L^2-Betti numbers of hypersurface complements

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Betti numbers

- Let M be a finite CW complex.
- Let $(C_*(M), \partial)$ be the cellular chain complex of M.
- $c_i(M) := \dim_{\mathbb{C}} C_i(M) \otimes \mathbb{C} =$ number of i-cells of M.
- The i-th Betti number of M is:

$$b_i(M) = \dim_{\mathbb{C}} H_i(M; \mathbb{C}).$$

- **Exercise:**

$$\chi(M) := \sum_i (-1)^i c_i(M) \overset{!}{=} \sum_i (-1)^i b_i(M).$$

Higher invariants

- Let Γ be a countable group.
- Let $M_\Gamma \to M$ be the regular cover of M defined by $\ker(\alpha : \pi_1(M) \to \Gamma)$, with α an epimorphism.
- e.g., if $\alpha = id_{\pi_1(M)}$, then M_Γ is the universal cover.
- Gromov: "Higher versions" of topological invariants of M should take π_1 into account.
- **Question:** How should one define "higher Betti numbers"?
- **First (wrong) guess:** Consider $H_i(M_\Gamma; \mathbb{C})$ as $\mathbb{C}[\Gamma]$-modules, and use dimension theory of such modules.
- **But** $\mathbb{C}[\Gamma]$ is not Noetherian, so dimension theory does not work.
- **Fix:** include $\mathbb{C}[\Gamma]$ into a larger ring that has a better dimension theory.

Von Neumann Algebra $\mathcal{N}(\Gamma)$

- $\ell^2(\Gamma) := \{f : \Gamma \to \mathbb{C} \mid \sum_{g\in\Gamma} |f(g)|^2 < \infty\}$ - Hilbert space.
- Γ acts on $\ell^2(\Gamma)$ by $(g \cdot f)(h) = f(hg)$.
- injective map $\mathbb{C}[\Gamma] \hookrightarrow \mathcal{B}(\ell^2(\Gamma))$=bounded operators on $\ell^2(\Gamma)$.
- view $\mathbb{C}[\Gamma]$ as a subset of $\mathcal{B}(\ell^2(\Gamma))$.
- **Def:** von Neumann algebra $\mathcal{N}(\Gamma)$ of Γ is the closure of $\mathbb{C}[\Gamma] \subset \mathcal{B}(\ell^2(\Gamma))$ w.r.t. pointwise convergence in $\mathcal{B}(\ell^2(\Gamma))$.
- **Example:** If Γ is finite, then $\mathbb{C}[\Gamma] = \ell^2(\Gamma) = \mathcal{N}(\Gamma)$.
- any $\mathcal{N}(\Gamma)$–module $\mathcal{M}$ has a dimension

$$\dim_{\mathcal{N}(\Gamma)}(\mathcal{M}) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

- $\mathcal{M} = 0 \iff \dim_{\mathcal{N}(\Gamma)}(\mathcal{M}) = 0$.

L^2-Betti numbers

- M – CW-complex, $\alpha : \pi_1(M) \to \Gamma$ homomorphism.
- $M_\Gamma :=$ regular cover of M defined by α.

Definition

To the pair (M, α) associate L^2-Betti numbers:

$$b_i^{(2)}(M,\alpha) := \dim_{\mathcal{N}(\Gamma)} H_i \left(C_*(M_\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{N}(\Gamma) \right) \in [0, \infty],$$

where $C_*(M_\Gamma)$ is the cellular (or singular) chain complex of M_Γ, with right Γ-action by deck transformations.

Properties of L^2-Betti numbers

- $b_i^{(2)}(M,\alpha)$ is a homotopy invariant of the pair (M,α).
- $b_0^{(2)}(M,\alpha) = \begin{cases} 0 & \text{if } Im(\alpha) \text{ is infinite} \\ \frac{1}{|Im(\alpha)|} & \text{if } Im(\alpha) \text{ is finite.} \end{cases}$
- can (and will) assume that α is *onto*, since if $Im(\alpha) \subset \widetilde{\Gamma} \subset \Gamma$:

$$b_i^{(2)}(M, \alpha : \pi_1(M) \to \widetilde{\Gamma}) = b_i^{(2)}(M, \alpha : \pi_1(M) \to \Gamma)$$

- Atiyah: if M is a *finite* CW-complex,

$$\sum_i (-1)^i b_i^{(2)}(M,\alpha) = \chi(M) = \sum_i (-1)^i b_i(M)$$

- L^2-Betti numbers tend to vanish more often than usual Betti numbers (e.g., $b_i^{(2)}$ are multiplicative for finite coverings).

Properties of L^2-Betti numbers

- L^2-Betti numbers provide obstructions to fibering over S^1, i.e.,

Lemma

Let M be a CW complex of finite type, and $f : M \to S^1$ a fibration with connected fiber F so that the epimorphism $f_* : \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ factors as $\pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} \mathbb{Z}$, with α and β epimorphisms . Then

$$b_i^{(2)}(M, \alpha) = 0 \ , \quad \text{for all } i \geq 0.$$

Atiyah's conjecture

Conjecture (Atiyah)

The L^2-Betti numbers $b_i^{(2)}(M, \alpha : \pi_1(M) \to \Gamma)$ of a finite CW-complex are rational. Moreover, if Γ is torsion-free, then $b_i^{(2)}(M, \alpha)$ are integers.

- known to be false for groups which are not finitely presented.
- proved for many groups (Linnell, Schick, etc.)
- Atiyah's conjecture $\Longrightarrow$ Kaplansky's conjecture: if Γ is torsion-free, then $\mathbb{C}[\Gamma]$ has no non-trivial zero-divisors.

Hopf-Singer Conjecture

Conjecture (Hopf)

Let M be a closed manifold of real dimension $2n$, with negative sectional curvature. Then $(-1)^n \cdot \chi(M) > 0$.

Theorem

If M^{2n} is a closed manifold with $-1 \leq sec(M) < -(1 - \frac{1}{n})^2$, then $(-1)^n \cdot \chi(M) > 0$.

Conjecture (Singer)

Let M^{2n} be a closed aspherical manifold. Then

$$b_i^{(2)}(M, id) = \begin{cases} 0, & \text{if } i \neq n, \\ (-1)^n \chi(M), & \text{if } i = n, \end{cases}$$

In particular, $(-1)^n \chi(M) \geq 0$.

Theorem (Jost-Zuo)

Let M be a compact Kähler manifold of complex dimension n, and non-positive sectional curvature. Then:

$$b_i^{(2)}(M, id) = \begin{cases} 0, & \text{for } i \neq n, \\ (-1)^n \chi(M), & \text{for } i = n, \end{cases}$$

Remark

If M carries a Riemannian metric with non-positive sectional curvature, then M is aspherical.

Affine hypersurface complements

- $f : \mathbb{C}^n \to \mathbb{C}$ square-free polynomial ($n \geq 2$).
- $X := f^{-1}(0) \subset \mathbb{C}^n$
- $M := \mathbb{C}^n \setminus X$.
- $\phi : \pi_1(M) \to \mathbb{Z}$ be the total linking number homomorphism:

$$\gamma \mapsto lk\#(\gamma, X).$$

- $\widetilde{M}$ = infinite cyclic cover of M defined by $\ker(\phi)$.
- $\alpha : \pi_1(M) \to \Gamma$ is called admissible if ϕ factors through α, i.e.,

$$\phi : \pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\widetilde{\phi}} \mathbb{Z}$$

- $\widetilde{\Gamma} := \ker(\widetilde{\phi})$
- get induced epimorphism $\widetilde{\alpha} : \pi_1(\widetilde{M}) \to \widetilde{\Gamma}$.
- Look at $b_i^{(2)}(M, \alpha)$ and $b_i^{(2)}(\widetilde{M}, \widetilde{\alpha})$.

Remark

- *$M := \mathbb{C}^n \setminus X$ is a n-dim. affine variety, so it has the homotopy type of a finite CW-complex of real dim. n. Hence*

$$b_i^{(2)}(M, \alpha) = 0 \ , \quad i > n.$$

- *$\widetilde{M}$ is an infinite CW complex, so its L^2-Betti numbers may be infinite.*
- **Theorem (M., 05)** *If X is "well-behaved at infinity", then $b_i(\widetilde{M})$ are finite for all $0 \leq i \leq n-1$.*
- *Here I give a "noncommutative" generalization of this fact.*

Weighted homogeneous hypersurfaces

Proposition

Let X be an affine hypersurface defined by a weighted homogeneous polynomial $f : \mathbb{C}^n \to \mathbb{C}$, with $M := \mathbb{C}^n \setminus X$. Then, for any admissible epimorphism $\alpha : \pi_1(M) \to \Gamma$, we have:

- *All L^2–Betti numbers $b_i^{(2)}(M, \alpha)$ of the complement M vanish.*
- *All L^2–Betti numbers $b_i^{(2)}(\widetilde{M}, \widetilde{\alpha})$ of the infinite cyclic cover $\widetilde{M}$ are finite, and $b_i^{(2)}(\widetilde{M}, \widetilde{\alpha}) = 0$ for $i \geq n$.*

Remark

$\chi(M) = 0$.

Proof.

- f is w.h., so there exist a global Milnor fibration

$$F = \{f = 1\} \hookrightarrow M = \mathbb{C}^n \setminus X \xrightarrow{f} \mathbb{C}^*.$$

- **Milnor:** F is connected and has the homotopy type of a finite CW-complex of dim. $n - 1$.
- M fibers over $S^1 \overset{Lemma}{\Longrightarrow} b_i^{(2)}(M, \alpha) = 0$, all i.
- $\widetilde{M} \simeq F$, hence $\widetilde{M}$ has the homotopy type of a *finite* CW complex of dimension $n - 1$.
- So $b_i^{(2)}(\widetilde{M}, \widetilde{\alpha})$ are finite, and $b_i^{(2)}(\widetilde{M}, \widetilde{\alpha}) = 0$ for $i \geq n$.

□

Hypersurfaces in general position at infinity

Theorem (M.)

Assume that the affine hypersurface $X \subset \mathbb{C}^n$ is in general position at infinity, i.e., the hyperplane at infinity $H \subset \mathbb{CP}^n$ is transversal in the stratified sense to the projective completion $\bar{X}$. Then, for any admissible epimorphism $\alpha : \pi_1(M) \to \Gamma$, we have:

- *The L^2–Betti numbers $b_i^{(2)}(\widetilde{M}, \widetilde{\alpha})$ of the infinite cyclic cover are finite for all $0 \leq i \leq n-1$.*
- *The L^2–Betti numbers of the complement M are computed by*

$$b_i^{(2)}(M, \alpha) = \begin{cases} 0, & \text{for } i \neq n, \\ (-1)^n \chi(M), & \text{for } i = n. \end{cases}$$

In particular,

$$(-1)^n \cdot \chi(M) \geq 0.$$

Remark

- *For plane curves ($n = 2$), the result was proved by Friedl-Leidy-M. (2009).*
- *For hyperplane arrangements, a similar result holds for $b_i^{(2)}(M, id)$, by Davis-Januszkiewitz-Leary (2007).*

Remark

Our result looks similar to Jost-Zuo's theorem, but it is independent of any metric considerations. Is there any relation between the two vanishing results?
(Note that $M = \mathbb{C}^n \setminus X$ is not aspherical in general.)

Idea of proof: reduce to the study of the "link at infinity"

- link at infinity: $X^\infty := X \cap S^\infty$, for S^∞ a large sphere in $\mathbb{C}^n$.
- $M^\infty := S^\infty \setminus X^\infty \subset M$.
- **Zariski-Lefschetz thm:** $\pi_i(M, M^\infty) = 0$, for $i \leq n-1$.
- so $\alpha^\infty : \pi_1(M^\infty) \to \pi_1(M) \overset{\alpha}{\to} \Gamma$ is onto.
- $\widetilde{M}^\infty :=$ infinite cyclic cover of M^∞ defined by α^∞.
- **Lefschetz hyperplane thm:** for all $i \leq n-1$ we have:

$$b_i^{(2)}(M, \alpha) \leq b_i^{(2)}(M^\infty; \alpha^\infty), \qquad b_i^{(2)}(\widetilde{M}, \widetilde{\alpha}) \leq b_i^{(2)}(\widetilde{M}^\infty; \widetilde{\alpha}^\infty)$$

- By general position at infinity, M^∞ is h.e. to the total space of a fibration over S^1 with connected fiber F. So, by Lemma: $b_i^{(2)}(M^\infty; \alpha^\infty) = 0$, all i.
- F is a finite dim CW complex and $\widetilde{M}^\infty \simeq F$. So $b_i^{(2)}(\widetilde{M}^\infty; \widetilde{\alpha}^\infty) < \infty$, all i.

Special case: plane curve complements

Theorem (Friedl-Leidy-M., 2009)

If $M = \mathbb{C}^2 \setminus \mathcal{C}$ for some curve $\mathcal{C}$ in general position at infinity, and $\alpha : \pi_1(X) \to \Gamma$ is admissible, then

$$b_i^{(2)}(M, \alpha) = \begin{cases} 0, & i \neq 2, \\ \chi(M), & i = 2. \end{cases}$$

Corollary

$b_i^{(2)}(M, \alpha)$ $(i \geq 0)$ depends only on the degree of $\mathcal{C}$ and on the local type of singularities, and is independent of α and of the position of singularities of $\mathcal{C}$. Indeed,

$$b_2^{(2)}(M, \alpha) = (d-1)^2 - \sum_{x \in \mathrm{Sing}(\mathcal{C})} \mu(\mathcal{C}, x),$$

where $\mu(\mathcal{C}, x)$ is the Milnor number of the germ near x.

Remark

$b_1^{(2)}(\widetilde{M}, \widetilde{\alpha})$ depends in general on the position of singularities of $\mathcal{C}$.

Example (Leidy-M.)

Let $\Gamma = \Gamma_n := \pi_1/\pi_1^{(n)}$ and $\alpha = \alpha_n : \pi_1 \to \Gamma_n$. Let $\bar{C} \subset \mathbb{CP}^2$ be a sextic with six cusps, L a generic line, and $M = \mathbb{CP}^2 \setminus (\bar{C} \cup L)$.

1. if the six cusps are on a conic, then

$$b_1^{(2)}(\widetilde{M}, \widetilde{\alpha_n}) = \begin{cases} 1, & n > 0 \\ 2, & n = 0. \end{cases}$$

2. if the six cusps are not on a conic, then $b_1^{(2)}(\widetilde{M}, \widetilde{\alpha_n}) = 0$, for all $n \geq 0$.

THANK YOU !!!