ALEXANDER INVARIANTS OF HYPERSURFACE COMPLEMENTS

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Abstract. These are notes I wrote for a series of lectures I gave at Tokyo University of Sciences and The University of Tokyo, Tokyo, Japan, May 2006.

1. Overview

• By analogy with knot theory, define global Alexander-type invariants for complex hypersurface complements.
• These were studied before in 'simple' cases: hypersurfaces with only isolated singularities (e.g. plane curves).
• **Goal:** study hypersurfaces with non-isolated singularities
• **Main result:** global Alexander invariants depend only on the local information encoded by link pairs of singular strata

2. Infinite cyclic Alexander invariants

• $V = \{f = f_1 \cdots f_s = 0\} \subset \mathbb{P}^{n+1}$ reduced, degree d hypersurface; $V_i = \{f_i = 0\}$; $d_i = \deg(V_i)$
Fix a hyperplane H, 'the hyperplane at infinity'; Set $\mathcal{U} := \mathbb{P}^{n+1} - (V \cup H)$, the 'complement'.
• $H_1(\mathcal{U}) \cong \mathbb{Z}^s$, generated by meridian loops γ_i about the non-singular part of each irreducible component V_i of V; If γ_∞ is the meridian loop about H, then in $H_1(\mathcal{U})$ there is a relation:

$$\sum_{i=1}^{s} d_i \gamma_i + \gamma_\infty = 0$$

• let $\mathcal{U}^c$ be the infinite cyclic cover of $\mathcal{U}$ corresponding to the kernel of the total linking #:

$$LK : \pi_1(\mathcal{U}) \to \mathbb{Z} \quad , \quad \alpha \mapsto \mathrm{lk}(\alpha, V \cup -dH)$$

i.e. $LK(\gamma_i) = 1$ and $LK(\gamma_\infty) = -d$.
• Under the deck group action, $H_i(\mathcal{U}^c;\mathbb{Q})$ become $\Gamma := \mathbb{Q}[t, t^{-1}]$-modules.
• $\{H_i(\mathcal{U}^c;\mathbb{Q})\}_{i\in\mathbb{Z}}$ are called **the homological Alexander modules of the hypersurface complement**;
• Γ is PID, so torsion Γ-modules have well-defined associated polynomials/orders;
• If $H_i(\mathcal{U}^c;\mathbb{Q})$ is torsion Γ-module, let $\Delta_i(t)$ be the associated polynomial;
• $\mathcal{U}$ is affine, hence h.e. to a finite CW-complex of dimension $n+1 \Rightarrow H_i(\mathcal{U}^c;\mathbb{Q})$ is of finite type over Γ (not over $\mathbb{Q}$). Moreover,

$$\begin{cases} H_i(\mathcal{U}^c;\mathbb{Q}) \cong 0, \ i > n+1 \\ H_{n+1}(\mathcal{U}^c;\mathbb{Q}) \text{ is free over } \Gamma \end{cases} \tag{2.1}$$

GOAL: Study the Γ-modules $H_i(\mathcal{U}^c;\mathbb{Q})$, for $i < n+1$.

From now on, we assume that V is transversal in the stratified sense to the hyperplane H at infinity.

2.1. **Isolated singularities.** • If V is *smooth*, then:

$$\widetilde{H}_i(\mathcal{U}^c;\mathbb{Z}) \cong 0 \text{ for } \ i < n+1$$

This follows from the general fact proved by Libgober: if V has *no codim* 1 *singularities*, then:

$$\begin{cases} \pi_1(\mathcal{U}) = \mathbb{Z} \\ \pi_i(\mathcal{U}) = 0, \text{ for } 1 < i \leq n-k-1 \end{cases} \tag{2.2}$$

where $k = \dim_{\mathbb{C}}(\mathrm{Sing(V)})$.
• If V has ***only isolated singularities***, then (Libgober):
(a)

$$\widetilde{H}_i(\mathcal{U}^c;\mathbb{Z}) \cong 0 \quad \text{for} \quad i < n$$

$$H_n(\mathcal{U}^c;\mathbb{Q}) \text{ is a torsion } \Gamma-\text{module}$$

(b) Divisibility Theorem: If $\Delta_n(t) := \mathrm{order} H_n(\mathcal{U}^c;\mathbb{Q})$, then $\Delta_n(t)$ divides (up to a power of (t-1)) the product:

$$\prod_{x \in \mathrm{Sing}(V)} \Delta_x(t)$$

of the local Alexander polynomials associated to the isolated singularities of V, so Δ_n depends on local type of singularities;
(c) The zeros of $\Delta_n(t)$ are roots of unity of order d, and $H_n(\mathcal{U}^c;\mathbb{C})$ is a semi-simple $\mathbb{C}[t,t^{-1}]$-module.
(d) $H_n(\mathcal{U}^c;\mathbb{C})$ has a natural mixed Hodge structure.

Example 2.1. Let $\bar{C} \subset \mathbb{P}^2$ is a degree d curve having only nodes and cusps as singular points. If $d \neq 0(\ 6)$, then $\Delta_C(t) = 1$ (note that for irreducible curves, we have $\Delta_C(1) = 1$).

Example 2.2. If $\bar{C} \subset \mathbb{P}^2$ is a (irreducible) sextic having only cusps singularities, and if $C = \bar{C} - L$ for L a generic line, then the global Alexander polynomial $\Delta_C(t)$ of the curve C is either 1 or a power of $t^2 - t + 1$. How to distinguish between them? If $\bar{C}$ has only 6 cusps singularities then (Zariski-Libgober):

(1) if $\bar{C}$ is in 'special *position*', i.e. the 6 cusps are on a conic, then $\Delta_C(t) = t^2 - t + 1$ (and $\pi_1(\mathbb{C}^2 - C) = B_3$).
(2) if $\bar{C}$ is in general position, i.e. the cusps are not on a conic, then $\Delta_C(t) = 1$ (and $\pi_1(\mathbb{C}^2 - C)$ is abelian).

2.2. **Non-isolated singularities.** Want to extend the previous results to non-isolated singularities case!
• Let $V \subset \mathbb{P}^{n+1}$ be reduced, degree d singular hypersurface. Assume that V is in general position at infinity, i.e. H is generic.
• *Note.* In general, for non-generic H, and V with more general singularities, the Alexander modules $H_i(\mathcal{U}^c;\mathbb{Q})$, $i \leq n$, are not torsion Γ-modules. There is a formula (Dimca, Nemethi) for their Γ-rank in terms of vanishing cycles at bifurcation points of polynomial defining $V - H$ in $\mathbb{C}^{n+1}$.

Theorem 2.3. *(M'05) For $i \leq n$, $H_i(\mathcal{U}^c;\mathbb{Q})$ is a finitely generated torsion Γ-module (and a finite dimensional $\mathbb{Q}$-vector space).*

Definition 2.4. Its order, $\Delta_i(t)$, is the characteristic polynomial of the generating covering transformation, and is called **the ith global Alexander polynomial of the hypersurface** V.

Corollary 2.5. $\mathit{rank}_{\Gamma} H_{n+1}(\mathcal{U}^c;\mathbb{Q}) = (-1)^{n+1}\chi(\mathcal{U})$.

Theorem 2.6. *(M'05) The zeros of $\Delta_i(t)$, $i \leq n$, are roots of unity of order d.*

Thm 2.3, Thm 2.6 are corollaries of the following application of Lefschetz hyperplane section theorem:

Theorem 2.7. *(M'05) Let S_∞ be a sphere of sufficiently large radius in $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} - H$. Let $\mathcal{U}^c_\infty$ be the infinite cyclic cover of $\mathcal{U}_\infty := S_\infty - (V \cap S_\infty)$ corresponding to the linking # with $V \cap S_\infty$. Then for $i < n$, there is an isomorphism of Γ-modules $H_i(\mathcal{U}^c;\mathbb{Q}) \cong H_i(\mathcal{U}^c_\infty;\mathbb{Q})$, and $H_n(\mathcal{U}^c;\mathbb{Q})$ is a quotient of $H_n(\mathcal{U}^c_\infty;\mathbb{Q})$.*

Proof. (Libgober)
First note that $S^\infty \setminus V$ is homotopy equivalent to $T(H) \setminus T(H) \cap (V \cup H)$ where $T(H)$ is the tubular neighbourhood of H for which S^∞ is the boundary. If L is a generic hyperplane in $\mathbb{CP}^{n+1}$, which we can assume belongs to $T(H)$, then by Lefschetz theorem we obtain that the composition:

$$\pi_i(L \setminus L \cap (V \cup H)) \to \pi_i(T(H) \setminus T(H) \cap (V \cup H)) \to \pi_i(\mathbb{CP}^{n+1} \setminus (V \cup H))$$

is isomorphism for $i < n$ and surjective for $i = n$. That is, $\pi_i(\mathcal{U}, \mathcal{U}_\infty) \cong 0$ for all $i \leq n$. Therefore, the same is true for any covering, in particular for the infinite cyclic coverings: $\pi_i(\mathcal{U}^c, \mathcal{U}^c_\infty) \cong 0$ for all $i \leq n$. Hence, by Hurewicz Theorem, the vanishing also holds for the relative homology groups, i.e., the maps of groups $H_i(\mathcal{U}^c_\infty) \to H_i(\mathcal{U}^c)$ are isomorphism for $i < n$ and onto for $i = n$. Since these maps are induced by an embedding, the above are morphisms of modules over the ring of Laurent polynomials in the variable t.

□

Definition 2.8. $\{H_i(\mathcal{U}^c_\infty;\mathbb{Q})\}_i$ are called *the Alexander modules at infinity.*

Proof. By Thm 2.7, STS Thm 2.3 & 2.6 for $H_i(\mathcal{U}^c_\infty;\mathbb{Q})$: Let $V \cap H = \{g = 0\} \subset H \cong \mathbb{P}^n$. Then

$S_\infty - (V \cap S_\infty) \simeq \mathbb{C}^{n+1} - \text{affine cone}(V \cap H) = \text{total space of Milnor fibration } F \hookrightarrow \mathbb{C}^{n+1} - \{g = 0\} \xrightarrow{g} \mathbb{C}^*$

Moreover: $\mathcal{U}^c_\infty \simeq F$ and the generator of the deck group corresponds to the monodromy of the Milnor fibration. Thus $H_i(\mathcal{U}^c_\infty;\mathbb{Q}) \cong H_i(F,\mathbb{Q})$ as Γ-modules, so for $i \leq n$ they are torsion, semi-simple, annihilated by $t^d - 1$. □

Remark 2.9. *Baby case: Projective hypersurface arrangements in $\mathbb{P}^n$*
If V is the projective cone on a degree d reduced hypersurface $Y = \{f = 0\} \subset \mathbb{CP}^n$, then there is a Γ-module isomorphism: $H_i(\mathcal{U}^c;\mathbb{Q}) \cong H_i(F;\mathbb{Q})$, where $F = f^{-1}(1)$ is the fiber of the global Milnor fibration $\mathcal{U} = \mathbb{C}^{n+1} - f^{-1}(0) \xrightarrow{f} \mathbb{C}^*$ associated to the homogeneous polynomial f, and the module structure on $H_i(F;\mathbb{Q})$ is induced by the monodromy action. Hence zeros of the global Alexander polynomials of V coincide with the eigenvalues of the monodromy operators acting on the homology of F. Since the monodromy homeomorphism has finite order d, all these eigenvalues are roots of unity of order d. So a polynomial in general position at infinity behaves much like a homogeneous polynomial.

Theorem 2.10. *(Dimca-Libgober '05) For $i \leq n$, there is a mixed Hodge structure on $H_i(\mathcal{U}^c;\mathbb{Q})$.*

2.2.1. *Divisibility.* Let $\mathcal{S}$ be a Whitney stratification of V, i.e. a decomposition of V into disjoint connected non-singular subvarieties $\{S_\alpha\}$, called *strata*, s.t. V is *uniformly singular* along each stratum. This yields a Whitney stratification of the pair $(\mathbb{P}^{n+1}, V)$, with $\mathcal{S}$ the set of singular strata. Fix $S \in \mathcal{S}$ a s-dim stratum of $(\mathbb{P}^{n+1}, V)$. A point $p \in S$ has a *distinguished neighborhood* in $(\mathbb{P}^{n+1}, V)$, which is homeo in a stratum-preserving way to

$$\mathbb{C}^s \times c^\circ(S^{2n-2s+1}(p), L(p))$$

The *link pair* $(S^{2n-2s+1}(p), L(p))$ has constant topological type along S, denoted $(S^{2n-2s+1}, L)$. This is a singular algebraic link, and has an associated local Milnor fibration:

$$F^s \hookrightarrow S^{2n-2s+1} - L \to S^1$$

with fibre F^s and monodromy $h^s : F^s \to F^s$. Let $\Delta^s_r(t) = \det(tI - (h^s)_* : H_r(F^s) \to H_r(F^s))$ be the r-th (local) Alexander polynomial associated to S.

Theorem 2.11. *(M'05) Fix an arbitrary irreducible component of V, say V_1, and fix $i \leq n$. Then the zeros of $\Delta_i(t)$ are among the zeros of polynomials $\Delta_r^s(t)$, associated to strata $S \subset V_1$, s.t. $n-i \leq s = dimS \leq n$, and r is in the range $2n-2s-i \leq r \leq n-s$. Moreover, if V has no codimension 1 singularities and is a* rational homology manifold*, then $\Delta_i(1) \neq 0$.*

Remark 2.12. 0-dim strata of V only contribute to Δ_n, 1-dim strata contribute to Δ_n and Δ_{n-1} and so on.

Corollary 2.13. (Vanishing of Alexander polynomials) *Let V be a degree d hypersurface in general position at infinity, which is rationally smooth and has no codimension 1 singularities. Assume that the local monodromies of link pairs of strata contained in some irreducible component V_1 of V have orders which are relatively prime to d (e.g., the transversal singularities along strata of V_1 are Brieskorn-type singularities, having all exponents relatively prime to d). Then $\Delta_i(t) \sim 1$, for $1 \leq i \leq n$.*

Example 2.14. *(One-dimensional singularities)* $V = \{y^2z + x^3 + tx^2 + v^3 = 0\} \subset \mathbb{P}^4 = \{(x:y:z:t:v)\}$, $H = \{t = 0\}$.

Theorem 2.15. *(M'05) If V has only isolated singularities, $\Delta_n(t)$ divides (up to a power of $(t-1)$) the product $\prod_{p \in V_1 \cap Sing(V)} \Delta_p(t)$ of local Alexander polynomials of links of the singular points p of V which are contained in V_1.*

Remark 2.16. Thm 2.15 shows the weakness of Alexander polynomial of plane curves: e.g. if C is a union of 2 curves that intersect transversally, then (Oka): $\Delta_C(t) = (t-1)^{s-1}$. To overcome this problem, study higher-order coverings of the complement.

Remark 2.17. *Projective hypersurface arrangements*
Apply the divisibility result to the case when V is the cone over a projective hypersurface arrangement in $\mathbb{P}^n$. Get a similar result for the characteristic polynomials of monodromy operators of the Milnor fiber of the arrangement.

Corollary 2.18. (Triviality of monodromy) *If $\lambda \neq 1$ is a d-th root of unity such that λ is not an eigenvalue of any of the local monodromies corresponding to link pairs of singular strata of Y_1 in a stratification of the pair $(\mathbb{P}^n, Y)$, then λ is not an eigenvalue of the monodromy operators acting on $H_q(F)$ for $q \leq n-1$.*

Remark 2.19. Let $P_q(t)$ be the characteristic polynomial of the monodromy operator $h_q : H_q(F) \to H_q(F)$. The polynomials $P_i(t)$, $i = 0, \cdots, n$, are related by the formula:

$$\prod_{q=0}^{n} P_q(t)^{(-1)^{q+1}} = (1-t^d)^{-\chi(F)/d}$$

where $\chi(F)$ is the Euler characteristic of the Milnor fiber. Therefore, it suffices to compute only the polynomials $P_0(t), \cdots, P_{n-1}(t)$ and the Euler characteristic of F.

Example 2.20. If $\bigcup_{i=1}^{s} Y_i$ is a normal crossing divisor at any point $x \in Y_1$, the monodromy action on $H_q(F;\mathbb{Q})$ is trivial for $q \leq n-1$.

2.2.2. *On the proof of Divisibility Thm: Intersection Alexander modules.* A Whitney stratification $\mathcal{S}$ of V induces stratifications of pairs $(\mathbb{P}^{n+1}, V)$ and $(\mathbb{P}^{n+1}, V \cup H)$. Define local system $\mathcal{L}$ on $\mathcal{U} = \mathbb{P}^{n+1} - V \cup H$:

$$\begin{cases} \text{stalk} : \Gamma = \mathbb{Q}[t, t^{-1}] \\ \text{for } \alpha \in \pi_1(\mathcal{U}), \gamma \in \Gamma : \alpha \times \gamma = t^{\mathrm{lk}(V \cup -dH, \alpha)}\gamma \end{cases} \tag{2.3}$$

• The intersection homology complex:

$$IC^{\bullet}_{\bar{p}} := IC^{\bullet}_{\bar{p}}(\mathbb{P}^{n+1}, \mathcal{L})$$

is defined for any perversity $\bar{p}$ by Deligne's axioms.

Definition 2.21. The modules

$$IH_i^{\bar{m}}(\mathbb{P}^{n+1};\mathcal{L}) := \mathbb{H}^{-i}(\mathbb{P}^{n+1}; IC^\bullet_{\bar{m}})$$

are called the *intersection Alexander modules of the hypersurface* V.

Lemma 2.22. *(M'05)*

$$IC^\bullet_{\bar{m}}|_{V\cup H} \cong 0$$

Theorem 2.23. *(M'05) There is an isomorphism of* Γ*-modules:*

$$IH_i^{\bar{m}}(\mathbb{P}^{n+1};\mathcal{L}) \cong H_i(\mathcal{U};\mathcal{L}) \cong H_i(\mathcal{U}^c;\mathbb{Q})$$

so the intersection Alexander modules are isomorphic to the Alexander modules of the hypersurface complement.

One can use derived categories, the sheaf-theoretical intersection homology and the functorial language for the study of the Alexander modules of hypersurface complements.

3. Universal abelian Alexander invariants (joint with Alex Dimca)

• By analogy with the case of links in S^3, Hironaka and Libgober defined new topological invariants of a plane curve complement: the sequence of characteristic varieties.
• The new invariants were mainly used to obtain information about all abelian covers of $\mathbb{P}^2$, branched along a curve.
• They were previously studied in relation with the cohomology support loci of rank 1 local systems defined on the complement of a hyperplane arrangement or a plane curve.
• **Goal:** Study universal abelian invariants of complements to arbitrary hypersurfaces.

3.1. **Definitions.** • Let R be a commutative ring with unit, which is Noetherian and UFD. Let A be a finitely generated R-module and M a presentation matrix of A given by the sequence

$$R^m \to R^n \to A \to 0$$

Definition 3.1. *(a) The* i***-th elementary ideal*** $\mathcal{E}_i(A)$ *of* A *is the ideal in* R *generated by the* $(n-i)\times(n-i)$ *minor determinants of* M*, with the convention that* $\mathcal{E}_i(A) = R$ *if* $i \geq n$*, and* $\mathcal{E}_i(A) = 0$ *if* $n-i > m$*.*
(b) The ***support*** $Supp(A)$ *of* A *is the reduced sub-scheme of* $Spec(R)$ *defined by the order ideal* $\mathcal{E}_0(A)$*. Equivalently, if* $P \in Spec(R)$ *then* $P \in Supp(A)$ *iff* $A_P \neq 0$*, i.e.,*

$$Supp(A) = \{P \in Spec(R), P \supset Ann(A)\}$$

(c) The i***-th characteristic variety*** $V_i(A)$ *of* A *is the reduced sub-scheme of* $Spec(R)$ *defined by the (*i*-th* Fitting ideal*) ideal* $\mathcal{E}_{i-1}(A)$*. Equivalently,*

$$V_i(A) = Supp(R/\mathcal{E}_{i-1}(A)) = Supp(\wedge^i A).$$

(note: $V_1(A) = Supp(A)$*).*

Note. $V_i(A)$ are invariants of the R-isomorphism type of A, i.e., independent (up to multiplication by a unit of R) of all choices.

3.2. **Hypersurface complements.** Recall that $H_1(\mathcal{U}) \cong \mathbb{Z}^s$, freely generated by meridian loops γ_i about the non-singular part of each irreducible component V_i of V;
• Let $\mathcal{U}^{ab}$ be the universal abelian cover of $\mathcal{U}$, i.e., given by the kernel of the linking #:

$$lk : \pi_1(\mathcal{U}) \to \mathbb{Z}^s$$

$$\alpha \mapsto (\mathrm{lk}(\alpha, V_1 \cup -d_1 H), \cdots, \mathrm{lk}(\alpha, V_s \cup -d_s H)),$$

i.e., $lk(\gamma_i) = e_i$ and $lk(\gamma_\infty) = (-d_1, \cdots, -d_s)$.
• The deck group of $\mathcal{U}^{ab}$ is $\mathbb{Z}^s$.
• Set $\Gamma_s := \mathbb{C}[\mathbb{Z}^s] \cong \mathbb{C}[t_1, t_1^{-1}, \cdots, t_s, t_s^{-1}]$. Then Γ_s is a regular Noetherian domain.
• Define a local coefficient system $\mathcal{L}$ on $\mathcal{U}$, with stalk Γ_s and action:

$$\pi_1(\mathcal{U}) \ni \alpha \mapsto \prod_{j=1}^{s} (t_j)^{\mathrm{lk}(\alpha, V_j \cup -d_j H)}.$$

In particular, $\gamma_j \mapsto t_j$, for $j = 1, \cdots, s$.
• let $\bar{\mathcal{L}}$ be the local system obtained from $\mathcal{L}$ by composing all module structures with the natural involution of Γ_s, sending t_j to t_j^{-1}, for $j = 1, \cdots, s$.

Definition 3.2. *The* ***universal homology k-th Alexander module*** *of* $\mathcal{U}$ *is* $A_k(\mathcal{U}) := H_k(\mathcal{U}, \mathcal{L})$, *i.e., the group* $H_k(\mathcal{U}^{ab}; \mathbb{C})$ *regarded as a* Γ_s*-module via the action of the deck group.*
The ***universal cohomology k-th Alexander module*** *of* $\mathcal{U}$ *is defined as* $A^k(\mathcal{U}) := H^k(\mathcal{U}; \bar{\mathcal{L}})$.

• *Note.* If C_* is the cellular complex of $\mathcal{U}^{ab}$, as $\mathbb{Z}[\mathbb{Z}^s]$-modules, and if $C_*^0 := C_* \otimes \mathbb{C}$, then:

$$A_k(\mathcal{U}) = H_k(C_*^0), \quad A^k(\mathcal{U}) = H_k(\mathrm{Hom}_{\Gamma_s}(C_*^0, \Gamma_s))$$

• Facts:
  - the modules $A^k(\mathcal{U})$ and resp. $A_k(\mathcal{U})$ are trivial for $k > n+1$.
  - $A_{n+1}(\mathcal{U})$ is a torsion-free Γ_s-module.
  - $A^k(\mathcal{U})$, $A_k(\mathcal{U})$ are Γ_s-modules of finite type, so have well-defined characteristic varieties, which are sub-varieties of the s-dimensional torus $\mathbb{T}^s = (\mathbb{C}^*)^s$ = the set of closed points in $\mathrm{Spec}(\Gamma_s)$.

Notation: For $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathbb{T}^s$, let m_λ the corresponding maximal ideal in Γ_s, and $\mathbb{C}_\lambda := \Gamma_s / m_\lambda \Gamma_s$. Note: $\mathbb{C}_\lambda \cong \mathbb{C}$ and the canonical projection $\rho_\lambda : \Gamma_s \to \Gamma_s / m_\lambda \Gamma_s = \mathbb{C}_\lambda$ corresponds to replacing each t_j by λ_j.
If A is a Γ_s-module, denote by A_λ the localization of A at m_λ. For $A = \Gamma_s$, use Γ_λ.
• If A is of finite type, then $A = 0$ iff $A_\lambda = 0$ for all $\lambda \in \mathbb{T}^s$. Hence

$$\mathrm{Supp}(A) = \{\lambda \in \mathbb{T}^s ; A_\lambda \neq 0\}$$

Note: $A_0(\mathcal{U}) = \mathbb{C}_{\mathbf{1}}$, where $\mathbf{1} = (1, \cdots, 1)$, so $\mathrm{Supp}(A_0(\mathcal{U})) = \{\mathbf{1}\}$.
• Let $V_{i,k}(\mathcal{U})$ the i-th characteristic variety of $A_k(\mathcal{U})$, and $V^{i,k}(\mathcal{U})$ that of $A^k(\mathcal{U})$. For each universal Alexander module, its characteristic varieties form a decreasing filtration of the character torus $\mathbb{T}^s$.
Note. All definitions work also in the local setting, i.e., when $\mathcal{U}$ is a complement of a hypersurface germ in a small ball.

Example 3.3. (a) The support of the universal homological Alexander module of a link complement in S^3 is the set of zeros of the multivariable Alexander polynomial.
(b) In the case of irreducible hypersurfaces, the infinite cyclic and universal abelian cover coincide, so the support of an Alexander module is the zero set of the associated one-variable polynomial.

3.3. **Study of Supports.**

3.3.1. *Homology vs. cohomology invariants.*

$$(\text{UCSS}): \quad \text{Ext}^q_{\Gamma_s}(A_p(\mathcal{U}), \Gamma_s) \Rightarrow A^{p+q}(\mathcal{U})$$

Relations b/w the corresponding characteristic varieties are obtained by localizing at any $\lambda \in \mathbb{T}^s$:

$$(\text{LUCSS}): \quad \text{Ext}^q_{\Gamma_\lambda}(A_p(\mathcal{U})_\lambda, \Gamma_\lambda) \Rightarrow A^{p+q}(\mathcal{U})_\lambda.$$

Notation: For a fixed $\lambda \in \mathbb{T}^s$, let:

$$k(\lambda) = \min\{m \in \mathbb{N}; A_m(\mathcal{U})_\lambda \neq 0\}$$

Proposition 3.4. *For any $\lambda \in \mathbb{T}^s$, $A^k(\mathcal{U})_\lambda = 0$ for $k < k(\lambda)$, and*

$$A^{k(\lambda)}(\mathcal{U})_\lambda = Hom(A_{k(\lambda)}(\mathcal{U})_\lambda, \Gamma_\lambda).$$

Example 3.5. (i) Let $\mathcal{U}$ be the complement of a NC divisor germ in a small ball. Then $\mathcal{U}^{ab}$ is contractible, so $A_0(\mathcal{U}) = \mathbb{C}_{\mathbf{1}}$ and $A_k(\mathcal{U}) = 0$ for $k > 0$. Moreover, for any $\lambda \neq \mathbf{1}$, $A^k(\mathcal{U})_\lambda = 0$, for any k.
(ii) Let $(Y, 0)$ be an INNC singularity germ at the origin of $\mathbb{C}^{n+1}$, and $\mathcal{U}$ be its complement in a small open ball centered at the origin in $\mathbb{C}^{n+1}$. Assume that $n \geq 2$.
Then $\mathcal{U}^{ab}$ is $(n-1)$-connected, being a bouquet of n-spheres (Dimca-Libgober). Hence $A_0(\mathcal{U}) = \mathbb{C}_{\mathbf{1}}$ and $A_k(\mathcal{U}) = 0$ for $k \neq n$. Moreover, for $\lambda \neq \mathbf{1}$, get $A^k(\mathcal{U})_\lambda = 0$ for $k < n$.

3.3.2. *Characterisctic varieties vs. (Co)homology support loci of rank* 1 *local systems.* Let $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathbb{T}^s$, and $\mathcal{L}_\lambda$ the local system on $\mathcal{U}$ with stalk $\mathbb{C} = \mathbb{C}_\lambda$ and action:

$$\pi_1(\mathcal{U}) \ni \alpha \mapsto \prod_{j=1}^{s} (\lambda_j)^{\text{lk}(\alpha, V_j \cup -d_j H)}$$

Definition 3.6. Define *topological characteristic varieties* by

$$V^t_{i,k}(\mathcal{U}) = \{\lambda \in \mathbb{T}^s; \dim_{\mathbb{C}} H_k(\mathcal{U}, \mathcal{L}_\lambda) > i\}$$

and similarly for cohomology.

• Note that

$$H_k(\mathcal{U}, \mathcal{L}_\lambda) = H_k(C^0_* \otimes_{\Gamma_s} \mathbb{C}_\lambda).$$

So by Künneth spectral sequence get:

$$E^2_{p,q} = Tor^{\Gamma_s}_p(A_q(\mathcal{U}), \mathbb{C}_\lambda) \Rightarrow H_{p+q}(\mathcal{U}, \mathcal{L}_\lambda).$$

• By localization and base change for Tor under $\Gamma_s \to \Gamma_\lambda$, get:

$$E^2_{p,q} = Tor^{\Gamma_\lambda}_p(A_q(\mathcal{U})_\lambda, \mathbb{C}_\lambda) \Rightarrow H_{p+q}(\mathcal{U}, \mathcal{L}_\lambda).$$

Proposition 3.7. *For any point $\lambda \in \mathbb{T}^s$:*
(1) min$\{m \in \mathbb{N},\ H_m(\mathcal{U}, \mathcal{L}_\lambda) \neq 0\}$ = min$\{m \in \mathbb{N},\ \lambda \in Supp(A_m(\mathcal{U}))\}$,
i.e., min$\{m \in \mathbb{N},\ \lambda \in V^t_{0,m}(\mathcal{U})\} = k(\lambda)$.
(2) dim$H_{k(\lambda)}(\mathcal{U}, \mathcal{L}_\lambda)$ = max$\{m \in \mathbb{N},\ \lambda \in V_{m,k(\lambda)}(\mathcal{U})\}$.

• There is also a spectral sequence

$$E_2^{p,q} = \text{Ext}^q_{\Gamma_\lambda}(A_p(\mathcal{U})_\lambda, \mathbb{C}_\lambda) \Rightarrow H^{p+q}(\mathcal{U}, \mathcal{L}_{\lambda^{-1}})$$

Thus: $H^m(\mathcal{U}, \mathcal{L}_{\lambda^{-1}}) = 0$ for $m < k(\lambda)$, and

$$H_{k(\lambda)}(\mathcal{U}, \mathcal{L}_\lambda)^* = H^{k(\lambda)}(\mathcal{U}, \mathcal{L}_{\lambda^{-1}})$$

Example 3.8. (cont. of Example 3.5)
(1) Let $\mathcal{U}$ be the complement of a NC divisor germ in a small ball. Then for all non-trivial characters $\lambda \in \mathbb{T}^s$ and all $k \in \mathbb{Z}$ we have that: $H_k(\mathcal{U};\mathcal{L}_\lambda) = H^{k(\lambda)}(\mathcal{U},\mathcal{L}_{\lambda^{-1}}) = 0$.
(2) Let $(Y,0)$ be an INNC singularity germ at the origin of $\mathbb{C}^{n+1}$, and $\mathcal{U}$ be its complement in a small open ball centered at the origin in $\mathbb{C}^{n+1}$. Assume that $n \geq 2$. Then for all non-trivial characters $\lambda \in \mathbb{T}^s$ and all $k < n$ we have that: $H_k(\mathcal{U};\mathcal{L}_\lambda) = H^{k(\lambda)}(\mathcal{U},\mathcal{L}_{\lambda^{-1}}) = 0$.

3.4. **Local versus Global.** Assume that V is transversal to the hyperplane at infinity H.

Theorem 3.9. *For $k \leq n$,*
$$Supp(A^k(\mathcal{U})) \subset \{t_1^{d_1} \cdots t_s^{d_s} - 1 = 0\},$$
thus has positive codimension in $\mathbb{T}^s$.

This generalizes a result of Libgober on supports of plane affine curves in general position at infinity, and is the analogue of the torsion property from the infinite cyclic case.

• For $x \in V$, let $\mathcal{U}_x = \mathcal{U} \cap B_x$, for B_x a small open ball at x in $\mathbb{P}^{n+1}$. Set $\mathcal{L}_x = \mathcal{L}_{|\mathcal{U}_x}$. Then:

Theorem 3.10. *Let $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathbb{T}^s$ and $\epsilon \in \mathbb{Z}_{\geq 0}$. Fix an irreducible component V_1 of V, and assume that $\lambda \notin Supp(H^q(\mathcal{U}_x, \bar{\mathcal{L}}_x))$ for all $q < n+1-\epsilon$ and all points $x \in V_1$. Then $\lambda \notin Supp(A^q(\mathcal{U}))$ for all $q < n+1-\epsilon$.*

The proofs of Theorems 3.9 and 3.10 rely in an essential way on the theory of perverse sheaves and Artin-type vanishing results.
• The tranversality at infinity, together with (UCSS) imply that $H^*(\mathcal{U}_x, \bar{\mathcal{L}}_x)$ can be expressed only in terms of the local universal homology Alexander modules $A_*(\mathcal{U}'_x)$, where $\mathcal{U}' := \mathbb{P}^{n+1} - V$ and $\mathcal{U}'_x := \mathcal{U}' \cap B_x$. The latter depend only on the singularity germ (V,x).

Corollary 3.11. NC & INNC
($\epsilon = 0$) If V is a NC divisor at any point $x \in V_1$, then: $Supp(A^k(\mathcal{U})) \subset \{\mathbf{1}\}$ for any $k < n+1$.
($\epsilon = 1$) If V is an INNC divisor at any point $x \in V_1$, then $Supp(A^k(\mathcal{U})) \subset \{\mathbf{1}\}$ for any $k < n$.

3.5. **Jumping loci of rank-one local systems.** For $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathbb{T}^s$, let $\mathcal{L}_\lambda$ be the corresponding local system on $\mathcal{U}$. Let $\alpha_j \in \mathbb{C}$ be s.t. $\exp(-2\pi i \alpha_j) = \lambda_j$ for $j = 1, ..., s$. $\{\alpha_j\}$ are called *the residues* of the connection ∇_α defined below. It follows that $\mathcal{L}_\lambda = \mathrm{Ker}(\nabla_\alpha)$, where:
$$\nabla_\alpha : \mathcal{O}_{\mathcal{U}} \to \Omega^1_{\mathcal{U}}, \quad \nabla_\alpha(u) = du + u \cdot \omega_\alpha,$$
for
$$\omega_\alpha = \sum_{j=1,s} \alpha_j \frac{dg_j}{g_j}$$
and $g_j(x_1, \cdots, x_{n+1}) = f_j(1, x_1, \cdots, x_{n+1})$.
Alternatively,
$$\omega_\alpha = \sum_{j=0,s} \alpha_j \frac{df_j}{f_j}$$
where $\alpha_0 = -\sum_{j=1,s} d_j \cdot \alpha_j$ and $f_0 = x_0$.

It follows that the complex $(\Omega^*_{\mathcal{U}}, \nabla_\alpha)$ is a soft resolution of $\mathcal{L}_\lambda$, therefore we have:
$$H^m(\mathcal{U}, \mathcal{L}_\lambda) = H^m(\Gamma(\mathcal{U}, \Omega^*_{\mathcal{U}}), \nabla_\alpha).$$
This result is not useful for explicit calculations, as the groups $\Gamma(\mathcal{U}, \Omega^*_{\mathcal{U}})$ are too large. Instead, we use *logarithmic connections*.
• Let $(Z,D) \xrightarrow{\pi} (\mathbb{P}^{n+1}, V \cup H)$ be an embedded resolution of singularities for the reduced divisor $V \cup H$. Set $\tilde{\nabla}_\alpha = \pi^*\nabla_\alpha$, and let ρ_i be the residue of $\tilde{\nabla}_\alpha$ along the component D_i of D.

Proposition 3.12. *Assume that $\alpha = (\alpha_0, \alpha_1, ..., \alpha_s)$ is an admissible choice of residues for $\mathcal{L}_\lambda$ (i.e. $\rho_i \notin \mathbb{N}_{>0}$, all i), and that $H^m(\mathcal{U})$ is pure of type (m,m) for all $m \leq k$. Then for all $m \leq k$:*

$$H^m(\mathcal{U}, \mathcal{L}_\lambda) = H^m(H^*(\mathcal{U}), \omega_\alpha \wedge)$$

Proof. First note that there is a Hodge-Deligne spectral seq.

$$E_1^{p,q} = H^q(Z, \Omega_Z^p(logD)) \Rightarrow H^{p+q}(\mathcal{U}, \mathbb{C})$$

degenerating at E_1 and inducing the Hodge filtration of the Deligne MHS on $H^{p+q}(\mathcal{U}, \mathbb{C})$. If the Deligne MHS on $H^m(\mathcal{U})$ is pure of type (m,m), then the only non-trivial graded piece in $H^m(\mathcal{U})$ is $gr_F^m gr_{2m}^W H^m(\mathcal{U}) = gr_F^m H^m(\mathcal{U}) = F^m H^m(\mathcal{U})$. Therefore, for all $q > 0$ and $p+q = m$ we have that $0 = F^p/F^{p+1} = H^q(Z, \Omega_Z^p(logD))$ and

$$H^m(\mathcal{U}) = gr_F^m H^m(\mathcal{U}) = \Gamma(Z, \Omega_Z^m(logD)).$$

Next, following Deligne, for any admissible choice of residues, there is a spectral sequence

$$E_1^{p,q} = H^q(Z, \Omega_Z^p(logD)) \Rightarrow H^{p+q}(\mathcal{U}, \mathcal{L}_\lambda)$$

whose differential d_1 is induced by $\tilde{\nabla}_\alpha = d + \omega_\alpha$. Now if $H^m(\mathcal{U})$ is pure of type (m,m) for $m \leq k$, then the first page of the spectral sequence above looks like:

$$E_1^{p,q} = 0 \text{ for } p+q = m \text{ and } q > 0$$

and

$$E_1^{m,0} = H^0(Z, \Omega_Z^m(logD)) = H^m(\mathcal{U}).$$

Also, in this range, the differential d_1 on E_1 reduces to $\wedge \omega_\alpha$ since $d = 0$ on closed forms. The conclusion follows.

□

Example 3.13. If V is either a hyperplane arrangement or a smooth rational curve arrangement (i.e. the irreducible components are lines or smooth conics), then $H^m(\mathcal{U})$ is pure of type (m,m) for all m. Get the results of [ESV], [STV], [Cog].

- More generally, let $\mathcal{U}_0 = \mathbb{P}^{n+1} - V$. Then $\sum_{i=1}^s d_i \gamma_i = 0$, so use the 1-form

$$\omega_\alpha = \sum_{j=1,s} \alpha_j \frac{df_j}{f_j}$$

where α satisfies: $\sum_{j=1,s} d_j \cdot \alpha_j = 0$. Then Prop. 3.12 holds in this setting.

Example 3.14. Let $s = 2$, $n > 1$ and assume that:
(i) each V_i has at most isolated singularities and is a $\mathbb{Q}$-manifold;
(ii) $V' = V_1 \cap V_2$ has at most isolated singularities (e.g., $d_1 < d_2$ and V_2 is smooth).
Then:

(1) $H^0(\mathcal{U}_0) = \mathbb{C}$ is pure of type (0,0),
$H^1(\mathcal{U}_0) = \mathbb{C}$ is pure of type (1,1), spanned by

$$\omega_1 = d_2 \cdot \frac{df_1}{f_1} - d_1 \cdot \frac{df_2}{f_2}$$

(note: $\omega_\alpha = a(\alpha)\omega_1$; indeed, $\omega_\alpha = \alpha_1 \frac{df_1}{f_1} + \alpha_2 \frac{df_2}{f_2}$ with $d_1\alpha_1 + d_2\alpha_2 = 0$).

(2) $H^k(\mathcal{U}_0) = 0$ for $1 < k < n$.

(3) $H^n(\mathcal{U}_0)$ is pure of weight $n+2$ and

$$b_n(\mathcal{U}_0) \leq \dim H_0^n(V').$$

If $d_1 < d_2$ and V_2 is smooth, then

$$H^n(\mathcal{U}_0) = 0.$$

(4) $H^{n+1}(\mathcal{U}_0)$ has weights $n+2$ and $n+3$.

Proof. Let $\mathcal{U}_i = \mathbb{CP}^{n+1} \setminus V_i$, $i=1,2$. So $\mathcal{U}_0 = \mathcal{U}_1 \cap \mathcal{U}_2$. Set $\mathcal{U}' = \mathbb{CP}^{n+1} \setminus V' = \mathcal{U}_1 \cup \mathcal{U}_2$. Then the Mayer-Vietoris sequence of the covering $\mathcal{U}'$ reads like

$$... \to H^{k-1}(\mathcal{U}_0) \to H^k(\mathcal{U}') \to H^k(\mathcal{U}_1) \oplus H^k(\mathcal{U}_2) \to H^k(\mathcal{U}_0) \to ... \tag{3.1}$$

Here and in the sequel the constant coefficients $\mathbb{C}$ are used unless stated otherwise.

Since V_i is a $\mathbb{Q}$-manifold with only isolated singularities, we have that

$$H^j(\mathcal{U}_i) = 0, \quad \text{for } 1 \leq j \leq n$$

for $i=1,2$. Indeed, here we need the following Lefschetz theorem: *Let V be a n-dimensional complete intersection in $\mathbb{P}^{n+c}$ and let $l = dim_{\mathbb{C}} Sing(V)$. Then $H^j(V) \cong H^j(\mathbb{P}^{n+c})$ for $n+l+2 \leq j \leq 2n$.* [1] In our case, $c=1$ and $l=-1$ since we work over $\mathbb{C}$. Therefore, for $i=1,2$, $H^j(V_i) = H^j(\mathbb{P}^{n+1})$ for $n+1 \leq j \leq 2n$. Now, by the long sequence of the pair $(\mathbb{P}^{n+1}, V_i)$, get $H^j(\mathbb{P}^{n+1}, V_i) = 0$ for $n+2 \leq j \leq 2n+1$. So, as $\mathbb{C}$-vector spaces (ignore the Hodge structures for the moment), we have that

$$H^j(\mathcal{U}_i) \cong H_j(\mathcal{U}_i)^\vee \overset{P.D.}{\cong} H^{2n+2-j}(\mathbb{P}^{n+1}, V)^\vee = 0$$

if $n+2 \leq 2n+2-j \leq 2n+1$, i.e., $1 \leq j \leq n$.

Now, from the sequence 3.1, it follows that $H^j(\mathcal{U}_0) \cong H^{j+1}(\mathcal{U}')$ for $1 \leq j \leq n-1$, and we have a monomorphism $0 \to H^n(\mathcal{U}_0) \to H^{n+1}(\mathcal{U}')$. Thus for proving (2), it suffices to show that $H^j(\mathcal{U}') = 0$ for $3 \leq j \leq n$. But this follows from the Lefschetz theorem mentioned above by noting that V' is a $n-1$-dimensional complete intersection in $\mathbb{P}^{n+1}$.

For proving (3) and (4), we write what's left of the sequence 3.1 in its upper part:

$$0 \to H^n(\mathcal{U}_0) \to H^{n+1}(\mathcal{U}') \to H^{n+1}(\mathcal{U}_1) \oplus H^{n+1}(\mathcal{U}_2) \to H^{n+1}(\mathcal{U}_0) \to H^{n+2}(\mathcal{U}') \to 0$$

Next, note that the Alexander duality isomorphism is compatible with the MHS after taking the Tate twist $(-n-1)$, so

$$H^j(\mathcal{U}') = H^{2n+2-j}(\mathbb{CP}^{n+1}, V')^\vee(-n-1) = H_0^{2n+1-j}(V')^\vee(-n-1) \tag{3.2}$$

Moreover, since V' has only isolated singularities, the group $H_0^k(V')$ has a pure Hodge structure of weight k and the Hodge numbers satisfy:

$$h^{p,q}(H^k(\mathcal{U}')) = h^{n+1-p,n+1-q}(H_0^{2n+1-k}(V')). \tag{3.3}$$

Therefore, $H^{n+1}(\mathcal{U}') \cong H_0^n(V')^\vee(-n-1)$ is pure of weight $n+2$, thus the same is true for $H^n(\mathcal{U}_0)$. The bound on $b_n(\mathcal{U}_0)$ is obvious.

Similarly, $H^{n+2}(\mathcal{U}')$ is pure HS of weight $n+3$, and $H^{n+1}(\mathcal{U}_i)$ $(i=1,2)$ is pure of weight $n+2$. The rest follows.

□

Corollary 3.15. *By the above proposition, for a non-trivial rank one local system $\mathcal{L}$ on $\mathcal{U}_0$ for which an admissible choice of residues $\alpha = (d_2 \cdot a(\alpha), -d_1 \cdot a(\alpha))$ exists, get:*

(1) $H^k(\mathcal{U}_0, \mathcal{L}) = 0$ *for* $k < n$.

(2) *If* $H_0^n(V') = 0$ *(e.g., if V' is a $\mathbb{Q}$-homology manifold) or if $d_1 < d_2$ and V_2 is smooth, then:* $H^n(\mathcal{U}_0, \mathcal{L}) = 0$.

[1] In general, for a hypersurface V we only have that $H^*(\mathbb{P}^{n+1}) \to H^*(V)$ is a monomorphism for $0 \leq * \leq 2n$. Thus, as $\mathbb{C}$-vector spaces:

$$H^j(\mathcal{U}) \cong H_j(\mathcal{U})^\vee \overset{P.D.}{\cong} H^{2n+2-j}(\mathbb{P}^{n+1}, V)^\vee \cong H_0^{2n+1-j}(V)^\vee,$$

where $H_0^*(V) := Coker\left(H^*(\mathbb{P}^{n+1}) \to H^*(V)\right)$.