TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
S 0002-9947(07)04241-9
Article electronically published on February 21, 2007

MULTIVARIABLE ALEXANDER INVARIANTS OF HYPERSURFACE COMPLEMENTS

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ABSTRACT. We start with a discussion on Alexander invariants, and then prove some general results concerning the divisibility of the Alexander polynomials and the supports of the Alexander modules, via Artin's vanishing theorem for perverse sheaves. We conclude with explicit computations of twisted cohomology following an idea already exploited in the hyperplane arrangement case, which combines the degeneration of the Hodge to de Rham spectral sequence with the purity of some cohomology groups.

1. INTRODUCTION

Alexander invariants in the form of Alexander modules, characteristic varieties and Alexander polynomials have been recently intensively studied, in particular in relation to the twisted cohomology of hypersurface arrangement complements; see for instance [1], [4], [5], [6], [18], [20], [21], [22], [25], [32], [33], [36], [42], [47].

In section 2, after giving the basic definitions introducing the *Alexander modules* $A^q(\mathcal{U})$ and $A_q(\mathcal{U})$ of an affine hypersurface arrangement complement $\mathcal{U}$, we investigate in Proposition 2.4 the relation between *the first nontrivial* Alexander polynomial in one variable and the corresponding Alexander polynomial in several variables. Proposition 2.5 expresses the relation between the characteristic varieties defined using the Fitting ideals and the characteristic varieties defined using the jumping loci of the cohomology with rank one local coefficients. Example 2.8 treats the simplest *local* situations: the normal crossing case and the case of isolated non-normal crossing singularities, whose study was initiated by A. Libgober in [36].

In section 3, Theorem 3.1 relates the Alexander invariants of the affine hypersurface arrangement complement $\mathcal{U} = \mathbb{C}^{n+1} \setminus X$ to the Alexander invariants of the complement $\mathcal{U}_\infty$ of the corresponding *link at infinity.* Theorems 3.2, 3.6 and Corollary 3.5 estimate the support of the Alexander modules $A^q(\mathcal{U})$ in terms of local properties of the projective closure $V = \overline{X}$.

In section 4, we recall and slightly extend the idea of combining the degeneration of the Hodge to de Rham spectral sequence with the purity of some cohomology groups (used first by Esnault, Schechtman and Viehweg in [25] and by Schechtman, Terao and Varchenko in [47]); see Corollary 4.1 and Proposition 4.5. Examples 4.8 and 4.10 illustrate this approach by looking at some arrangements of lines and conics in the plane. Though these examples may be treated using the results by

Received by the editors August 25, 2005.
2000 *Mathematics Subject Classification.* Primary 32S20, 32S22, 32S35, 32S60; Secondary 14J70, 14F17, 14F45.

Cogolludo in [4], we feel that our approach is more general and hence more likely to extend to other situations.

In the last section we consider the complement $\mathcal{U}_0$ of an arbitrary *projective* hypersurface arrangement V, and, after a short general discussion, we revisit from a new perspective a useful result by Randell saying what happens to the twisted cohomology of a plane curve complement when we add an extra line; see Corollary 5.1. Coming back to dimension $n \geq 2$, Example 5.3 discusses the already interesting case when V is irreducible and has only isolated singularities. This case leads, in particular, to examples where for $m = n, n+1$ and some rank one local coefficients $\mathcal{L}_\beta$ on $\mathcal{U}_0$ one has

$$\dim H^m(\mathcal{U}_0, \mathcal{L}_\beta) > \dim H^m(\mathcal{U}_0, \mathbb{C}).$$

By the minimality property of hyperplane arrangement complements, it is known that the above inequality is impossible for such type of complements, [23]. We conclude by a detailed study of the case when V has two irreducible components, each of them having only isolated singularities.

Throughout the paper we usually work with complex coefficients $\mathbb{C}$, although the study of finite field coefficients is very important, due for instance to torsion open questions; see [8], [41]. Our choice is imposed by the analytic tools used in the last two sections. Most of the results in the previous sections hold over arbitrary fields.

2. Multivariable Alexander invariants

2.1. **Algebraic preliminaries.** Let R be a commutative ring with unit, which is Noetherian and a unique factorization domain (e.g., the ring of complex Laurent polynomials in s variables, $s \geq 1$). Let A be a finitely generated R-module, and M an $(n \times m)$ presentation matrix of A associated to an exact sequence

$$R^m \to R^n \to A \to 0.$$

The i-th *elementary ideal* $\mathcal{E}_i(A)$ of A is the ideal in R generated by the $(n-i)\times(n-i)$ minor determinants of M, with the convention that $\mathcal{E}_i(A) = R$ if $i \geq n$, and $\mathcal{E}_i(A) = 0$ if $n - i > m$. Let $\Delta_i(A)$ be the generator of the smallest principal ideal in R containing $\mathcal{E}_i(A)$, i.e., the greatest common divisor of all elements of $\mathcal{E}_i(A)$. $\Delta_i(A)$ is called the i-th *characteristic polynomial* of A. Note that $\Delta_{i+1}(A)$ divides $\Delta_i(A)$ in R for all i since $\mathcal{E}_i(A) \subset \mathcal{E}_{i+1}(A)$. In particular, if R is a principal ideal domain (e.g., the ring of complex Laurent polynomials in one variable), then $\mathcal{E}_i(A)$ is a principal ideal generated exactly by $\Delta_i(A)$.

As an example, for any ring R, assume that $A = R^s \oplus R/(\lambda_1) \oplus \cdots \oplus R/(\lambda_r)$, where λ_j $(j = 1, 2, \cdots, r)$ are nonzero elements in R such that $\lambda_{j+1}|\lambda_j$. Then we have $\Delta_i(A)$ is 0, $\lambda_{i-s+1} \cdots \lambda_r$, or 1, according to whether $0 \leq i \leq s-1$, $s \leq i \leq s+r-1$, or $s + r \leq i$.

The *support* $\mathrm{Supp}(A)$ of A is the reduced subscheme of $\mathrm{Spec}(R)$ defined by (the *order ideal*) $\mathcal{E}_0(A)$. Since

$$\sqrt{\mathcal{E}_0(A)} = \sqrt{\mathrm{Ann}(A)},$$

this is the usual notion of support in algebraic geometry based on the *annihilator ideal* $\mathrm{Ann}(A)$ of the module A. In particular, for a prime ideal $P \subset R$, $P \in \mathrm{Supp}(A)$ if and only if the localized module A_P is nonzero.

The support $\mathrm{Supp}(A)$ is also called the *first characteristic variety* of A, and we define the i-th *characteristic variety* $V_i(A)$ of A to be the reduced subscheme of $\mathrm{Spec}(R)$ defined by the (i-th *Fitting ideal*) ideal $\mathcal{E}_{i-1}(A)$.

Note that $\mathrm{codim} V_i(A) > 1$ implies $\Delta_{i-1}(A) = 1$; i.e., the corresponding Alexander polynomial carries no information.

All definitions above are independent (up to multiplication by a unit of R) of the choices involved; thus the characteristic varieties and polynomials of A are invariants of the R-isomorphism type of A.

We state for future reference the following "divisibility" properties of the polynomials and characteristic varieties (for proofs, see [50] and [35]):

Lemma 2.1.
- *If A, B are finitely generated R-modules, then*
$$\Delta_0(A \oplus B) = \Delta_0(A) \times \Delta_0(B).$$
- *If A and B are finitely generated R-modules, then*
$$\mathrm{Supp}(A \otimes_R B) = \mathrm{Supp}(A) \cap \mathrm{Supp}(B).$$
- *If A is a submodule of B, then for all i, $\Delta_i(A)$ divides $\Delta_i(B)$.*
- *If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated R-modules, then the following hold:*
  1. $\Delta_0(B) = \Delta_0(A) \times \Delta_0(C)$*;*
  2. *for all i, $\Delta_i(B)$ divides $\Delta_i(A) \times \Delta_0(C)$;*
  3. *if $\Delta_0(C) = 1$, then $\Delta_i(A) = \Delta_i(B)$ for all i;*
  4. $\mathrm{Supp}(B) = \mathrm{Supp}(A) \cup \mathrm{Supp}(C)$*;*
  5. *for $i \geq 2$: $V_i(C) \subset V_i(B) \subset V_i(C) \cup (V_{i-1}(C) \cap \mathrm{Supp}(A))$.*

2.2. **Alexander invariants of hypersurface complements.** Let V be a reduced hypersurface in $\mathbb{CP}^{n+1}$, defined by a homogeneous equation: $f = f_1 \cdots f_s = 0$, where the f_i are the irreducible factors of f, and $V_i = \{f_i = 0\}$ the irreducible components of V. We fix a hyperplane H in $\mathbb{CP}^{n+1}$, which we call "the hyperplane at infinity". Let $\mathcal{U}$ be the (affine) hypersurface complement $\mathcal{U} = \mathbb{CP}^{n+1} \setminus (V \cup H)$. (Alternatively, $\mathcal{U}$ may be regarded as the complement of a hypersurface in the affine space $\mathbb{C}^{n+1}$.) Then $H_1(\mathcal{U}) \cong \mathbb{Z}^s$ ([16], (4.1.3), (4.1.4)), generated by the meridian loops γ_i about the nonsingular part of each irreducible component V_i, for $i = 1, \cdots, s$. If γ_∞ denotes the meridian about the hyperplane at infinity, then in $H_1(\mathcal{U})$ there is a relation: $\gamma_\infty + \sum d_i \gamma_i = 0$, where $d_i = \deg(V_i)$.

Note that $\mathcal{U}$ is affine, therefore has the homotopy type of a finite CW complex. Let $\mathcal{U}^{ab}$ be the universal abelian cover of $\mathcal{U}$, i.e. the covering associated to the commutator subgroup of $\pi_1(\mathcal{U})$, or equivalently, the covering associated to the kernel of the linking number homomorphism $lk : \pi_1(\mathcal{U}) \to \mathbb{Z}^s$, which maps a loop α to $(\mathrm{lk}(\alpha, V_1 \cup -d_1 H), \cdots, \mathrm{lk}(\alpha, V_s \cup -d_s H))$. The group of covering transformations of $\mathcal{U}^{ab}$ is isomorphic to $\mathbb{Z}^s$ and acts on the covering space. By choosing fixed lifts of the cells of $\mathcal{U}$ to $\mathcal{U}^{ab}$, we obtain a free basis for C_*, the cellular cell complex of $\mathcal{U}^{ab}$, as a $\mathbb{Z}[\mathbb{Z}^s]$-module. The isomorphism determined by the meridians $\{\gamma_i\}$ enables us to identify $\mathbb{Z}[\mathbb{Z}^s]$ with $\mathbb{Z}[t_1, t_1^{-1}, \cdots, t_s, t_s^{-1}]$, the ring of integral Laurent polynomials in s variables. When $s = 1$ we set $t_1 = t$.

For reasons that will become transparent later, our base ring will always be the ring of complex Laurent polynomials in s variables, $\mathbb{C}[t_1, t_1^{-1}, \cdots, t_s, t_s^{-1}]$, which we denote by R_s. Note that R_s is a regular Noetherian domain, and in particular it is factorial. As a group ring, R_s has a natural involution denoted by an overbar, sending each t_i to $\bar{t}_i := t_i^{-1}$. To an R_s-module A, we associate the conjugate R_s-module, still denoted by A, with the same underlying abelian group but with the R_s-action given by $(r, a) \mapsto \bar{r} \cdot a$, for $a \in A$ and $r \in R_s$.

Remark 2.2. Though the ring R_s is commutative, it should be regarded as a quotient ring of $\mathbb{C}[\pi_1(\mathcal{U})]$, which is non-commutative in general. Because of that, one should be careful to distinguish the right from the left R_s-modules. If, for instance, A is a left R_s-module, then the associated right R_s-module is the module conjugate to A, whose module structure is given by

$$a \cdot r := \bar{r} \cdot a$$

for all $a \in A$ and $r \in R_s$. This corresponds to regarding any left $\mathbb{C}[\pi_1(\mathcal{U})]$-module A as a right $\mathbb{C}[\pi_1(\mathcal{U})]$-module by setting $a \cdot \gamma = \gamma^{-1} \cdot a$, for all $a \in A$ and $\gamma \in \pi_1(\mathcal{U})$, and extending by linearity. Following [11], p. 97, we regard in this paper $C_*^0 = C_* \otimes \mathbb{C}$ as a complex of right R_s-modules.

Define a local coefficient system $\mathcal{L}$ on $\mathcal{U}$, with stalk R_s and action of a loop $\alpha \in \pi_1(\mathcal{U})$ determined by (left) multiplication by $\prod_{j=1}^{s} (t_j)^{\mathrm{lk}(\alpha, V_j \cup -d_j H)}$. In particular, the action of the meridian γ_i is given by multiplication by t_i. Let $\mathcal{L}^\vee$ be the dual local system, whose stalk at a point $y \in \mathcal{U}$ is $\mathcal{L}_y^\vee := \mathrm{Hom}(\mathcal{L}_y, R_s)$, and let $\alpha \in \pi_1(\mathcal{U}, y)$ act on $\varphi \in \mathcal{L}_y^\vee$ by

$$(\alpha \cdot \varphi)(m) := \varphi(\alpha^{-1} \cdot m)\ , \quad m \in \mathcal{L}_y.$$

We denote by $\bar{\mathcal{L}}$ the local system obtained from $\mathcal{L}$ by composing all module structures with the involution of R_s (i.e., by changing the stalks of $\mathcal{L}$ from left into right R_s-modules). The perfect pairing

$$\bar{\mathcal{L}} \otimes_{R_s} \mathcal{L} \to R_s,$$

given by

$$(f, g) \mapsto \bar{f} \cdot g$$

on the stalk over a basepoint, tells us that there is an isomorphism of local systems on $\mathcal{U}$:

$$\mathcal{L}^\vee \simeq \bar{\mathcal{L}}.$$

The *universal homology k-th Alexander invariant* $A_k(\mathcal{U})$ of $\mathcal{U}$ is by definition the R_s-module $H_k(C_*^0)$, or equivalently $H_k(\mathcal{U}; \mathcal{L})$. This is the group $H_k(\mathcal{U}^{ab}; \mathbb{C})$ considered as an R_s-module via the covering transformations (see [29], Example 3H.2). Similarly, the *universal cohomology k-th Alexander invariant* $A^k(\mathcal{U})$ of $\mathcal{U}$ is by definition the k-th cohomology module of the dual complex $\mathrm{Hom}_{R_s}(C_*^0, R_s)$. Here R_s is considered with the induced right R_s-module structure as explained in Remark 2.2. Based on our previous considerations on local systems, $A^k(\mathcal{U})$ is just $H^k(\mathcal{U}; \mathcal{L}^\vee)$. This may also be regarded as the k-th cohomology with compact support and complex coefficients of $\mathcal{U}_b^{ab}$, where $\mathcal{U}_b$ is the compact manifold with boundary obtained from $\mathbb{CP}^{n+1}$ by removing a small open regular neighborhood of the divisor $V \cup H$ (compare [29], Prop. 3H.5).

Note that, since $\mathcal{U}$ is an $(n+1)$-dimensional affine variety, the modules $A^k(\mathcal{U})$ and resp. $A_k(\mathcal{U})$ are trivial for $k > n+1$. Moreover, since the stalks of $\mathcal{L}$ are torsion-free, $A_{n+1}(\mathcal{U})$ is also a torsion-free R_s-module (see [48], Example 6.0.6).

As in classical knot theory, by using a deformation retract argument, one could define the universal abelian invariants above after replacing $\mathcal{U}$ by the manifold with boundary $\mathcal{U}_b$, obtained from $\mathbb{CP}^{n+1}$ by removing a small open regular neighborhood of the divisor $V \cup H$. Now, since the chain complex $C_*(\mathcal{U}_b^{ab})$ is of finite type, and since R_s is Noetherian, this implies that all these universal Alexander modules are finitely generated. Hence their characteristic varieties and polynomials are well defined. The associated characteristic varieties, in particular the supports, become subvarieties of the s-dimensional torus $\mathbb{T}^s = (\mathbb{C}^*)^s$, which is regarded as the set of closed points in $\mathrm{Spec}(R_s)$. More precisely, for $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathbb{T}^s$, we denote by m_λ the corresponding maximal ideal in R_s and by $\mathbb{C}_\lambda$ the quotient $R_s/m_\lambda R_s$. This quotient is isomorphic to $\mathbb{C}$, and the canonical projection

$$\rho_\lambda : R_s \to R_s/m_\lambda R_s = \mathbb{C}_\lambda \tag{2.1}$$

corresponds to replacing t_j by λ_j for $j = 1, ..., s$. Here we regard $\mathbb{C}_\lambda$ as a (left) R_s-module, with an involution given by complex conjugation (which is compatible with the one induced from R_s since $\lambda_j \in \mathbb{T}^1$).

If A is an R_s-module, we denote be A_λ the localization of A at the maximal ideal m_λ. For $A = R_s$, we use the simpler notation R_λ when there is no danger of confusion. If A is of finite type, then $A = 0$ if and only if $A_\lambda = 0$ for all $\lambda \in \mathbb{T}^s$. More precisely

$$\mathrm{Supp}(A) = \{\lambda \in \mathbb{T}^s; A_\lambda \neq 0\}.$$

In particular $A_0(\mathcal{U}) = \mathbb{C}_{\mathbf{1}}$, where $\mathbf{1} = (1, \cdots, 1)$ and hence

$$\mathrm{Supp}(A_0(\mathcal{U})) = \{\mathbf{1}\}. \tag{2.2}$$

We denote by $V_{i,k}(\mathcal{U})$ the i-th characteristic variety associated with the homological Alexander module $A_k(\mathcal{U})$, and similarly denote by $\Delta_{i,k}(\mathcal{U})$ the associated characteristic polynomials. The notation $V^{i,k}(\mathcal{U})$ and $\Delta^{i,k}(\mathcal{U})$ denote the similar objects associated with the cohomological Alexander invariants $A^k(\mathcal{U})$.

2.3. **Homology versus cohomology Alexander modules.** It is natural to ask what are the relations between the homology and the cohomology universal Alexander modules, or to find the relations between $V_{i,k}(\mathcal{U})$ and $V^{i,k}(\mathcal{U})$; and between $\Delta_{i,k}(\mathcal{U})$ and $\Delta^{i,k}(\mathcal{U})$.

Some answers to this question can be given as follows. The cohomology modules may be related to the homology modules by the Universal Coefficient spectral sequence (see [30], p.20 or [31], Thm. 2.3):

$$\mathrm{Ext}^q_{R_s}(A_p(\mathcal{U}), R_s) \Rightarrow A^{p+q}(\mathcal{U}). \tag{2.3}$$

Using the exactness of the localization (see [51], p. 76), we get the following spectral sequence for any $\lambda \in \mathbb{T}^s$:

$$\mathrm{Ext}^q_{R_\lambda}(A_p(\mathcal{U})_\lambda, R_\lambda) \Rightarrow A^{p+q}(\mathcal{U})_\lambda. \tag{2.4}$$

For a fixed $\lambda \in \mathbb{T}^s$, we define

$$k(\lambda) = \min\{m \in \mathbb{N}; A_m(\mathcal{U})_\lambda \neq 0\}. \tag{2.5}$$

Then the spectral sequence (2.4) implies the following.

Proposition 2.3. *For any $\lambda \in \mathbb{T}^s$, $A^k(\mathcal{U})_\lambda = 0$ for $k < k(\lambda)$ and*

$$A^{k(\lambda)}(\mathcal{U})_\lambda = \mathrm{Hom}(A_{k(\lambda)}(\mathcal{U})_\lambda, R_\lambda). \tag{2.6}$$

This equality shows in particular that one may have $A^{k(\lambda)}(\mathcal{U})_\lambda = 0$, even when $A_{k(\lambda)}(\mathcal{U})_\lambda \neq 0$, e.g. when the last module is torsion, which is often the case, e.g. see (2.2).

2.4. **Multivariable versus one variable Alexander modules.** Consider a family of integral weights $\mathbf{e} = (e_1, \cdots, e_s) \in \mathbb{Z}^s$, and let

$$q := \text{g.c.d.}(e_1, \cdots, e_s).$$

Consider the morphism $p(\mathbf{e}) : R_s \to R_1$ defined by $t_i \mapsto t^{e_i}$, inducing a (left) R_s-module structure on R_1. Let $\mathcal{L}(\mathbf{e})$ be the local system on $\mathcal{U}$ with stalk R_1 and monodromy action for a loop $\alpha \in \pi_1(\mathcal{U})$ given by multiplication by $t^{\sum e_j \text{lk}(\alpha, V_j \cup -d_j H)}$.

The corresponding homology groups $H_k(\mathcal{U}, \mathcal{L}(\mathbf{e})) = H_k(C_*^0 \otimes_{R_s} R_1)$ are finite type R_1-modules, and hence they have associated characteristic varieties $V_{i,k}(\mathcal{U}, \mathbf{e})$ and Alexander polynomials $\Delta_{i,k}(\mathcal{U}, \mathbf{e})$.

It is natural to ask under which conditions the equalities

$$\Delta_{i,k}(\mathcal{U}, \mathbf{e})(t) = (t^q - 1)\Delta_{i,k}(\mathcal{U})(t^{e_1}, \cdots, t^{e_s})$$

do hold? Something like this works in classical knot theory, more precisely for oriented multilinks in S^3 with at least 2 components, where the case $i = 0$, $k = 1$ is considered (see [24], Prop. 5.1, and also [43], Lemma 10.1 for the case of weight $(1, \cdots, 1)$).

For the weight $\mathbf{1} = (1, 1, ..., 1)$, we call the corresponding Alexander polynomials the usual (or, univariable) Alexander polynomials and we denote them by $\Delta_{i,k}^T(\mathcal{U})$ (see below for some explanation).

If the equality in Question 2 holds for all but finitely many multi-indices $\mathbf{e}$, then the 1-variable polynomials $\Delta_{i,k}(\mathcal{U}, \mathbf{e})$ determine (up to a unit in R_s) the multivariable polynomial $\Delta_{i,k}(\mathcal{U})$ (see [3], Lemma 2.2).

Some insight into this question can be obtained as follows. We consider only the simplest case, namely $\mathbf{e} = \mathbf{1}$, and leave the other cases to the interested reader.

Note that the universal abelian covering $\mathcal{U}^{ab} \to \mathcal{U}$ corresponds to the kernel K^{ab} of the abelianization morphism

$$\pi_1(\mathcal{U}) \to H_1(\mathcal{U}).$$

The total linking number covering $\mathcal{U}^T \to \mathcal{U}$ corresponds to the kernel K^T of the morphism

$$\pi_1(\mathcal{U}) \to H_1(\mathcal{U}) = \mathbb{Z}^s \to \mathbb{Z},$$

where the second morphism is $\sum c_j \gamma_j \mapsto \sum c_j$. It follows that $\mathcal{U}^{ab} \to \mathcal{U}^T$ is a covering with deck transformation group $G = K^T / K^{ab}$ identified to the subgroup

$$\{c \in \mathbb{Z}^s; \sum c_j = 0\}.$$

The complex C_*^0 is a complex of free R_s-modules of finite rank, and the derivatives are R_s-linear. It follows that we can regard this complex as being a complex $\mathcal{C}_*^0$ of free $\mathcal{O}_{\mathbb{T}^s}$-modules on the affine variety $\mathbb{T}^s$.

Since $\mathcal{U}^T = \mathcal{U}^{ab}/G$, it follows that the complex of singular chains of $\mathcal{U}^T$ is

$$C_*(\mathcal{U}^T) = C_*(\mathcal{U}^{ab})_G = (C_*^0)_G \tag{2.7}$$

(see [51], p.204). Here

$$(C_p^0)_G = C_p^0 / \langle gm - m;\ g \in G,\ m \in C_p^0 \rangle. \tag{2.8}$$

Using the fact that the group G is generated by the elements having a 1 as the i-th coordinate, a -1 as the j-th coordinate (for $i < j$) and all the other coordinates zero, we see that $(C_p^0)_G$ is the quotient of C_p^0 by the submodule

$$\langle (t_i - t_j)m;\ m \in C_p^0 \rangle.$$

It follows that the associated sheaf $(\mathcal{C}_p^0)_G$ is just the restriction (as a coherent sheaf) of $\mathcal{C}_p^0$ to the 1-dimensional subtorus $S = \{(t,t,...,t) \in \mathbb{T}^s\}$, i.e. $(\mathcal{C}_p^0)_G = \mathcal{C}_p^0 \otimes_{\mathcal{O}_{\mathbb{T}^s}} \mathcal{O}_S$. Unfortunately, the inclusion $S \to \mathbb{T}^s$ is not a flat morphism (see [28], p. 254), and hence the restriction to S does not commute with taking homology.

However, by our discussion above,

$$(C_p^0)_G = C_p^0 \otimes_{R_s} R_1,$$

with the (left) R_s-module structure on R_1 induced by $p(\mathbf{1})$. Use now the Künneth spectral sequence (see [51], p.143), and get

$$E^2_{p,q} = Tor_p^{R_s}(A_q(\mathcal{U}), R_1) \Rightarrow H_{p+q}((C_*^0)_G) = A^T_{p+q}(\mathcal{U}). \tag{2.9}$$

For $a \in \mathbb{T}^1 = S = \{(t,t,...,t) \in \mathbb{T}^s\}$, we get by localization a new Künneth spectral sequence, namely,

$$E^2_{p,q} = Tor_p^{R_a}(A_q(\mathcal{U})_a, R_{1,a}) \Rightarrow H_{p+q}((C_*^0)_G)_a. \tag{2.10}$$

In particular we get the following.

Proposition 2.4. *For any $a \in \mathbb{T}^1$, $A_k^T(\mathcal{U})_a = 0$ for $k < k(a)$ and*

$$A_{k(a)}(\mathcal{U})_a \otimes_{R_a} R_{1,a} = A^T_{k(a)}(\mathcal{U})_a. \tag{2.11}$$

In particular, for any $a \in \mathbb{T}^1 = S$, the multiplicity of the root $t = a$ in the polynomials $\Delta^T_{i,k(a)}(\mathcal{U})(t)$ and $\Delta_{i,k(a)}(\mathcal{U})(t, \cdots, t)$ is the same.

Proof. To get the second claim, note that any presentation

$$R_a^m \to R_a^n \to A_{k(a)}(\mathcal{U})_a \to 0$$

yields by tensor product a presentation

$$R_{1,a}^m \to R_{1,a}^n \to A^T_{k(a)}(\mathcal{U})_a \to 0.$$

□

2.5. **Characteristic varieties as jumping loci of rank-1 local systems.** Let $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathbb{T}^s$ and denote by $\mathcal{L}_\lambda$ the local coefficient system on $\mathcal{U}$ with stalk $\mathbb{C} = \mathbb{C}_\lambda$ and action of a loop $\alpha \in \pi_1(\mathcal{U})$ determined by multiplication by $\prod_{j=1}^s (\lambda_j)^{\mathrm{lk}(\alpha, V_j \cup -d_j H)}$. We let $\mathcal{L}_\lambda^\vee \simeq \mathcal{L}_{\lambda^{-1}}$ be the dual local system, where $\lambda^{-1} := (\lambda_1^{-1}, \cdots, \lambda_s^{-1}) \in \mathbb{T}^s$.

One can define new *topological characteristic varieties* by setting

$$V^t_{i,k}(\mathcal{U}) = \{\lambda \in \mathbb{T}^s ; \dim H_k(\mathcal{U}, \mathcal{L}_\lambda) > i\}$$

and

$$V_t^{i,k}(\mathcal{U}) = \{\lambda \in \mathbb{T}^s ; \dim H^k(\mathcal{U}, \mathcal{L}_\lambda) > i\}.$$

It is natural to investigate the relations between the two types of characteristic varieties. Some cases are considered in [35], [36].

Here is a general approach to this question. It is known that

$$H_k(\mathcal{U}, \mathcal{L}_\lambda) = H_k(C_*^0 \otimes_{R_s} \mathbb{C}_\lambda).$$

Using the Künneth spectral sequence, we get

$$E^2_{p,q} = Tor_p^{R_s}(A_q(\mathcal{U}), \mathbb{C}_\lambda) \Rightarrow H_{p+q}(\mathcal{U}, \mathcal{L}_\lambda). \tag{2.12}$$

Now since the localization is exact, the base change for Tor under $R_s \to R_\lambda$ (see [51], p. 144), yields a new spectral sequence

$$E^2_{p,q} = Tor_p^{R_\lambda}(A_q(\mathcal{U})_\lambda, \mathbb{C}_\lambda) \Rightarrow H_{p+q}(\mathcal{U}, \mathcal{L}_\lambda). \tag{2.13}$$

This proves the first claim of the next result.

Proposition 2.5. *For any point $\lambda \in \mathbb{T}^s$, one has the following:*
(i) $\min\{m \in \mathbb{N},\ H_m(\mathcal{U}, \mathcal{L}_\lambda) \neq 0\} = \min\{m \in \mathbb{N},\ \lambda \in \mathrm{Supp}(A_m(\mathcal{U}))\} = k(\lambda)$*;*
(ii) $\dim H_{k(\lambda)}(\mathcal{U}, \mathcal{L}_\lambda) = \max\{m \in \mathbb{N},\ \lambda \in V_{m,k(\lambda)}(\mathcal{U})\}$*.*

Proof. To prove the second claim, note that the spectral sequence (2.13) yields

$$H_{k(\lambda)}(\mathcal{U}, \mathcal{L}_\lambda) = A_{k(\lambda)}(\mathcal{U})_\lambda / m_\lambda A_{k(\lambda)}(\mathcal{U})_\lambda.$$

Let n be the dimension of these two vector spaces. Then by Nakayama's Lemma, the module $A_{k(\lambda)}(\mathcal{U})_\lambda$ is generated by n elements over the local ring R_λ. In other words, there is the presentation

$$R_\lambda^m \to R_\lambda^n \to A_{k(\lambda)}(\mathcal{U})_\lambda \to 0.$$

Moreover, the first morphism is given by a matrix M whose entries m_{ij} are all in the maximal ideal m_λ. The second claim now follows by the definition of the characteristic varieties. □

Remark 2.6. Note that there is also a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^q_{R_\lambda}(A_p(\mathcal{U})_\lambda, \mathbb{C}_\lambda) \Rightarrow H^{p+q}(\mathcal{U}, \mathcal{L}_{\lambda^{-1}}). \tag{2.14}$$

Here $\mathbb{C}_\lambda$ is considered with the right R_s-module structure as indicated in Remark 2.2. This is why in the abutment of the spectral sequence (2.14), we obtain a cohomology with coefficients in the dual local system $\mathcal{L}_\lambda^\vee \simeq \mathcal{L}_{\lambda^{-1}}$. The above spectral sequence yields that $H^m(\mathcal{U}, \mathcal{L}_{\lambda^{-1}}) = 0$ for $m < k(\lambda)$ and $H^{k(\lambda)}(\mathcal{U}, \mathcal{L}_{\lambda^{-1}}) = \mathrm{Hom}_{R_\lambda}(A_{k(\lambda)}(\mathcal{U})_\lambda, \mathbb{C}_\lambda)$. However

$$\mathrm{Hom}_{R_\lambda}(A_{k(\lambda)}(\mathcal{U})_\lambda, \mathbb{C}_\lambda) = \mathrm{Hom}_{\mathbb{C}}(A_{k(\lambda)}(\mathcal{U})_\lambda / m_\lambda A_{k(\lambda)}(\mathcal{U})_\lambda, \mathbb{C}_\lambda)$$

and hence

$$H_{k(\lambda)}(\mathcal{U}, \mathcal{L}_\lambda)^* = H^{k(\lambda)}(\mathcal{U}, \mathcal{L}_{\lambda^{-1}}) \tag{2.15}$$

(compare [18], p.50 and p. 69). The case $k = 1$ of this useful formula was established in [41], Remark 5.2. Note that this formula holds over arbitrary fields, with the same proof as above.

Remark 2.7. All the results in this section so far hold for the local setting as well, i.e., when $\mathcal{U}$ is the complement of a hypersurface germ in a small ball. The first part of the example below corresponds to the germ of a normal crossing divisor. The second part of the example below corresponds to isolated non-normal crossing divisors (for short INNC); see [22], [36], [37].

Similarly, instead of localizing at a point, one may localize along the hyperplane H at infinity, i.e. replace $\mathcal{U}$ by $\mathcal{U}_\infty = \mathcal{U} \cap S_\infty$, where S_∞ is a large enough sphere in $\mathbb{C}^{n+1}$; see Theorem 3.1 below.

Example 2.8. (i) Let $\mathcal{U} = (\mathbb{C}^*)^s \times \mathbb{C}^{n+1-s}$ for some integer $0 \leq s \leq n+1$. Then the universal abelian covering $\mathcal{U}^{ab}$ is contractible, and then $A_0(\mathcal{U}) = \mathbb{C}_{\mathbf{1}}$ and $A_k(\mathcal{U}) = 0$ for $k > 0$. Therefore, by the spectral sequence (2.3) we get $A^k(\mathcal{U}) \cong \mathrm{Ext}^k_{R_s}(\mathbb{C}_{\mathbf{1}}, R_s)$ for all $k \geq 0$. Using the free resolution of $\mathbb{C}_{\mathbf{1}}$ given by the Koszul complex of the regular sequence $\{x_j = t_j - 1\}_{j=1,\dots,s}$ in the ring R_s ([51], Cor. 4.5.5), we obtain that

$A^k(\mathcal{U}) = 0$ for $k \neq s$ and $A^s(\mathcal{U}) = \mathbb{C}_{\mathbf{1}}$ ([51], Ex. 4.5.2 and Cor 4.5.4). Therefore, for any $\lambda \neq \mathbf{1}$, Proposition 2.3 shows that the corresponding cohomology Alexander modules satisfy $A^k(\mathcal{U})_\lambda = 0$ for any k. Moreover $H_k(\mathcal{U}, \mathcal{L}_\lambda) = H^k(\mathcal{U}, \mathcal{L}_{\lambda^{-1}}) = 0$ for any k.

(ii) Let $(Y, 0)$ be an INNC singularity at the origin of $\mathbb{C}^{n+1}$. Set $\mathcal{U}(Y,0) = B \setminus Y$, where B is a small open ball centered at the origin in $\mathbb{C}^{n+1}$. Assume that $n \geq 2$. Then the universal abelian cover $\mathcal{U}(Y,0)^{ab}$ of $\mathcal{U}(Y,0)$ is $(n-1)$-connected; see Libgober [36]. More precisely, it is a bouquet of n-spheres, see [22], and hence $A_0(\mathcal{U}(Y,0)) = \mathbb{C}_{\mathbf{1}}$ and $A_k(\mathcal{U}(Y,0)) = 0$ for $k \neq n$. As in (i) above, we get $A^k(\mathcal{U}(Y,0)) \cong \mathrm{Ext}^k_{R_s}(\mathbb{C}_{\mathbf{1}}, R_s)$ for all $k < n$. For $\lambda \neq \mathbf{1}$ this yields $A^k(\mathcal{U}(Y,0))_\lambda = 0$ for $k < n$, and therefore $H^k(\mathcal{U}(Y,0), \mathcal{L}_\lambda) = 0$ for any $k < n$.

3. Divisibility results and characteristic varieties

In this section we give an algebraic-geometrical interpretation for the multi-variable Alexander invariants of the hypersurface complement, similar in flavor to the one-variable case described in [42], but see also the reformulation of these results in [21]. We will use an approach based on the general theory of perverse sheaves, close to the one presented in [21] (see also [9] and [18]). Note that the supports and characteristic polynomials Δ_0 of the multi-variable Alexander modules are the analogue of the set of roots of the Alexander polynomials and respectively Alexander polynomials in the one-variable case (cf. [42], [21]).

The first result is an extension of [34], Theorem 3.2, to arbitrary hypersurface singularities . Let S_∞ be a sphere of sufficiently large radius in $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} \setminus H$ (or equivalently, the boundary of a sufficiently small tubular neighborhood of H in $\mathbb{CP}^{n+1}$). Let $V_\infty = S_\infty \cap V$ be the link of V at infinity, and $\mathcal{U}_\infty = S_\infty \setminus V_\infty$ its complement.

Theorem 3.1. *For all i, and all $k \leq n$: $V_{i,k}(\mathcal{U}) \subset V_{i,k}(\mathcal{U}_\infty)$, and $\Delta_{i,k}(\mathcal{U}) | \Delta_{i,k}(\mathcal{U}_\infty)$. Moreover, for $k < n$, these inclusions and divisibility conditions are replaced by equalities.*

Proof. The case $n = 1$ is considered in [34]. In fact in this situation one sets, for $i \leq 1$ and $k \leq 1$, $V_{i,k}(\mathcal{U}_\infty)$ to be the k-th characteristic variety of the i-th homology module of the covering space of $\mathcal{U}_\infty$ corresponding to the kernel of the composition

$$\pi_1(\mathcal{U}_\infty) \to \pi_1(\mathcal{U}) \to H_1(\mathcal{U}).$$

For $n \geq 2$, the theorem is an easy consequence of the Lefschetz hyperplane theorem. Indeed, as in the proof of Theorem 4.5 of [32], it follows that $\pi_1(\mathcal{U}) \cong \pi_1(\mathcal{U}_\infty)$, and more generally $\pi_k(\mathcal{U}, \mathcal{U}_\infty) \cong 0$ for all $k \leq n$. Therefore, the same is true for any covering, in particular for the universal abelian coverings: $\pi_k(\mathcal{U}^{ab}, \mathcal{U}^{ab}_\infty) \cong 0$ for all $k \leq n$. Hence, by the Hurewicz Theorem, the vanishing also holds for the relative homology groups, i.e., the maps of groups $H_k(\mathcal{U}^{ab}_\infty) \to H_k(\mathcal{U}^{ab})$ are isomorphisms for $k < n$ and onto for $k = n$. Since these maps are induced by an embedding (recall $n \geq 2$), the above are morphisms of modules over the ring of Laurent polynomials in s variables. The statement of the theorem follows now from Lemma 2.1. □

From now on to the end of this section, we will make the assumption that the hyperplane H at infinity is *transversal* (in the stratified sense) to the hypersurface V. With this assumption, we show that the global cohomological Alexander invariants of the hypersurface complement are entirely determined by the degrees of the irreducible components on the one hand, and by the local topological information encoded by the singularities of V on the other hand. In particular, these invariants depend on the local type of singularities of the hypersurface.

First, we need some notation. Recall from §2.2 that $A^q(\mathcal{U}) \cong H^q(\mathcal{U}, \mathcal{L}^\vee)$. For $x \in V$, we let $\mathcal{U}_x = \mathcal{U} \cap B_x$, for B_x a small open ball at x in $\mathbb{CP}^{n+1}$. Denote by $\mathcal{L}_x$ the restriction of the local coefficient system $\mathcal{L}$ to $\mathcal{U}_x$. Then the groups $H^*(\mathcal{U}_x, \mathcal{L}_x^\vee)$ inherit an R_s-module structure.

Theorem 3.2. *Let $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathbb{T}^s$ and $\epsilon \in \mathbb{Z}_{\geq 0}$. Fix an irreducible component V_1 of V, and assume that $\lambda \notin \operatorname{Supp}(H^q(\mathcal{U}_x, \mathcal{L}_x^\vee))$ for all $q < n+1-\epsilon$ and all points $x \in V_1$. Then $\lambda \notin \operatorname{Supp}(A^q(\mathcal{U}))$ for all $q < n+1-\epsilon$.*

Proof. Let $\mathcal{U}_1 = \mathbb{CP}^{n+1} \setminus V_1$, and let $i : \mathcal{U} \hookrightarrow \mathcal{U}_1$ and $j : \mathcal{U}_1 \hookrightarrow \mathbb{CP}^{n+1}$ be the two inclusions. Then $\mathcal{L}^\vee[n+1] \in \operatorname{Perv}(\mathcal{U})$, since $\mathcal{U}$ is smooth. Moreover $\mathcal{F} := Ri_*(\mathcal{L}^\vee[n+1]) \in \operatorname{Perv}(\mathcal{U}_1)$, since i is a quasi-finite affine morphism (see [48], Theorem 6.0.4). But $\mathcal{U}_1$ is affine $(n+1)$-dimensional, and $\mathcal{F} \in \operatorname{Perv}(\mathcal{U}_1)$; therefore by Artin's vanishing theorem for perverse sheaves (see [48], Corollary 6.0.4), the following hold:

$$\mathbb{H}^k(\mathcal{U}_1, \mathcal{F}) = 0, \text{ for all } k > 0,$$
$$\mathbb{H}^k_c(\mathcal{U}_1, \mathcal{F}) = 0, \text{ for all } k < 0.$$

Let $a : \mathbb{CP}^{n+1} \to point$ be the constant map. Then

$$\mathbb{H}^k(\mathcal{U}_1, \mathcal{F}) \cong H^{k+n+1}(\mathcal{U}, \mathcal{L}^\vee) \cong H^k(Ra_*Rj_*\mathcal{F})$$

and

$$\mathbb{H}^k_c(\mathcal{U}_1, \mathcal{F}) \cong H^k(Ra_!Rj_!\mathcal{F}).$$

Note that since a is a proper map, we have $Ra_! = Ra_*$.

Now consider the canonical morphism $Rj_!\mathcal{F} \to Rj_*\mathcal{F}$ and extend it to the distinguished triangle

$$Rj_!\mathcal{F} \to Rj_*\mathcal{F} \to \mathcal{G} \xrightarrow{[1]}$$

in $D^b_c(\mathbb{CP}^{n+1})$. Since $j^*j_! \cong id \cong j^*j_*$, the complex $\mathcal{G}$ is supported on V_1. Apply $Ra_! = Ra_*$ to the above distinguished triangle and obtain

$$Ra_!Rj_!\mathcal{F} \to Ra_*Rj_*\mathcal{F} \to Ra_*\mathcal{G} \xrightarrow{[1]} .$$

Upon applying the cohomology functor to this triangle, and using the above vanishing, we obtain that

$$H^{k+n+1}(\mathcal{U}, \mathcal{L}^\vee) \cong \mathbb{H}^k(\mathbb{CP}^{n+1}, \mathcal{G}) \cong \mathbb{H}^k(V_1, \mathcal{G}) \text{ for } k < -1,$$

and $H^n(\mathcal{U}, \mathcal{L}^\vee)$ is a submodule of $\mathbb{H}^{-1}(V_1, \mathcal{G})$.

Therefore, by Lemma 2.1, in order to prove the theorem it suffices to show that, under our assumptions, $\lambda \notin \operatorname{Supp}(\mathbb{H}^k(V_1, \mathcal{G}))$ for all $k < -\epsilon$. This follows from the local calculation and the hypercohomology spectral sequence. Indeed, for $x \in V_1$, we have

$$\begin{aligned}\mathcal{H}^q(\mathcal{G})_x &\cong \mathcal{H}^q(Rj_*\mathcal{F})_x \cong \mathcal{H}^{q+n+1}(Rj_*Ri_*\mathcal{L}^\vee)_x \cong \mathbb{H}^{q+n+1}(B_x, R(j \circ i)_*\mathcal{L}^\vee) \\ &\cong H^{q+n+1}(\mathcal{U}_x, \mathcal{L}_x^\vee),\end{aligned}$$

where $\mathcal{U}_x = \mathcal{U} \cap B_x$, for B_x a small open ball at x in $\mathbb{CP}^{n+1}$, and $\mathcal{L}_x$ is the restriction of the local coefficient system $\mathcal{L}$ to $\mathcal{U}_x$. Therefore, for a fixed $x \in V_1$ the assumption that $\lambda \notin \operatorname{Supp}(H^q(\mathcal{U}_x, \mathcal{L}_x^\vee))$ for all $q < n+1-\epsilon$ is equivalent to the assumption

$\lambda \notin \mathrm{Supp}(\mathcal{H}^q(\mathcal{G})_x)$ for all $q < -\epsilon$. Next note that $\mathbb{H}^k(V_1, \mathcal{G})$ is the abutment of a spectral sequence with the E_2-term defined by $E_2^{p,q} = H^p(V_1, \mathcal{H}^q(\mathcal{G}))$. Moreover, if $\lambda \notin \mathrm{Supp}(\mathcal{H}^q(\mathcal{G})_x)$ for all $q < -\epsilon$ and for all $x \in V_1$, then $\lambda \notin \mathrm{Supp}(H^p(V_1, \mathcal{H}^q(\mathcal{G})))$ for $p + q = k < -\epsilon$ (since $E_2^{p,q}$ is nontrivial only if $p \geq 0$). Thus, from the spectral sequence, it follows that $\lambda \notin \mathrm{Supp}(\mathbb{H}^k(V_1, \mathcal{G}))$ for all $k < -\epsilon$. This finishes the proof of the theorem. □

Remark 3.3. Theorem 3.2 leads to vanishing-type results for the global Alexander invariants of the hypersurface, as a consequence of the vanishing of (supports of) local Alexander invariants at singular points (e.g., see Corollary 3.5 below). In order to derive such a result, we need to clarify the relationship between the local modules $H^*(\mathcal{U}_x, \mathcal{L}_x^\vee)$ that appear in the statement of Theorem 3.2 on the one hand, and the local universal Alexander modules at a singular point x on the other hand. The latter are defined as in §2.2. More precisely, let $\mathcal{U}_0$ denote the hypersurface complement $\mathbb{CP}^{n+1} \setminus V$, and for $x \in V_1$ set $\mathcal{U}'_x = \mathcal{U}_0 \cap B_x$, for B_x a small open ball at x in $\mathbb{CP}^{n+1}$. Let $\mathcal{U}_x^{ab}$ and $(\mathcal{U}'_x)^{ab}$ be the universal abelian covers of $\mathcal{U}_x$ and $\mathcal{U}'_x$, respectively, and denote by $A_*(\mathcal{U}_x)$ and respectively $A_*(\mathcal{U}'_x)$ the associated universal homological Alexander modules. The modules $A_*(\mathcal{U}'_x)$ are called *the local universal homological Alexander modules at* x, as they depend only on the singularity germ (V, x).

We first relate $H^*(\mathcal{U}_x, \mathcal{L}_x^\vee)$ to the modules $A_*(\mathcal{U}_x)$, then express the latter in terms of the local universal Alexander modules at x.

If $i_x : \mathcal{U}_x \hookrightarrow \mathcal{U}$ denotes the inclusion map, then the local system $\mathcal{L}_x$ on $\mathcal{U}_x$ is induced via the composition of maps

$$\phi : \pi_1(\mathcal{U}_x) \overset{(i_x)_\#}{\to} \pi_1(\mathcal{U}) \overset{\mathrm{lk}}{\to} H_1(\mathcal{U}) \to \mathrm{Aut}(R_s).$$

On the other hand, by the naturality of the Hurewicz morphism, ϕ factors through $\mathrm{lk}_x : \pi_1(\mathcal{U}_x) \to H_1(\mathcal{U}_x)$, R_s becoming in this way a (left) $\mathbb{C}[H_1(\mathcal{U}_x)]$-module. Then, by [18], p. 50, it follows that $H^*(\mathcal{U}_x, \mathcal{L}_x^\vee)$ is the homology of the equivariant Hom:

$$C^*(\mathcal{U}_x, \mathcal{L}_x^\vee) = \mathrm{Hom}_{\mathbb{C}[H_1(\mathcal{U}_x)]}(C_*^0(\mathcal{U}_x^{ab}), R_s),$$

where R_s is regarded now as a right $\mathbb{C}[H_1(\mathcal{U}_x)]$-module using the involution on the group ring as in Remark 2.2, and as a left R_s-module. By [31], p.6, there is a spectral sequence converging to $H^*(\mathcal{U}_x, \mathcal{L}_x^\vee)$ with the E_2-term given by

$$E_2^{p,q} = \mathrm{Ext}^q_{\mathbb{C}[H_1(\mathcal{U}_x)]}(A_p(\mathcal{U}_x), R_s). \tag{3.1}$$

Thus each module $H^*(\mathcal{U}_x, \mathcal{L}_x^\vee)$ is built up entirely from information carried by the modules $A_*(\mathcal{U}_x)$. The latter are related to the local Alexander modules $A_*(\mathcal{U}'_x)$ by the following observations. For points $x \in V_1 \setminus (V_1 \cap H)$ we have $\mathcal{U}'_x = \mathcal{U}_x$; thus $A_*(\mathcal{U}_x) = A_*(\mathcal{U}'_x)$. For $x \in V_1 \cap H$, the transversality assumption implies that $\mathcal{U}_x$ is homotopy equivalent to $\mathcal{U}'_x \times S^1$. It follows that $\mathcal{U}_x^{ab} \simeq (\mathcal{U}'_x)^{ab} \times \mathbb{R}$; thus by the homological Künneth formula we obtain that the group $A_p(\mathcal{U}_x)$ is isomorphic to $H_p((\mathcal{U}'_x)^{ab}, \mathbb{C}) \otimes H_0(\mathbb{R}, \mathbb{C}) \cong A_p(\mathcal{U}'_x)$. When considering the $\mathbb{C}[H_1(\mathcal{U}_x)]$-module structure, the isomorphism can be written as (see [6], Prop. 1.8):

$$A_p(\mathcal{U}_x) \cong (A_p(\mathcal{U}'_x) \otimes_{\mathbb{C}[H_1(\mathcal{U}'_x)]} \mathbb{C}[H_1(\mathcal{U}_x)]) \otimes_{\mathbb{C}[\mathbb{Z}]} \mathbb{C}.$$

Together with the spectral sequence (3.1), this yields the desired relationship.

Remark 3.4. If S is an s-dimensional stratum in a Whitney stratification of V such that $x \in S$, then $A_p(\mathcal{U}'_x) = 0$ if $p > n - s$. Indeed, $\mathcal{U}'_x$ has the homotopy type of the link complement $S_x^{2n-2s+1} \setminus L_x$, where $S_x^{2n-2s+1}$ is a small sphere at x in a submanifold of $\mathbb{CP}^{n+1}$ which meets S transversally at x (and no other point), and $(S_x^{2n-2s+1}, L_x)$ is the link pair of the stratum S in the pair $(\mathbb{CP}^{n+1}, V)$. Since $S_x^{2n-2s+1} \setminus L_x$ admits a cyclic cover which has the homotopy type of a CW complex of dimension $n-s$ (i.e., the fiber of the Milnor fibration associated to the algebraic link $(S_x^{2n-2s+1}, L_x)$), it follows that the universal abelian cover $(\mathcal{U}'_x)^{ab}$ has the homotopy type of an $(n-s)$-dimensional CW complex, thus proving the claim.

The following consequence of Theorem 3.2, Remark 3.3, and of Example 2.8 is similar to some results in [22], [36], [37].

Corollary 3.5. (i) (*Case* $\epsilon = 0$) *With the notation in the above theorem, assume in addition that V is a normal crossing divisor at any point of the component V_1. Then* $\mathrm{Supp}(A^k(\mathcal{U})) \subset \{\mathbf{1}\}$ *for any $k < n+1$.*

(ii) (*Case* $\epsilon = 1$) *With the notation in the above theorem, assume in addition that V is an INNC divisor at any point of the component V_1. Then* $\mathrm{Supp}(A^k(\mathcal{U})) \subset \{\mathbf{1}\}$ *for any $k < n$.*

Using a similar argument (see also [21]) we obtain the following result.

Theorem 3.6. *Assume that the hypersurface V is transversal (in the stratified sense) to the hyperplane H at infinity. Then for $k \leq n$,* $\mathrm{Supp}(A^k(\mathcal{U}))$ *is contained in the zero set of the polynomial $t_1^{d_1} \cdots t_s^{d_s} - 1$, thus has positive codimension in $\mathbb{T}^s$.*

The positive codimension property of supports in the universal abelian case should be regarded as the analogue of the torsion property in the infinite cyclic case (cf. [42], [21]). Example 5.6 below shows that transversality except at finitely many points is not enough to get Theorem 3.6.

Proof. As in the proof of the previous theorem, after replacing $\mathcal{U}_1$ by the affine space $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} \setminus H$, it follows that for $k \leq -1$, $H^{k+n+1}(\mathcal{U}, \mathcal{L}^\vee)$ is a submodule of $\mathbb{H}^k(\mathbb{CP}^{n+1}, \mathcal{G})$, where $\mathcal{G}$ is now a complex of sheaves supported on H. Therefore, by Lemma 2.1, it suffices to prove the theorem for the supports of the modules $\mathbb{H}^k(H, \mathcal{G})$ with $k \leq -1$.

As in the previous theorem, for $x \in H$, the local calculation on stalks yields $\mathcal{H}^q(\mathcal{G})_x \cong H^{q+n+1}(\mathcal{U}_x, \mathcal{L}_x^\vee)$, where $\mathcal{U}_x = \mathcal{U} \cap B_x$, for B_x a small open ball at x in $\mathbb{CP}^{n+1}$. If $x \in H \setminus H \cap V$, then $\mathcal{U}_x$ is homotopy equivalent to $\mathbb{C}^*$, and the corresponding local system $\mathcal{L}_x^\vee$ is defined by the action of γ_∞, i.e., by multiplication by $\prod_{j=1}^s (t_j)^{d_j}$. On the other hand, if $x \in V \cap H$, then due to the transversality assumption, $\mathcal{U}_x$ is homotopy equivalent to a product $(B'_x \setminus V \cap B'_x) \times \mathbb{C}^*$, with B'_x a small open ball centered at x in H, and the local system $\mathcal{L}_x^\vee$ is an external tensor product, the second factor being defined by multiplication by $\prod_{j=1}^s (t_j)^{d_j}$. Thus, by the Kunneth spectral sequence, the stalk cohomology groups of $\mathcal{G}$ along H, i.e. $\mathcal{H}^q(\mathcal{G})_{x \in H}$, have supports contained in the zero set of the polynomial $t_1^{d_1} \cdots t_s^{d_s} - 1$. Then by the hypercohomology spectral sequence, the same is true for the supports of the hypercohomology groups $\mathbb{H}^k(H, \mathcal{G})$. □

4. Explicit computations via logarithmic connections

We review a general method used to determine the characteristic varieties in the case of hyperplane arrangements, see [25] and [47], and show that essentially the same method applies to more general situations as well.

Let $\pi : (Z, D) \to (\mathbb{CP}^{n+1}, V \cup H)$ be an embedded resolution of singularities for the reduced divisor $V \cup H$. In particular,

(i) D is a normal crossing divisor with smooth irreducible components;

(ii) $\pi : Z \setminus D \to \mathcal{U}$ is an isomorphism.

In this setting there is a Hodge–Deligne spectral sequence

$$E_1^{p,q} = H^q(Z, \Omega_Z^p(\log D)) \Rightarrow H^{p+q}(\mathcal{U}, \mathbb{C}) \tag{4.1}$$

degenerating at E_1 and inducing the Hodge filtration F of the Deligne mixed Hodge structure on $H^{p+q}(\mathcal{U}, \mathbb{C})$; see [13].

Corollary 4.1. *If the Deligne mixed Hodge structure on some cohomology space $H^m(\mathcal{U})$ is pure of type (m, m), then*

(i) $H^0(Z, \Omega_Z^m(\log D)) = H^m(\mathcal{U})$ *and*

(ii) $H^q(Z, \Omega_Z^p(\log D)) = 0$ *for* $p + q = m$ *and* $q > 0$.

We list below several cases when this property holds.

Example 4.2. (a) When V is a hyperplane arrangement, the cohomology space $H^m(\mathcal{U})$ is pure of type (m, m) for all $m \geq 0$; see [19].

(b) When V is a smooth rational curve arrangement in the projective plane (i.e., any irreducible component of V is either a line or a smooth conic), the cohomology space $H^m(\mathcal{U})$ is pure of type (m, m) for all $m \geq 0$ (easy exercise for the reader).

(c) $H^m(\mathcal{U})$ is always pure of type (m, m) for all $m \leq 1$. This follows from the fact that $g = (g_1, ..., g_s) : \mathcal{U} \to \mathbb{T}^s$ induces an isomorphism at the H^m-level for all $m \leq 1$. Here we look at $\mathcal{U}$ as a subset of $\mathbb{C}^{n+1}$ and we set $g_j(x_1, ..., x_{n+1}) = f_j(1, x_1, ..., x_{n+1})$.

For $\lambda = (\lambda_1, ..., \lambda_s) \in \mathbb{T}^s$, let $\mathcal{L}_\lambda$ be the corresponding local system on $\mathcal{U} = Z \setminus D$. Let $\alpha_j \in \mathbb{C}$ be such that $\exp(-2\pi i \alpha_j) = \lambda_j$ for $j = 1, ..., s$. Then $\mathcal{L}_\lambda$ is the local system of horizontal sections of the connection

$$\nabla_\alpha : \mathcal{O}_\mathcal{U} \to \Omega_\mathcal{U}^1$$

given by $\nabla_\alpha(u) = du + u \cdot \omega_\alpha$, where

$$\omega_\alpha = \sum_{j=1,s} \alpha_j \frac{dg_j}{g_j}.$$

Alternatively, if we look at $\mathcal{U}$ as a subset of $\mathbb{CP}^{n+1}$, then we can use the formula

$$\omega_\alpha = \sum_{j=0,s} \alpha_j \frac{df_j}{f_j} \tag{4.2}$$

where we set $\alpha_0 = -\sum_{j=1,s} d_j \cdot \alpha_j$. Recall that $f_0 = x_0$.

Using the fact that $\mathcal{U}$ is affine and our connection is regular, it follows that

$$H^m(\mathcal{U}, \mathcal{L}_\lambda) = H^m(H^0(\mathcal{U}, \Omega_\mathcal{U}^*), \nabla_\alpha) \tag{4.3}$$

just as in [18, Thm. 3.4.18] or, for complete proofs, [12]. However, this result is not so useful to perform explicit computations since the groups $H^0(\mathcal{U}, \Omega_\mathcal{U}^*)$ are too large.

There is a second approach to computing $H^m(\mathcal{U}, \mathcal{L}_\lambda)$, this time using *logarithmic connections*. It has the advantage of reducing the size of the spaces $H^0(\mathcal{U}, \Omega_\mathcal{U}^*)$, but one has to be more careful about the *residues* α_j. More precisely, the pull-back of the connection ∇_α under the embedded resolution π is a logarithmic connection

$\tilde{\nabla}_\alpha$ on Z with poles along D. Let ρ_i be the residue of the connection $\tilde{\nabla}_\alpha$ along the irreducible component D_i of D. When D_i is the proper transform of some component V_j of V one has $\rho_i = \alpha_j$.

Definition 4.3. A choice of residues $\alpha = (\alpha_0, \alpha_1, ..., \alpha_s)$ for $\mathcal{L}_\lambda$ as above is an *admissible choice of residues* for $\mathcal{L}_\lambda$ if $\rho_i \notin \mathbb{N}_{>0}$ for all irreducible components D_i of D. A rank one local system $\mathcal{L}_\lambda$ is *admissible* if there is some admissible choice of residues for it.

Remark 4.4. It is easy to see, using Hironaka's embedded resolution of singularities and by blowing-up smooth subvarieties, that for any i there is a relation

$$\rho_i = \sum_{j=1,s} n_{ij}\alpha_j$$

with $n_{ij} \in \mathbb{Z}$ (see [25] for similar formulas and note that negative coefficients occur due to the presence of the hyperplane at infinity). The condition $\rho_i \notin \mathbb{N}_{>0}$ is clearly satisfied if all α_j are sufficiently small. In other words, there is a neighborhood $U(\mathbf{1})$ of the trivial local system $\mathbf{1} \in \mathbb{T}^s$ formed entirely by admissible local systems.

If we move away from the trivial local system, it is not clear whether all the local systems are admissible. The answer to this question is negative for some hyperplane arrangements; see [7, Example 4.4], [5, Example 3.4], [38] and [49]. On the other hand, for not very complicated arrangements, see Examples 4.8 and 4.10 below, the answer is positive.

For an admissible choice of residues one has an E_1-spectral sequence

$$E_1^{p,q} = H^q(Z, \Omega_Z^p(\log D)) \Rightarrow H^{p+q}(\mathcal{U}, \mathcal{L}_\lambda) \tag{4.4}$$

whose differential d_1 is induced by $\tilde{\nabla}_\alpha$; see [18, Thm. 3.4.11 (i)]. The above discussion proves the following.

Proposition 4.5. *Assume that $\alpha = (\alpha_0, \alpha_1, ..., \alpha_s)$ is an admissible choice of residues for $\mathcal{L}_\lambda$ and that the cohomology groups $H^m(\mathcal{U})$ are pure of type (m, m) for all $m \leq k$. Then*

$$H^m(\mathcal{U}, \mathcal{L}_\lambda) = H^m(H^*(\mathcal{U}), \omega_\alpha \wedge)$$

for all $m \leq k$ and $H^{k+1}(H^*(\mathcal{U}), \omega_\alpha \wedge)$ is a subspace in $H^{k+1}(\mathcal{U}, \mathcal{L}_\lambda)$.

When $\mathcal{U}$ is a hyperplane arrangement complement, this is exactly the argument used in [25] and [47]. Proposition 4.5, Remark 4.4 and Example 4.2 yield the following.

Corollary 4.6. *If $\mathcal{U}$ is an affine hypersurface arrangement complement, then there is a neighborhood $U(\mathbf{1})$ of the trivial local system $\mathbf{1} \in \mathbb{T}^s$ such that*

$$H^1(\mathcal{U}, \mathcal{L}_\lambda) = H^1(H^*(\mathcal{U}), \omega_\alpha \wedge)$$

for any local system $\mathcal{L}_\lambda \in U(\mathbf{1})$, α being an arbitrary choice of admissible residues for $\mathcal{L}_\lambda$.

Corollary 4.7. *If $\mathcal{U} = M(\mathcal{A})$ is a hyperplane arrangement complement, then there is a neighborhood $U(\mathbf{1})$ of the trivial local system $\mathbf{1} \in \mathbb{T}^s$ such that*

$$H^m(\mathcal{U}, \mathcal{L}_\lambda) = H^m(H^*(\mathcal{U}), \omega_\alpha \wedge) \tag{4.5}$$

for any $m \in \mathbb{N}$, and any local system $\mathcal{L}_\lambda \in U(\mathbf{1})$, α being an arbitrary choice of admissible residues for $\mathcal{L}_\lambda$.

In relation to the isomorphism (4.5), we note that in the case of hyperplane arrangement complements the following inequality holds for any $m \in \mathbb{N}$ and any character $\lambda \in \mathbb{T}^s$ ([38], Proposition 4.2):

$$\dim H^m(\mathcal{U}, \mathcal{L}_\lambda) \geq \dim H^m(H^*(\mathcal{U}), \omega_\alpha \wedge).$$

However, the opposite inequality is false in general (see [49], Example 4.1).

Example 4.8. In the projective plane $\mathbb{CP}^2$ consider the hypersurface V having as irreducible components $V_1 : x = 0$, $V_2 : y = 0$, $V_3 : x^2 - yz = 0$. Let $H = V_0$ be the line at infinity given by $z = 0$ and note that H is *not transverse* in a stratified sense to V . Consider the connection ∇_λ whose residues are $\alpha = (\alpha_0, \alpha_1, ..., \alpha_3)$ with

$$\alpha_0 = -\alpha_1 - \alpha_2 - 2\alpha_3.$$

Let $A = V_1 \cap V_2 \cap V_3 = (0 : 0 : 1)$ and $B = V_1 \cap V_0 \cap V_3 = (0 : 1 : 0)$. To construct the embedded resolution of $V \cup H$ we first blow up the points A and B, creating thus two exceptional divisors, D_A and respectively D_B. The corresponding residues along D_A and D_B are easily computable and we get $\alpha_A = \alpha_1 + \alpha_2 + \alpha_3$ and respectively $\alpha_B = \alpha_1 + \alpha_0 + \alpha_3 = -\alpha_2 - \alpha_3$. Let $P = D_A \cap V_2' \cap V_3'$ and $Q = D_B \cap V_0' \cap V_3'$, where $'$ denotes the proper transform of a divisor. To get the embedded resolution of $V \cup H$ we just have to blow up the points P and Q, creating thus two new exceptional divisors, D_P and respectively D_Q. The corresponding residues are $\alpha_P = -\alpha_Q = \alpha_1 + 2\alpha_2 + 2\alpha_3$. Therefore the choice of residues $\alpha = (\alpha_0, \alpha_1, ..., \alpha_3)$ is admissible if and only if none of the residues

$$\alpha_1, \alpha_2, \alpha_3, -\alpha_1 - \alpha_2 - 2\alpha_3, \alpha_1 + \alpha_2 + \alpha_3, -\alpha_2 - \alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, -(\alpha_1 + 2\alpha_2 + 2\alpha_3)$$

is a strictly positive integer.

Lemma 4.9. *In the situation of Example* 4.8*, any rank one local system is admissible.*

Proof. It is clearly enough to consider the case of real residues α_j. Otherwise, we just look at the corresponding real parts.

We divide the possibilities into the following two cases.

Case 1. ($\alpha_1 + 2\alpha_2 + 2\alpha_3 \notin \mathbb{Z}$).

Suppose first that, in addition, $\alpha_1 + \alpha_2 + \alpha_3 \notin \mathbb{Z}$. Then the choice with $\alpha_j \in [0, 1)$ for $j = 1, 2, 3$ is admissible.

Now suppose that $\alpha_1 + \alpha_2 + \alpha_3 \in \mathbb{Z}$. It follows that $\alpha_2 + \alpha_3 \notin \mathbb{Z}$. Then the choice with $\alpha_j \in [0, 1)$ for $j = 2, 3$ and $\alpha_1 < 0$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 0$ is admissible.

Case 2. ($\alpha_1 + 2\alpha_2 + 2\alpha_3 \in \mathbb{Z}$).

Then we have to choose $\alpha_1 = -2\alpha_2 - 2\alpha_3$. The residues in this case are just

$$-2(\alpha_2 + \alpha_3),\ -(\alpha_2 + \alpha_3),\ \alpha_2,\ \alpha_3.$$

Hence it is enough to take $\alpha_j \in [0, 1)$ for $j = 2, 3$. □

Now we continue Example 4.8 by applying Example 4.2 and Proposition 4.5 to get $H^m(\mathcal{U}, \mathcal{L}_\lambda) = H^m(H^*(\mathcal{U}), \omega_\alpha \wedge)$ for all m. In order to perform this computation, we need a precise description of the cohomology algebra $H^*(\mathcal{U})$ (with $\mathbb{C}$ coefficients), and this can be obtained in this example from the local considerations in [16, pp. 47-49]. The result can be described as follows:

(i) $H^0(\mathcal{U}) = \mathbb{C}$ and the generator is 1;

(ii) $H^1(\mathcal{U}) = \mathbb{C}^3$ and a basis is given by $\eta_1 = \frac{dx}{x}$, $\eta_2 = \frac{dy}{y}$ and $\eta_3 = \frac{d(x^2-y)}{x^2-y}$;

(iii) $H^2(\mathcal{U}) = \mathbb{C}^2$ and a basis is given by $\eta_{12} = \eta_1 \wedge \eta_2$ and $\eta_{23} = \eta_2 \wedge \eta_3$. The multiplication is given by the relation

$$2\eta_1 \wedge \eta_3 = 2\eta_{12} + \eta_{23}.$$

(iv) Since $\mathcal{U}$ is affine, $H^m(\mathcal{U}) = 0$ for $m > 2$.

The computation of $H^m(H^*(\mathcal{U}), \omega_\lambda \wedge)$ falls into 3 cases.

Case 1. $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\mathcal{L}_\lambda = \mathbb{C}$ is the constant local system. Then of course $H^m(\mathcal{U}, \mathcal{L}_\lambda) = H^m(\mathcal{U})$ for all m.

Case 2. $\alpha_1+2\alpha_2+2\alpha_3 = 0$. Then a direct computation shows that $H^0(\mathcal{U}, \mathcal{L}_\lambda) = 0$ and $\dim H^1(\mathcal{U}, \mathcal{L}_\lambda) = \dim H^2(\mathcal{U}, \mathcal{L}_\lambda) = 1$.

Case 3. $\alpha_1+2\alpha_2+2\alpha_3 \neq 0$. Again a direct computation shows that $H^0(\mathcal{U}, \mathcal{L}_\lambda) = H^1(\mathcal{U}, \mathcal{L}_\lambda) = H^2(\mathcal{U}, \mathcal{L}_\lambda) = 0$.

The above computations yield the following equalities.

$$V_t^{0,1}(\mathcal{U}) = V_t^{0,2}(\mathcal{U}) = \{\lambda \in \mathbb{T}^3; \quad \lambda_1\lambda_2^2\lambda_3^2 = 1\},$$
$$V_t^{1,1}(\mathcal{U}) = V_t^{2,1}(\mathcal{U}) = V_t^{1,2}(\mathcal{U}) = \{\mathbf{1}\},$$

$V_t^{m,1}(\mathcal{U}) = \emptyset$ for $m > 2$ and $V_t^{m,2}(\mathcal{U}) = \emptyset$ for $m > 1$.

These results are consistent with the general results by Arapura [1]. See also Suciu [49] for a related discussion.

Note that the above 2-dimensional subtorus $\mathbb{T} = \{\lambda \in \mathbb{T}^3; \quad \lambda_1\lambda_2^2\lambda_3^2 = 1\}$ is different from the 2-dimensional subtorus predicted by Theorem 3.6 in the case of a divisor V transverse to the line at infinity.

A special class of local systems is formed by the equimonodromical local systems $\mathcal{L}_\lambda$ such that $\lambda_0 = \lambda_1 = ... = \lambda_3$. Then, for $\lambda_0^5 = 1$, the dimension of the cohomology space $H^m(\mathcal{U}, \mathcal{L}_\lambda)$ is exactly the multiplicity of the root $t = \lambda_0$ in the characteristic polynomial

$$\Delta^m(t) = \det(t \cdot Id - h^m),$$

where $F : xyz(x^2-yz) = 1$ is the associated Milnor fiber of $xyz(x^2-yz)$ in $\mathbb{C}^3$ and $h : F \to F$ is the monodromy operator; see for instance [18, 6.4.6]. To compute the cohomology of such an equimonodromical local system $\mathcal{L}_\lambda$, one should start by an admissible choice for the residues $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. For instance, the obvious choice $\alpha = (\frac{-4}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ is not admissible. A good choice here is $\alpha = (\frac{1}{5}, \frac{-4}{5}, \frac{1}{5}, \frac{1}{5})$. Using this choice, we get the following characteristic polynomials in this situation:

$$\Delta^0(t) = t-1, \ \Delta^1(t) = (t-1)^2(t^5-1), \ \Delta^2(t) = (t-1)(t^5-1).$$

The following example is similar to the previous one, but it exhibits a curve V which is transversal to the line H at infinity and it needs a different approach for the computation of the cohomology algebra $H^*(\mathcal{U})$. Moreover, in this case the cohomology algebra $H^*(\mathcal{U})$ is not spanned by the degree one part $H^1(\mathcal{U})$.

Example 4.10. In the projective plane $\mathbb{CP}^2$ consider the hypersurface V having as irreducible components $V_1 : x = 0$, $V_2 : y = 0$, $V_3 : x^2 - y^2 + yz = 0$. Let $H = V_0$ be the line at infinity given by $z = 0$ and note that H is *transverse* in a stratified sense to V (i.e., each irreducible component of V is smooth, H is transverse to each of them and avoids the intersection points). Consider the connection ∇_λ whose residues are $\alpha = (\alpha_0, \alpha_1, ..., \alpha_3)$ with

$$\alpha_0 = -\alpha_1 - \alpha_2 - 2\alpha_3.$$

Let $A = V_1 \cap V_2 \cap V_3 = (0:0:1)$. To construct the embedded resolution of $V \cup H$ we first blow up the point A, creating an exceptional divisor D_A. The corresponding residue along D_A is $\alpha_A = \alpha_1 + \alpha_2 + \alpha_3$. Let $P = D_A \cap V_2' \cap V_3'$, where $'$ denotes the proper transform of a divisor. To get the embedded resolution of $V \cup H$ we just have to blow up the point P, creating a new exceptional divisor D_P. The corresponding residue is $\alpha_P = \alpha_1 + 2\alpha_2 + 2\alpha_3$. Therefore the choice of residues $\alpha = (\alpha_0, \alpha_1, ..., \alpha_3)$ is admissible in this case if and only if none of the residues

$$\alpha_1, \alpha_2, \alpha_3, -\alpha_1 - \alpha_2 - 2\alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3$$

is a strictly positive integer. It can be shown, exactly as in Lemma 4.9 above, that in this situation any rank one local system is admissible.

It follows that we can apply Example 4.2 and Proposition 4.5 to get $H^m(\mathcal{U}, \mathcal{L}_\lambda) = H^m(H^*(\mathcal{U}), \omega_\lambda \wedge)$ for all m. To get a precise description of the cohomology algebra $H^*(\mathcal{U})$ we can proceed as follows.

(i) $H^0(\mathcal{U}) = \mathbb{C}$ and the generator is 1;

(ii) $H^1(\mathcal{U}) = \mathbb{C}^3$ and a basis is given by $\eta_1 = \frac{dx}{x}$, $\eta_2 = \frac{dy}{y}$ and $\eta_3 = \frac{d(x^2-y^2+y)}{x^2-y^2+y}$;

(iii) To compute $H^2(\mathcal{U})$ is the first difficulty. This can be done by setting $\mathcal{U}^0 = \mathbb{CP}^2 \setminus (V_0 \cup V_1 \cup V_2)$, $V_3^0 = V_3 \setminus (V_0 \cup V_1 \cup V_2)$ and considering the Gysin sequence

$$H^1(\mathcal{U}) \to H^0(V_3^0) \to H^2(\mathcal{U}^0) \to H^2(\mathcal{U}) \to H^1(V_3^0) \to 0.$$

The first morphism, given by the Poincaré–Leray residue R, is clearly surjective, i.e. $R(\eta_3) = 1$. Then $\dim H^2(\mathcal{U}^0) = 1$ and a generator is $\eta_{12} = \eta_1 \wedge \eta_2$. The affine curve V_3^0 is isomorphic to $\mathbb{C} \setminus \{-1, 0, 1\}$ under the parametrization

$$x = \frac{t}{t^2 - 1},\ y = \frac{t^2}{t^2 - 1}.$$

Using this parametrization, we can identify $H^1(V_3^0)$ with $\mathbb{C}^3$ by sending a rational differential form to its residues at the points $\{-1, 0, 1\}$. Some explicit computations involving the last nonzero morphism in the exact sequence above (which is again given by the Poincaré–Leray residue R) show that $R(\eta_{13})$ and $R(\eta_{23})$ are linearly independent in $H^1(V_3^0) = \mathbb{C}^3$, where $\eta_{13} = \eta_1 \wedge \eta_3$ and $\eta_{23} = \eta_2 \wedge \eta_3$. It follows that η_{12}, η_{13} and η_{23} are linearly independent in $H^2(\mathcal{U})$, which is 4-dimensional.

It follows that the following cases are possible in this example.

Case 1. $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\mathcal{L}_\lambda = \mathbb{C}$ is the constant local system. Then of course $H^m(\mathcal{U}, \mathcal{L}_\lambda) = H^m(\mathcal{U})$ for all m.

Case 2. $(\alpha_1, \alpha_2, \alpha_3) \neq (0,0,0)$. Then a direct computation shows that $H^0(\mathcal{U}, \mathcal{L}_\lambda) = H^1(\mathcal{U}, \mathcal{L}_\lambda) = 0$ and $\dim H^2(\mathcal{U}, \mathcal{L}_\lambda) = 2$.

The above computations yield the following equalities:

$$V_t^{0,2}(\mathcal{U}) = V_t^{1,2}(\mathcal{U}) = \mathbb{T}^3$$

(hence here the support has 0 codimension),

$$V_t^{1,1}(\mathcal{U}) = V_t^{2,1}(\mathcal{U}) = V_t^{2,2}(\mathcal{U}) = V_t^{3,2}(\mathcal{U}) = \{\mathbf{1}\},$$

$V_t^{m,1}(\mathcal{U}) = \emptyset$ for $m > 2$ and $V_t^{m,2}(\mathcal{U}) = \emptyset$ for $m > 3$. Note that the inclusion in Theorem 3.6 is strict in this case.

Consider as in the above example the associated Milnor fiber $F : xyz(x^2 - y^2 + yz) = 1$ and the monodromy operator $h : F \to F$. A good choice of residue is again given by $\alpha = (\frac{1}{5}, \frac{-4}{5}, \frac{1}{5}, \frac{1}{5})$. Using this choice, we get the following characteristic polynomials in this situation:

$$\Delta^0(t) = t - 1, \ \Delta^1(t) = (t-1)^3, \ \Delta^2(t) = (t-1)^2(t^5-1)^2.$$

Remark 4.11. In order to apply Theorem 3.2, we have to check the vanishing of some local cohomology groups. When the hypersurface germs occurring in these local complements are quasi-homogeneous, then we can globalize the local situation and compute the corresponding local cohomology groups using the ideas explained in this section. For instance, Example 4.8 covers the case of a plane curve singularity consisting of 3 smooth branches $(C_1, 0)$, $(C_2, 0)$ and $(C_3, 0)$ such that the intersection multiplicities are given by $(C_1, C_2) = 1$, $(C_1, C_3) = 1$ and $(C_2, C_3) = 2$. This follows from the topological classification of the plane curve germs; see [16, p. 45].

5. A more general setting

In this section we define multi-variable Alexander invariants in a more general setting (see below) and attempt to relate them to the invariants previously defined.

Assume that the hypersurface V in $\mathbb{CP}^{n+1}$ has s irreducible components V_i with degrees $\deg(V_i) = d_i$ for $i = 1, \cdots, s$. Denote by $\mathcal{U}_0$ the complement $\mathbb{CP}^{n+1} \setminus V$, and let $d = \text{g.c.d.}(d_1, \cdots, d_s)$. Then

$$H_1(\mathcal{U}_0) = \mathbb{Z}^{s-1} \oplus (\mathbb{Z}/d\mathbb{Z})$$

is generated by the meridians γ_i about the nonsingular part of each component V_i, for $i = 1, \cdots, s$ (cf. [16], (4.1.3)). These meridians satisfy a single relation, namely,

$$\sum_{i=1}^{s} d_i \gamma_i = 0.$$

Now fix a hyperplane H and set, as before, $\mathcal{U} = \mathbb{CP}^{n+1} \setminus (V \cup H)$. Recall that $H_1(\mathcal{U}) = \mathbb{Z}^s$, freely generated by the meridians γ_i, $i = 1, \cdots, s$. Let $i : \mathcal{U} \hookrightarrow \mathcal{U}_0$ be the inclusion map, and denote by $\mathcal{U}_0^{ab}$ and $\mathcal{U}^{ab}$ the universal abelian covers of $\mathcal{U}_0$ and $\mathcal{U}$ respectively, and by p_0 and p the corresponding covering projections.

The invariants we are interested in are those associated with $\mathcal{U}_0^{ab}$, and they are regarded as modules over the quotient ring

$$\mathbb{C}[H_1(\mathcal{U}_0)] = \mathbb{C}[t_1^{\pm 1}, \cdots, t_s^{\pm 1}]/(t_1^{d_1} \cdots t_s^{d_s} - 1).$$

It is a natural question to find the relation between the universal abelian invariants associated with the complement of V, and those associated with the complement of $V \cup H$.

For a topological space X, let $\mathcal{L}(X)$ denote the set of rank one complex local systems on X. When $X = \mathcal{U}$, then $\mathcal{L}(\mathcal{U})$ is naturally identified with the s-dimensional complex torus $\mathbb{T}^s$. For $X = \mathcal{U}_0$, the set $\mathcal{L}(\mathcal{U}_0)$ corresponds to the subset in $\mathbb{T}^s$ given by

$$\{\lambda = (\lambda_1, ..., \lambda_s) \in \mathbb{T}^s \ | \lambda_1^{d_1} \cdots \lambda_s^{d_s} = 1\}.$$

With the notation above, let $d_j = d \cdot d'_j$ and consider the $(s-1)$-dimensional complex subtorus

$$\mathbb{T} = \{\lambda = (\lambda_1, ..., \lambda_s) \in \mathbb{T}^s \ | \lambda_1^{d'_1} \cdots \lambda_s^{d'_s} = 1\}.$$

For each d-root of unity β, let $\lambda(\beta)$ be one point in the hypersurface in $\mathbb{T}^s$ given by the equation

$$\lambda_1^{d'_1} \cdots \lambda_s^{d'_s} = \beta.$$

Then $\mathcal{L}(\mathcal{U}_0)$ is precisely the disjoint union of translated tori given by

$$\mathcal{L}(\mathcal{U}_0) = \bigcup_{\beta} \lambda(\beta)\mathbb{T}.$$

The discussion in the previous section relating local systems to connections can be extended to this setting in an obvious way. For instance, we should now use the 1-form

$$\omega_\alpha = \sum_{j=1,s} \alpha_j \frac{df_j}{f_j}, \tag{5.1}$$

where the residues α satisfy the condition $\sum_{j=1,s} d_j \cdot \alpha_j = 0$, which is a necessary condition in order to have a 1-form on $\mathcal{U}_0$.

A different way of looking at a local system $\mathcal{L}$ in $\mathcal{L}(\mathcal{U}_0)$ is by considering it as a local system in $\mathcal{L}(\mathcal{U})$ (given by the obvious restriction $\mathcal{L}|\mathcal{U}$) such that the action of the elementary loop about the hyperplane H is trivial. This viewpoint yields the following exact sequence:

$$\cdots \to H^k(\mathcal{U}_0, \mathcal{L}) \to H^k(\mathcal{U}, \mathcal{L}) \to H^{k-1}(\mathcal{U}_0 \cap H, \mathcal{L}) \to H^{k+1}(\mathcal{U}_0, \mathcal{L}) \to \cdots;$$

for details on this see [18, pp. 221-222]. The following consequence should be compared to [46], [39, Proposition 1.3]. The higher-dimensional case, but with a generic hyperplane H at infinity, was considered in [32, Lemmas 1.5, 1.11 and 1.13].

Corollary 5.1. *Assume that V is a plane curve arrangement, i.e. $n = 1$. Then, for any rank one local system $\mathcal{L} = \mathcal{L}_\lambda$ on $\mathcal{U}_0$ and any choice of the line H at infinity, one has*

$$\dim H^1(\mathcal{U}, \mathcal{L}) = \dim H^1(\mathcal{U}_0, \mathcal{L}) + \epsilon.$$

Here $\epsilon \in \{0, 1\}$ and $\epsilon = 0$ if there is a point $p \in V \cap H$ such that

$$\prod_{j=1,s} \lambda_j^{k_j} \neq 1,$$

where $k_j = mult_p(V_j, H)$ is the intersection multiplicity of the component V_j and the line H at the point p.

Proof. We use the above exact sequence and get

$$0 \to H^1(\mathcal{U}_0, \mathcal{L}) \to H^1(\mathcal{U}, \mathcal{L}) \to H^0(\mathcal{U}_0 \cap H, \mathcal{L}) \to H^2(\mathcal{U}_0, \mathcal{L}) \to ...$$

The existence of a point p as stated implies that $H^0(\mathcal{U}_0 \cap H, \mathcal{L}) = 0$; hence clearly $\epsilon = 0$ as well. If there is no such point p, then the local system $\mathcal{L}|(\mathcal{U}_0 \cap H)$ is the trivial rank one local system $\mathbb{C}$ and hence $H^0(\mathcal{U}_0 \cap H, \mathcal{L}) = \mathbb{C}$. □

Remark 5.2. (added in proof) We point out that the converse statement in the second part of the above corollary is not true. Here is a counter-example. In $\mathbb{CP}^2$, let V be the union of the following four lines: $x = 0$, $x - z = 0$, $y = 0$ and $y - z = 0$. Choose the line at infinity H to be $z = 0$. Let $\mathcal{L}$ be the local system with monodromy -1 about any of the first four lines, and trivial monodromy 1 about H. Then the product of the monodromies at each of the two intersection points $a = (0 : 1 : 0)$ and $b = (1 : 0 : 0)$ is 1. On the other hand, both cohomology groups are trivial. For $H^1(\mathcal{U}_0, \mathcal{L})$, we can use our Theorem 3.2 above, while for $H^1(\mathcal{U}, \mathcal{L})$ we can use the Künneth Formula (see Theorem 4.3.14 in [18]) with $X = Y = \mathbb{C} \setminus \{0, 1\}$ and $\mathcal{F} = \mathcal{G}$ the rank one local system with monodromy -1 about the points 0 and 1.

One case which is already well explored is the following.

Example 5.3. Assume that $n > 1$, $s = 1$ and that $V = V_1$ is a hypersurface of degree d having only isolated singularities. Then $\pi_1(\mathcal{U}_0) = \mathbb{Z}/d\mathbb{Z}$ and hence a local system $\mathcal{L} = \mathcal{L}_\beta$ corresponds to a choice of a d-root of unity β. For $\beta = 1$ we get $H^0(\mathcal{U}_0; \mathbb{C}) = \mathbb{C}$ and $H^j(\mathcal{U}_0; \mathbb{C}) = 0$ for $0 < j < n$. When V is a $\mathbb{Q}$-manifold, one also has $H^n(\mathcal{U}_0; \mathbb{C}) = 0$. The computation of $H^n(\mathcal{U}_0; \mathbb{C}) \approx H_0^{n+1}(V)$ is quite difficult in general, as it may depend on the position of the singularities; see [16], [32]. Here $H_0^*(V)$ denotes the primitive cohomology of V, i.e., the cokernel of the natural monomorphism $H^*(\mathbb{CP}^{n+1}) \to H^*(V)$ induced by the inclusion of V into $\mathbb{CP}^{n+1}$.

For $\beta \neq 1$, one can use the isomorphism $H^m(\mathcal{U}_0, \mathcal{L}) = H^m(F, \mathbb{C})_\beta$, the β-eigenspace of the monodromy acting on the Milnor fiber F associated to V. In particular, $H^m(\mathcal{U}_0, \mathcal{L}) = 0$ for $m < n$. It is possible to construct examples such that for $m \in \{n, n+1\}$ one has

$$\dim H^m(\mathcal{U}_0, \mathcal{L}) > \dim H^m(\mathcal{U}_0, \mathbb{C}).$$

Indeed, consider the polynomials in [16, p. 148], which have a monodromy operator without the eigenvalue 1 on all the reduced cohomology groups $\tilde{H}^m(F, \mathbb{C})$ (equivalently, V has the same rational cohomology as $\mathbb{CP}^n$). It is not possible that $\tilde{H}^m(F, \mathbb{C}) = 0$ for all $m \in \mathbb{N}$, by A'Campo's result on the Lefschetz number of the monodromy; see [18, p. 174]. Hence there is some integer m and some d-root of unity $\beta \neq 1$ such that $\dim H^m(\mathcal{U}_0, \mathcal{L}_\beta) > 0 = \dim H^m(\mathcal{U}_0, \mathbb{C})$. Using the Euler characteristic equality $\chi(\mathcal{U}_0, \mathbb{C}) = \chi(\mathcal{U}_0, \mathcal{L}_\beta)$, it follows that the inequality should hold for the two possible values of m. By the minimality property of hyperplane arrangement complements, it is known that the above inequality is impossible for such complements, [23].

5.1. **Some 2-component arrangements.** We consider now in detail the case of hypersurface arrangements V with $s = 2$ irreducible components. We assume moreover that:

(i) $n > 1$ and each V_i has at most isolated singularities and is a $\mathbb{Q}$-manifold;

(ii) $V' = V_1 \cap V_2$ has at most isolated singularities; this condition is automatically fulfilled when $d_1 < d_2$ and V_2 is smooth, see [10].

Let $\mathcal{U}_i = \mathbb{CP}^{n+1} \setminus V_i$. Then the Mayer–Vietoris sequence of the covering $\mathcal{U}' = \mathcal{U}_1 \cup \mathcal{U}_2$ reads like

$$\text{(5.2)} \qquad ... \to H^{k-1}(\mathcal{U}_0) \to H^k(\mathcal{U}') \to H^k(\mathcal{U}_1) \oplus H^k(\mathcal{U}_2) \to H^k(\mathcal{U}_0) \to ....$$

Here and in the sequel the constant coefficients $\mathbb{C}$ are used unless stated otherwise. Using Example 5.3 to handle the cohomology groups $H^*(\mathcal{U}_i)$ for $i = 1, 2$ and the Alexander duality isomorphism (which is compatible with the MHS after taking the Tate twist $(-n-1)$, see for details [26])

$$\text{(5.3)} \qquad H^k(\mathcal{U}') = H^{2n+2-k}(\mathbb{CP}^{n+1}, V')^\vee(-n-1) = H_0^{2n+1-k}(V')^\vee(-n-1),$$

we get the following result.

Proposition 5.4. *With the above notation and assumptions, the following hold.*

(i) $H^0(\mathcal{U}_0) = \mathbb{C}$ *is pure of type* $(0,0)$ *and* $H^1(\mathcal{U}_0) = \mathbb{C}$ *is pure of type* $(1,1)$ *and it is spanned by the* 1*-form*

$$\omega_1 = d_2 \cdot \frac{df_1}{f_1} - d_1 \cdot \frac{df_2}{f_2}.$$

(ii) $H^k(\mathcal{U}_0) = 0$ *for* $1 < k < n$.

(iii) $H^n(\mathcal{U}_0)$ *is pure of weight* $n+2$ *and* $b_n(\mathcal{U}_0) \leq \dim H_0^n(V')$. *Moreover* $H^n(\mathcal{U}_0) = 0$ *if* $d_1 < d_2$ *and* V_2 *is smooth.*

(iv) $H^{n+1}(\mathcal{U}_0)$ *has weights* $\geq n+2$ *and one has an isomorphism of MHS*

$$H^{n+1}(\mathcal{U}_0)/W_{n+2}H^{n+1}(\mathcal{U}_0) = H_0^{n-1}(V')^{\vee}(-n-1).$$

Proof. The vanishing of $H^n(\mathcal{U}_0)$ in the third claim follows from an unexpected source. Indeed, the Gysin sequence of the smooth divisor $X_2 = V_2 \cap \mathcal{U}_1$ in $\mathcal{U}_1$ gives a monomorphism $H^n(\mathcal{U}_0) \to H^{n-1}(X_2)$. But this latter group $H^{n-1}(X_2)$ is trivial by some general connectivity results recently obtained by the first author; see [15].

Indeed, Theorem 1.1 in [15] applied to $V = V_2$ and $H = V_1$ yields the $(n-1)$-connectivity of the relative Milnor fiber

$$F_{rel} = \{x \in \mathbb{C}^{n+2} \mid f_2(x) = 0,\ f_1(x) = 1\}.$$

Since X_2 is the quotient of F_{rel} under the obvious action of the group of d_1-roots of unity, we get $H^{n-1}(X_2) = 0$.

The examples given in [15] show that the case $d_1 = d_2$ is much more complicated; in particular, the group $H^{n-1}(X_2)$ can be nonzero. Example 5.6 below shows that the assumption V_2 smooth cannot be relaxed to V_2 with isolated singularities and a $\mathbb{Q}$-manifold. The key point here is that the singularities of V_2 are situated on V_1, a situation not covered by the results in [15].

The only other claims that are not obvious are those on the MHS. They follow from the fact that $H_0^n(V')$ has a pure HS of weight n (the singularities of V' being isolated) and the following consequence of the Alexander duality (5.3)

$$h^{p,q}(H^k(\mathcal{U}')) = h^{n+1-p,n+1-q}(H_0^{2n+1-k}(V')). \tag{5.4}$$

□

For a rank one local system $\mathcal{L} \in \mathcal{L}(\mathcal{U}_0)$, we can choose the corresponding form ω_α to be a multiple $a(\alpha)\omega_1$ of the 1-form ω_1 introduced above. Then Propositions 4.5 and 5.4 yield the following.

Corollary 5.5. *For a nontrivial rank one local system* $\mathcal{L} \in \mathcal{L}(\mathcal{U}_0)$ *for which an admissible choice of residues* $\alpha = (d_2 \cdot a(\alpha), -d_1 \cdot a(\alpha))$ *exists, the following hold:*

(i) $H^k(\mathcal{U}_0, \mathcal{L}) = 0$ *for* $k < n$;

(ii) *if* $H_0^n(V') = 0$ *or if* $d_1 < d_2$ *and* V_2 *is smooth, then* $H^n(\mathcal{U}_0, \mathcal{L}) = 0$.

Note that the first claim above holds by Corollary 3.5, since V' has only INNC singularities.

The vanishing of $H_0^n(V')$ holds when V' is a $\mathbb{Q}$-homology manifold, but also in many other cases; see for instance the discussion in [16, pp. 207-216]. There one considers only the case when V_1 is a hyperplane. Indeed, any hypersurface W having only isolated singularities in $\mathbb{CP}^n$ can be obtained as the intersection of a smooth hypersurface V_2 in $\mathbb{CP}^{n+1}$ with the hyperplane $H = \mathbb{CP}^n$; see [14, p. 206].

However, this situation is usually uninteresting according to the second claim of the above corollary.

We conclude with an example where V_1 is a hyperplane and V_2 is singular, so that it may have been considered already in the previous section (in such a case $\mathcal{U}_0$ from this section is exactly $\mathcal{U}$ from the previous section, but for the hypersurface $V = V_2$!).

Example 5.6. In $\mathbb{CP}^3$ (with homogeneous coordinates $(x : y : z : t)$) consider the hyperplane $H = V_1 : t = 0$ and the surface $V_2 : xyz - t^3 = 0$. Then V_2 has exactly 3 singularities of type A_2; hence it is a $\mathbb{Q}$-manifold. Moreover, H is transverse to V_2, except at the 3 singular points of V_2.

To compute the cohomology of the complement $\mathcal{U}_0$, we use the Gysin exact sequence of the smooth divisor $D = V_2 \setminus V_1$ in the affine space $\mathbb{CP}^3 \setminus V_1$ (with coordinates (x, y, z)) and get

$$H^k(\mathcal{U}_0) = H^{k-1}(D)(-1)$$

for $k = 2, 3$, where (-1) denotes the Tate twist. Now D is given by the equation $xyz = 1$; hence it is a 2-dimensional torus. It follows that

(i) $H^2(\mathcal{U}_0) = \mathbb{C}^2$ is pure of type (2,2);

(ii) $H^3(\mathcal{U}_0) = \mathbb{C}$ is pure of type (3,3). Moreover, as explained in the fourth section, the 1-form ω_1 is a multiple of $\frac{dg}{g}$ with $g = xyz - 1$.

Let $g_0 = g + 1 = xyz$, and note that $F_{g_0} = D$ is the Milnor fiber of the homogeneous polynomial g_0. Since $\mathcal{U}_0 = \mathbb{C}^3 \setminus F_{g_0}$, we may use the description of the cohomology groups of $\mathcal{U}_0$ using Remark (2.11) in [16], p.192. By taking

$$\eta_i = A_i x dy \wedge dz - B_i dx \wedge dz + C_i dx \wedge dy$$

in the formula (2.12) loc. cit. with $(A_1, B_1, C_1) = (x, -y, 0)$ and $(A_2, B_2, C_2) = (0, -y, z)$ we get a basis of $H^2(\mathcal{U}_0)$. A direct computation then shows that $\omega_1 \wedge \eta_i = 0$ in $H^3(\mathcal{U}_0) = H^2(D)$. To see this, note that $d\eta_i = dg \wedge \eta_i = 0$ and hence the Poincaré–Leray residue of the form

$$\omega_1 \wedge \eta_i = \frac{dg}{g} \wedge \eta_i$$

is the form η_i. Since $d\eta_i = 0$ on $\mathbb{C}^3$, it follows that $\eta_i = d\eta_i'$, for some 1-forms η_i' on $\mathbb{C}^3$. Hence the cohomology class of η_i in $H^2(D)$ is trivial.

It follows that, for a nontrivial rank one local system $\mathcal{L} \in \mathcal{L}(\mathcal{U}_0)$ for which an admissible choice of residues α exists, one has $H^*(\mathcal{U}_0, \mathcal{L}) = H^*(H^*(\mathcal{U}_0), \omega_\alpha)$. Therefore we get the following equalities:

$$\dim H^0(\mathcal{U}_0, \mathcal{L}) = \dim H^1(\mathcal{U}_0, \mathcal{L}) = 0,\ \dim H^2(\mathcal{U}_0, \mathcal{L}) = 2,\ \text{and} \dim H^3(\mathcal{U}_0, \mathcal{L}) = 1.$$

In particular, $\mathrm{Supp} A^2(\mathcal{U}_0)$ coincides with the character torus $\mathbb{T}^1$. This follows since $\mathrm{Supp} A^2(\mathcal{U}_0)$ is a Zariski closed subset, with a nonempty interior by Remark 4.4, in the irreducible algebraic variety $\mathbb{T}^1$. The reader should compare this fact to Theorem 3.6 above.

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