Characteristic classes of mixed Hodge modules and applications

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Part I: Mixed Hodge modules. Examples

- X - complex algebraic variety.
- MHM(X)= *algebraic mixed Hodge modules on X*
- If $X = pt$ is a point, then
$$\mathrm{MHM}(pt) = \mathrm{MHS}^p = \text{(polarizable) } \mathbb{Q} - \mathrm{MHS}$$
- If X is *smooth*, then $\mathrm{MHM}(X) \ni M = ((\mathcal{M}, F, W), (K, W))$, with
  - $(\mathcal{M}, F)$ a regular holonomic filtered $\mathcal{D}_X$-module, with F a *good* filtration.
  - K a perverse sheaf
  - isomorphism $\alpha : \mathrm{DR}(\mathcal{M})^{an} \simeq K \otimes_{\mathbb{Q}_X} \mathbb{C}_X$ compatible with W.
- If X is *singular*, use suitable local embeddings into manifolds and filtered $\mathcal{D}$-modules supported on X.

Basic example: *good variations of MHS*

- X – complex algebraic *manifold* of pure complex dimension n.
- (L, F, W) – *good* (i.e., admissible, with quasi-unipotent monodromy at infinity) variation of MHS on X.
- $(\mathcal{L} := L \otimes_{\mathbb{Q}_X} \mathcal{O}_X, \nabla)$ is a holonomic (left) $\mathcal{D}$-module.
- Hodge filtration F on $\mathcal{L}$ induces by Griffiths' transversality a good filtration $F_p(\mathcal{L}) := F^{-p}\mathcal{L}$ on $\mathcal{L}$ as a filtered $\mathcal{D}$-module.
- $\alpha : \mathrm{DR}(\mathcal{L})^{an} \simeq L[n]$, with shifted de Rham complex

$$\mathrm{DR}(\mathcal{L}) := [\mathcal{L} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{L} \otimes_{\mathcal{O}_X} \Omega^n_X]$$

  with $\mathcal{L}$ in degree $-n$.
- α is compatible with the induced filtration W defined by $W^i(L[n]) := W^{i-n}L[n]$ and $W^i(\mathcal{L}) := (W^{i-n}L) \otimes_{\mathbb{Q}_X} \mathcal{O}_X$
- This data defines a mixed Hodge module $L^H[n]$ on X.

Grothendieck groups of MHM

- $K_0(\mathrm{MHM}(X)) \simeq K_0(D^b\mathrm{MHM}(X))$ – Grothendieck group of (complexes of) MHM on X
- $K_0(\mathrm{MHM}(X))$ is a unitary $K_0(\mathrm{MHM}(pt))$-module, with multiplication induced by exact external product with pt:

$$\boxtimes : \mathrm{MHM}(X) \times \mathrm{MHM}(pt) \to \mathrm{MHM}(X \times \{pt\}) \simeq \mathrm{MHM}(X)$$

and

$$M \otimes \mathbb{Q}_X^H \simeq M \boxtimes \mathbb{Q}_{pt}^H \simeq M$$

- $K_0(\mathrm{MHM}(X))$ is generated by $f_*[j_*L^H]$ (or, alternatively, by $f_*[j_!L^H]$), with $f : Y \to X$ a proper morphism from a complex algebraic manifold Y, $j : U \hookrightarrow Y$ the inclusion of a Zariski open and dense subset U with complement D a sncd, and L a good variation of mixed Hodge structures on U.

Filtered de Rham complexes

Theorem (Saito)

For any variety X, there is a functor of triangulated categories

$$Gr^F_p DR : D^b MHM(X) \longrightarrow D^b_{\mathrm{coh}}(X)$$

commuting with proper pushforward, with $Gr^F_p DR(M) = 0$ for almost all p and M fixed.

(a) *If X is a (pure) n-dimensional complex algebraic manifold, then for $M \in MHM(X)$, $Gr^F_p DR(M)$ is the complex associated to the de Rham complex of the underlying algebraic left $\mathcal{D}$-module $\mathcal{M}$ with its integrable connection ∇:*

$$DR(\mathcal{M}) = [\mathcal{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^n_X]$$

with $\mathcal{M}$ in degree $-n$, filtered by

$$F_p DR(\mathcal{M}) = [F_p\mathcal{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} F_{p+n}\mathcal{M} \otimes_{\mathcal{O}_X} \Omega^n_X]$$

Theorem (Filtered de Rham complexes, cont'd)

(b) *$\bar{X}$ – smooth partial compactification of the algebraic manifold X with complement D a sncd, and open inclusion $j : X \hookrightarrow \bar{X}$.*

For a good variation (L, F, W) of MHS on X, $(DR(j_*L^H), F)$ is filtered quasi-isomorphic to the logarithmic de Rham complex

$$DR_{\log}(\mathcal{L}) := [\overline{\mathcal{L}} \xrightarrow{\overline{\nabla}} \cdots \xrightarrow{\overline{\nabla}} \overline{\mathcal{L}} \otimes_{\mathcal{O}_{\bar{X}}} \Omega^n_{\bar{X}}(\log(D))]$$

with increasing filtration $F_{-p} := F^p$ given by

$$F^p DR_{log}(\mathcal{L}) = [F^p\overline{\mathcal{L}} \xrightarrow{\overline{\nabla}} \cdots \xrightarrow{\overline{\nabla}} F^{p-n}\overline{\mathcal{L}} \otimes_{\mathcal{O}_{\bar{X}}} \Omega^n_{\bar{X}}(\log(D))]$$

where $\overline{\mathcal{L}}$ is the canonical Deligne extension of $\mathcal{L} := L \otimes_{\mathbb{Q}_X} \mathcal{O}_X$.

In particular, $Gr^F_{-p} DR(j_*L^H)$ is quasi-isomorphic to

$$Gr^p_F DR_{log}(\mathcal{L}) = [Gr^p_F\overline{\mathcal{L}} \xrightarrow{Gr\,\overline{\nabla}} \cdots \xrightarrow{Gr\,\overline{\nabla}} Gr^{p-n}_F\overline{\mathcal{L}} \otimes_{\mathcal{O}_{\bar{X}}} \Omega^n_{\bar{X}}(\log(D))]$$

(c) *For $(DR(j_!L^H), F)$, consider instead the logarithmic de Rham complex associated to the Deligne extension $\overline{\mathcal{L}} \otimes \mathcal{O}(-D)$ of $\mathcal{L}$.*

Hirzebruch classes of mixed Hodge modules

The transformations $Gr_p^F \mathrm{DR}$ induce group homomorphisms

$$Gr_p^F \mathrm{DR} : K_0(\mathrm{MHM}(X)) \longrightarrow K_0(X) \simeq K_0(D^b_{\mathrm{coh}}(X))$$

Definition (Brasselet-S.-Yokura)

- The *Hodge-Chern class transformation* of a variety X is:

$$\mathrm{DR}_y : K_0(\mathrm{MHM}(X)) \longrightarrow K_0(X) \otimes \mathbb{Z}[y^{\pm 1}]$$
$$\mathrm{DR}_y([M]) := \sum_{i,p} (-1)^i \left[\mathcal{H}^i Gr_F^p \mathrm{DR}(M)\right] \cdot (-y)^p$$

- The *un-normalized Hirzebruch class transformation* is:

$$T_{y*} := td_* \circ \mathrm{DR}_y : K_0(\mathrm{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}]$$

with $td_* : K_0(X) \to H_*(X) \otimes \mathbb{Q}$ the *Todd class transformation* of the *singular (G-R-R) thm* of Baum-Fulton-MacPherson, linearly extended over $\mathbb{Z}[y^{\pm 1}]$, and $H_*(X) := H^{BM}_{2*}(X)$.

Definition (Brasselet-S.-Yokura)

- The *normalized Hirzebruch class transformation* is:
$\widehat{T}_{y*} := td_{(1+y)*} \circ \mathrm{DR}_y : K_0(\mathrm{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}[y, \frac{1}{y(y+1)}]$
where
$$td_{(1+y)*} : K_0(X) \otimes \mathbb{Z}[y^{\pm 1}] \to H_*(X) \otimes \mathbb{Q}[y, \tfrac{1}{y(y+1)}]$$
is the scalar extension of td_* together with the multiplication by $(1+y)^{-k}$ on the degree k component.
- *Homology Hirzebruch characteristic classes* of a complex algebraic variety X are defined by evaluating at the (class of the) constant Hodge module $\mathbb{Q}_X^H$:
$T_{y*}(X) := T_{y*}([\mathbb{Q}_X^H]), \ \widehat{T}_{y*}(X) := \widehat{T}_{y*}([\mathbb{Q}_X^H]) \in H_*(X) \otimes \mathbb{Q}[y]$.

Properties

- The transformations DR_y and (by Riemann-Roch) T_{y*} and $\widehat{T}_{y*}$ *commute with proper pushforward*.
-
$$\widehat{T}_{y*}([M]) \in H_*(X) \otimes \mathbb{Q}[y^{\pm 1}],$$
and for $y = -1$:
$$\widehat{T}_{-1*}([M]) = c_*([\mathrm{rat}(M)]) \in H_*(X) \otimes \mathbb{Q}$$
is the *MacPherson-Chern class* of the constructible complex $\mathrm{rat}(M)$ (i.e., the MacPherson-Chern class of the *constructible function* defined by taking stalkwise the Euler characteristic).
- If X is *Du Bois*, i.e., the canonical map
$$\mathcal{O}_X \xrightarrow{\sim} Gr_F^0\mathrm{DR}(\mathbb{Q}_X^H) \in D^b_{\mathrm{coh}}(X)$$
is a quasi-isomorphism (cf. *Saito*), then
$$T_{0*}(X) = \widehat{T}_{0*}(X) = td_*([\mathcal{O}_X]) =: td_*(X)$$
for td_* the Todd class transformation.

Example: $\bar{X}$ – smooth partial compactification of the algebraic manifold X with complement D a sncd, and open inclusion $j : X \hookrightarrow \bar{X}$.

- *Recall*: if (L, F, W) is a good variation of MHS on X, then
$$\left(\mathrm{DR}(j_* L^H), F_{-\cdot}\right) \simeq \left(\mathrm{DR}_{log}(\mathcal{L}), F^{\cdot}\right)$$
with the $F_{-p} := F^p$ induced by Griffiths' transversality.
- *Define*
$$\mathrm{DR}^y(Rj_* L) := \sum_p [Gr^p_F(\overline{\mathcal{L}})] \cdot (-y)^p \in K^0(\bar{X})[y^{\pm 1}],$$
with $K^0(\bar{X})$= Grothendieck group of algebraic vector bundles
- Get
$$\mathrm{DR}_y([j_* L^H]) = \mathrm{DR}^y(Rj_* L) \cap \left(\Lambda_y\left(\Omega^1_{\bar{X}}(\log(D))\right) \cap [\mathcal{O}_{\bar{X}}]\right).$$
- Similarly, for
$$\mathrm{DR}^y(j_! L) := \sum_p [\mathcal{O}_{\bar{X}}(-D) \otimes Gr^p_F(\overline{\mathcal{L}})] \cdot (-y)^p \in K^0(\bar{X})[y^{\pm 1}],$$
get
$$\mathrm{DR}_y([j_! L^H]) = \mathrm{DR}^y(j_! L) \cap \left(\Lambda_y\left(\Omega^1_{\bar{X}}(\log(D))\right) \cap [\mathcal{O}_{\bar{X}}]\right),$$
- For $j = id : X \to X$, get the *Atiyah-Meyer type formula*:
$$\mathrm{DR}_y([L^H]) = DR^y(L) \cap (\lambda_y(T^*_X) \cap [\mathcal{O}_X])$$

Corollary (Properties of Hirzebruch classes)

- *Normalization: if X is smooth, then*

$$T_{y*}(X) = T_y^*(T_X) \cap [X]\,, \quad \widehat{T}_{y*}(X) = \widehat{T}_y^*(T_X) \cap [X]$$

with $T_y^*(T_X)$ and $\widehat{T}_y^*(T_X)$ defined by power series

$$Q_y(\alpha) := \frac{\alpha(1+ye^{-\alpha})}{1-e^{-\alpha}}\,,\ \widehat{Q}_y(\alpha) := \frac{\alpha(1+ye^{-\alpha(1+y)})}{1-e^{-\alpha(1+y)}} \in \mathbb{Q}[y][[\alpha]]$$

- *Degree: If X is compact:*

$$\int_X T_{y*}(X) = \int_X \widehat{T}_{y*}(X) = \sum_{j,p} (-1)^j \dim \mathrm{Gr}_F^p H^j(X;\mathbb{C}) \cdot (-y)^p =: \chi_y(X)$$

- *Multiplicativity: DR_y, T_{y*}, $\widehat{T}_{y*}$ commute with external products, e.g.,:*

$$DR_y([M \boxtimes M']) = DR_y([M]) \boxtimes DR_y([M'])$$

for $M \in D^bMHM(Z)$ and $M' \in D^bMHM(Z')$.

Additivity of Hodge-Chern and Hirzebruch classes

Proposition (M.-Saito-S.)

For a complex variety X, fix $M \in D^b MHM(X)$ with $K := \mathrm{rat}(M)$. Let $\mathcal{S} = \{S\}$ be a complex algebraic stratification of X so that for any $S \in \mathcal{S}$: S is smooth, $\bar{S} \setminus S$ is a union of strata, and the sheaves $L_{S,\ell} := \mathcal{H}^\ell K|_S$ are local systems on S for any ℓ.

If $j_S : S \overset{i_{S,\bar{S}}}{\hookrightarrow} \bar{S} \overset{i_{\bar{S},X}}{\hookrightarrow} X$ is the inclusion map of a stratum $S \in \mathcal{S}$, then:

$$[M] = \sum_{S,\ell} (-1)^\ell \left[(j_S)_! L^H_{S,\ell}\right] = \sum_{S,\ell} (-1)^\ell (i_{\bar{S},X})_* \left[(i_{S,\bar{S}})_! L^H_{S,\ell}\right]$$

In particular,

$$DR_y([M]) = \textstyle\sum_{S,\ell} (-1)^\ell (i_{\bar{S},X})_* DR_y \left[(i_{S,\bar{S}})_! L^H_{S,\ell}\right]$$

$$T_{y*}(M) = \textstyle\sum_{S,\ell} (-1)^\ell (i_{\bar{S},X})_* T_{y*}\left((i_{S,\bar{S}})_! L^H_{S,\ell}\right).$$

Explicit computation of summands $\mathrm{DR}_y\left[(i_{S,\bar{S}})_! L^H_{S,\ell}\right]$

Theorem (M.-Saito-S.)

Let L be a good variation of MHS on a stratum S and $i_{S,Z} : S \hookrightarrow Z$ a smooth partial compactification of S so that $D := Z \setminus S$ is a sncd and $i_{S,\bar{S}} = \pi_Z \circ i_{S,Z}$ for a proper morphism $\pi_Z : Z \to \bar{S}$. Then:

$$DR_y(\left[(i_{S,\bar{S}})_! L^H\right]) = (\pi_Z)_* \left[DR^y((i_{S,Z})_! L^H) \cap \lambda_y \left(\Omega^1_Z(\log(D)\right)\right].$$

In particular, if $\overline{\mathcal{L}}$ is the canonical Deligne extension on Z of $\mathcal{L} := L \otimes_{\mathbb{Q}_S} \mathcal{O}_S$, then:

$$T_{y*}((i_{S,\bar{S}})_! L^H) = \sum_{p,q} (-1)^q (\pi_Z)_* td_* \left[\mathcal{O}_Z(-D) \otimes Gr^p_F \overline{\mathcal{L}} \otimes \Omega^q_Z(\log D)\right](-y)^{p+q}.$$

Application: Hirzebruch classes of toric varieties

Theorem (M.-S.)

Let X_Σ be the toric variety defined by the fan Σ. For any cone $\sigma \in \Sigma$, with orbit O_σ and inclusion $i_\sigma : O_\sigma \hookrightarrow \overline{O}_\sigma = V_\sigma$, have:

$$DR_y(\left[(i_\sigma)_! \mathbb{Q}^H_{O_\sigma}\right]) = (1+y)^{\dim(O_\sigma)} \cdot [\omega_{V_\sigma}],$$

where ω_{V_σ} is the canonical sheaf on the toric variety V_σ.

Corollary

Let X_Σ be the toric variety defined by the fan Σ. Then:

$$DR_y(X_\Sigma) = \sum_{\sigma \in \Sigma} (1+y)^{\dim(O_\sigma)} \cdot [\omega_{V_\sigma}].$$

$$T_{y*}(X_\Sigma) = \sum_{\sigma \in \Sigma} (1+y)^{\dim(O_\sigma)} \cdot td_*([\omega_{V_\sigma}]).$$

$$\widehat{T}_{y*}(X_\Sigma) = \sum_{\sigma,k} (1+y)^{\dim(O_\sigma)-k} \cdot td_k([\omega_{V_\sigma}]).$$

Corollary

(*a*) *(Ehler's formula) The (rational) MacPherson-Chern class* $c_*(X_\Sigma) := c_*([\mathbb{Q}_{X_\Sigma}])$ *of a toric variety* X_Σ *is computed by:*

$$c_*(X_\Sigma) = \widehat{T}_{-1*}(X_\Sigma) = \sum_{\sigma \in \Sigma} [V_\sigma].$$

(*b*) *The Todd class* $td_*(X_\Sigma)$ *of a toric variety is computed by:*

$$td_*(X_\Sigma) = T_{0*}(X_\Sigma) = \sum_{\sigma \in \Sigma} td_*([\omega_{V_\sigma}]).$$

Application: Weighted lattice point counting

Corollary (*Generalized Pick's formula*)

If X_P is the projective toric variety associated to a full-dimensional lattice polytope $P \subset M_{\mathbb{R}} \cong \mathbb{R}^n$, and $\ell \in \mathbb{Z}_{>0}$ then:

$$\sum_{Q \preceq P} (1+y)^{\dim(Q)} \cdot \#\big(\mathrm{Relint}(\ell Q) \cap M\big) = \int_{X_P} e^{\ell [D_P]} \cap T_{y*}(X_P)$$

$$\overset{n=2}{=} (1+y)^2 \cdot Area(P) + \frac{1-y^2}{2} \#(\partial P \cap M) + \chi_y(P).$$

Remark ($y=0$, *Danilov*)

$$\#(\ell P \cap M) = \sum_{Q \preceq P} \#\big(\mathrm{Relint}(\ell Q) \cap M\big) = \int_{X_P} e^{\ell [D_P]} \cap td_*(X_P).$$

Part II: Equivariant Setting

- Let G a finite group acting algebraically on a quasi-projective variety X.
- A G-equivariant object in $D^b\mathrm{MHM}(X)$ is an object M with a G-action given by isomorphisms

$$\psi_g : M \to g_* M \quad (g \in G),$$

  such that $\psi_{id} = id$ and $\psi_{gh} = g_*(\psi_h) \circ \psi_g$ for all $g, h \in G$.
- Let $D^{b,G}\mathrm{MHM}(X)$ be the category of G-equivariant objects in the derived category $D^b\mathrm{MHM}(X)$. Similarly for $D^{b,G}_{\mathrm{coh}}(X)$.
- $D^{b,G}(-)$ is not triangulated in general, hence it is different from $D^b_G(-)$ of Bernstein-Lunts.
- Using *equivariant distinguished triangles* in $D^b(-)$, can define a Grothendieck group $K_0(D^{b,G}(-))$ so that:

$$K_0(D^{b,G}\mathrm{MHM}(X)) = K_0(\mathrm{MHM}^G(X)), \;\; K_0(D^{b,G}_{\mathrm{coh}}(X)) = K_0^G(X).$$

Equivariant Hodge-Chern and Hirzebruch classes

Saito's transformations $Gr_p^F \mathrm{DR} : D^b\mathrm{MHM}(X) \to D^b_{\mathrm{coh}}(X)$ commute with the push-forward g_* induced by each $g \in G$, so get induced equivariant transformations

$$Gr_p^F \mathrm{DR}^G : D^{b,G}\mathrm{MHM}(X) \to D^{b,G}_{\mathrm{coh}}(X).$$

The *G-equivariant Hodge-Chern class transformation* is:

$$\mathrm{DR}_y^G : K_0(\mathrm{MHM}^G(X)) \to K_0^G(X) \otimes \mathbb{Z}[y^{\pm 1}]$$

$$\mathrm{DR}_y^G([M]) := \sum_{i,p} (-1)^i \left[\mathcal{H}^i(Gr_F^p \mathrm{DR}^G(M))\right] \cdot (-y)^p.$$

The (un-normalized) *Atiyah–Singer class transformation* is:

$$T_{y*}(-;g) := td_*(-;g) \circ \mathrm{DR}_y^G : K_0(\mathrm{MHM}^G(X)) \to H_*(X^g) \otimes \mathbb{C}[y^{\pm 1}]$$

with $td_*(-;g) : K_0^G(X) \to H_*(X^g) \otimes \mathbb{C}$ the *Lefschetz–Riemann –Roch transformation* of Baum-Fulton-Quart and Moonen.
Equivariant classes of X are defined by evaluating at $[\mathbb{Q}_X^H]$.

Properties of equivariant classes

- DR_y^G and $T_{y_*}(-;g)$ *commute with proper pushforward and* $\boxtimes$.
- If $G = \{id\}$, then $\mathrm{DR}_y^G = \mathrm{DR}_y$ and $T_{y_*}(-;id) = T_{y_*}(-) \otimes \mathbb{C}$.
- If X is *Du Bois*, then $T_{0*}(X;g) = td_*([\mathcal{O}_X];g) =: td_*(X;g)$.
- *Degree*: If X is projective, then:

$$\begin{aligned} \int_{[X^g]} T_{y_*}(X;g) &= \sum_{j,p} (-1)^j trace(g|Gr_F^p H^j(X;\mathbb{C}))\,(-y)^p \\ &=: \chi_y(X;g). \end{aligned}$$

Relation with non-equivariant classes

Let G act *trivially* on the quasi-projective variety X.
Define the *projector*

$$(-)^G := \frac{1}{|G|} \sum_{g \in G} \psi_g$$

acting on the categories $D^{b,G}\mathrm{MHM}(X)$ and $D^{b,G}_{\mathrm{coh}}(X)$, for ψ_g the isomorphism induced from the action of $g \in G$.
The projector $(-)^G$ is well-defined since $D^b\mathrm{MHM}(X)$ and $D^b_{\mathrm{coh}}(X)$ are *Karoubian* $\mathbb{Q}$*-linear additive* categories.
The exact functor $(-)^G$ induces functors

$$[-]^G : K_0(\mathrm{MHM}^G(X)) \to K_0(\mathrm{MHM}(X)), \ [-]^G : K_0^G(X) \to K_0(X).$$

Proposition

$$[-]^G \circ DR_y^G = DR_y \circ [-]^G : K_0(MHM^G(X)) \to K_0(X) \otimes \mathbb{Z}[y^{\pm 1}]$$

Hirzebruch classes of global quotients

Theorem (Cappell-M.-S.-Shaneson)

Let G be a finite group acting algebraically on the quasi-projective variety X. Let $\pi^g := \pi \circ i^g : X^g \to X/G$, with projection $\pi : X \to X/G$ and inclusion $i^g : X^g \hookrightarrow X$. Then, for any $M \in D^{b,G}MHM(X)$, we have:

$$T_{y*}([\pi_* M]^G) = \frac{1}{|G|} \sum_{g \in G} \pi_*^g T_{y*}([M]; g).$$

Corollary

In the above notations, and for $M = \mathbb{Q}_X^H$, get:

$$T_{y*}(X/G) = \frac{1}{|G|} \sum_{g \in G} \pi_*^g T_{y*}(X; g).$$

Symmetric Products of Mixed Hodge Modules

Definition

The n-th symmetric product of a quasi-projective variety X is:

$$X^{(n)} := X^n / \Sigma_n.$$

If $\pi_n : X^n \to X^{(n)}$ is the projection map, the n-th symmetric product of $M \in D^b\mathrm{MHM}(X)$ is defined as:

$$M^{(n)} := (\pi_{n*} M^{\boxtimes n})^{\Sigma_n} \in D^b\mathrm{MHM}(X^{(n)}),$$

where

- $M^{\boxtimes n} \in D^b\mathrm{MHM}(X^n)$ is the n-th external product of M with the induced Σ_n-action (Saito).
- $(-)^{\Sigma_n}$ is the projector on the Σ_n-invariant sub-object.

Example

- if $M = \mathbb{Q}^H_X$ then:

$$\left(\mathbb{Q}^H_X\right)^{(n)} = \mathbb{Q}^H_{X^{(n)}}$$

- if $M = IC'^H_X := IC^H_X[-\dim X]$ then:

$$\left(IC'^H_X\right)^{(n)} = IC'^H_{X^{(n)}}$$

- for $f : Y \to X$ a proper map and $M := f_{*(!)}\mathbb{Q}^H_Y$, the *equivariant Künneth formula* (Saito) yields:

$$\left(f_{*(!)}\mathbb{Q}^H_Y\right)^{(n)} = f^{(n)}_{*(!)}\mathbb{Q}^H_{Y^{(n)}}$$

Theorem (M.-Saito-S.)

There is a canonical isomorphisms of graded MHS

$$H^{\bullet}_{(c)}(X^{(n)}; M^{(n)}) \simeq H^{\bullet}_{(c)}(X^n; \boxtimes^n M)^{\Sigma_n} \simeq \left(\bigotimes^n H^{\bullet}_{(c)}(X; M) \right)^{\Sigma_n},$$

in a compatible way with the corresponding isomorphisms of the underlying $\mathbb{Q}$-complexes.

Theorem (M.-Saito-S.)

For X a quasi-projective variety and $M \in D^bMHM(X)$, let

$$h^{p,q,k}_{(c)}(X,M) := h^{p,q}(H^k_{(c)}(X;M)) := \dim_{\mathbb{C}} Gr^p_F Gr^W_{p+q} H^k_{(c)}(X;M)$$

be the corresponding Hodge numbers. Then:

$$\sum_{n \geq 0} \left(\sum_{p,q,k} h^{p,q,k}_{(c)}(X^{(n)}, M^{(n)}) \cdot y^p x^q (-z)^k \right) \cdot t^n$$

$$= \prod_{p,q,k} \left(\frac{1}{1 - y^p x^q z^k t} \right)^{(-1)^k \cdot h^{p,q,k}_{(c)}(X,M)}$$

Corollary

$$\sum_{n \geq 0} \chi^{(c)}_{-y}(X^{(n)}, M^{(n)}) \cdot t^n = \exp \left(\sum_{r \geq 1} \chi^{(c)}_{-y^r}(X,M) \cdot \frac{t^r}{r} \right)$$

Theorem (Cappell-M.-S.-Shaneson-Yokura)

Let X be a complex quasi-projective variety with Pontrjagin ring

$$PH_*(X) := \sum_{n=0}^{\infty} \left(H_*(X^{(n)}) \otimes \mathbb{Q}[y^{\pm 1}]\right) \cdot t^n := \prod_{n=0}^{\infty} H_*(X^{(n)}) \otimes \mathbb{Q}[y^{\pm 1}].$$

For $M \in D^bMHM(X)$, the following identity holds in $PH_*(X)$:

$$\sum_{n \geq 0} T_{(-y)_*}(M^{(n)}) \cdot t^n = \exp\left(\sum_{r \geq 1} \Psi_r \left(d^r_* T_{(-y)_*}(M)\right) \cdot \frac{t^r}{r}\right),$$

where:

(a) $d^r : X \to X^{(r)}$ *is the composition of the diagonal embedding* $i_r : X \simeq \Delta_r(X) \hookrightarrow X^r$ *with the projection* $\pi_r : X^r \to X^{(r)}$.

(b) Ψ_r *is the r-th homological Adams operation, which on* $H_k(X^r) \otimes \mathbb{Q} := H^{BM}_{2k}(X^r; \mathbb{Q})$ ($k \in \mathbb{Z}$) *is defined by multiplication by* $\frac{1}{r^k}$, *together with* $y \mapsto y^r$.

Corollary

Let X be a complex quasi-projective variety. Then

$$\sum_{n\geq 0} T_{(-y)_*}(X^{(n)})\cdot t^n = \exp\left(\sum_{r\geq 1}\Psi_r\left(d_*^r T_{(-y)_*}(X)\right)\cdot \frac{t^r}{r}\right).$$

Corollary (Ohmoto)

After a suitable re-normalization, for $y=1$ get

$$\sum_{n\geq 0} c_*(X^{(n)})\cdot t^n = \exp\left(\sum_{r\geq 1} d_*^r c_*(X)\cdot \frac{t^r}{r}\right).$$

Part III: Specialization of Hirzebruch classes

- Let $X \overset{i}{\hookrightarrow} Y$ be a complex algebraic hypersurface (or lci) in a complex algebraic manifold Y, with *normal bundle* $N_X Y$.
- If X is *smooth*: $0 \to T_X \to T_Y|_X \to N_X Y \to 0$.
- A normal bundle $N_X Y$ exists even if X is *singular* !
- The *virtual tangent bundle* of X is:

$$T_X^{\mathrm{vir}} := [T_Y|_X - N_X Y] \in K^0(X)$$

- T_X^{vir} is *independent of the embedding* in Y, thus it is a well-defined element in $K^0(X)$, the Grothendieck group of algebraic vector bundles on X.
- If X *smooth*: $T_X^{\mathrm{vir}} = [T_X] \in K^0(X)$.

Characteristic classes

- Let R be a commutative ring with unit, and

$$cl^* : \left(K^0(X), \oplus\right) \to (H^*(X) \otimes R, \cup)$$

  a *multiplicative characteristic class theory* of complex algebraic vector bundles, with $H^*(X) = H^{2*}(X; \mathbb{Z})$.
- Associate to a hypersurface (or lci) X an *intrinsic* homology class (i.e., independent of the embedding $X \hookrightarrow Y$):

$$cl_*^{\mathrm{vir}}(X) := cl^*(T_X^{\mathrm{vir}}) \cap [X] \in H_*(X) \otimes R,$$

  where $[X] \in H_*(X)$ is the fundamental class of X.
- Let $cl_*(-)$ be a functorial *homology characteristic class theory* for complex algebraic varieties, so that
  - *Normalization*: if X smooth, then $cl_*(X) = cl^*(T_X) \cap [X]$.

Example

(a) *Todd classes*
$cl^* = td^*$=*Todd class*, and $td_* : K_0(X) \to H_*(X) \otimes \mathbb{Q}$, the *Baum-Fulton-MacPherson Todd class transformation*, with $td_*(X) := td_*([\mathcal{O}_X])$.

(b) *Chern classes*
$cl^* = c^*$=*Chern class*, and $c_* : K_0(D^b_c(X)) \to F(X) \to H_*(X)$ the functorial *Chern class transformation of MacPherson*, with $c_*(X) := c_*(\mathbb{Q}_X)$. (Here $F(X)$ is the group of *constructible functions* on X.)

(c) *Hirzebruch classes*
$cl^* = \widehat{T}^*_y$=*Hirzebruch class*, and
$\widehat{T}_{y*} : K_0(\mathrm{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}]$ the normalized homology Hirzebruch class transformation, with $\widehat{T}_{y*}(X) := \widehat{T}_{y*}(\mathbb{Q}^H_X)$.

- If X is *smooth*:

$$cl_*^{\mathrm{vir}}(X) \stackrel{\mathrm{def}}{:=} cl^*(T_X^{\mathrm{vir}}) \cap [X] \stackrel{\mathrm{sm}}{=} cl^*(T_X) \cap [X] \stackrel{nor}{=} cl_*(X)\,.$$

- If X is *singular*, the difference

$$\mathcal{M}cl_*(X) := cl_*^{\mathrm{vir}}(X) - cl_*(X)$$

depends in general on the singularities of X.

- By the exact sequence

$$H_*(X_{\mathrm{sing}}) \otimes R \xrightarrow{i_*} H_*(X) \otimes R \xrightarrow{j^*} H_*(X_{\mathrm{reg}}) \otimes R$$

and $j^*\mathcal{M}cl_*(X) = \mathcal{M}cl_*(X_{\mathrm{reg}}) = 0$, get:

$$\mathcal{M}cl_*(X) \in \mathrm{Image}(i_*)$$

- **Corollary**: $cl_k^{\mathrm{vir}}(X) = cl_k(X)$, for $k > \dim X_{\mathrm{sing}}$.

Problem: *Describe $\mathcal{M}cl_*(X)$ in terms of the geometry of the singular locus $X_{\rm sing}$ of X.*

Upshot: Compute the (very) *complicated* homology class $cl_*(X)$ in terms of the simpler (cohomological) virtual class and invariants of the singularities of X.

Specialization for globally defined hypersurfaces

- Let Y be a smooth complex algebraic variety, and $f : Y \to \mathbb{C}$ an algebraic function, with $X := \{f = 0\}$ of codimension one.
- Let $X \stackrel{i}{\hookrightarrow} Y$, so $N_X Y$ is a trivial line bundle.
- Let $\psi_f, \varphi_f : D^b_c(Y) \to D^b_c(X)$ be Deligne's *nearby* and resp. *vanishing cycle* functors.

Theorem (Verdier)

(a)

$$td_* \circ i^!_K = i^! \circ td_* : K_0(Y) \to H_{*-1}(X) \otimes \mathbb{Q}$$

with $i^!_K : K_0(Y) \to K_0(X)$ *induced from* Li^**, and* $i^! : H_*(Y) \to H_{*-1}(X)$ *the corresponding Gysin morphisms.*

(b)

$$c_* \circ \psi_f = i^! \circ c_* : K_0(D^b_c(Y)) \to H_{*-1}(X)$$

Corollary (of Verdier specialization)

(a)

$$\mathcal{M}td_*(X) := td_*^{\mathrm{vir}}(X) - td_*(X) = 0$$

(b)

$$c_*^{\mathrm{vir}}(X) = c_*(\psi_f(\mathbb{Q}_Y))$$

hence

$$\mathcal{M}_*(X) := \mathcal{M}c_*(X) := c_*^{\mathrm{vir}}(X) - c_*(X) = c_*(\varphi_f(\mathbb{Q}_Y))$$

with $c_*(\varphi_f(\mathbb{Q}_Y)) \in H_*(X_{\mathrm{sing}})$.

Recall:

$$\mathcal{H}^k(\psi_f \mathbb{Q}_Y)_x \simeq H^k(F_x;\mathbb{Q}) , \quad \mathcal{H}^k(\varphi_f \mathbb{Q}_Y)_x \simeq \widetilde{H}^k(F_x;\mathbb{Q})$$

for F_x the Milnor fiber of f at $x \in X$.

Example

If X has only *isolated* singularities, then

$$\mathcal{M}_*(X) = \sum_{x \in X_{\mathrm{sing}}} (-1)^n \mu_x,$$

for μ_x the *Milnor number* of the IHS $(X,x) \subset (\mathbb{C}^{n+1},0)$.
In general, $\mathcal{M}_*(X)$ is called the *Milnor class* of X.

Specialization of Hodge-Chern and Hirzebruch classes

- ψ_f, φ_f admit lifts $\psi_f^H[1], \varphi_f^H[1]$ to Saito's mixed Hodge modules.
- Aim: Search for the counterpart of Verdier's specialization for DR_y and $\widehat{T}_{y*}$.

Motivating example

- Let $i : X := \{f = 0\} \hookrightarrow Y$ be a *smooth* hypersurface inclusion, and L a *good variation of MHS* on Y.
- *Atiyah-Meyer*: $\mathrm{DR}_y([L^H]) = \mathrm{DR}^y(L) \cap (\Lambda_y(T^*_Y) \cap [\mathcal{O}_Y])$
- $$\begin{aligned} i^! \mathrm{DR}_y([L^H]) &= i^* \left(\mathrm{DR}^y(L) \cup \Lambda_y(T^*_Y)\right) \cap i^!([\mathcal{O}_Y]) \\ &= \left(\mathrm{DR}^y(i^*L) \cup \Lambda_y(i^*T^*_Y)\right) \cap [\mathcal{O}_X] \\ &= \Lambda_y(N^*_X Y) \cap \mathrm{DR}_y([i^*L^H]) \\ &= (1+y) \cdot \mathrm{DR}_y(i^*[L^H]) \\ &= -(1+y) \cdot \mathrm{DR}_y([\psi^H_f(L^H)]) \end{aligned}$$
- Claim: This identity holds for a singular hypersurface X and any $M \in \mathrm{MHM}(Y)$.

Specialization of Hirzebruch classes

Theorem (S.)

Let Y be a smooth complex algebraic variety, and $f : Y \to \mathbb{C}$ an algebraic function, with $X := \{f = 0\} \stackrel{i}{\hookrightarrow} Y$ of codimension one. Then

(a)

$$-(1+y) \cdot DR_y(\ \psi_f^H(-)\ ) = i^! DR_y(-)$$

as transformations $K_0(MHM(Y)) \to K_0(X)[y^{\pm 1}]$.

(b)

$$-\widehat{T}_{y*}(\psi_f^H(-)) = i^! \widehat{T}_{y*}(-) : K_0(MHM(Y)) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}].$$

Remark

The proof uses the algebraic theory of nearby and vanishing cycles given by the V-filtration of Malgrange-Kashiwara in the context of strictly specializable filtered $\mathcal{D}$-modules, together with a specialization result about the filtered de Rham complex of the filtered $\mathcal{D}$-module underlying a mixed Hodge module.

Corollary

1. $$T_{y_*}^{\mathrm{vir}}(X) := T_y^*(T^{\mathrm{vir}}X) \cap [X] = -T_{y_*}(\psi_f^H([\mathbb{Q}_Y^H]))$$

2. $$\mathcal{M}T_{y_*}(X) := T_{y_*}^{\mathrm{vir}}(X) - T_{y_*}(X) = -T_{y_*}(\varphi_f^H([\mathbb{Q}_Y^H]))$$

Example (Isolated singularities)

If the n-dimensional hypersurface X has *only isolated singularities*, then

$$\mathcal{M}T_{y_*}(X) = \sum_{x \in X_{\mathrm{sing}}} (-1)^n \chi_y([\widetilde{H}^n(F_x;\mathbb{Q})]),$$

where F_x is the Milnor fiber of the IHS (X, x).

Remark

The Hodge polynomial

$$\chi_y([\widetilde{H}^n(F_x;\mathbb{Q})]) := \sum_p \dim_{\mathbb{C}} Gr_F^p \widetilde{H}^n(F_x;\mathbb{C}) \cdot (-y)^p$$

is just a special case of the *Hodge spectrum* of (X, x).

Theorem (Cappell-M.-S.-Shaneson)

Let $X = \{f = 0\} \subset Y$, for $f : Y \to \mathbb{C}$ an algebraic function on a complex algebraic manifold Y. Let $\mathcal{S}_0$ be a partition of the singular locus $X_{\rm sing}$ into disjoint locally closed algebraic submanifolds S, such that the restrictions $\varphi_f(\mathbb{Q}_Y)|_S$ have constant cohomology sheaves (e.g., these are locally constant sheaves on each S, and the pieces S are simply-connected). For each $S \in \mathcal{S}_0$, let F_s be the Milnor fiber of a point $s \in S$. Then:

$$\mathcal{M}T_{y_*}(X) = \sum_{S \in \mathcal{S}_0} \underbrace{\left(T_{y_*}(\bar{S}) - T_{y_*}(\bar{S} \setminus S)\right)}_{\text{horizontal info}} \cdot \underbrace{\chi_y([\widetilde{H}^*(F_s;\mathbb{Q})])}_{\text{vertical info}}$$

The case $y = -1$

- $$\mathcal{M}T_{y_*}(X)|_{y=-1} = \mathcal{M}_*(X) \otimes \mathbb{Q} = c_*(\varphi_f(\mathbb{Q}_Y)).$$
- Hence, for $y = -1$, the previous formula holds without any monodromy assumptions.

The case $y = 0$

Assume X has only Du Bois (e.g., rational) singularities. Then:

1. $T_{y_*}(X)|_{y=0} = td_*(X)$.
2. Hence: $\mathcal{M}T_{y_*}(X)|_{y=0} = \mathcal{M}td_* = 0$, a class version of Dolgachev-Steenbrink cohomological insignificance.
3. E.g., If X has *only isolated Du Bois singularities*, get:

$$\dim_{\mathbb{C}} Gr_F^0 \widetilde{H}^n(F_x;\mathbb{C}) = 0, \quad \text{for all} \quad x \in X_{\mathrm{sing}}$$

4. Ishii:

$$x \in X_{\mathrm{sing}} \text{ is an isolated Du Bois singularity}$$
$$\Updownarrow$$
$$\dim_{\mathbb{C}} Gr_F^0 \widetilde{H}^n(F_x;\mathbb{C}) = 0$$

More details to follow in the discussion session!