A guided introduction to intersection homology and applications

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Preliminaries. Notations

♣ Intersection (co)homology was initially introduced in topology, for studying properties of spaces with singularities (like Poincaré duality).

♣ Since its inception, intersection (co)homology found applications in many fields where singular spaces play an important role, like: algebraic geometry, combinatorics, or representation theory.

♣ We will work with complex projective varieties $X \subset \mathbb{C}P^N$.

♣ $\mathbb{C}P^N = \frac{\mathbb{C}^{N+1}-\{0\}}{\mathbb{C}-\{0\}} = \{\text{complex lines in } \mathbb{C}^{N+1}\}$.

♣ $X = \{[z_0 : \ldots : z_N] \in \mathbb{C}P^N \mid f_j(z_0, \ldots, z_N) = 0,\ 1 \leq j \leq m\}$, where $f_1, \ldots, f_m$ are *homogeneous* polynomials in $N+1$ variables.

♣ X is smooth (or non-singular) if the Jacobian matrix $\left(\frac{\partial f_j}{\partial z_i}\right)$ has rank m. In this case, X is a complex submanifold of $\mathbb{C}P^N$.

♣ Otherwise X is said to be singular.

♣ (Singular/simplicial) Homology (with $\mathbb{C}$-coefficients):

$$H_*(X) := H_*(X;\mathbb{C}) = H_*(C_*(X) \otimes \mathbb{C}),$$

with $C_*(X)$ the chain complex of singular/PL chains in X.

♣ Cohomology (with $\mathbb{C}$-coefficients):

$$H^*(X) := H^*(X;\mathbb{C}) \cong H_*(X)^\vee.$$

Motivation: *Kähler package for the cohomology of complex projective manifolds*

Theorem (Kähler package)

Assume $X \subset \mathbb{C}P^N$ is a complex projective manifold, $\dim_{\mathbb{C}}(X) = n$. Then $H^*(X) := H^*(X;\mathbb{C})$ satisfies the following properties:

(a) *Poincaré duality:*

$$H^i(X) \cong H^{2n-i}(X)^\vee$$

for all $i \in \mathbb{Z}$. In particular, the Betti numbers of X in complementary degrees coincide: $b_i(X) = b_{2n-i}(X)$.

(b) *Hodge decomposition:*

$$H^i(X) \cong \bigoplus_{p+q=i} H^{p,q}(X),$$

with $H^{q,p}(X) = \overline{H^{p,q}(X)}$. In particular, the odd Betti numbers of X are even.

Theorem (Kähler package, cont'd)

(c) *Lefschetz hyperplane section theorem (Weak Lefschetz): If H is a generic hyperplane in $\mathbb{C}P^N$, the restriction homomorphism*

$$H^i(X) \longrightarrow H^i(X \cap H)$$

is an isomorphism for $i < n-1$, and it is injective if $i = n-1$. In particular, generic hyperplane sections of X are connected if $n \geq 2$.

(d) *Hard Lefschetz theorem: If H is a generic hyperplane in $\mathbb{C}P^N$, there is an isomorphism*

$$H^{n-i}(X) \xrightarrow{\cup [H]^i} H^{n+i}(X)$$

for all $i \geq 0$, where $[H] \in H^2(X)$ is the Poincaré dual of $[X \cap H] \in H_{2n-2}(X)$. In particular, the Betti numbers of X are unimodal: $b_{i-2}(X) \leq b_i(X)$ for all $i \leq n/2$.

Application to combinatorics

♣ Let $p(i, d, n-d)$ be the number of partitions of the integer i whose Young diagrams fit inside a $d \times (n-d)$ box (i.e., partitions of i into $\leq d$ parts, with largest part $\leq n-d$).

♣ Show that the sequence

$$p(0, d, n-d),\ p(1, d, n-d), \cdots,\ p(d(n-d), d, n-d)$$

is symmetric and unimodal.

♣ Let $X = \mathbf{G}_d(\mathbb{C}^n)$ be the Grassmann variety of d-planes in $\mathbb{C}^n$; this is a complex projective manifold of dimension $d(n-d)$.

♣ X has an algebraic cell decomposition (i.e., all cells are complex affine spaces), so all of its cells appear in even real dimensions. Hence the odd Betti numbers of X vanish.

♣ The even Betti numbers of X are computed as

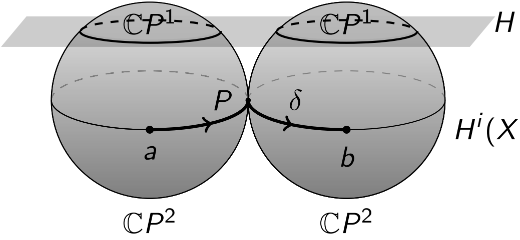
$$b_{2i}(X) = p(i, d, n-d)$$

♣ The assertion about $p(i, d, n-d)$ follows by applying the Kähler package to $H^*(X)$.

Singular context: *Kähler package fails in cohomology!*

Example

$X = \mathbb{C}P^2 \cup_P \mathbb{C}P^2 \subset \mathbb{C}P^4$, $\mathrm{Sing}(X) = \{P\}$.

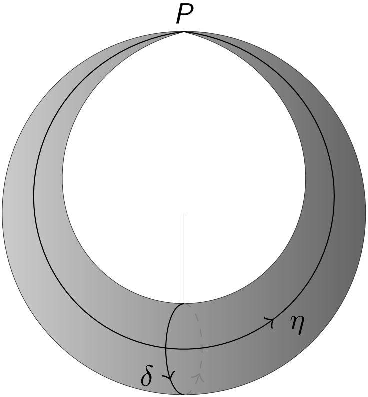


$$H^i(X) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i = 1 \\ \mathbb{C} \oplus \mathbb{C} & i = 2 \\ 0 & i = 3 \\ \mathbb{C} \oplus \mathbb{C} & i = 4. \end{cases}$$

- the 0-cycles $[a]$ and $[b] \in C_0(X)$ cobound a 1-chain δ passing through P. So $[a] = [b] \in H_0(X) \cong H^0(X)^\vee$.
- if H is a generic hyperplane in $\mathbb{C}P^4$ then $X \cap H = \mathbb{C}P^1 \sqcup \mathbb{C}P^1$ is not connected, so the Lefschetz hyperplane section theorem fails.
- $H^0(X) = \mathbb{C} \ncong \mathbb{C} \oplus \mathbb{C} = H^4(X)$, so Poincaré duality and Hard Lefschetz also fail for the singular space X.

Example: Nodal cubic

$$X = \{x_0^3 + x_1^3 = x_0 x_1 x_2\} \subset \mathbb{C}P^2, \ \ \mathrm{Sing}(X) = \{P = [0:0:1]\}.$$



We have

$$H_1(X) = \mathbb{C} = \langle \eta \rangle \cong H^1(X),$$

where η is a longitude in X. (The meridian δ is a boundary in X.) As the first Betti number $b_1(X)$ is odd, there cannot exist a Hodge decomposition for $H^1(X)$.

Intersection homology: *chain definition*

♣ To restore the Kähler package for singular varieties, one has to replace cohomology by (middle-perversity) *intersection cohomology* $IH^*(X)$.

♣ Homologically, this is a theory of allowable chains

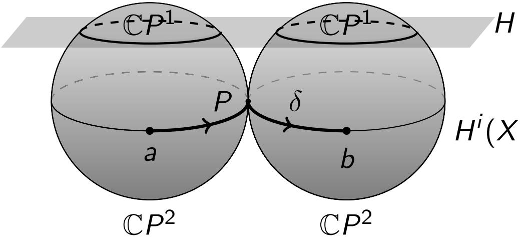
$$IC_*(X) \subset C_*(X)$$

(with induced boundary maps).

♣ Allowability controls the defect of transversality of intersections of chains with the singular strata of X.

Example

$X = \mathbb{C}P^2 \cup_P \mathbb{C}P^2 \subset \mathbb{C}P^4$, $\mathrm{Sing}(X) = \{P\}$.

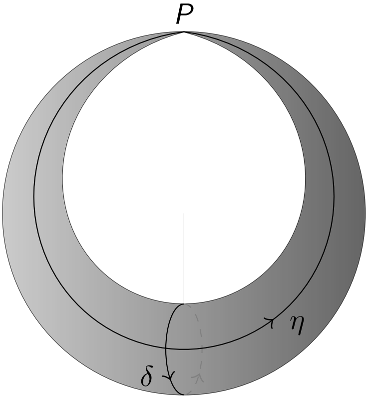


$$H^i(X) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i = 1 \\ \mathbb{C} \oplus \mathbb{C} & i = 2 \\ 0 & i = 3 \\ \mathbb{C} \oplus \mathbb{C} & i = 4. \end{cases}$$

- to restore the symmetry of Betti numbers, 1-chains should not be allowed to pass through singularities.
- the 1-chain δ connecting the 0-cycles $[a]$ and $[b]$ will not be allowed, hence $[a] \neq [b]$ in $IH_0(X) = \mathbb{C} \oplus \mathbb{C}$.

Example: Nodal cubic

$$X = \{x_0^3 + x_1^3 = x_0x_1x_2\} \subset \mathbb{C}P^2, \ \ \mathrm{Sing}(X) = \{P = [0:0:1]\}.$$



- $H_1(X) = \mathbb{C} = \langle \eta \rangle$.
- If we do not allow 1-chains to pass through P, the 1-chain η is not allowed, but 2-chains are allowed to go through P (so δ is still a boundary).
- $[\delta] = 0$ in $IH_1(X) = 0$.

For simplicity, include here the chain definition of intersection homology only for complex varieties with isolated singularities. Everything works with coefficients in an arbitrary noetherian ring A (e.g., $\mathbb{Z}$ or a field), but use $\mathbb{C}$ for convenience.

Definition (Goresky-MacPherson)

Let X be a pure-dimensional (e.g., irreducible) complex algebraic variety with only isolated singularities, with $\dim_{\mathbb{C}}(X) = n$.
If ξ is a PL i-chain on X with support $|\xi|$ (in a sufficiently fine triangulation of X compatible with the natural stratification $\mathrm{Sing}(X) \subset X$), then:

$$\xi \in IC_i(X) \iff \begin{cases} \dim(|\xi| \cap \mathrm{Sing}(X)) < i - n \\ \dim(|\partial\xi| \cap \mathrm{Sing}(X)) < i - n - 1. \end{cases}$$

with boundary $\partial : IC_i(X) \to IC_{i-1}(X)$ induced from ∂ of $C_*(X)$. Get a chain complex $(IC_*(X), \partial)$ of allowable chains whose homology is the (middle-perversity) intersection homology $IH_*(X)$.

Since low-dimensional chains are not allowed to meet the singular points and there are no restriction on higher chains, we get:

Proposition

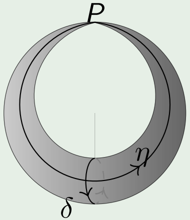
Let X be a pure-dimensional complex algebraic variety, $\dim_{\mathbb{C}}(X) = n$, with only one isolated singular point P. Then,

$$IH_i(X) = \begin{cases} H_i(X - \{P\}), & i < n, \\ \mathrm{Image}(H_n(X - \{P\}) \to H_n(X)), & i = n, \\ H_i(X), & i > n. \end{cases}$$

Example (Nodal cubic)

For the nodal cubic, $X - \{P\}$ deformation retracts to δ, a boundary in X. So:

$$IH_i(X) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i = 1, \\ \mathbb{C}, & i = 2. \end{cases}$$



Remarks

♣ For a projective variety X of complex pure dimension n and with arbitrary singularities, we start with a Whitney (pseudomanifold) stratification of X and the associated filtration

$$X = X_n \supseteq X_{n-1} \supseteq \dots \supseteq X_0 \supseteq \emptyset,$$

where X_i denotes the (closed) union of strata of complex dimension $\leq i$, and impose conditions on how chains and their boundaries meet *all* singular strata:

$$\xi \in IC_i(X) \iff \forall k \geq 1, \begin{cases} \dim(|\xi| \cap X_{n-k}) < i - k \\ \dim(|\partial\xi| \cap X_{n-k}) < i - k - 1. \end{cases}$$

Similar constructions apply to real pseudomanifolds, e.g., (open) cones on manifolds, etc.

Remarks

♣ If X is a compact pseudomanifold of real dimension m (e.g., a complex projective variety of dimenison $m/2$), McCrory showed that

$$H^{m-i}(X) \cong H_i(C_*^{tr}(X))$$

is the homology of the complex of *transverse chains* (which meet the singular strata in the expected dimension).
Since $H_i(X) = H_i(C_*(X))$ is the homology of *all* chains, the intersection homology $IH_*(X)$ splits the difference, so the cap product map

$$\cap[X] : H^{m-i}(X) \to H_i(X)$$

factors through $IH_i(X)$.

Remarks

♣ If X is not compact, we can also work with locally finite allowable chains $IC_i^{lf}(X)$, which compute the Borel-Moore version of intersection homology, $IH_*^{BM}(X)$. This theory is good for sheafification.

♣ IH_* is **not** a homotopy invariant (e.g., if L is a real manifold, then $IH_*(\mathring{c}L)$ is the same as $H_*(L)$ in low degrees; recall that low dimensional chains cannot go through the cone/singular point.)

♣ IH_* is **not** functorial.

♣ IH_* is independent of the stratification and PL structure used to define it.

♣ IH_* is a topological invariant.

♣ If X is a (rational homology) manifold, then $IH_*(X) = H_*(X)$.

Remarks

♣ A *singular* version of intersection homology was developed by King. An allowable singular i-simplex on X is a singular i-simplex $\sigma : \Delta_i \to X$ satisfying

$$\sigma^{-1}(X_{n-k} - X_{n-k-1}) \subseteq (i-k)\text{-skeleton of } \Delta_i$$

for all $k \geq 1$ (again, k denotes here the complex codimension). A singular i-chain is allowable if it is a (locally finite) combination of allowable singular i-simplices. In order to form a subcomplex of allowable chains, need to ask that boundaries of allowable singular chains are allowable.

Sheafification of allowable chains.
Deligne's IC-complex

Setup

♣ For simplicity, work with field coefficients A, e.g., $\mathbb{Q}$ or $\mathbb{C}$.

♣ X pure-dimensional complex algebraic variety, $\dim_{\mathbb{C}}(X) = n$.

♣ X has a Whitney (pseudomanifold) stratification which yields a filtration

$$X = X_n \supseteq X_{n-1} \supseteq \ldots \supseteq X_0 \supseteq \emptyset,$$

with X_i the (closed) union of strata of complex dimension $\leq i$.

♣ X admits a PL structure compatible with the stratification (i.e., each X_i is a union of simplices).

♣ If $U \subseteq X$ is an open subset, then U gets an induced PL structure, so $IC_i(U)$ and $IC_i^{lf}(U)$ can be defined.

Sheafification of allowable chains

Definition

For every integer i, define a (soft) sheaf $IC^{-i} \in Sh_A(X)$ whose sections on each open subset U of X are given by

$$IC^{-i}(U) := IC_i^{lf}(U),$$

i.e., the allowable locally finite i-chains on U (with A-coefficients). Differentials

$$d^{-i} : IC^{-i} \to IC^{-i+1}$$

are induced by the boundary maps $\partial_i : IC_i \to IC_{i-1}$.
This defines a bounded complex of sheaves of A-modules $IC^{\bullet}_{top}$, called the *intersection cohomology complex* of X.

Proposition

We have:

$$IH_i^{BM}(X) := H_i(IC_\bullet^{lf}(X)) = H^{-i}(IC_{-\bullet}^{lf}(X)) = H^{-i}\Gamma(X, IC_{top}^\bullet)$$
$$\cong \mathbb{H}^{-i}(X; IC_{top}^\bullet).$$

Similarly,

$$IH_i(X) \cong \mathbb{H}_c^{-i}(X; IC_{top}^\bullet).$$

Remark

One can start with a local system $\mathscr{L}$ on $X - X_{n-1}$, get (soft) sheaves $IC^{-i}(\mathscr{L})$ and a complex $IC_{top}^\bullet(\mathscr{L})$ whose hypercohomology computes $IH_i^{BM}(X; \mathscr{L})$.

♣ One can now employ the 6-functor formalism (in the derived category $D^b(X)$) and homological algebra to recast old, and prove new, results about IH^*.

Theorem

$IC^{\bullet}_{top}(\mathscr{L})$ is uniquely characterized in the derived category $D^b(X)$ of bounded complexes by a set of axioms (derived from local chain calculations), and can be constructed by Deligne's recipe, consisting of a sequence of (derived) pushforwards and truncations, starting with $\mathscr{L}[2n]$ on $X - X_{n-1}$.

Remark

No PL structure is involved into Deligne's construction of $IC^{\bullet}_{top}$, so $IH_*(X)$, $IH^{BM}_*(X)$ are independent of the underlying PL structure. To get topological independence, it suffices to show that $IC^{\bullet}_{top}$ is independent of the stratification. For this, one can rephrase the axioms in a way that depends minimally on the stratification. (This involves (co)supports, like for defining perverse sheaves.)

Proposition

$IC_X := IC^{\bullet}_{top}[-n]$ is a (simple) perverse sheaf on X.

Definition

Intersection cohomology groups with A-coefficients are defined by

$$IH^i(X) := \mathbb{H}^{i-n}(X; IC_X) = IH^{BM}_{2n-i}(X),$$

$$IH^i_c(X) := \mathbb{H}^{i-n}_c(X; IC_X) = IH_{2n-i}(X).$$

♣ For X of pure complex dimension n, there is a canonical map

$$\mathbb{Q}_X[n] \longrightarrow IC_X,$$

induced by the isomorphism on the smooth locus of X, hence inducing a homomorphism

$$H^i(X; \mathbb{Q}) \longrightarrow IH^i(X; \mathbb{Q}), \text{ for } i \in \mathbb{Z}.$$

Theorem

X is a rational homology manifold if and only if $IC_X = \mathbb{Q}_X[n]$.

Theorem (Poincaré Duality for IH^*)

For A a field and X a pure-dimensional complex projective variety of $\dim_{\mathbb{C}}(X) = n$, there is a non-degenerate intersection pairing

$$IH^i(X) \otimes IH^{2n-i}(X) \longrightarrow A$$

induced from the quasi-isomorphism in $D^b(X)$:

$$\mathcal{D}_X(IC_X) \simeq IC_X$$

Proof.

The dual complex $\mathcal{D}_X(IC_X)$ satisfies the axioms for IC_X. □

♣ Geometrically, the intersection pairing is defined by an appropriate count of intersection points of cycles of complementary dimensions, upon perturbing them so that these intersections happen in the nonsingular locus of X.

Theorem (Lefschetz hyperplane section theorem for IH^*)

Assume A is a field. Let $X \subset \mathbb{C}P^N$ be a pure n-dimensional closed algebraic subvariety with a Whitney stratification $\mathscr{X}$. Let $H \subset \mathbb{C}P^N$ be a generic hyperplane (transversal to all strata of $\mathscr{X}$). Then for $0 \leq i \leq n-2$ the natural homomorphism

$$IH^i(X;A) \longrightarrow IH^i(X \cap H;A)$$

is an isomorphism, and it is a monomorphism for $i = n-1$.

Proof.

For $D = X \cap H$ and $U = X - D$, one has a long exact sequence

$$\cdots \longrightarrow IH^k_c(U) \longrightarrow IH^k(X) \longrightarrow IH^k(D) \longrightarrow \cdots$$

Stratified Morse theory (Goresky-MacPherson) for U affine yields:

$$IH^k_c(U) = 0, \;\; \forall \; k < n.$$

□

♣ Hodge structures and Hard Lefschetz for $IH^*(X)$ are much more involved and follow from work of Beinlinson-Bernstein-Deligne (by positive characteristic methods), Saito (via mixed Hodge modules) and/or de Cataldo-Migliorini (by classical Hodge theory).

Theorem (Hodge decomposition)

If X is a complex projective variety of pure complex dimension n, then $IH^i(X;\mathbb{Q})$ has a pure Hodge structure of weight i. Hence, the odd intersection cohomology Betti numbers of X are even.

Remark

$$\mathrm{Ker}\ \left(H^i(Y;\mathbb{Q}) \to IH^i(Y;\mathbb{Q})\right) = W_{\leq i-1}H^i(Y;\mathbb{Q}),$$

where $W_{\leq i-1}H^i(Y;\mathbb{Q})$ is the vector subspace of $H^i(Y;\mathbb{Q})$ consisting of classes of Deligne weight $\leq i-1$.

Theorem (Hard Lefschetz theorem for intersection cohomology)

Let X be a complex projective variety of pure complex dimension n, with $[H] \in H^2(X;\mathbb{Q})$ the first Chern class of an ample line bundle on X. Then there are isomorphisms

$$\cup[H]^i : IH^{n-i}(X;\mathbb{Q}) \xrightarrow{\cong} IH^{n+i}(X;\mathbb{Q})$$

for every integer $i > 0$, induced by the cup product by $[H]^i$. Hence, the intersection cohomology Betti numbers of X are unimodal.

Theorem (Decomposition theorem)

Let $f : X \to Y$ be a proper map of irreducible complex algebraic varieties, and let $\mathscr{Y}$ be the set of connected components of strata of Y in a stratification of f. There is an isomorphism in $D^b(Y)$:

$$Rf_* IC_X \simeq \bigoplus_{i \in \mathbb{Z}} \bigoplus_{S \in \mathscr{Y}} IC_{\overline{S}}(\mathscr{L}_{i,S})[-i],$$

where the local systems $\mathscr{L}_{i,S}$ on S are semi-simple.
In particular, for every $j \in \mathbb{Z}$ there is a splitting:

$$IH^j(X;\mathbb{Q}) \cong \bigoplus_{i \in \mathbb{Z}} \bigoplus_{S \in \mathscr{Y}} IH^{j-\dim_{\mathbb{C}} X+\dim_{\mathbb{C}} S-i}(\overline{S};\mathscr{L}_{i,S}).$$

Example

If F is a compact variety, the decomposition for the projection $Y \times F \to Y$ yields Künneth formula for intersection cohomology.

♣ The set of supports of f is the collection of subvarieties $\bar{S}$ (closure of strata) appearing in the above decomposition with a non-zero local system $\mathscr{L}_{i,S}$. E.g., $f(X)$ is in the support of f. This set is difficult to determine in general, but it can be described explicitly for toric maps (de Cataldo-Migliorini-Mustaţă), for certain Hilbert-Chow maps (Migliorini-Schende, Maulik-Yun), etc.

♣ The following consequence of the decomposition theorem was used by Ngô in the proof of his *support theorem*, which gives a sharp condition for the absence of supports in the case of abelian fibrations (a key step in his proof of the fundamental lemma in the *Langlands' program*):

Theorem

Let $f : X \to Y$ be a proper surjective map of complex algebraic varieties, with X smooth. Assume f has pure relative dimension d (i.e., all fibers of f have pure complex dimension d). Let S be a subvariety of Y appearing in the decomposition of $Rf_*\mathbb{Q}_X[\dim_{\mathbb{C}} X]$. Then $\operatorname{codim}_Y(S) \leq d$.

More generally, one has the following application of the decomposition theorem:

Theorem

Let $f : X \to Y$ be a proper surjective map of complex irreducible algebraic varieties. Then $IH^j(Y;\mathbb{Q})$ is a direct summand of $IH^j(X;\mathbb{Q})$ for every integer j.
More precisely, if $d = \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y$ is the relative dimension of f, then $IC_Y[d]$ is a direct summand of Rf_*IC_X.

Corollary

The intersection cohomology $IH^j(Y;\mathbb{Q})$ of an irreducible complex algebraic variety Y is a direct summand of the cohomology $H^j(X;\mathbb{Q})$ of a resolution of singularities X of Y.

Application: *Stanley's proof of McMullen's conjecture*

♣ Stanley used intersection cohomology and its Kähler package to prove McMullen's conjecture, giving an if and only if condition for the existence of a simplicial polytope with a prescribed face vector.

♣ Summary of Stanley's idea:

- to a simplicial polytope P one associates a projective (toric) variety X_P so that McMullen's combinatorial conditions for P translate into properties of the Betti numbers of X_P.
- X_P is in general singular, but its singularities are mild (finite quotient singularities), making X_P into a rational homology manifold. Hence $H^*(X_P;\mathbb{Q}) \cong IH^*(X_P;\mathbb{Q})$.
- the assertions about the Betti number of X_P follow from the Poincaré duality and the Hard Lefschetz theorem for its intersection cohomology.

McMullen's conjecture

♣ Let P be an n-dimensional convex polytope.

♣ The *face vector* of P collects the numbers $f_i = f_i(P)$ of its i-dimensional faces, $0 \leq i \leq n-1$, into a string

$$f(P) := (f_0, \cdots, f_{n-1}).$$

♣ **Realization Problem**: find conditions by which one can recognize if a given string of natural numbers is the f-vector of a convex polytope or not.

♣ Obvious obstructions, e.g., the face vector $f(P)$ of P satisfies the *generalized Euler formula*:

$$f_0 - f_1 + f_2 - \cdots + (-1)^{n-1} f_{n-1} = 1 + (-1)^{n-1}.$$

♣ McMullen (1971): conjectural description of f-vectors of *simplicial* polytopes (i.e., convex polytopes whose faces are all simplices).

McMullen's conjecture

♣ The *h-vector* $h(P) := (h_0, \cdots, h_n)$ of an n-dimensional simplicial polytope P with face vector $f(P)$ is defined by the coefficients of the *h-polynomial*

$$h(P,t) = \sum_{i=0}^{n} h_i t^i := (t-1)^n + f_0(t-1)^{n-1} + \cdots + f_{n-1},$$

i.e., with $f_{-1} := 1$,

$$h_i = \sum_{j=0}^{i} \binom{n-j}{n-i} (-1)^{i-j} f_{j-1}.$$

Conjecture (McMullen)

$f = (f_0, \cdots, f_{n-1}) \in \mathbb{N}^n$ is the face vector $f(P)$ of an n-dimensional simplicial polytope P if and only if the following conditions hold:

(1) *(Dehn–Sommerville)* $h_i = h_{n-i}$ *for all* $0 \leq i \leq n$;

(2) *there is a graded commutative* $\mathbb{Q}$*-algebra* $R = \bigoplus_{i \geq 0} R_i$, *with* $R_0 = \mathbb{Q}$, *generated by* R_1, *and with* $\dim_{\mathbb{Q}} R_i = h_i - h_{i-1}$ *for* $1 \leq i \leq \lfloor n/2 \rfloor$. *(In particular,* $h_{i-1} \leq h_i$ *for* $1 \leq i \leq \lfloor n/2 \rfloor$*.)*

Stanley's proof of McMullen's conjecture–necessity part

♣ The sufficiency part of McMullen's conjecture was verified by Billera–Lee (1980). Stanley proved the necessity part.
♣ By a slight perturbation and translation (which don't change $f(P)$), can assume P is *rational* (i.e., its vertices are in the rational points of the given lattice) and contains the origin is in its interior.
♣ Define the fan $\Sigma(P)$ consisting of the cones (with vertex at the origin) over the proper faces of P, and let $X_P := X_{\Sigma(P)}$ be the associated toric variety.
♣ X_P is projective and simplicial with $\dim_{\mathbb{C}} X_P = n$, hence a rational homology manifold. Moreover, for every $0 \leq i \leq n$,

$$b_{2i+1}(X_P) = 0 \quad \text{and} \quad b_{2i}(X_P) = h_i(P).$$

♣ $H^*(X_P;\mathbb{Q}) \cong IH^*(X_P;\mathbb{Q})$ satisfies Poincaré duality and Hard Lefschetz.
♣ the Dehn–Sommerville relation (1) follows from Poincaré duality, and the unimodality of the h_i's in (2) follows by Hard Lefschetz.

Remark

♣ *If the polytope P is non-simplicial, the h-polynomial $h(P, t)$ may have negative coefficients and the Dehn–Sommerville relations do not hold. Moreover, assuming P is rational, $H^*(X_P;\mathbb{Q})$ can exist in odd degrees, and the corresponding Betti numbers are not invariants of the combinatorics of faces.*

♣ *To generalize the above to the rational non-simplicial context, one needs to replace $H^*(X_P;\mathbb{Q})$ by $IH^*(X_P;\mathbb{Q})$. The generalized h-polynomial $h(P,t) = \sum_{i=0}^{n} h_i(P)t^i$ is defined as*

$$h_i(P) := \dim_{\mathbb{Q}} IH^{2i}(X_P;\mathbb{Q}),$$

and its coefficients satisfy the Dehn–Sommerville relations and unimodality by Poincaré duality and, resp., Hard Lefschetz for IH^*.

♣ *The generalized h-polynomial is a combinatorial invariant (can be defined only in terms of the partially ordered set of faces of P).*

Application: *Dowling-Wilson and Rota conjectures*

♣ Let $E = \{v_1, \cdots, v_d\}$ be a spanning subset of an n-dimensional complex vector space V.

♣ Let $w_i(E)$ be the number of i-dimensional subspaces spanned by subsets of E.

Conjecture (Dowling-Wilson top-heavy conjecture)

For all $i < n/2$ one has:

$$w_i(E) \leq w_{n-i}(E).$$

Conjecture (Rota's unimodal conjecture)

There is some j so that

$$w_0(E) \leq \cdots \leq w_{j-1}(E) \leq w_j(E) \geq w_{j+1}(E) \geq \cdots \geq w_n(E).$$

♣ Huh-Wang used the Kähler package on intersection cohomology to prove the Dowling-Wilson top-heavy conjecture, and of the unimodality of the "lower half" of the sequence $\{w_i(E)\}$:

Theorem (Huh-Wang)

For all $i < n/2$, the following properties hold:

(a) *(top heavy)* $w_i(E) \leq w_{n-i}(E)$.

(b) *(unimodality)* $w_i(E) \leq w_{i+1}(E)$.

The proof rests on two key steps:

(1) There exists a (highly singular) complex n-dimensional projective variety Y such that for every $0 \leq i \leq n$ one has:

$$H^{2i+1}(Y;\mathbb{Q}) = 0 \quad \text{and} \quad \dim_{\mathbb{Q}} H^{2i}(Y;\mathbb{Q}) = w_i(E).$$

(2) There exists a resolution of singularities $\pi : X \to Y$ of Y such that the induced homomorphism

$$\pi^* : H^*(Y;\mathbb{Q}) \longrightarrow H^*(X;\mathbb{Q})$$

is injective in each degree.

♣ To define the variety Y of Step (1), use $E = \{v_1, \cdots, v_d\}$ to construct a map $i_E : V^\vee \to \mathbb{C}^d$ by regarding each $v_i \in E$ as a linear map on the dual vector space $V^\vee$. Precomposing i_E with the open inclusion $\mathbb{C}^d \hookrightarrow (\mathbb{C}P^1)^d$ yields a map

$$f : V^\vee \to (\mathbb{C}P^1)^d.$$

The matroid Schubert variety of E is defined as

$$Y := \overline{\mathrm{Im}\ (f)} \subset (\mathbb{C}P^1)^d.$$

♣ Ardila-Boocher showed that the variety Y has an algebraic cell decomposition, the number of $\mathbb{C}^i$'s appearing in the decomposition of Y being exactly $w_i(E)$. This completes Step (1) by cellular cohomology.

♣ Having defined Y, the resolution X is a sequence of blow-ups (a *wonderful model*) associated to a certain canonical stratification of Y. The cohomology rings of both Y and X are well-understood combinatorially and Step (2) can be checked directly.

♣ Assuming (1) and (2), note that π^* factorizes as

$$\pi^* : H^*(Y;\mathbb{Q}) \xrightarrow{\alpha} IH^*(Y;\mathbb{Q}) \overset{\beta}{\hookrightarrow} H^*(X;\mathbb{Q}).$$

♣ Since π^* is injective by Step (2), get that

$$\alpha : H^*(Y;\mathbb{Q}) \to IH^*(Y;\mathbb{Q})$$

is injective.

♣ The injectivity of α can be shown directly without using a resolution, by making use of Hodge theory instead. This is a consequence of two facts:

(i) $\mathrm{Ker}\ \left(\alpha : H^i(Y;\mathbb{Q}) \to IH^i(Y;\mathbb{Q})\right) = W_{\leq i-1}H^i(Y;\mathbb{Q})$.

(ii) the MHS on $H^i(Y;\mathbb{Q})$ is pure of weight i (since Y has an algebraic cell decomposition).

Proof of Huh-Wang theorem

♣ Aim to follow the pattern of Stanley's proof of McMullen's conjecture. But the space Y whose Betti numbers encode $\{w_i(E)\}$ is singular, so $H^*(Y;\mathbb{Q})$ does not satisfy the Kähler package.

♣ For $i < n/2$, consider the following commutative diagram:

$$\begin{array}{ccc} H^{2i}(Y;\mathbb{Q}) & \xhookrightarrow{\alpha} & IH^{2i}(Y;\mathbb{Q}) \\ \cup[H]^{n-2i}\big\downarrow & & \cong\big\downarrow\cup[H]^{n-2i} \\ H^{2n-2i}(Y;\mathbb{Q}) & \xhookrightarrow{\alpha} & IH^{2n-2i}(Y;\mathbb{Q}) \end{array}$$

where the right-hand vertical arrow is the Hard Lefschetz isomorphism for $IH^*(Y;\mathbb{Q})$. Since the α's are injective, get that

$$H^{2i}(Y;\mathbb{Q}) \xrightarrow{\cup[H]^{n-2i}} H^{2n-2i}(Y;\mathbb{Q})$$

is also injective. Hence, for every $i < n/2$:

$$w_i(E) = \dim_{\mathbb{Q}} H^{2i}(Y;\mathbb{Q}) \leq \dim_{\mathbb{Q}} H^{2n-2i}(Y;\mathbb{Q}) = w_{n-i}(E)$$

♣ Part (b) follows similarly, using the unimodality of the intersection cohomology Betti numbers.

Remark

Dowling-Wilson and Rota conjectures concern matroids. The proof by Huh-Wang discussed above is applicable only for matroids realizable over some field. A more general proof was obtained recently by Huh et co. for any matroid.

Many more applications

♣ Topology of Hilbert schemes of points.

♣ Representation theory, e.g., Kazhdan-Lusztig polynomials are computed from the IC-complex of Schubert varieties.

♣ Characteristic classes for singular varieties.

♣ Higher (rational and Du Bois) singularities via Hodge theory.

♣ ......

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Thank you !