On the Alexander invariants of hypersurface complements

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We survey few of the recent developments in the study of Alexander-type invariants associated to complex hypersurface complements, and point out the dependence of such invariants on the local type and position of singularities of the hypersurface.

Keywords: hypersurface complement; non-isolated singularities; Alexander polynomial; intersection homology; monodromy; characteristic varieties.

1. Introduction

In this mostly expository note, we survey few of the recent developments in the study of the topology of hypersurface complements. Most of the results outlined here are contained in [33] and [14].

The study of plane singular curves is a subject going back to the work of Zariski, who observed that the position of singularities has an influence on the topology of the curve, and this phenomena can be detected by the fundamental group of the complement. However, the fundamental group of a plane curve complement is in general highly non-commutative, thus difficult to handle. On the other hand, Alexander invariants of the complement are more manageable, and turn out to be also sensitive to the position of singularities.

Alexander invariants appeared first in the classical knot theory, where it was noted that in order to study a knot, it is useful to consider the topology of its complement. By analogy with knot theory, Libgober [20–23] introduced and studied Alexander-type invariants for the total linking number infinite cyclic cover of complements to affine complex hypersurfaces. For hypersurfaces with only isolated singularities, he showed that there is essentially only one interesting global Alexander invariant, which depends on the local type and the position of singularities. In [5], twisted Alexander invariants of plane algebraic curves are shown to have a similar property, and examples of curves with trivial Alexander polynomial, but non-trivial twisted Alexander polynomials are given.

It is a natural question to ask how such invariants behave if the hypersurface is allowed to have more general singularities. If the hypersurface is reducible, one has to distinguish between Alexander-type invariants associated to an infinite cyclic cover of the complement on one hand, and to the universal abelian cover on the other hand. The relation between invariants of the total linking number infinite cyclic cover and the topology of the polynomial function defining the (affine) hypersurface, mainly reflected by the monodromy and the vanishing cycles, was also considered in [11].

In [33], we extend Libgober's results on the infinite cyclic Alexander invariants to the case of hypersurfaces with non-isolated singularities and in general position at infinity. It turns out that the infinite cyclic Alexander modules of the complement can be realized as intersection homology modules of the ambient projective space (obtained by adding the hyperplane at infinity), with a certain local coefficient system defined on the complement. This new approach allows the use of techniques from homological algebra (e.g., derived categories and perverse sheaves) in showing that most of these Alexander modules are torsion over the ring of complex Laurent polynomials. Moreover, the associated global Alexander polynomials are entirely determined by the local topological information encoded by the link pairs of singular strata of the hypersurface. Similar methods can be used to obtain obstructions on the eigenvalues of the monodromy operators associated to the Milnor fiber of a projective hypersurface arrangement. The Alexander invariants of the total linking number infinite cyclic cover are further studied in [12], where it is shown that there is a natural mixed Hodge structure on the Alexander modules of the complement.

The analogy with link complements in S^3 is reflected in [28], where invariants of the universal abelian cover of a plane curve complement are considered. In was observed in [25] that such invariants depend on the local type of singularities, and they are calculated in terms of position of singularities of the curve in the plane. More general situations are studied in [13,26,27]. In [14], we show that the universal abelian invariants of complements to hypersurfaces with any kind of singularities, are also determined by the corresponding local invariants associated with singular strata of the hypersurface.

2. Infinite cyclic Alexander invariants of the complement

In this section we study Alexander invariants associated to the total linking number infinite cyclic cover of a hypersurface complement. We first recall Libgober's results for the case of hypersurfaces with only isolated singularities, and then show how to extend his results to hypersurfaces with arbitrary singularities and in general position at infinity (for complete details, see [33]).

2.1. *Preliminaries*

To fix notations for the rest of the paper, let V be a reduced hypersurface in $\mathbb{CP}^{n+1}$, defined by a degree d homogeneous equation: $f = 0$. Let f_i, $i = 1, \cdots, s$ be the irreducible factors of f and $V_i = \{f_i = 0\}$ the corresponding irreducible components of V. Throughout this paper, we will assume that V is *in general position at infinity*, that is, we choose a generic hyperplane H_∞ (transversal to all singular strata in a stratification of V) which we call 'the hyperplane at infinity'. Let $\mathcal{U}$ be the (affine) hypersurface complement: $\mathcal{U} = \mathbb{CP}^{n+1} - (V \cup H_\infty)$.

Then $H_1(\mathcal{U}) \cong \mathbb{Z}^s$ (cf. [9], (4.1.3), (4.1.4)), generated by meridian loops γ_i about the non-singular part of each irreducible component V_i, $i = 1, \cdots, s$. Moreover, if γ_∞ denotes a meridian about the hyperplane at infinity, then there is a relation in $H_1(\mathcal{U})$: $\gamma_\infty + \sum d_i \gamma_i = 0$, where $d_i = deg(V_i)$.

We consider the infinite cyclic cover $\mathcal{U}^c$ of $\mathcal{U}$ defined by the kernel of the total linking number homomorphism $Lk : \pi_1(\mathcal{U}) \to \mathbb{Z}$, which maps all meridian generators to 1, and thus any loop α to the linking number $\mathrm{lk}(\alpha, V \cup -dH_\infty)$ of α with the divisor $V \cup -dH_\infty$ in $\mathbb{CP}^{n+1}$. By definition, the *infinite cyclic Alexander modules of the hypersurface complement* are the homology groups $H_i(\mathcal{U}^c; \mathbb{C})$, regarded as $\Gamma := \mathbb{C}[t, t^{-1}]$-modules, where t acts as the canonical covering transformation.

Note that Γ is a principal ideal domain, hence any torsion Γ-module M of finite type has a well-defined associated order (see [35]). This is called the *Alexander polynomial* of M, and is denoted by $\Delta_M(t)$. We regard the trivial module as a torsion module whose associated polynomial is 1. It is easy to see that if $f : M \to N$ is an epimorphism of Γ-modules and M is torsion of finite type, then N is also torsion of finite type and $\Delta_N(t)$ divides $\Delta_M(t)$.

In studying the Alexander modules of the complement we first note that, since $\mathcal{U}$ has the homotopy type of a finite CW complex of dimension

$\leq n+1$ (e.g., see [9] (1.6.7), (1.6.8)), all the associated Alexander modules are of finite type over Γ, but in general not over $\mathbb{C}$. It also follows that $H_i(\mathcal{U}^c;\mathbb{C}) = 0$ for $i > n+1$, and $H_{n+1}(\mathcal{U}^c;\mathbb{C})$ is free over Γ. Thus, of particular interest are the Alexander modules $H_i(\mathcal{U}^c;\mathbb{C})$ for $i < n+1$.

In [21], Libgober showed that if V is a hypersurface with only isolated singularities, then $\tilde{H}_i(\mathcal{U}^c;\mathbb{Z}) = 0$ for $i < n$, and $H_n(\mathcal{U}^c;\mathbb{C})$ is a torsion Γ-module. Moreover, if $\Delta_n(t)$ is the polynomial associated to the torsion module $H_n(\mathcal{U}^c;\mathbb{C})$, then $\Delta_n(t)$ divides (up to a power of $(t-1)$) the product $\prod_{x \in \mathrm{Sing(V)}} \Delta_x(t)$ of the Alexander polynomials of link pairs around the isolated singular points $x \in V$. This shows the dependence of the Alexander polynomial on the local type of singularities of V. If V is a rational homology manifold, then $\Delta_n(1) \neq 0$. As in the case of a homogeneous isolated hypersurface singularity germ, the zeros of $\Delta_n(t)$ are roots of unity of order $d = deg(V)$ and $H_n(\mathcal{U}^c;\mathbb{C})$ is a semi-simple Γ-module (cf. [21]).

As shown by Zariski, the fundamental group of curve complements in $\mathbb{CP}^2$ is sensitive also to the position of singularities. (Note that by a Zariski-Lefchetz type theorem, cf. [9] p. 25, the class of fundamental groups to curve complements coincides with the class of fundamental groups of the complements to hypersurfaces in a projective space.) In [21–24], Libgober observed that the Alexander invariant of an irreducible curve in $\mathbb{CP}^2$ (or more generally, the 'first non-trivial' Alexander invariant of a hypersurface complement) exhibits a similar property.
As an example, let $C \subset \mathbb{CP}^2$ be a sextic with only cusps singularities. Then by the above divisibility result, the global Alexander polynomial $\Delta_1(t)$ of the curve C is either 1 or a power of t^2-t+1. The influence of the position of singularities can be seen as follows: if C has only 6 cusps then ([21,22]):

(1) if C is in 'special position', i.e., the 6 cusps are on a conic, then $\Delta_1(t) = t^2 - t + 1$.
(2) if C is in 'general position', i.e., the cusps are not on a conic, then $\Delta_1(t) = 1$.

Note. Libgober's divisibility theorem ([21], Theorem 4.3) holds for hypersurfaces with isolated singularities, *including at infinity.* However, for non-generic H_∞ and for hypersurfaces with more general singularities, the Alexander modules $H_i(\mathcal{U}^c;\mathbb{C})$ $(i \leq n)$ are not torsion in general. Their Γ-rank is calculated in [11]. One of the main results in [33] asserts that if V is a reduced hypersurface in general position at infinity, then the modules $H_i(\mathcal{U}^c;\mathbb{C})$ are torsion Γ-modules for all $i \leq n$. We will discuss this aspect and related results in the next section.

2.2. *Intersection homology approach*

Our approach to the study of the infinite cyclic Alexander invariants of the complement makes use of intersection homology theory ([3,15,16]) and the foundational work of Cappell-Shaneson [4] on the study of pseudomanifolds, pl-embedded in codimension two into a manifold. We will use freely the background material from these references (but see also [32], §2, for a quick overview).

Following [4], it is possible to think of a n-dimensional projective hypersurface V as the singular locus of $\mathbb{CP}^{n+1}$, which is now regarded as a filtered space stratified by V and the strata of its singularities. This yields a regular stratification of the pair $(\mathbb{CP}^{n+1}, V)$. Due to the transversality assumption we may also consider the induced stratification for the pair $(\mathbb{CP}^{n+1}, V \cup H_\infty)$. Let $\mathcal{L}$ be a locally constant sheaf on $\mathcal{U}$, with stalk $\Gamma := \mathbb{C}[t, t^{-1}]$ and action by an element $\alpha \in \pi_1(\mathcal{U})$ determined by multiplication by $t^{\mathrm{lk}(\alpha, V \cup -dH_\infty)}$. Then, for any perversity $\bar{p}$, the intersection homology complex $\mathcal{IC}^\bullet_{\bar{p}} := \mathcal{IC}^\bullet_{\bar{p}}(\mathbb{CP}^{n+1}, \mathcal{L})$ is defined by Deligne's axiomatic construction as in [3,16]. (Through this section, we make use of the indexing convention of [16].) The *intersection Alexander modules of the hypersurface* V are then defined as hypercohomology groups of the middle-perversity intersection homology complex:

$$IH^{\bar{m}}_i(\mathbb{CP}^{n+1}; \mathcal{L}) := \mathbb{H}^{-i}(\mathbb{CP}^{n+1}; \mathcal{IC}^\bullet_{\bar{m}}), \quad i \in \mathbb{Z}.$$

Note that, in our setting, the following *superduality isomorphism* holds (cf. [4], Theorem 3.3):

$$\mathcal{IC}^\bullet_{\bar{m}} \cong \mathcal{DIC}^{\bullet\, op}_{\bar{l}}[2n+2] \tag{1}$$

(here $\mathcal{D}(\mathcal{A}^\bullet)$ is the Verdier-dual to the complex $\mathcal{A}^\bullet$, and A^{op} is the Γ-module obtained from the Γ-module A by composing all module structures with the involution $t \to t^{-1}$.) Recall that the middle and logarithmic perversities are defined by: $\bar{m}(s) = [(s-1)/2]$ and $\bar{l}(s) = [(s+1)/2]$.

The assumption on the position with respect to the hyperplane at infinity is crucial in proving the following technical but important fact:

Lemma 2.1. *([33]) If $i : V \cup H_\infty \hookrightarrow \mathbb{CP}^{n+1}$ is the inclusion, then $i^*\mathcal{IC}^\bullet_{\bar{m}}$ is quasi-isomorphic to the zero complex, i.e. the cohomology stalks of the complex $\mathcal{IC}^\bullet_{\bar{m}}$ vanish at points in $V \cup H_\infty$.*

As a corollary, we obtain the intersection homology realization of the infinite cyclic Alexander modules of hypersurface complements:

Theorem 2.1. *([33]) There is an isomorphism of* Γ*-modules:*

$$IH_*^{\bar{m}}(\mathbb{CP}^{n+1};\mathcal{L}) \cong H_*(\mathcal{U};\mathcal{L}) \cong H_*(\mathcal{U}^c;\mathbb{C}).$$

So the intersection Alexander modules of the hypersurface are isomorphic to the infinite cyclic Alexander modules of the hypersurface complement.

At this point, we can use freely the language of derived categories, derived functors etc., in order to describe the infinite cyclic Alexander invariants. Our first main application is the following (recall that V is assumed to be transversal to the hyperplane at infinity):

Theorem 2.2. *([33]) For any* $i \leq n$*, the group* $H_i(\mathcal{U}^c;\mathbb{C})$ *is a finitely generated torsion* Γ*-module.*

Proof. First, by the superduality isomorphism (1) and Lemma 2.1 we obtain the quasi-isomorphism: $i^!\mathcal{IC}^\bullet_{\bar{l}} \overset{q.i.}{\cong} 0$. Therefore:

$$IH_i^{\bar{l}}(\mathbb{CP}^{n+1};\mathcal{L}) \cong H_i^{BM}(\mathcal{U};\mathcal{L}) \cong 0 \quad \text{if} \quad i < n+1, \tag{2}$$

where H_*^{BM} stands for the Borel-Moore homology. The vanishing in (2) follows by Artin's theorem ([36], Example 6.0.6) applied to the $(n+1)$-dimensional affine variety $\mathcal{U}$.

Now, recall that the *peripheral complex*, $\mathcal{R}^\bullet$, associated to the finite local type embedding $V \cup H_\infty \subset \mathbb{CP}^{n+1}$, is a torsion complex (i.e. the cohomology stalks $\mathcal{H}^q(\mathcal{R}^\bullet)_x$ are finite dimensional $\mathbb{C}$-vector spaces, for all $x \in \mathbb{CP}^{n+1}$) and its hypercohomology fits into a long exact sequence (for more details, see [4], p. 339-340):

$$\cdots \to \mathbb{H}^q(\mathbb{CP}^{n+1};\mathcal{IC}^\bullet_{\bar{m}}) \to \mathbb{H}^q(\mathbb{CP}^{n+1};\mathcal{IC}^\bullet_{\bar{l}}) \to \mathbb{H}^q(\mathbb{CP}^{n+1};\mathcal{R}^\bullet) \to$$
$$\to \mathbb{H}^{q+1}(\mathbb{CP}^{n+1};\mathcal{IC}^\bullet_{\bar{m}}) \to \cdots$$

By the hypercohomology spectral sequence, the groups $\mathbb{H}^*(\mathbb{CP}^{n+1};\mathcal{R}^\bullet)$ are finite dimensional complex vector spaces, hence torsion Γ-modules. Thus, our claim follows from the above long exact sequence and the vanishing for the logarithmic complex $\mathcal{IC}^\bullet_{\bar{l}}$ in (2). □

Note that if $i \leq n$, the Γ-module $H_i(\mathcal{U}^c;\mathbb{C})$ is actually a finite dimensional complex vector space, thus its order coincides with the characteristic polynomial of the $\mathbb{C}$-linear map induced by a generator of the group of covering transformations (see [35]). It is shown in [12,33] that this map is $\mathbb{C}$-diagonalizable, thus the Γ-module $H_i(\mathcal{U}^c;\mathbb{C})$ is semi-simple.

Definition 2.1. For $i \leq n$, we denote by $\Delta_i(t)$ the polynomial associated to the torsion Γ-module $H_i(\mathcal{U}^c;\mathbb{C})$, and call it the i-th global Alexander polynomial of the hypersurface V.

These polynomials are well-defined up to multiplication by ct^k ($c \in \mathbb{C}$, $k \in \mathbb{Z}$).

As a consequence of Theorem 2.2, for hypersurfaces in general position at infinity we may now calculate the rank of the free Γ-module $H_{n+1}(\mathcal{U}^c;\mathbb{C})$ in terms of the Euler characteristic of the complement:

Corollary 2.1.

$$rank_{\Gamma} H_{n+1}(\mathcal{U}^c;\mathbb{C}) = (-1)^{n+1}\chi(\mathcal{U}).$$

Another interesting property of the infinite cyclic Alexander invariants is that they depend on the degree d of the hypersurface. More precisely:

Theorem 2.3. *([33], Theorem 4.1)*
For $i \leq n$, all zeros of the global Alexander polynomial $\Delta_i(t)$ are roots of unity of order d.

This is a generalization of a similar result obtained by Libgober in the case of hypersurfaces with only isolated singularities (cf. [21], Corollary 4.8).

The last two theorems, 2.2 and 2.3, show a striking similarity between the case of hypersurfaces in general position at infinity and that of a homogeneous singularity germ (in which case the total linking number infinite cyclic cover may be replaced by the Milnor fiber). We will discuss this relation in some detail in the next section.

But perhaps the most important consequence of Theorem 2.1 is the dependence of the infinite cyclic cover Alexander invariants on the local type of singularities of the hypersurface. This is in the spirit of early results of Zariski for the fundamental group of the complement, and those of Libgober for the Alexander invariants of hypersurfaces with only isolated singularities (see [20–24]).
We first need some notation. Let $\mathcal{S}$ be a Whitney stratification of V, and consider the induced Whitney stratification of the pair $(\mathbb{CP}^{n+1}, V)$, with $\mathcal{S}$ the set of singular strata. If $S \in \mathcal{S}$ is an s-dimensional stratum of $(\mathbb{CP}^{n+1}, V)$, then a point $p \in S$ has a distinguished neighborhood W in $(\mathbb{CP}^{n+1}, V)$, which is homeomorphic in a stratum-preserving way to

$\mathbb{C}^s \times c^\circ(S^{2n-2s+1}(p), L(p))$, for $S^{2n-2s+1}(p)$ a small sphere at p in a normal slice for S and $L(p) = S^{2n-2s+1}(p) \cap V$. The link pair $(S^{2n-2s+1}(p), L(p))$ has constant topological type along the stratum S, which we denote by $(S^{2n-2s+1}, L)$.

Now fix an arbitrary irreducible component of V, say V_1. For $S \in \mathcal{S}$, an s-dimensional stratum contained in V_1, let $(S^{2n-2s+1}, L)$ be its link pair in $(\mathbb{CP}^{n+1}, V)$. This is a (possibly singular) algebraic link, and has an associated local Milnor fibration ([34]):

$$F^s \hookrightarrow S^{2n-2s+1} - L \to S^1$$

with fibre F^s and monodromy homeomorphism $h^s : F^s \to F^s$. Let $\Delta_r^s(t) = \det(tI - (h^s)_* : H_r(F^s) \to H_r(F^s))$ be the r-th (local) Alexander polynomial associated to S. Then we have the following divisibility result:

Theorem 2.4. *([33], Theorem 4.2)*
Fix $i \leq n$, and let V_1 be a fixed irreducible component of V. Then the prime divisors of $\Delta_i(t)$ are among the divisors of the local polynomials $\Delta_r^s(t)$ associated to strata $S \subset V_1$, such that $n - i \leq s = dimS \leq n$, and with r satisfying $2n - 2s - i \leq r \leq n - s$. Moreover, if V is a rational homology manifold and has no codimension one singularities (e.g., V is normal), then $\Delta_i(1) \neq 0$.

Note. It follows that the zero-dimensional strata of V may only contribute to $\Delta_n(t)$, the one-dimensional singular strata may only contribute to $\Delta_n(t)$ and $\Delta_{n-1}(t)$, and so on. This observation is crucial in obtaining obstructions on the eigenvalues of the monodromy operators of a hypersurface arrangement (see Theorem 2.6 of the next section). It is not clear at this point what is the role played by the position of singularities in the study of Alexander invariants for hypersurfaces with non-isolated singularities. The position itself is yet to be understood.

The proof of Theorem 2.4 uses the intersection homology realization of the infinite cyclic Alexander modules, together with the superduality isomorphism of Cappell-Shaneson and the properties of the associated peripheral complex. In the case of hypersurfaces with only isolated singularities and in general position at infinity, the theorem can be refined as follows (compare [21], Theorem 3.1):

Theorem 2.5. *If V has only isolated singularities, then $\Delta_n(t)$ divides (up* *to a power of* $(t-1)$*) the product*

$$\prod_{p\in V_1\cap Sing(V)} \Delta_p(t)$$

of the local Alexander polynomials of links of the singular points p *of* V *which are contained in* V_1*.*

We will sketch a proof of the isolated singularities case, the general case being treated in a similar manner.

Proof. (sketch)
Assume V has only isolated singularities. If $j:\mathbb{CP}^{n+1}-V_1\hookrightarrow\mathbb{CP}^{n+1}$ and $i:V_1\hookrightarrow\mathbb{CP}^{n+1}$ denote the inclusion maps, then the hypercohomology long exact sequence associated to the distinguished triangle:

$$i_*i^!\mathcal{IC}^\bullet_{\bar{m}}\to\mathcal{IC}^\bullet_{\bar{m}}\to j_*j^*\mathcal{IC}^\bullet_{\bar{m}}\overset{[1]}{\to}$$

yields:

$$\to\mathbb{H}^{-n}_{V_1}(\mathbb{CP}^{n+1};\mathcal{IC}^\bullet_{\bar{m}})\to IH^{\bar{m}}_n(\mathbb{CP}^{n+1};\mathcal{L})\to\mathbb{H}^{-n}(\mathbb{CP}^{n+1}-V_1;j^*\mathcal{IC}^\bullet_{\bar{m}})\to$$

But $\mathbb{CP}^{n+1}-V_1$ is a $(n+1)$-dimensional affine variety, thus by Artin's vanishing theorem we obtain

$$\mathbb{H}^{-n}(\mathbb{CP}^{n+1}-V_1;j^*\mathcal{IC}^\bullet_{\bar{m}})\cong IH^{\bar{m}}_n(\mathbb{CP}^{n+1}-V_1;\mathcal{L})\cong 0.$$

So $IH^{\bar{m}}_n(\mathbb{CP}^{n+1};\mathcal{L})$ is a quotient of

$$\mathbb{H}^{-n}_{V_1}(\mathbb{CP}^{n+1};\mathcal{IC}^\bullet_{\bar{m}})\overset{Sd}{\cong}\mathbb{H}^{-n-1}(V_1;i^*\mathcal{IC}^{\bullet\,op}_{\bar{l}}),\tag{3}$$

the isomorphism (3) being a consequence of the Cappell-Shaneson superduality isomorphism (1).

Now let $\Sigma_0:=V_1\cap\operatorname{Sing}(V)$ and consider the long exact sequence:

$$\to\mathbb{H}^{-n-1}_c(V_1-\Sigma_0;\mathcal{IC}^{\bullet\,op}_{\bar{l}})\to\mathbb{H}^{-n-1}(V_1;\mathcal{IC}^{\bullet\,op}_{\bar{l}})\to\mathbb{H}^{-n-1}(\Sigma_0;\mathcal{IC}^{\bullet\,op}_{\bar{l}})\to\tag{4}$$

By local calculation and superduality for link pairs ([4], Corollary 3.4), we have that:

$$\mathbb{H}^{-n-1}(\Sigma_0;\mathcal{IC}^{\bullet\,op}_{\bar{l}})\cong\oplus_{p\in\Sigma_0}H_n(S^{2n+1}_p-S^{2n+1}_p\cap V;\Gamma)\tag{5}$$

where $(S^{2n+1}_p,S^{2n+1}_p\cap V)$ is the link pair of the singular point $p\in\Sigma_0$, and Γ denotes the induced local coefficient system on the link complement. By the hypercohomology spectral sequence, the modules $\mathbb{H}^*_c(V_1-\Sigma_0;\mathcal{IC}^{\bullet\,op}_{\bar{l}})$

are annihilated by powers of $t-1$. The theorem follows by observing that for $p \in \Sigma_0$ we have an isomorphism of Γ-modules:

$$H_n(S_p^{2n+1} - S_p^{2n+1} \cap V; \Gamma) \cong H_n(\widetilde{S_p^{2n+1} - S_p^{2n+1} \cap V}; \mathbb{C}) \cong H_n(F_p, \mathbb{C}),$$

where $\widetilde{S_p^{2n+1} - S_p^{2n+1} \cap V}$ is the total linking number infinite cyclic cover of the algebraic link complement, and F_p is the local Milnor fiber at p. The module structure on the group $H_n(F_p, \mathbb{C})$ is induced by the action of the local monodromy homeomorphism at p. □

Example 2.1. Let V be a degree d reduced projective hypersurface, in general position at infinity, such that V is a rational homology manifold with no codimension one singularities. Assume that the local monodromies of link pairs of strata contained in some irreducible component V_1 of V have orders which are relatively prime to d (e.g., the transversal singularities along strata of V_1 are Brieskorn-type singularities, having all exponents relatively prime to d). Then, by Theorem 2.3 and Theorem 2.4, it follows that $\Delta_i(t) = 1$ for all $1 \leq i \leq n$.

2.3. *The Milnor fiber of a projective hypersurface arrangement*

As an application of the previous results, by a conning construction we obtain obstructions on the eigenvalues of the monodromy operators acting on the homology of the Milnor fiber of a projective hypersurface arrangement.

Let $Y = \{f = 0\}$ be a reduced degree d hypersurface in $\mathbb{CP}^n$, defining a projective hypersurface arrangement $\mathcal{A} = (Y_i)_{i=1,s}$, where Y_i are the irreducible components of Y. Associated to the homogeneous polynomial f there is the global Milnor fibration $f : \mathcal{U} = \mathbb{C}^{n+1} - f^{-1}(0) \to \mathbb{C}^*$, whose fiber $F = f^{-1}(1)$ is called the Milnor fiber of the arrangement $\mathcal{A}$. The monodromy homeomorphism $h : F \to F$ of the Milnor fibre is explicitely described by the mapping $h(x) = \tau \cdot x$, where $\tau = \exp(2\pi i/d)$. Denote by $P_q(t)$ the characteristic polynomial of the monodromy operator $h_q : H_q(F) \to H_q(F)$. Since $h^d = id$, the zeros of $P_q(t)$ are roots of unity of order d.

Note that $\mathcal{U}$ is the complement of a central arrangement $A = \{f^{-1}(0)\}$ in $\mathbb{C}^{n+1}$, namely the cone on $\mathcal{A}$. Moreover, it's easy to see that the projective completion of A in $\mathbb{CP}^{n+1}$ is in general position at infinity. The key observation for what follows is that the Milnor fiber F is homotopy equivalent to the infinite cyclic cover $\mathcal{U}^c$ of $\mathcal{U}$, corresponding to the kernel of the total linking number homomorphism and, with this identification, the monodromy homeomorphism h corresponds to a generator of the group of covering transformations (see [9], p. 106-107).

Theorem 2.4, when applied to the projective cone on Y (i.e., the hypersurface $V = \{f = 0\} \subset \mathbb{CP}^{n+1}$), translates into divisibility results for the characteristic polynomials of the monodromy operators of F, thus showing the dependence of the monodromy of the arrangement $\mathcal{A}$ on the local monodromy operators associated with singular strata in a stratification of Y. With the notations from Theorem 2.4, we can now state the following:

Theorem 2.6. *([33])*
Fix an arbitrary component of the arrangement, say Y_1, and let $\mathcal{Y}$ be the set of (open) singular strata of a stratification of the pair $(\mathbb{CP}^n, Y)$. Then for fixed $q \leq n-1$, a d^{th} root of unity λ is a zero of $P_q(t)$ only if λ is a zero of one of the local polynomials $\Delta_r^s(t)$ associated with strata $\mathcal{V} \in \mathcal{Y}$ of complex dimension s, for $n-q-1 \leq s \leq n-1$, such that $\mathcal{V} \subset Y_1$ and $2(n-1) - 2s - q \leq r \leq n-s-1$.

This theorem provides obstructions on the eigenvalues of the monodromy operators, similar to those obtained by Libgober in the case of hyperplane arrangements [30], or Dimca in the case of curve arrangements [10]. For the special case of an arrangement with only normal crossing singularities along one of its components, we deduce from Theorem 2.6 the following result (compare [8], Corollary 16):

Corollary 2.2. *Let $\mathcal{A} = (Y_i)_{i=1,s}$ be a hypersurface arrangement in $\mathbb{CP}^n$, and fix one irreducible component, say Y_1. Assume that $\bigcup_{i=1,s} Y_i$ is a normal crossing divisor at any point $x \in Y_1$. Then the monodromy action on $H_q(F;\mathbb{C})$ is trivial for $q \leq n-1$.*

3. Universal abelian Alexander invariants of the complement

In [28] (and later in [25]), Libgober introduced new topological invariants of the complement to plane algebraic curves: the sequence of characteristic varieties. These invariants were also considered in E. Hironaka's doctoral thesis, but see also [18]. Characteristic varieties were originally used to obtain information about all abelian covers of the complex projective plane, branched along a curve (see [25], §1.3). In the context of complex hyperplane arrangements, characteristic varieties of the first homology group of the universal abelian cover of the complement are considered in [6,7,29], and studied in relation with the cohomology support loci of rank one local systems defined on the complement (see also [1] for the study of the latter).

Here we consider (co)homological universal abelian invariants of complements to arbitrary hypersurfaces, as they are described in [14].

3.1. *Definition of Characteristic varieties.*

In this section, characteristic varieties are defined, first for general noetherian modules, then in the context of complex hypersurface complements.

Let R be a commutative ring with unit, which is Noetherian and a unique factorization domain. Let A be a finitely generated R-module, and M a $(m \times n)$ presentation matrix of A associated to an exact sequence: $R^m \to R^n \to A \to 0$.

Definition 3.1. The i-th elementary ideal $\mathcal{E}_i(A)$ of A is the ideal in R generated by the $(n-i) \times (n-i)$ minor determinants of M, with the convention that $\mathcal{E}_i(A) = R$ if $i \geq n$, and $\mathcal{E}_i(A) = 0$ if $n - i > m$.

Definition 3.2. The support $\mathrm{Supp}(A)$ of A is the reduced sub-scheme of $\mathrm{Spec}(R)$ defined by the order ideal $\mathcal{E}_0(A)$. Equivalently, if P is a prime ideal of R then $P \in \mathrm{Supp}(A)$ if and only if the localized module A_P is non-zero.

The support $\mathrm{Supp}(A)$ is also called the *first characteristic variety* of A, and we define the i-th *characteristic variety* $V_i(A)$ of A to be the reduced sub-scheme of $\mathrm{Spec}(R)$ defined by the (i-th *Fitting ideal*) ideal $\mathcal{E}_{i-1}(A)$.

All definitions above are independent (up to multiplication by a unit of R) of the choices involved, thus the characteristic varieties are invariants of the R-isomorphism type of A.

Now let V be a reduced hypersurface in $\mathbb{CP}^{n+1}$, and H_∞ be the hyperplane at infinity. As in § 2, we let $\mathcal{U}$ be the complement $\mathbb{CP}^{n+1} - (V \cup H_\infty)$. We denote by $\mathcal{U}^{ab}$ the universal abelian cover of $\mathcal{U}$, or equivalently, the covering associated to the kernel of the homomorphism:

$$Lk^{ab} : \pi_1(\mathcal{U}) \to \mathbb{Z}^s, \quad \alpha \mapsto (\mathrm{lk}(\alpha, V_1 \cup -d_1 H_\infty), \cdots, \mathrm{lk}(\alpha, V_s \cup -d_s H_\infty)).$$

The group of covering transformations of $\mathcal{U}^{ab}$ is isomorphic to $\mathbb{Z}^s$ and acts on the covering space. Let Γ_s be the group ring $\mathbb{C}[\mathbb{Z}^s]$, which is identified with the ring of complex Laurent polynomials in s variables, $\mathbb{C}[t_1, t_1^{-1}, \cdots, t_s, t_s^{-1}]$. Note that Γ_s is a regular Noetherian domain, and in particular it is factorial. As a group ring, Γ_s has a natural involution, denoted by an overbar, sending each t_i to $\bar{t}_i := t_i^{-1}$.

Define a local coefficient system $\mathcal{L}^{ab}$ on $\mathcal{U}$, with stalk Γ_s and action of a loop $\alpha \in \pi_1(\mathcal{U})$ determined by multiplication by $\prod_{j=1}^{s} (t_j)^{\mathrm{lk}(\alpha, V_j \cup -d_j H_\infty)}$. In particular, the action of the meridian γ_i is given by multiplication by t_i. We let $\bar{\mathcal{L}^{ab}}$ be the local system obtained from $\mathcal{L}^{ab}$ by composing all module structures with the natural involution of Γ_s.

Definition 3.3. The universal homology k-th Alexander module of $\mathcal{U}$ is by definition $A_k(\mathcal{U}) := H_k(\mathcal{U}, \mathcal{L}^{ab})$, that is, the group $H_k(\mathcal{U}^{ab}; \mathbb{C})$ considered as a Γ_s-module via the action of covering transformations. Similarly, the universal cohomology k-th Alexander module of $\mathcal{U}$ is defined as $A^k(\mathcal{U}) := H^k(\mathcal{U}; \bar{\mathcal{L}^{ab}})$.

Remark 3.1. If C_* is the cellular complex of $\mathcal{U}^{ab}$, as $\mathbb{Z}[\mathbb{Z}^s]$-modules, and if $C_*^0 := C_* \otimes \mathbb{C}$ denotes the complexified complex, then: $A_k(\mathcal{U}) = H_k(C_*^0)$ and $A^k(\mathcal{U}) = H_k(\mathrm{Hom}_{\Gamma_s}(C_*^0, \Gamma_s))$.

As in §2, the modules $A^k(\mathcal{U})$ and resp. $A_k(\mathcal{U})$ are trivial for $k > n+1$. Moreover, $A_{n+1}(\mathcal{U})$ is a torsion-free Γ_s-module.

It is easy to see that the universal abelian Alexander modules are of finite type over Γ_s. Hence their characteristic varieties are well-defined. The associated characteristic varieties, in particular the supports, become subvarieties of the s-dimensional torus $\mathbb{T}^s = (\mathbb{C}^*)^s$, which is regarded as the set of closed points in $\mathrm{Spec}(\Gamma_s)$. More precisely, for $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathbb{T}^s$, we denote by m_λ the corresponding maximal ideal in Γ_s, and by $\mathbb{C}_\lambda$ the quotient $\Gamma_s / m_\lambda \Gamma_s$. This quotient is isomorphic to $\mathbb{C}$ and the canonical projection $\rho_\lambda : \Gamma_s \to \Gamma_s / m_\lambda \Gamma_s = \mathbb{C}_\lambda$ corresponds to replacing t_j by λ_j for $j = 1, \cdots, s$. If A is a Γ_s-module, we denote be A_λ the localization of A at the maximal ideal m_λ. For $A = \Gamma_s$, we use the simpler notation Γ_λ when there is no danger of confusion. Note that if A is of finite type, then $A = 0$ if and only if $A_\lambda = 0$ for all $\lambda \in \mathbb{T}^s$. It follows that

$$\mathrm{Supp}(A) = \{\lambda \in \mathbb{T}^s; A_\lambda \neq 0\}$$

In particular $A_0(\mathcal{U}) = \mathbb{C}_{\mathbf{1}}$, where $\mathbf{1} = (1, \cdots, 1)$. Hence $\mathrm{Supp}(A_0(\mathcal{U})) = \{\mathbf{1}\}$.

We denote by $V_{i,k}(\mathcal{U})$ the i-th characteristic variety associated to the homological Alexander module $A_k(\mathcal{U})$, and by $V^{i,k}(\mathcal{U})$ that associated to the cohomological Alexander module $A^k(\mathcal{U})$. Note that for each universal Alexander module, its characteristic varieties form a decreasing filtration of the character torus $\mathbb{T}^s$. This follows from the fact that for a noetherian R-

module A of finite type, the elementary ideals form an increasing filtration of R.

All definitions in this section work also in the local setting, i.e., when $\mathcal{U}$ is a complement of a hypersurface germ in a small ball.

Remark 3.2. The invariants defined above originate in the classical knot theory (see [17]), where it follows directly from definition that the support of the universal homological Alexander module of a link complement in S^3 is the set of zeros of the multivariable Alexander polynomial. In the case of irreducible hypersurfaces, where the infinite cyclic and universal abelian cover coincide, the support of an Alexander module is simply the zero set of the associated one-variable polynomial.

3.2. *Further study of Supports.*

The results mentioned here are taken from [14].

First note that the cohomology modules may be related to the homology modules by the Universal Coefficient spectral sequence (see [17], p. 20, or [19], Theorem 2.3).

$$\mathrm{Ext}^q_{\Gamma_s}(A_p(\mathcal{U}), \Gamma_s) \Rightarrow A^{p+q}(\mathcal{U}). \tag{6}$$

Relations between the corresponding characteristic varieties are consequences of the spectral sequence obtained by localizing at any $\lambda \in \mathbb{T}^s$:

$$\mathrm{Ext}^q_{\Gamma_\lambda}(A_p(\mathcal{U})_\lambda, \Gamma_\lambda) \Rightarrow A^{p+q}(\mathcal{U})_\lambda. \tag{7}$$

If for a fixed $\lambda \in \mathbb{T}^s$, we define

$$k(\lambda) = \min\{m \in \mathbb{N}; A_m(\mathcal{U})_\lambda \neq 0\}, \tag{8}$$

then the spectral sequence (7) yields the following:

Proposition 3.1.

For any $\lambda \in \mathbb{T}^s$, $A^k(\mathcal{U})_\lambda = 0$ for $k < k(\lambda)$ and

$$A^{k(\lambda)}(\mathcal{U})_\lambda = Hom(A_{k(\lambda)}(\mathcal{U})_\lambda, \Gamma_\lambda). \tag{9}$$

As a simple application of these facts, we obtain the following:

Example 3.1.

(i) Let $\mathcal{U}$ be the complement of a normal crossing divisor germ in a small ball. Then the universal abelian covering $\mathcal{U}^{ab}$ is contractible, so $A_0(\mathcal{U}) = \mathbb{C}_{\mathbf{1}}$ and $A_k(\mathcal{U}) = 0$ for $k > 0$. Moreover, for any $\lambda \neq \mathbf{1}$, the cohomology Alexander modules satisfy $A^k(\mathcal{U})_\lambda = 0$, for any k.

(ii) Let $(Y, 0)$ be an isolated non-normal crossing singularity at the origin of $\mathbb{C}^{n+1}$ (shortly, INNC), that is, each component of Y is nonsingular outside the origin and, moreover, the union of components in a neighborhood of a point outside the origin is a normal crossing divisor. Let $\mathcal{U}$ be its complement in a small open ball centered at the origin in $\mathbb{C}^{n+1}$ and assume that $n \geq 2$. Then the universal abelian cover $\mathcal{U}^{ab}$ is $(n-1)$-connected (see [26]). More precisely, it is a bouquet of n-spheres (see [13]), hence $A_0(\mathcal{U}) = \mathbb{C}_{\mathbf{1}}$ and $A_k(\mathcal{U}) = 0$ for $k \neq n$. Moreover, for $\lambda \neq \mathbf{1}$, we obtain $A^k(\mathcal{U})_\lambda = 0$ for $k < n$.

In relation with the infinite cyclic Alexander invariants, we note that $\mathcal{U}^{ab} \to \mathcal{U}^c$ is a covering map, and there is a spectral sequence:

$$E^2_{p,q} = Tor_p^{\Gamma_s}(A_q(\mathcal{U}), \Gamma_1) \Rightarrow H_{p+q}(\mathcal{U}^c; \mathbb{C}), \tag{10}$$

where the Γ_s-module structure on Γ_1 is defined by sending each t_i to t. For $a \in \mathbb{T}^1 = \{(t, t, ..., t) \in \mathbb{T}^s\}$, we get by localization a new Künneth spectral sequence, namely

$$E^2_{p,q} = Tor_p^{\Gamma_a}(A_q(\mathcal{U})_a, \Gamma_{1,a}) \Rightarrow H_{p+q}(\mathcal{U}^c; \mathbb{C})_a. \tag{11}$$

In particular we obtain:

Proposition 3.2.
For any $a \in \mathbb{T}^1$, $H_k(\mathcal{U}^c; \mathbb{C})_a = 0$ for $k < k(a)$ and

$$A_{k(a)}(\mathcal{U})_a \otimes_{\Gamma_a} \Gamma_{1,a} = H_{k(a)}(\mathcal{U}^c; \mathbb{C})_a. \tag{12}$$

In connection with the (co)homology support loci of rank one local systems on $\mathcal{U}$, we mention the following: Let $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathbb{T}^s$ and denote by $\mathcal{L}_\lambda$ the local coefficient system on $\mathcal{U}$ with stalk $\mathbb{C} = \mathbb{C}_\lambda$ and action of a loop $\alpha \in \pi_1(\mathcal{U})$ determined by multiplication by $\prod_{j=1}^s (\lambda_j)^{\mathrm{lk}(\alpha, V_j \cup -d_j H_\infty)}$. One can define new *topological characteristic varieties* by setting

$$V^t_{i,k}(\mathcal{U}) = \{\lambda \in \mathbb{T}^s; \dim_{\mathbb{C}} H_k(\mathcal{U}, \mathcal{L}_\lambda) > i\}$$

and similarly for cohomology. It is known that

$$H_k(\mathcal{U}, \mathcal{L}_\lambda) = H_k(C^0_* \otimes_{\Gamma_s} \mathbb{C}_\lambda).$$

Therefore, by the Künneth spectral sequence, we get

$$E^2_{p,q} = Tor_p^{\Gamma_s}(A_q(\mathcal{U}), \mathbb{C}_\lambda) \Rightarrow H_{p+q}(\mathcal{U}, \mathcal{L}_\lambda). \tag{13}$$

Since the localization is exact, the base change for Tor under $\Gamma_s \to \Gamma_\lambda$ yields a new spectral sequence

$$E^2_{p,q} = Tor_p^{\Gamma_\lambda}(A_q(\mathcal{U})_\lambda, \mathbb{C}_\lambda) \Rightarrow H_{p+q}(\mathcal{U}, \mathcal{L}_\lambda). \tag{14}$$

This is used in proving the next result:

Proposition 3.3. *([14]) For any point $\lambda \in \mathbb{T}^s$, one has the following:*

(i) $min\{m \in \mathbb{N},\ H_m(\mathcal{U}, \mathcal{L}_\lambda) \neq 0\} = min\{m \in \mathbb{N},\ \lambda \in Supp(A_m(\mathcal{U}))\}$.

(ii) $dim H_{k(\lambda)}(\mathcal{U}, \mathcal{L}_\lambda) = max\{m \in \mathbb{N},\ \lambda \in V_{m,k(\lambda)}(\mathcal{U})\}$.

3.3. *Dependence on the local data*

In [25], characteristic varieties of plane curve complements are described in terms of local type of singularities and dimensions of linear systems which are attached to the configuration of singularities of the curve. In [14] we obtain a different type of dependence on the local data, leading to vanishing results.

Again, we assume that the hyperplane at infinity H_∞ is transversal in the stratified sense to the hypersurface V. Under this assumption, the universal cohomological Alexander invariants of the complement are entirely determined by the degrees of the irreducible components on one hand, and by the local topological information encoded by the singularities of V on the other hand. In particular, these invariants depend on the local type of singularities of the hypersurface.

First, we need some notations. For $x \in V$, we let $\mathcal{U}_x = \mathcal{U} \cap B_x$, for B_x a small open ball at x in $\mathbb{CP}^{n+1}$. Denote by $\mathcal{L}^{ab}_x$ the restriction of the local coefficient system $\mathcal{L}^{ab}$ to $\mathcal{U}_x$. Then:

Theorem 3.1. *([14]) Let $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathbb{T}^s$ and $\epsilon \in \mathbb{Z}_{\geq 0}$. Fix an irreducible component V_1 of V, and assume that $\lambda \notin Supp(H^q(\mathcal{U}_x, \bar{\mathcal{L}^{ab}_x}))$ for all $q < n+1-\epsilon$ and all points $x \in V_1$. Then $\lambda \notin Supp(A^q(\mathcal{U}))$ for all $q < n+1-\epsilon$.*

The assumption on the hyperplane at infinity, together with the universal coefficient spectral sequence imply that the modules $H^*(\mathcal{U}_x, \bar{\mathcal{L}^{ab}_x})$ in Theorem 3.1 can be expressed in terms of the local universal homological Alexander modules $A_*(\mathcal{U}'_x)$, where $\mathcal{U}' := \mathbb{CP}^{n+1} - V$ and $\mathcal{U}'_x := \mathcal{U}' \cap B_x$. The latter depend only on the hypersurface singularity germ (V, x), and are defined as in § 3.1. For complete details, see [14].

The following consequence of Theorem 3.1 and of Example 3.1 is similar to some results in [13,26,27].

Corollary 3.1. *(i) (Case $\epsilon = 0$) With the notation in the above theorem, assume in addition that V is a normal crossing divisor at any point of the component V_1. Then Supp($A^k(\mathcal{U})$) $\subset \{\mathbf{1}\}$ for any $k < n+1$.*

(ii) (Case $\epsilon = 1$) With the notation in the above theorem, assume in addition that V is an INNC divisor at any point of the component V_1. Then Supp($A^k(\mathcal{U})$) $\subset \{\mathbf{1}\}$ for any $k < n$.

The dependence on the degrees of components of V is reflected by the following result, a generalization of a similar result from [28]:

Theorem 3.2. *([14]) For $k \leq n$, Supp($A^k(\mathcal{U})$) is contained in the zero set of the polynomial $t_1^{d_1} \cdots t_s^{d_s} - 1$, thus has positive codimension in $\mathbb{T}^s$.*

The positive codimension property of supports in the universal abelian case should be regarded as the analogue of the torsion property in the infinite cyclic case (cf. [12,33]).

Remark 3.3. In proving the results of this section, the general theory of perverse sheaves is used (cf. [2,10,31,36]). The use of intersection homology as in §2 is constrained by lacking the superduality isomorphism (1), which only holds over a Dedekind domain.

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