathcal{L}_i$ is the admissible variation of mixed Hodge structures on B with stalk $IH^{dimF+i}(F;\mathbb{Q})$ and with associated smooth mixed Hodge module $H^i(f_* IC^H_E) \in$ MHM(B).

PROOF. The following equation in $K_0(\text{MHM}(B))$ is a consequence of the identities (2.8) and (2.9):

$$[f_* IC^H_E] = \sum_i (-1)^i \left[H^i(f_* IC^H_E)\right] = (-1)^{\dim(B)} \cdot \sum_i (-1)^i \left[H^i(f_* IC^H_E)[-\dim(B)]\right]. \tag{4.14}$$

Note that $H^i(f_* IC^H_E) \in \text{MHM}(B)$ is the smooth mixed Hodge module on B whose underlying rational complex is

$$rat(H^i(f_* IC^H_E)) = {}^p\mathcal{H}^i(Rf_* IC_E) = (R^{i-\dim B} f_* IC_E)[\dim B], \tag{4.15}$$

where the second equality above follows since B is smooth (hence smooth perverse sheaves are, up to a shift, just local systems on B). In particular, each of the local systems $\mathcal{L}_i := R^{i-\dim B} f_* IC_E$ ($i \in \mathbb{Z}$) underlies an admissible variation of mixed Hodge structures.

By applying the natural transformation MHT_y to the equation (4.14), and using the fact that MHT_y commutes with f_* (since f is proper), we obtain the formula in equation (4.13).

It remains to identify the stalks of the local systems $\mathcal{L}_i$ ($i \in \mathbb{Z}$). Let $b \in B$ with $i_b : \{b\} \hookrightarrow B$ the inclusion map. Then $\{f=b\}$ is the (general) fiber F of f, so it is locally normally nonsingular embedded in E. It follows that we have a quasi-isomorphism $IC_E|_F \simeq IC_F[\text{codim}F]$ (e.g., see [**GM**], §5.4.1). Then by proper base change we obtain that

$$\begin{aligned}(\mathcal{L}_i)_b &= (R^{i-\dim B} f_* IC_E)_b = \mathcal{H}^{i-\dim B}(i_b^* Rf_* IC_E)\\ &= IH^{i-\dim B+\dim E}(F;\mathbb{Q}) = IH^{i+\dim F}(F;\mathbb{Q})\,.\end{aligned}$$

□

Each term in the right hand side of equation (4.13) can be computed by formula (4.2). Let $\mathcal{V}_i$ be the flat bundle with connection associated to the admissible variation of mixed Hodge structures $\mathcal{L}_i := R^{i-\dim B} f_* IC_E$, that is $\mathcal{V}_i := \mathcal{L}_i \otimes_{\mathbb{Q}} \mathcal{O}_B$. Recall that this comes equipped with a filtration by holomorphic sub-bundles satisfying Griffiths' transversality. Define the $I\chi_y$-characteristic of f by

$$I\chi_y(f) := \sum_i (-1)^{i+\dim F} \cdot \chi_y(\mathcal{V}_i). \tag{4.16}$$

Then as a consequence of (4.2), the above proposition yields the following

Corollary 4.8. *Under the notations and assumptions of Propositions 4.7, we obtain*

$$f_* IT_{y_*}(E) = ch^*_{(1+y)}(I\chi_y(f)) \cap T_{y_*}(B). \tag{4.17}$$

In particular, if $\pi_1(B) = 0$, *then* $f_* IT_{y_*}(E) = I\chi_y(F) \cdot T_{y_*}(B)$.

The last assertion of the corollary follows since, under the trivial monodromy assumption, we have that

$$ch^*_{(1+y)}(I\chi_y(f)) = I\chi_y(F) \in H^0(B; \mathbb{Q}) \, .$$

Similar considerations apply to genera. This is a very special case of the stratified multiplicative property studied in detail in [**CMSb**] and summarized in Section §3 above.

4.1. Atiyah-Meyer formulae in intersection homology. We conclude this report with a result from work in progress ([**CMSS**]) on the computation of twisted intersection homology genera. The following theorem can be regarded as a Hodge-theoretic analogue of the Banagl-Cappell-Shaneson formula ([**BCS**]):

Theorem 4.9. *(*[**CMSS**]*) Assume* $i : Z \hookrightarrow M$ *is the closed inclusion of an irreducible (or pure-dimensional) algebraic subvariety into the smooth algebraic manifold* M, *with* $\mathcal{L}$ *a local system on* M *underlying an admissible variation of mixed Hodge structures with associated flat bundle* $(\mathcal{V}, \mathcal{F}^\bullet)$. *Then one has the formula:*
(4.18)

$$IT_{y_*}(Z; i^*\mathcal{L}) = i^*(ch^*_{(1+y)}(\chi_y(\mathcal{V}))) \cap IT_{y_*}(Z) = ch^*_{(1+y)}(i^*(\chi_y(\mathcal{V}))) \cap IT_{y_*}(Z).$$

Proof. One has for the underlying perverse sheaves the equality:

$$IC_Z(\mathcal{L}) = IC_Z \otimes i^*\mathcal{L} \quad \text{and, after shifting,} \quad IC'_Z(\mathcal{L}) = IC'_Z \otimes i^*\mathcal{L} \, .$$

And this implies on the level of (shifted) mixed Hodge modules that:

$$IC^H_Z(\mathcal{L}) = IC^H_Z \otimes i^*\mathcal{L}^H \quad \text{and resp.} \quad IC'^H_Z(\mathcal{L}) = IC'^H_Z \otimes i^*\mathcal{L}^H \, .$$

So the stated formula is a special case of the following more general result for any $M \in K_0(\mathrm{MHM}(Z))$:

$$MHT_y([M \otimes i^*\mathcal{L}^H]) = ch^*_{(1+y)}(i^*(\chi_y(\mathcal{V}))) \cap MHT_y([M]) \, . \tag{4.19}$$

By resolution of singularities, we see that the Grothendieck group $K_0(\mathrm{MHM}(Z))$ is generated by elements of the form $[p_*(j_*\mathcal{L}'^H)]$ with $p : X \to Z$ a proper algebraic map from a smooth algebraic manifold X, $j : U = X \setminus D \hookrightarrow X$ the open inclusion of the complement of a normal crossing divisor D with smooth irreducible components, and $\mathcal{L}'$ an admissible variation of mixed Hodge structures on U. But MHT_y

commutes with proper pushdown, and $ch^*_{(1+y)}$ commutes with pullbacks, so that by the projection formula it is enough to show that:

$$MHT_y([j_*\mathcal{L}'^H \otimes p^*i^*\mathcal{L}^H]) = ch^*_{(1+y)}(p^*i^*(\chi_y(\mathcal{V}))) \cap MHT_y([j_*\mathcal{L}'^H]). \tag{4.20}$$

At this point we can use the identity of formula (4.7), already discussed in Remark 4.4:

$$td_*\left(MHC_*([j_*\mathcal{L}'^H])\right) = ch^*(\chi_y(\bar{\mathcal{V}}')) \cup ch^*(\lambda_y(\Omega^1_X(\log D))) \cap td_*(X)\,, \tag{4.21}$$

with $td_*(X) = td^*(T_X) \cap [X]$, and $\bar{\mathcal{V}}'$ the Hodge bundle of the Deligne extension of $\mathcal{L}'$ to (X, D). Moreover, we also have that:

$$Rj_*(\mathcal{L}') \otimes p^*i^*\mathcal{L} = Rj_*(\mathcal{L}' \otimes j^*p^*i^*\mathcal{L})$$

and similarly on the level of (shifted) mixed Hodge modules, so that

$$ch^*(\chi_y(\bar{\mathcal{V}}' \otimes j^*p^*i^*\mathcal{V})) = ch^*(\chi_y(\bar{\mathcal{V}}')) \cup ch^*(\chi_y(p^*i^*\mathcal{V})).$$

From here the stated formula follows (as in Theorem 4.1) by the usual recalculation in terms of $ch^*_{(1+y)}$. □

Remark 4.10. The formula (4.21) can be also be used for showing the following important facts (cf. [**CMSS**]):

(1) The motivic Hirzebruch transformation MHT_y commutes with exterior products.
(2) The limit $IT_{y_*}(Z;\mathcal{L})$ for $y = -1$ always exists, as well as $IT_{y_*}(Z;\mathcal{L})$ for $y = 0$, if $\mathcal{L}$ is of non-negative weight, e.g. $\mathcal{L} = \mathbb{Q}_Z$.
(3) More generally the limit $MHT_y([M])$ for $y = -1$ always exists for any mixed Hodge module M on Z, with

$$MHT_{-1}([M]) = c_*([rat(M)]) \otimes \mathbb{Q}$$

the rationalized MacPherson-Chern class of the underlying perverse sheaf, i.e. of the corresponding constructible function given by the Euler characteristics of the stalks.

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