# Lefschetz properties in the constructible context and applications

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**I**. *Kähler package for complex projective manifolds*

## Theorem (Kähler package)

*Let $X \subset \mathbb{C}P^N$ be an n-dimensional complex projective manifold. Then $H^*(X) := H^*(X; \mathbb{C})$ satisfies the following properties:*

(a) *Poincaré duality:*

$$H^k(X) \cong H^{2n-k}(X)^\vee$$

*for all $k \in \mathbb{Z}$. In particular, the Betti numbers of $X$ in complementary degrees coincide: $b_k(X) = b_{2n-k}(X)$.*

(b) *Hodge structure: $H^k(X)$ has a pure Hodge structure of weight $k$. In fact,*

$$H^k(X) \cong H^k_{DR}(X) \cong \bigoplus_{p+q=k} H^{p,q}(X),$$

*with $H^{q,p}(X) = \overline{H^{p,q}(X)}$. In particular, the odd Betti numbers of $X$ are even.*

## Theorem (Kähler package, cont'd)

(c) *Weak Lefschetz (Lefschetz hyperplane section theorem): If $H$ is a generic hyperplane in $\mathbb{C}P^N$, the homomorphism*

$$H^k(X) \longrightarrow H^k(X \cap H)$$

*induced by restriction is an isomorphism for $k < n-1$, and it is injective if $k = n-1$. In particular, generic hyperplane sections of $X$ are connected if $n \geq 2$.*

(d) *Hard Lefschetz: If $H$ is a generic hyperplane in $\mathbb{C}P^N$, there is an isomorphism*

$$H^{n-k}(X) \xrightarrow{\cup [H]^k} H^{n+k}(X)$$

*for all $k \geq 0$, where $[H] \in H^2(X)$ is the Poincaré dual of $[X \cap H] \in H_{2n-2}(X)$. In particular, the Betti numbers of $X$ are unimodal: $b_{k-2}(X) \leq b_k(X)$ for all $k \leq n/2$.*

## Example

Let $X = \mathbf{G}_d(\mathbb{C}^n)$ be the Grassmann variety of $d$-planes in $\mathbb{C}^n$, a complex projective manifold of complex dimension $d(n-d)$.
$X$ has an algebraic cell decomposition by complex affine spaces, so the odd Betti numbers of $X$ vanish.
The even Betti numbers are computed as

$$b_{2k}(X) = p(k, d, n-d),$$

where $p(k, d, n-d)$ is the number of partitions of the integer $k$ whose Young diagrams fit inside a $d \times (n-d)$ box (i.e., partitions of $k$ into $\leq d$ parts, with largest part $\leq n-d$).
The Kähler package implies that the sequence

$$p(0, d, n-d),\ p(1, d, n-d), \cdots,\ p(d(n-d), d, n-d)$$

is symmetric and unimodal.

**Remark**

- "Non-abelian" Hard Lefschetz due to Simpson (1992), replaces constant coefficients by a semi-simple local system.
- $\ell$-adic version for smooth projective varieties defined over finite fields, due to Deligne (1980).

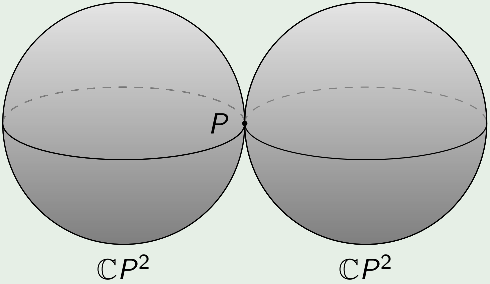
**II**. *Kähler package for singular varieties*

## Example

Let $X = \mathbb{C}P^2 \cup \mathbb{C}P^2 \subset \mathbb{C}P^4 = \{[x_0 : x_1 : \cdots : x_4]\}$, where the two copies of $\mathbb{C}P^2$ in $X$ meet at a point $P$. So

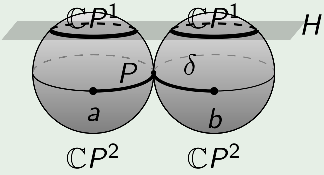
$$X = \{x_i x_j = 0 \mid i \in \{0,1\}, j \in \{3,4\}\},$$

with $\mathrm{Sing}(X) = \{P = [0 : 0 : 1 : 0 : 0]\}$.



## Example (cont'd)

The Mayer-Vietoris sequence yields:




$$H^k(X) = \begin{cases} \mathbb{C} & k=0 \\ 0 & k=1 \\ \mathbb{C}\oplus\mathbb{C} & k=2 \\ 0 & k=3 \\ \mathbb{C}\oplus\mathbb{C} & k=4. \end{cases}$$

The 0-cycles $[a], [b] \in C_0(X)$ cobound a 1-chain $\delta$ passing through the singular point $P$. In particular, $[a] = [b] \in H_0(X) \cong H^0(X)^\vee$. If $H$ is a generic hyperplane in $\mathbb{C}P^4$, then $X \cap H = \mathbb{C}P^1 \sqcup \mathbb{C}P^1$, which is not connected, so Weak Lefschetz *fails*. Moreover,

$$H^0(X) = \mathbb{C} \not\cong \mathbb{C}\oplus\mathbb{C} = H^4(X),$$

so Poincaré duality and Hard Lefschetz also *fail* for $X$.

To restore the Kähler package in the singular setting, one has to replace cohomology $H^*(X)$ by (middle-perversity) intersection cohomology $IH^*(X)$. Homologically, this is a theory of "allowable chains", controlling the defect of transversality of intersections of chains with the singular strata. In the above example, 1-chains are not allowed to pass through singularities. So the 1-chain $\delta$ connecting the 0-cycles $[a]$ and $[b]$ is not allowed, hence $[a] \neq [b]$ in $IH_0(X)$. More generally,

## Proposition

*Let $X$ be a complex algebraic variety of pure complex dimension $n$, with only isolated singularities. Let $U = X_{\mathrm{reg}} = X \setminus \mathrm{Sing}(X)$ be the nonsingular locus of $X$. Then (with $\mathbb{C}$-coefficients):*

$$IH^k(X) = \begin{cases} H^k(U), & k < n, \\ \mathrm{Image}\left(H^n(X) \to H^n(U)\right), & k = n, \\ H^k(X), & k > n. \end{cases}$$

$IH^*(X)$ is computed by Deligne's IC-complex $IC_X$, which is uniquely characterized (up to quasi-isomorphism) by a set of axioms. If $X$ is pure $n$-dimensional, then

$$IH^k(X) = \mathbb{H}^{k-n}(X; IC_X).$$

Checking the $IC$-axioms for the Verdier dual $\mathcal{D}(IC_X)$, one gets:

**Theorem (Poincaré Duality for $IH^*$)**

*If $X$ is a pure n-dimensional complex projective variety, there is a non-degenerate intersection pairing*

$$IH^k(X) \otimes IH^{2n-k}(X) \longrightarrow \mathbb{C}$$

*induced from the quasi-isomorphism*

$$\mathcal{D}(IC_X) \simeq IC_X.$$

All other statements of the Kähler package hold for the intersection cohomology groups of a complex projective variety.

# Weak Lefschetz

Weak Lefschetz holds, more generally, for any perverse sheaf $\mathcal{F}$ on a projective variety $X$ (e.g., $IC_X$):

**Theorem (Weak Lefschetz Theorem for Perverse Sheaves)**

*If $X$ is a complex projective variety and $i : D \hookrightarrow X$ is the inclusion of a hyperplane section, then for every $\mathcal{F} \in Perv(X)$ the restriction map $\mathbb{H}^k(X;\mathcal{F}) \to \mathbb{H}^k(D; i^*\mathcal{F})$ is an isomorphism for $k < -1$ and is injective for $k = -1$.*

The proof follows from Artin's vanishing theorem for the affine inclusion $j : U = X \setminus D \hookrightarrow X$, with $j^*\mathcal{F} \in Perv(U)$, i.e.,

$$\mathbb{H}^k_c(U; j^*\mathcal{F}) = 0 \ \text{ for } k < 0.$$

# WL for $IH^*$

Note that, if $D \stackrel{i}{\hookrightarrow} X$ is a *generic* hyperplane section of $X$ and $\mathcal{F} \in Perv(X)$, then $i^*\mathcal{F}^\bullet[-1] \in Perv(D)$. E.g., $i^* IC_X \simeq IC_D[1]$. This gives (WL) for $IH^*$.

**Theorem (Lefschetz hyperplane section theorem for $IH^*$)**

*Let $X \subset \mathbb{C}P^N$ be a pure n-dimensional closed algebraic subvariety with a Whitney stratification $\mathcal{X}$. Let $H \subset \mathbb{C}P^N$ be a generic hyperplane (i.e., transversal to all strata of $\mathcal{X}$). Then the natural homomorphism*

$$IH^k(X;\mathbb{C}) \longrightarrow IH^k(X \cap H;\mathbb{C})$$

*is an isomorphism for $0 \leq k \leq n-2$ and a monomorphism for $k = n-1$.*

# WL for $H^*$

**Theorem**

*Let $X$ be a projective variety of pure complex dimension $n$, and let $D$ be a hyperplane section of $X$ chosen so that $U = X \setminus D$ is smooth. Then the inclusion $D \hookrightarrow X$ induces isomorphisms*

$$H^k(X;\mathbb{C}) \longrightarrow H^k(D;\mathbb{C})$$

*for all $k < n-1$ and a monomorphism for $k = n-1$.*

This uses Artin vanishing for the smooth affine variety $U$, with $\underline{\mathbb{C}}_U[n] \in Perv(U)$.

# WL for hypersurfaces

**Theorem (M.-Păunescu-Tibăr)**

*If* $X \subset \mathbb{C}P^{n+1}$ *is a hypersurface, then* $\mathcal{F} = \underline{\mathbb{C}}_X[n] \in Perv(X)$*, so the inclusion* $i \colon D \hookrightarrow X$ *of any hyperplane section induces isomorphisms*

$$H^k(X;\mathbb{C}) \longrightarrow H^k(D;\mathbb{C})$$

*for all* $k < n-1$ *and a monomorphism for* $k = n-1$*. Moreover, if* $X$ *is reduced with* $s = \dim_{\mathbb{C}} \mathrm{Sing}(X)$*, and* $D$ *is generic, then* $H^k(X, D;\mathbb{C}) = 0$ *for* $n+s+2 \leq k < 2n$*, and* $H^{2n}(X, D;\mathbb{C}) = \mathbb{C}^r$*, where* $r$ *is the number of irreducible components of* $X$*.*

This can be used inductively (M.-Păunescu-Tibăr) to prove:

**Corollary (Kato)**

*If* $X \subset \mathbb{C}P^{n+1}$ *is a reduced hypersurface with* $s = \dim_{\mathbb{C}} \mathrm{Sing}(X)$*, then* $H^k(X;\mathbb{C}) \cong H^k(\mathbb{C}P^n;\mathbb{C})$ *for* $k < n$ *and* $n+s+2 \leq k \leq 2n$*.*

# Hard Lefschetz for $IH^*$

Hodge structures and Hard Lefschetz for $IH^*$ follow from work of Beinlinson-Bernstein-Deligne, Saito and/or de Cataldo-Migliorini.

(HL) for $IH^*$ is a consequence of the *Relative Hard Lefschetz* for projective morphisms, applied to the constant map $X \to point$.

($IH^*$ is not a ring, but a module over $H^*$.)

**Theorem (Hard Lefschetz theorem for intersection cohomology)**

*Let $X$ be a complex projective variety of pure complex dimension $n$, with $[H] \in H^2(X;\mathbb{Q})$ the first Chern class of an ample line bundle on $X$. Then there are isomorphisms (of pure HS)*

$$\cup[H]^i : IH^{n-i}(X;\mathbb{Q}) \xrightarrow{\cong} IH^{n+i}(X;\mathbb{Q})$$

*for every integer $i > 0$, induced by the cup product by $[H]^i$. In particular, the intersection cohomology Betti numbers of $X$ are unimodal, i.e.,* $\dim IH^{i-2}(X;\mathbb{Q}) \leq \dim IH^i(X;\mathbb{Q})$ *for all $i \leq n/2$.*

# Hodge index theorem for $IH^*$

Extending the classical Hodge index theorem for Kähler manifolds, one has:

**Theorem (M.-Saito-Schürmann)**

*Let $X$ be a complex projective variety of pure complex dimension $n$, and let $Ih^{p,q}(X)$ be the Hodge numbers of the pure HS on $IH^*(X)$. Then the Goresky-MacPherson signature $\sigma(X)$, which is defined by Poincare duality on $IH^n(X)$, is computed by:*

$$\sigma(X) = \sum_{p,q} (-1)^q \cdot Ih^{p,q}(X).$$

# Further generalizations of (HL)

(HL) for $IH^*$ is a special case (for $\mathcal{F} = IC_X$) of the (HL) for semisimple perverse sheaves:

**Theorem (Mochizuki)**

*Under the previous assumptions, if $\mathcal{F} \in Perv(X)$ is semisimple, then*

$$\cup[H]^i : \mathbb{H}^{-i}(X;\mathcal{F}) \xrightarrow{\cong} \mathbb{H}^i(X;\mathcal{F})$$

*is an isomorphism for every integer $i > 0$.*

**Remark**

Mochizuki's theorem extends both classical and non-abelian versions of the Hard Lefschetz from the smooth context.

## **III**. *Applications of the Kähler package*

- Large number of applications to geometry & topology, algebra, combinatorics (e.g., McMullen's g-conjecture, Dowling-Wilson & Rota conjectures, etc.), representation theory (Kazhdan-Lusztig conjecture).
- Geometric results motivated the development of combinatorial intersection cohomology theories for convex polytopes and matroids.

## Lemma

*Let $X \subset \mathbb{C}P^N$ be a projective variety of pure dimension n, which has an algebraic cell decomposition (i.e., all cells are $\mathbb{C}^i$'s). Then*

$$\dim H^{2k}(X;\mathbb{C}) \leq \dim H^{2n-2k}(X;\mathbb{C}), \text{ for all } k \leq n/2.$$

## Proof.

Using Hodge theory, one can show that the map

$$\alpha : H^*(X;\mathbb{C}) \to IH^*(X;\mathbb{C})$$

is injective. The lemma follows from the following commutative diagram, together with (HL) for $IH^*(X;\mathbb{C})$:

$$\begin{array}{ccc} H^{2k}(X;\mathbb{C}) & \overset{\alpha}{\hookrightarrow} & IH^{2k}(X;\mathbb{C}) \\ \downarrow {\scriptstyle \cup[H]^{n-2k}} & & {\scriptstyle \cong}\downarrow {\scriptstyle \cup[H]^{n-2k}} \\ H^{2n-2k}(X;\mathbb{C}) & \overset{\alpha}{\hookrightarrow} & IH^{2n-2k}(X;\mathbb{C}) \end{array}$$

□

# Dowling-Wilson and Rota conjectures

Let $E = \{v_1, \cdots, v_d\}$ be a spanning subset of a $n$-dimensional complex vector space $V$, and let $w_k(E)$ be the number of $k$-dimensional subspaces spanned by subsets of $E$.

**Conjecture (Dowling-Wilson top-heavy conjecture)**

*For all $k < n/2$ one has:*

$$w_k(E) \leq w_{n-k}(E).$$

**Conjecture (Rota's unimodal conjecture)**

*There is some $j$ so that*

$$w_0(E) \leq \cdots \leq w_{j-1}(E) \leq w_j(E) \geq w_{j+1}(E) \geq \cdots \geq w_n(E).$$

Huh-Wang used the previous lemma to prove the Dowling-Wilson top-heavy conjecture.

The proof uses the fact that there exists a complex $n$-dimensional projective variety $X$ such that for every $0 \leq k \leq n$ one has:

$$H^{2k+1}(X;\mathbb{C}) = 0 \text{ and } \dim_{\mathbb{C}} H^{2k}(X;\mathbb{C}) = w_k(E).$$

To define $X$, use $E = \{v_1, \cdots, v_d\}$ to construct a map $i_E : V^\vee \to \mathbb{C}^d$ by regarding each $v_k \in E$ as a linear map $V^\vee \to \mathbb{C}$. Precomposing $i_E$ with the $\mathbb{C}^d \hookrightarrow (\mathbb{C}P^1)^d$ yields a map $f : V^\vee \to (\mathbb{C}P^1)^d$. Set

$$X := \overline{\mathrm{Im}\ (f)} \subset (\mathbb{C}P^1)^d.$$

Ardilla-Boocher showed that the variety $X$ has an algebraic cell decomposition, the number of $\mathbb{C}^k$'s appearing in the decomposition of $X$ being exactly $w_k(E)$. This shows the top-heavy property of the sequence $\{w_k(E)\}$.

The unimodality of the "lower half" of the sequence $\{w_k(E)\}$ follows similarly, using the unimodality of intersection cohomology Betti numbers of $X$.

The Dowling-Wilson and Rota conjectures were initially formulated for *matroids*, with the previous discussion corresponding to the case of matroids realizable over $\mathbb{C}$. The general case was proved more recently by Braden-Huh-Matherne-Proudfoot-Wang by mimicking the above geometric picture in combinatorial terms.

**THANK YOU !!!**