Fundamental groups of complex quasiprojective manifolds and their Alexander type invariants

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Motivation

- Every finitely presented group occurs as the fundamental group of a smooth closed real manifold of dimension ≥ 4.
- Serre's problem: Which groups are *(quasi-)projective groups*, i.e., can be realized as fundamental groups of complex (quasi-)projective *manifolds*?
- By a Zariski-Lefschetz type theorem, one can restrict to the class of fundamental groups of smooth complex (quasi-)projective *surfaces*.
- Kapovich (2013): (if one allows mild singularities) every finitely presented group is the fundamental group of a complex *irreducible* projective surface whose singularities are simple normal crossings and Whitney umbrellas.
- Kapovich-Kollár (2014): every finitely presented group is the fundamental group of a complex (non-irreducible) projective surface with simple normal crossing singularities.
- Here we focus on π_1's of smooth varieties.

- Similar questions apply to *Kähler groups*.
- There is no reduction to the surface case in the Kähler context, though Kodaira showed that any compact complex analytic surface with even first Betti number b_1 (e.g., a compact Kähler surface) deforms to a projective one (this is not true in higher dimensions, as shown by Voisin).
- (Consequence of Kodaira's theorem) If π is a finitely presented group, the following are equivalent:
  - π is π_1 of a smooth complex projective variety
  - π is π_1 of a smooth complex projective surface
  - π is π_1 of a Kähler compact complex surface
  - π is π_1 of a compact complex surface with even b_1.

Related open questions

- is any Kähler group a projective group?
  - Kodaira: true in dimension 2.
  - Claudon (2018): true for "linear" groups.
  - Claudon-Höring-Lin (2019): true in dimension 3.
  - Voisin (2004): there is a compact Kähler manifold which is not of the real homotopy type of any complex projective manifold.
- Singer-Hopf Conjecture: if $\pi = \pi_1(M)$ is an aspherical Kähler group, then $(-1)^n \chi(M) \geq 0$, where $n = \dim_{\mathbb{C}} M$.
- find obstructions on *homotopy types* of complex quasi-projective manifolds:
  - complex algebraic varieties are h.e. to finite CW complexes.
  - Andreotti-Frankel: a complex n-dim. *affine* variety has the homotopy type of a finite CW complex of dimension $\leq n$.
  - structure results for the *cohomology jump loci* of rank-one $\mathbb{C}$-local systems (Beauville, Green-Lazarsfeld, Simpson, Arapura, Campana, Libgober, Dimca-Papadima-Suciu, Budur-Wang, etc.)

Some History/Facts

- Serre: every *finite* group is a *projective group*, i.e., π_1 of a complex projective manifold.
- However, most finitely presented groups (e.g., free abelian groups of *odd* rank) are *not projective groups*.
  - by Hodge theory, the first Betti number b_1 (i.e., rank of abelianization of π_1) of a Kähler/projective group is even!
- Gromov: projective groups can't split as nontrivial free products
- Carlson-Toledo: If M is a closed real hyperbolic n-manifold ($n \geq 3$), then $\pi_1(M)$ is not projective.
- By contrast, Taubes (1992) showed that *every* finitely presented group is π_1 of a compact complex manifold (of $\dim_{\mathbb{C}} = 3$).

Some exercises: quasi-projective case

- **Exercise:** All *cyclic* groups are quasi-projective.
  - Hint 1: If $C \subset \mathbb{CP}^2$ is a smooth plane curve of degree d, then $\pi_1(\mathbb{CP}^2 \setminus C) = \mathbb{Z}/d$.
  - Hint 2: If L_1, L_2 are lines in $\mathbb{CP}^2$, then $\pi_1(\mathbb{CP}^2 \setminus (L_1 \cup L_2)) = \mathbb{Z}$.
- **Exercise:** All *abelian* groups are quasi-projective.
  - Hint: If $C_0, \ldots, C_r \subset \mathbb{CP}^2$ are irreducible smooth curves of degree $d_i = \deg C_i$ intersecting transversally, then $$\pi_1(\mathbb{CP}^2 \setminus (C_0 \cup \ldots \cup C_r)) = \mathbb{Z}^r \oplus \mathbb{Z}/d,$$ where $d = \gcd(d_0, \ldots, d_r)$.
- **Exercise:** All *free* groups of finite rank are quasi-projective.
  - Hint: What is π_1 of $\mathbb{CP}^1$ minus a finite set of points?
- **Exercise:** Finite index subgroups of (quasi-)projective groups are (quasi-)projective.
- **Exercise:** Direct products of (quasi-)projective groups are (quasi-)projective.
- Morgan (1978), Kapovich-Milson (1997), etc. found *infinitely many* non-isomorphic examples of *non-quasi-projective groups*.

Some exercises: projective/Kähler

- **Exercise:** $\mathbb{Z}^{2n}$ is projective (hence Kähler).
  - Hint: $\mathbb{Z}^{2n} = \pi_1(A)$, where $A \cong \mathbb{C}^n/\mathbb{Z}^{2n}$ is an abelian variety of complex dimension n.
- **Exercise:** Direct products of Kähler groups are Kähler.
  - Hint: a product of Kähler manifolds is Kähler.
- **Exercise:** Finite index subgroups of Kähler groups are Kähler.
  - Hint: Kähler metrics can be lifted to finite coverings.
- **Exercise:** Non-trivial free groups are not Kähler.
  - Hint: such groups have subgroups of finite index and odd rank.
- **Exercise:** $\mathbb{Z}/2 * \mathbb{Z}/2$ is not Kähler.
  - Hint: it contains $\mathbb{Z}$ with index 2.
- **Exercise:** $SL(2,\mathbb{Z})$ is not Kähler.
  - Hint: it contains subgroups of finite index which are free.
- $SL(n,\mathbb{Z})$ is not Kähler for $n > 2$ (non-abelian Hodge theory).

- A group may contain a Kähler/projective group without being itself Kähler/projective: e.g., π_1 of the Klein bottle is not projective (why?), but it contains $\mathbb{Z}^2$ with index 2.

- One may also study sub-classes of (quasi-)projective/Kähler groups.
- Delzant (2010): a solvable Kähler group is virtually nilpotent.
- Donaldson-Goldman Conjecture (proved by Dimca-Suciu, Kotschick,...): If π is a Kähler group and $\pi = \pi_1(M)$ for M a closed 3-manifold, then π is a finite group.
- Hence, infinite 3-manifold groups are not Kähler!
  E.g., $\pi_1(S \times \mathbb{Z})$ is not Kähler (with S a closed orientable real surface).
- Blasco-Garcia, Cogolludo: studied quasi-projectivity of Artin groups (cf. also Dimca-Papadima-Suciu for the case of right-angled Artin groups).

π_1's of hypersurface complements

- From now on, consider only the sub-class of π_1 of complex quasi-projective manifolds which are complements to hypersurfaces in $\mathbb{C}^n$ (or $\mathbb{CP}^n$).
- Reduction to a low-dimensional topology problem: by a Zariski-Lefschetz type theorem, possible π_1's of complements to hypersurfaces in $\mathbb{C}^n$ (or $\mathbb{CP}^n$) are precisely the fundamental groups of complements to plane curves in $\mathbb{C}^2$ (resp. $\mathbb{CP}^2$), a class of groups already considered by Zariski and Van Kampen in 1930s.
- Question: What groups can be π_1 of complements to curves in $\mathbb{C}^2$ (resp. $\mathbb{CP}^2$)? What obstructions are there?
- E.g., many *knot groups* cannot be realized as $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ for a plane curve $\mathcal{C}$ (to be justified later).

- This question is also motivated by Zariski's problem:

  Find examples of Zariski pairs, i.e., pairs of plane curves in $\mathbb{CP}^2$ which have homeo tubular neighborhoods (hence same type of singularities), but non-homeo complements (due to the position of their singularities.)

- E.g., Let $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ be a *sextic with* 6 *cusps* (i.e., locally defined by $x^2 = y^3$). Zariski showed that the moduli space of such curves has (at least) two connected components, representatives in each component being distinguished by π_1 of their complements.

- A lot of work on Zariski pairs done by Artal-Bartolo, Tokunaga, Oka, Shimada, Eyral, Cogolludo, etc.

- π_1's of plane curve complements are difficult to handle.
- Zariski's conjecture: *If* $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ *has only nodal singularities (i.e., locally defined by* $x^2 = y^2$*), then* $\pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}})$ *is abelian.* (proved by Deligne and Fulton, cf. also Orevkov, Oka,...)
- Nori: If $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ has a nodes and b cusps, and $2a + 6b < d^2$ ($d = \deg \bar{\mathcal{C}}$), then $\pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}})$ is abelian.
- Tokunaga: If $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ has a nodes and b cusps, and $2a + 6b > 2d^2 - 6d + 6$, then $\pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}})$ is not abelian.
- There are *Zariski pairs* of degree $d = 6$ with a nodes ($a = 0, .., 3$) and 6 cusps which fail one test or the other.

An interesting example of π_1

Example (Oka, Nemethi)

Let $p, q \in \mathbb{Z}$, $p, q \geq 2$, with $(p, q) = 1$. Consider

$$\bar{\mathcal{C}}_{p,q} : (x^p + y^p)^q + (y^q + z^q)^p = 0$$

Then

$$\pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}}_{p,q}) = \mathbb{Z}/p * \mathbb{Z}/q$$

- $p = 2, q = 3$: Zariski's sextic curve with six cusps on a conic.
- recent generalizations by Cogolludo-Elduque.

- It is natural to look for invariants of π_1 which are easier to handle that π_1, and still capture a lot of the topology of the curve. For instance, one may consider
  - Alexander-type invariants (polynomials, modules)
  - Novikov-Betti numbers
  - L^2 Betti numbers
- Rigidity properties for such invariants impose lots of obstructions on π_1's of curve complements.

I. *Alexander-type invariants of plane curve complements*

Plane curve complements: Setting

- Let $\bar{\mathcal{C}} = \{F(x, y, z) = 0\}$ be a degree d reduced curve in $\mathbb{CP}^2$.
- Let $L_\infty = \{z = 0\} \subset \mathbb{CP}^2$ be a generic line, i.e., $\bar{\mathcal{C}} \pitchfork L_\infty$.
- Let $f(x, y) := F(x, y, 1)$, and
$$\mathcal{C} = \{f(x, y) = 0\} = \bar{\mathcal{C}} \setminus L_\infty \subset \mathbb{C}^2.$$
- Zariski: there is a *central extension*:
$$0 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 \setminus \mathcal{C}) \to \pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}}) \to 0,$$
so $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ and $\pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}})$ carry the "same" info.
- it is more convenient to work with $\mathbb{C}^2 \setminus \mathcal{C}$, which has a larger π_1 than $\mathbb{CP}^2 \setminus \bar{\mathcal{C}}$.
- Set $M = \mathbb{C}^2 \setminus \mathcal{C}$, with $\pi = \pi_1(M)$.
- M is h.e. to a finite CW complex of *real* dimension 2.
- $H_1(M) = H_1(\pi) = \mathbb{Z}^r$, for $r = \#$ of irred. components of $\mathcal{C}$.

(a) Classical Alexander polynomials

- $M = \mathbb{C}^2 \setminus \mathcal{C} = \mathbb{CP}^2 \setminus (\bar{\mathcal{C}} \cup L_\infty)$, with $\bar{\mathcal{C}} \pitchfork L_\infty$.
- $f_* : \pi = \pi_1(M) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}$ induces a $\mathbb{Z}$-fold cover M^c of M, the pullback of the universal cover $\mathbb{C} \to \mathbb{C}^*$.
- $H_i(M^c; \mathbb{C})$ is a finitely generated $\mathbb{C}[\mathbb{Z}] \simeq \mathbb{C}[t^{\pm 1}]$-module.

Theorem (Libgober)

$H_1(M^c; \mathbb{C})$ is a torsion $\mathbb{C}[t^{\pm 1}]$-module.

Definition

$\Delta_{\mathcal{C}}(t) :=$ *order* $H_1(M^c; \mathbb{C})$ is *the Alexander polynomial of $\mathcal{C}$ (or π)*.

- $\Delta_{\mathcal{C}}(t)$ can be computed by Fox calculus from a presentation of π (e.g., obtained via *braid monodromy*).
- *Rigidity* properties of $\Delta_{\mathcal{C}}(t)$ impose obstructions on π.

Relation to Milnor fiber of F

- Let $M_F := F^{-1}(1)$ be the *Milnor fiber* of the degree d homogeneous polynomial $F(x,y,z)$ which defines $\overline{\mathcal{C}} \subset \mathbb{CP}^2$, with *monodromy* $h : M_F \to M_F$.
- $h^d = id$, and $M_F/\langle h \rangle \simeq \mathbb{CP}^2 \setminus \bar{\mathcal{C}}$.
- Randell: The Alexander polynomial $\Delta_{\mathcal{C}}(t)$ equals the characteristic polynomial of monodromy $h_* : H_1(M_F) \circlearrowleft$
- Consequences:
  - $\deg \Delta_{\mathcal{C}}(t) = b_1(M_F)$.
  - if $\mathcal{C}$ is irreducible, then $\Delta_{\mathcal{C}}(t) = 1 \iff H_1(M_F)$ is at most a finite group.
  - if $\pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}})$ is a finite group, then $\Delta_{\mathcal{C}}(t) = 1$.
  - the multiplicity of the factor $(t-1)$ in $\Delta_{\mathcal{C}}(t)$ is $r-1$, i.e., $\mathrm{rank} H_1(M_F)_1$, with $r = \#$ of irred. components of $\mathcal{C}$.
  - Libgober: $\Delta_{\mathcal{C}}(t)$ divides $(t^d-1)^{d-2}(t-1)$.
    - $\Delta_{\mathcal{C}}(t)$ is a product of cyclotomic polynomials.
    - roots of $\Delta_{\mathcal{C}}(t)$ are d-th roots of unity.

Example

- Many knot groups, e.g. that of *figure eight knot* (whose Alexander polynomial is $t^2 - 3t + 1$), cannot be of the form $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$.
- However, the class of possible π_1 of plane curve complements includes *braid groups*, or groups of *torus knots* of type (p, q).

Example

Let $p, q \in \mathbb{Z}$, $p, q \geq 2$, with $(p, q) = 1$. Consider Oka's curve

$$\bar{\mathcal{C}}_{p,q} : (x^p + y^p)^q + (y^q + z^q)^p = 0$$

with $\mathcal{C}_{p,q} = \bar{\mathcal{C}}_{p,q} \setminus L_\infty$. Then $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}_{p,q})$ is π_1 of a torus knot of type (p, q), and

$$\Delta_{\mathcal{C}_{p,q}}(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}.$$

Relation to Milnor fiber, cont'd

- Let $M_F := F^{-1}(1)$ be the Milnor fiber of the degree d homogeneous polynomial $F(x,y,z)$, with monodromy h.
- Recall: $\deg \Delta_{\mathcal{C}}(t) = b_1(M_F)$.
- $H_1(M_F) = H_1(M_F)_1 \oplus H_1(M_F)_{\neq 1}$
- $\mathrm{rank} H_1(M_F)_1 = \mathrm{rank} H_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}}) = r-1$
- $\mathrm{rank} H_1(M_F)_{\neq 1} = b_3(V)$, where V is a d-fold cover of $\mathbb{CP}^2$ branched along $\bar{\mathcal{C}}$. (This non-unipotent piece is the *jump*.)
- Problem: compute $b_1(M_F)$.
- Main difficulty: as we will see, $b_1(M_F)$ depends on the position of singularities of $\bar{\mathcal{C}}$ in $\mathbb{CP}^2$.
- Conjecture: if F defines a *line arrangement* in $\mathbb{CP}^2$, then $b_1(F)$ and $h_* : H_1(M_F) \circlearrowleft$ are combinatorially determined. (Progress: Dimca, Papadima, Suciu, Libgober, etc.)

Divisibility results for Alexander polynomials

- For each $x \in \mathrm{Sing}(\mathcal{C})$, let $L_x := S^3_x \cap \mathcal{C}$ be the *link* of x, with (local) complement $M_x := S^3_x \setminus L_x$.
- Milnor: There is a locally trivial fibration $F_x \hookrightarrow M_x \to S^1$
- The *Milnor fibre* F_x is homotopy equivalent to a join of circles, their number being equal to the *Milnor number* $\mu(\mathcal{C}, x)$.
- Let $h_x : F_x \to F_x$ be the *monodromy homeomorphism*.
- The *local Alexander polynomial at x* is defined by

$$\Delta_x(t) := \det \left(tI - (h_x)_* : H_1(F_x) \to H_1(F_x) \right)$$

- Monodromy theorem: the zeros of $\Delta_x(t)$ are roots of 1.

Theorem (Libgober)

$\Delta_{\mathcal{C}}(t)$ *divides* $(t-1)^{r-1} \cdot \prod_{x \in Sing(\mathcal{C})} \Delta_x(t)$

Theorem (M.)

For any irreducible component $\mathcal{C}_i$ *of* $\mathcal{C}$, $\Delta_{\mathcal{C}}(t)$ *divides* $(t-1)^{r-1} \cdot \prod_{x \in Sing(\mathcal{C}) \cap \mathcal{C}_i} \Delta_x(t)$

Corollary

$\Delta_{\mathcal{C}}(t)$ *is a product of cyclotomic polynomials.*

Corollary

Let $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ *be an irreducible degree* d *curve with only nodes and cusps as its singularities. If* $d \not\equiv 0 \pmod 6$, *then* $\Delta_{\mathcal{C}}(t) = 1$.

Theorem (Budur-Liu-Wang)

If $\pi = \pi_1(M)$ *is a quasi-projective group, and* $\epsilon : \pi \to \mathbb{Z}$ *an epimorphism, then* $\Delta_\pi(t)$ *is a product of cyclotomic polynomials, where* Δ_π *is the order of the torsion part of* $H_1(M_\epsilon; \mathbb{C})$.

- The divisibility results for $\Delta_{\mathcal{C}}(t)$ show that the *local type* of singularities affects the topology of $\mathcal{C}$.
- Zariski showed that the *position* of singularities has effect on the topology of $\mathcal{C}$.
- Moreover, Libgober noticed that $\Delta_{\mathcal{C}}(t)$ is already sensitive to the position of singularities.

Example (Zariski's sextics with 6 cusps)

Let $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ be an irreducible *sextic* with only *6 cusps*.
Set $\mathcal{C} := \bar{\mathcal{C}} \setminus L_\infty$, for L_∞ a generic line at infinity in $\mathbb{CP}^2$.

- If *the 6 cusps are on a conic*, then $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ is isomorphic to π_1 of the trefoil knot, and has Alexander polynomial $\Delta_{\mathcal{C}}(t) = t^2 - t + 1$. (In fact, $\pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$)
- If *the 6 cusps are not on a conic*, then $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ is abelian, so $\Delta_{\mathcal{C}}(t) = 1$. (In fact, $\pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}}) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$)

Example

Moishezon showed that Artin's braid group on k strands

$$B_k = \langle \sigma_1, \ldots, \sigma_{k-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$$

appears as $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}_k)$, where $\mathcal{C}_k$ is an affine curve in general position at infinity, whose projective completion $\bar{\mathcal{C}}_k$ is the branching locus of a generic projection $V_k \to \mathbb{CP}^2$, with V_k a degree k smooth surface in $\mathbb{CP}^3$.
Then $\bar{\mathcal{C}}_k$ is an irreducible curve of degree $k(k-1)$ with $k(k-1)(k-2)(k-3)/2$ nodes and $k(k-1)(k-2)$ cusps.
E.g., if $k=3$, $\bar{\mathcal{C}}_3$ is the six-cuspidal sextic with all cusps on a conic, and recall that B_3 is π_1 of the trefoil knot.
For $k \geq 5$, one computes (e.g., using Reidemeister-Schreier) $B'_k/B''_k = 0$, hence $\mathcal{C}_k$ has a trivial Alexander polynomial.
The Alexander polynomial of $\mathcal{C}_4$ is $t^2 - t + 1$.

- Several important algebro-geometric descriptions of $\Delta_{\mathcal{C}}(t)$ have been obtained by Libgober, Loeser-Vaquié, Esnault, Artal-Bartolo, etc.
- Cogolludo-Libgober: If $\bar{\mathcal{C}} \subset \mathbb{CP}^2$ is a degree d irreducible curve with only nodes and cusps singularities, then

$$\deg \Delta_{\mathcal{C}}(t) \leq \frac{5}{3}d - 2.$$

Mixed Hodge structure on the Alexander module

- Libgober, Kulikov-Kulikov: the Alexander module $H_i(M^c;\mathbb{Q})$ carries a canonical mixed Hodge structure.
- Libgober: generalization to higher dimensional hypersurfaces with only isolated singularities.
- M., Dimca-Libgober: generalization to higher dimensional hypersurfaces with arbitrary singularities.
- Elduque-Geske-Herradon-M.-Wang: constructed MHS on the torsion parts of the Alexander modules of a complex quasi-projective manifold X, induced via an algebraic map $f\colon X \to \mathbb{C}^*$.

"Weakness" of Alexander polynomial

- Assume $\bar{\mathcal{C}} = \bar{\mathcal{C}}_1 \cup \bar{\mathcal{C}}_2 \subset \mathbb{CP}^2$, so that $\bar{\mathcal{C}}_1$, $\bar{\mathcal{C}}_2$ are reduced and *intersect transversally*.
- Fix a generic line L_∞ at infinity, and let $\mathcal{C} = \bar{\mathcal{C}} \setminus L_\infty$, $\mathcal{C}_i = \bar{\mathcal{C}}_i \setminus L_\infty$ $(i = 1, 2)$.
- Oka-Sakamoto: There is an isomorphism (induced by inclusions $\mathbb{C}^2 \setminus \mathcal{C} \subset \mathbb{C}^2 \setminus \mathcal{C}_i$):

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) \xrightarrow{\cong} \pi_1(\mathbb{C}^2 \setminus \mathcal{C}_1) \times \pi_1(\mathbb{C}^2 \setminus \mathcal{C}_2).$$

- $\Delta_{\mathcal{C}}(t) = (t-1)^{r-1}$, where $r = \#$ of irred. components of $\mathcal{C}$.
- so, while $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ retains information about $\bar{\mathcal{C}}_1$, $\bar{\mathcal{C}}_2$, the Alexander polynomial $\Delta_{\mathcal{C}}(t)$ does not!
- to overcome this "weakness", can study twisted versions of Alexander polynomials or higher coverings of the plane curve complement (e.g., universal abelian cover, solvable covers, etc.).

(b) Twisted Alexander invariants

- introduced by Lin, Wada, Kirk-Livingston in 1990s.
- striking applications to the study of real closed 3-manifolds by Friedl-Vidussi.
- adapted to the study of plane curve complements by Cogolludo-Florens, who found *new examples of Zariski pairs* which can be detected by the *twisted Alexander polynomial*, but which have the same classical Alexander polynomial.

Setup

- M := path-connected finite CW complex, $\pi := \pi_1(M)$.
- $\varepsilon : \pi \to \mathbb{Z}$ homomorphism.
- M_ε := infinite cyclic cover of M defined by $\bar{\pi} := \ker(\varepsilon)$.
- $\mathbb{V}$ - finite dim. $\mathbb{C}$-vector space, $\ell := \dim_{\mathbb{C}} \mathbb{V}$.
- $\rho : \pi \to GL(\mathbb{V})$ representation, denoted by $\mathbb{V}_\rho$.

Definition (Twisted Alexander modules)

The i-th twisted Alexander module of (M, ε, ρ) is:

$$H_i^{\varepsilon,\rho}(M; \mathbb{C}[t^{\pm 1}]) = H_i(M_\varepsilon; \mathbb{V}_\rho) := H_i\left(C_*(M_\varepsilon, \mathbb{V}_\rho)\right),$$

where $C_*(M_\varepsilon, \mathbb{V}_\rho) := \mathbb{V} \otimes_{\mathbb{C}[\bar{\pi}]} C_*(M_\varepsilon)$ is the twisted chain complex of M_ε.

Twisted Alexander invariants of plane curve complements

- Assume $M = \mathbb{C}^2 \setminus \mathcal{C}$, with $\bar{\mathcal{C}} \pitchfork L_\infty$, and let $\pi = \pi_1(M)$.

Theorem (M.-Wong)

For any pair (ε, ρ), the twisted Alexander modules $H_i^{\varepsilon,\rho}(M; \mathbb{C}[t^{\pm 1}])$ of $M = \mathbb{C}^2 \setminus \mathcal{C}$ are torsion $\mathbb{C}[t^{\pm 1}]$-modules, for $i = 0, 1$.

Remark

If $\varepsilon = lk$, $\mathbb{V} = \mathbb{C}$ and $\rho =$ trivial, get back the classical Alexander modules $H_i(M^c; \mathbb{C})$ of M.

Definition

$\Delta_{\mathcal{C}}^{\varepsilon,\rho}(t) =$ *order* $H_1^{\varepsilon,\rho}(M; \mathbb{C}[t^{\pm 1}])$ is *the twisted Alexander polynomial of $(\mathcal{C}, \varepsilon, \rho)$*.

Roots of twisted Alexander polynomials

- $M = \mathbb{C}^2 \setminus \mathcal{C} = \mathbb{CP}^2 \setminus (\bar{\mathcal{C}} \cup L_\infty)$.
- Let γ_∞ be the meridian in $\pi = \pi_1(M)$ about L_∞.

Theorem (M.-Wong)

Assume $\varepsilon = lk$, and $\rho : \pi \to GL(\mathbb{V})$ an arbitrary representation. Let $\lambda_1, \cdots, \lambda_\ell$ be the eigenvalues of $\rho(\gamma_\infty)^{-1}$. Then the roots of $\Delta_{\mathcal{C}}^{\varepsilon,\rho}(t)$ are contained in the splitting field of $\prod_{i=1}^{\ell}(t^d - \lambda_i)$ over $\mathbb{Q}$, which is cyclotomic over $\mathbb{Q}(\lambda_1, \cdots, \lambda_\ell)$.

Divisibility for twisted Alexander polynomials

- If $x \in \mathrm{Sing}(\mathcal{C})$, let $L_x = S^3_x \cap \mathcal{C}$ be the link of x, with local complement $M_x = S^3_x \setminus L_x$.
- Let $\pi_x := \pi_1(M_x)$.
- Let (ε_x, ρ_x) be the pair induced by (ε, ρ) on π_x via $\pi_x \to \pi$.

Theorem (M.-Wong)

If $x \in \mathrm{Sing}(\mathcal{C})$, the local twisted Alexander modules at x, i.e., $H_i^{\varepsilon_x,\rho_x}(M_x; \mathbb{C}[t^{\pm 1}])$, are torsion $\mathbb{C}[t^{\pm 1}]$-modules for $i = 0, 1$.

Definition

$\Delta_x^{\varepsilon_x,\rho_x}(t) :=$ *order* $H_1^{\varepsilon_x,\rho_x}(M_x; \mathbb{C}[t^{\pm 1}])$ is the *local twisted Alexander polynomial at x*.

Theorem (Cogolludo-Florens, M.-Wong)

divisibility for twisted Alexander polynomials, relating the local and global ones.

II. *Novikov homology*

Novikov-Betti and Novikov-torsion numbers

- M :=connected topological space, h.e. to a *finite* CW complex, $\pi := \pi_1(M)$.
- fix $\xi \in H^1(M;\mathbb{R}) \cong \mathrm{Hom}(\pi,\mathbb{R})$.
- $\Gamma_\xi := Im(\pi \stackrel{\xi}{\to} \mathbb{R}) \hookrightarrow \mathbb{R}$, so $\Gamma_\xi \cong \mathbb{Z}^s$, for some $s = rk(\xi) \geq 0$.
- M_ξ := covering of M defined by $\ker(\xi)$, so $H_i(M_\xi;\mathbb{Z})$ are finitely generated $\mathbb{Z}[\Gamma_\xi]$-modules.
- the *i-th Novikov-Betti number* $b_i(M,\xi)$ of (M,ξ) is the $\mathbb{Z}[\Gamma_\xi]$-rank of $H_i(M_\xi;\mathbb{Z})$:

$$b_i(M,\xi) := \dim_{\mathbb{Q}_\xi} \mathbb{Q}_\xi \otimes_{\mathbb{Z}[\Gamma_\xi]} H_i(M_\xi;\mathbb{Z}) = rk_{R\Gamma_\xi} H_i(M;R\Gamma_\xi),$$

  where $\mathbb{Q}_\xi := Frac(\mathbb{Z}[\Gamma_\xi])$, and $R\Gamma_\xi$ is the *rational Novikov ring* of Γ_ξ (a certain PID localization of $\mathbb{Z}[\Gamma_\xi]$).
- the *i-th Novikov-torsion number* $q_i(M,\xi)$ is the minimal number of generators of $Tors(H_i(M;R\Gamma_\xi))$.

Theorem (Properties of Novikov-Betti numbers)

- $\chi(M) = \sum_i (-1)^i b_i(M, \xi)$.
- $b_i(M, \xi) \leq b_i(M)$, *for any* $\xi \in H^1(M; \mathbb{R})$.
- $b_i(M, 0) = b_i(M)$.

Novikov-type invariants of plane curve complements

- Assume $M = \mathbb{C}^2 \setminus \mathcal{C}$, with $\bar{\mathcal{C}} \pitchfork L_\infty$, and let $\pi = \pi_1(M)$

Definition

$\xi \in H^1(M;\mathbb{R})$ is called *positive* if $\xi : \pi \to \mathbb{R}$ takes strictly positive values on each positively oriented meridian about the irreducible components of $\mathcal{C}$.

Theorem (Friedl-M.)

For any positive $\xi \in H^1(M;\mathbb{R})$, we have:

- $b_i(M,\xi) = \begin{cases} 0, & i \neq 2, \\ \chi(M), & i = 2. \end{cases}$
- $q_i(M,\xi) = 0$ *for all* $i \geq 0$.

Remark

The above result holds more generally, for twisted Novikov-type invariants.

III. *L^2-Betti numbers*

To any CW complex M, countable group Γ, and group homomorphism $\alpha : \pi_1(M) \to \Gamma$, one associates *$L^2$-Betti numbers*

$$b_i^{(2)}(M,\alpha) := \dim_{\mathcal{N}(\Gamma)} H_i\big(C_*(M_\alpha) \otimes_{\mathbb{Z}\Gamma} \mathcal{N}(\Gamma)\big) \in [0,\infty],$$

where M_α is the covering of M defined by α, and $\mathcal{N}(\Gamma)$ is the von Neumann algebra of Γ (a certain completion of $\mathbb{C}[\Gamma]$), so that

- $b_i^{(2)}(M,\alpha)$ is a homotopy invariant of the pair (M,α).
- if M is a finite CW-complex,

$$\chi(M) = \sum_{i \geq 0} (-1)^i \cdot b_i^{(2)}(M,\alpha)$$

Remark (Friedl-M.)

Novikov-Betti numbers associated to $\xi \in H^1(M;\mathbb{R})$ are special cases of L^2-Betti numbers (but torsion-Novikov numbers do not have an L^2 interpretation):

$$b_i(M,\xi) = b_i^{(2)}(M, \pi_1(M) \overset{\xi}{\twoheadrightarrow} Im(\xi))$$

Back to Curves

- $\mathcal{C} = \{f(x,y) = 0\}$, $M = \mathbb{C}^2 \setminus \mathcal{C}$, $\pi = \pi_1(M)$.
- $\alpha : \pi \to \Gamma$ is called *admissible* if $f_* : \pi \to \mathbb{Z}$ factors through α.
- For admissible α, let $\bar{\pi} = \ker(f_*)$, with corresponding covering M^c, and $\bar{\Gamma} := \mathrm{Im}(\bar{\pi} \hookrightarrow \pi \xrightarrow{\alpha} \Gamma)$ with induced map $\bar{\alpha} : \bar{\pi} \to \bar{\Gamma}$.
- Consider $b_p^{(2)}(M, \alpha)$ and $b_p^{(2)}(M^c, \bar{\alpha})$.
- A priori, there is no reason to expect $b_1^{(2)}(M^c, \bar{\alpha})$ to be finite (as M^c is not a finite CW complex).

Obstructions on the L^2-Betti numbers of curves

Theorem (Friedl-Leidy-M.)

If $\alpha : \pi_1(M) \to \Gamma$ is admissible, then

$$b_i^{(2)}(M, \alpha) = \begin{cases} 0, & i \neq 2, \\ \chi(M), & i = 2. \end{cases}$$

Corollary

$b_i^{(2)}(M, \alpha)$ $(i \geq 0)$ depends only on the degree of $\mathcal{C}$ and on the local type of singularities, and is independent on α and on the position of singularities of $\mathcal{C}$. In fact,

$$b_2^{(2)}(M, \alpha) = (d-1)^2 - \sum_{x \in \mathrm{Sing}(\mathcal{C})} \mu(\mathcal{C}, x).$$

Obstructions on the L^2-Betti numbers of curves, cont'd

Theorem (Friedl-Leidy-M.)

If $\alpha : \pi_1(M) \to \Gamma$ is admissible, then $b_1^{(2)}(M^c, \bar{\alpha})$ is finite, and an upper bound is determined by the local type of singularities of $\mathcal{C}$:

$$b_1^{(2)}(M^c, \bar{\alpha}) \leq \sum_{x \in \mathrm{Sing}(\mathcal{C})} (\mu(\mathcal{C}, x) + n_x - 1) + 2g + d,$$

where n_x is the number of branches through $x \in \mathrm{Sing}(\mathcal{C})$ and g is the genus of the normalization of $\mathcal{C}$.

Remark

$b_1^{(2)}(M^c, \bar{\alpha})$ *depends in general on the position of singularities of $\mathcal{C}$ (this can be checked on Zariski's example of sextics with 6 cusps).*

Consequences of finiteness property

Free groups $\mathbb{F}_m$ with $m \geq 2$ cannot be of the form $\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$, for $\mathcal{C}$ a curve in general position at infinity, and similarly for groups of (complements of) boundary links (i.e., those links whose components admit mutually disjoint Seifert surfaces). Equivalently, such groups cannot be central extensions by $\mathbb{Z}$ of groups of the form $\pi_1(\mathbb{CP}^2 \setminus \bar{\mathcal{C}})$.

Concluding remarks

- All invariants of plane curve complements discussed in this lecture are dominated by the corresponding invariants of the link of $\mathcal{C}$ at infinity (i.e., Hopf link on d components) and, resp., by those of the boundary manifold of $\mathcal{C}$.
- All the above finiteness/torsion/rigidity results for homological-type invariants (Alexander modules and polynomials, various types of Betti numbers etc.) admit higher dimensional generalizations to complements of hypersurfaces in $\mathbb{C}^n$ (or $\mathbb{CP}^n$) with arbitrary singularities. Proofs are more involved (use intersection homology, perverse sheaves, etc.).
- One can prove similar statements even after relaxing mildly the transversality assumption (in works of Libgober, Elduque-M.)

THANK YOU !!!

Happy Birthday
Enrique and Alejandro !!!