

Math 754  
Chapter I: Homotopy Theory

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# 1 Homotopy Groups

**Definition 1.1.** For each  $n \geq 0$  and  $X$  a topological space with  $x_0 \in X$ , the  $n$ -th homotopy group of  $X$  is defined as

$$\pi_n(X, x_0) = \{f : (I^n, \partial I^n) \rightarrow (X, x_0)\} / \sim$$

where  $I = [0, 1]$  and  $\sim$  is the usual homotopy of maps.

**Remark 1.2.** Note that we have the following diagram of sets:

$$\begin{array}{ccc} (I^n, \partial I^n) & \xrightarrow{f} & (X, x_0) \\ & \searrow & \nearrow g \\ & (I^n / \partial I^n, \partial I^n / \partial I^n) & \end{array}$$

with  $(I^n / \partial I^n, \partial I^n / \partial I^n) \simeq (S^n, s_0)$ . So we can also define

$$\pi_n(X, x_0) = \{g : (S^n, s_0) \rightarrow (X, x_0)\} / \sim .$$

**Remark 1.3.** If  $n = 0$ , then  $\pi_0(X)$  is the set of connected components of  $X$ . Indeed, we have  $I^0 = \text{pt}$  and  $\partial I^0 = \emptyset$ , so  $\pi_0(X)$  consists of homotopy classes of maps from a point into the space  $X$ .

Now we will prove several results analogous to the case  $n = 1$ , which corresponds to the fundamental group.

**Proposition 1.4.** If  $n \geq 1$ , then  $\pi_n(X, x_0)$  is a group with respect to the operation  $+$  defined as:

$$(f + g)(s_1, s_2, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, \dots, s_n) & \frac{1}{2} \leq s_1 \leq 1. \end{cases}$$

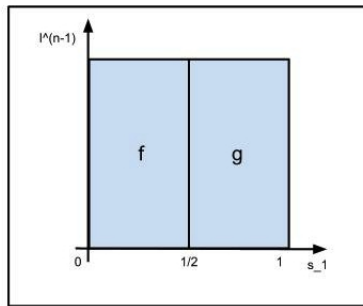


Figure 1:  $f + g$

(Note that if  $n = 1$ , this is the usual concatenation of paths/loops.)

*Proof.* First note that since only the first coordinate is involved in this operation, the same argument used to prove that  $\pi_1$  is a group is valid here as well. Then the identity element is the constant map taking all of  $I^n$  to  $x_0$  and the inverse element is given by

$$-f(s_1, s_2, \dots, s_n) = f(1 - s_1, s_2, \dots, s_n).$$

□

**Proposition 1.5.** *If  $n \geq 2$ , then  $\pi_n(X, x_0)$  is abelian.*

Intuitively, since the  $+$  operation only involves the first coordinate, if  $n \geq 2$ , there is enough space to “slide  $f$  past  $g$ ”.

*Proof.* Let  $n \geq 2$  and let  $f, g \in \pi_n(X, x_0)$ . We wish to show that  $f + g \simeq g + f$ . Consider the following visualization:

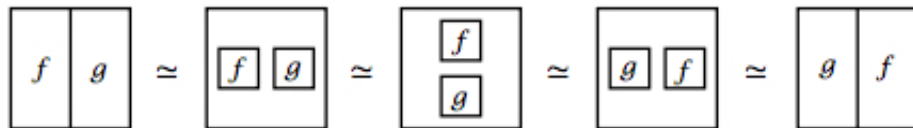


Figure 2:  $f + g \simeq g + f$

We first shrink the domains of  $f$  and  $g$  to smaller cubes inside  $I^n$  and map the remaining region to the base point  $x_0$ . Note that this is possible since both  $f$  and  $g$  map to  $x_0$  on the boundaries, so the resulting map is continuous. Then there is enough room to slide  $f$  past  $g$  inside  $I^n$ . We then enlarge the domains of  $f$  and  $g$  back to their original size and get  $g + f$ . So we have “constructed” a homotopy between  $f + g$  and  $g + f$ , and hence  $\pi_n(X, x_0)$  is abelian. □

**Remark 1.6.** If we view  $\pi_n(X, x_0)$  as homotopy classes of maps  $(S^n, s_0) \rightarrow (X, x_0)$ , then we have the following visual representation of  $f + g$  (one can see this by collapsing boundaries in the above cube interpretation).

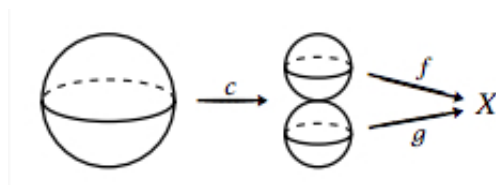


Figure 3:  $f + g$ , revisited

Next recall that if  $X$  is path-connected and  $x_0, x_1 \in X$ , then there is an isomorphism

$$\beta_\gamma : \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$$

where  $\gamma$  is a path from  $x_1$  to  $x_0$ , i.e.,  $\gamma : [0, 1] \rightarrow X$  with  $\gamma(0) = x_1$  and  $\gamma(1) = x_0$ . The isomorphism  $\beta_\gamma$  is given by

$$\beta_\gamma([f]) = [\bar{\gamma} * f * \gamma]$$

for any  $[f] \in \pi_1(X, x_1)$ , where  $\bar{\gamma} = \gamma^{-1}$  and  $*$  denotes path concatenation. We next show a similar fact holds for all  $n \geq 1$ .

**Proposition 1.7.** *If  $n \geq 1$  and  $X$  is path-connected, then there is an isomorphism  $\beta_\gamma : \pi_n(X, x_1) \rightarrow \pi_n(X, x_0)$  given by*

$$\beta_\gamma([f]) = [\gamma \cdot f],$$

where  $\gamma$  is a path in  $X$  from  $x_1$  to  $x_0$ , and  $\gamma \cdot f$  is constructed by first shrinking the domain of  $f$  to a smaller cube inside  $I^n$ , and then inserting the path  $\gamma$  radially from  $x_1$  to  $x_0$  on the boundaries of these cubes.

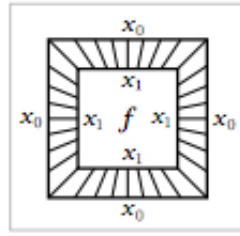


Figure 4:  $\beta_\gamma$

*Proof.* It is easy to check that the following properties hold:

1.  $\gamma \cdot (f + g) \simeq \gamma \cdot f + \gamma \cdot g$
2.  $(\gamma \cdot \eta) \cdot f \simeq \gamma \cdot (\eta \cdot f)$ , for  $\eta$  a path from  $x_0$  to  $x_1$
3.  $c_{x_0} \cdot f \simeq f$ , where  $c_{x_0}$  denotes the constant path based at  $x_0$ .
4.  $\beta_\gamma$  is well-defined with respect to homotopies of  $\gamma$  or  $f$ .

Note that (1) implies that  $\beta_\gamma$  is a group homomorphism, while (2) and (3) show that  $\beta_\gamma$  is invertible. Indeed, if  $\bar{\gamma}(t) = \gamma(1 - t)$ , then  $\beta_{\bar{\gamma}}^{-1} = \beta_\gamma$ .  $\square$

So, as in the case  $n = 1$ , if the space  $X$  is path-connected, then  $\pi_n$  is independent of the choice of base point. Further, if  $x_0 = x_1$ , then (2) and (3) also imply that  $\pi_1(X, x_0)$  acts on  $\pi_n(X, x_0)$  as:

$$\begin{aligned} \pi_1 \times \pi_n &\rightarrow \pi_n \\ (\gamma, [f]) &\mapsto [\gamma \cdot f] \end{aligned}$$

**Definition 1.8.** We say  $X$  is an abelian space if  $\pi_1$  acts trivially on  $\pi_n$  for all  $n \geq 1$ .

In particular, this implies that  $\pi_1$  is abelian, since the action of  $\pi_1$  on  $\pi_1$  is by inner-automorphisms, which must all be trivial.

We next show that  $\pi_n$  is a functor.

**Proposition 1.9.** A map  $\phi : X \rightarrow Y$  induces group homomorphisms  $\phi_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, \phi(x_0))$  given by  $[f] \mapsto [\phi \circ f]$ , for all  $n \geq 1$ .

*Proof.* First note that, if  $f \simeq g$ , then  $\phi \circ f \simeq \phi \circ g$ . Indeed, if  $\psi_t$  is a homotopy between  $f$  and  $g$ , then  $\phi \circ \psi_t$  is a homotopy between  $\phi \circ f$  and  $\phi \circ g$ . So  $\phi_*$  is well-defined. Moreover, from the definition of the group operation on  $\pi_n$ , it is clear that we have  $\phi \circ (f + g) = (\phi \circ f) + (\phi \circ g)$ . So  $\phi_*([f + g]) = \phi_*([f]) + \phi_*([g])$ . Hence  $\phi_*$  is a group homomorphism.  $\square$

The following is a consequence of the definition of the above induced homomorphisms:

**Proposition 1.10.** The homomorphisms induced by  $\phi : X \rightarrow Y$  on higher homotopy groups satisfy the following two properties:

1.  $(\phi \circ \psi)_* = \phi_* \circ \psi_*$ .
2.  $(id_X)_* = id_{\pi_n(X, x_0)}$ .

We thus have the following important consequence:

**Corollary 1.11.** If  $\phi : (X, x_0) \rightarrow (Y, y_0)$  is a homotopy equivalence, then  $\phi_* : \pi_n(X, x_0) \rightarrow \pi_n(Y, \phi(x_0))$  is an isomorphism, for all  $n \geq 1$ .

**Example 1.12.** Consider  $\mathbb{R}^n$  (or any contractible space). We have  $\pi_i(\mathbb{R}^n) = 0$  for all  $i \geq 1$ , since  $\mathbb{R}^n$  is homotopy equivalent to a point.

The following result is very useful for computations:

**Proposition 1.13.** If  $p : \tilde{X} \rightarrow X$  is a covering map, then  $p_* : \pi_n(\tilde{X}, \tilde{x}) \rightarrow \pi_n(X, p(\tilde{x}))$  is an isomorphism for all  $n \geq 2$ .

*Proof.* First we show that  $p_*$  is surjective. Let  $x = p(\tilde{x})$  and consider  $f : (S^n, s_0) \rightarrow (X, x)$ . Since  $n \geq 2$ , we have that  $\pi_1(S^n) = 0$ , so  $f_*(\pi_1(S^n, s_0)) = 0 \subset p_*(\pi_1(\tilde{X}, \tilde{x}))$ . So  $f$  admits a lift to  $\tilde{X}$ , i.e., there exists  $\tilde{f} : (S^n, s_0) \rightarrow (\tilde{X}, \tilde{x})$  such that  $p \circ \tilde{f} = f$ . Then  $[f] = [p \circ \tilde{f}] = p_*([\tilde{f}])$ . So  $p_*$  is surjective.

$$\begin{array}{ccc} & & (\tilde{X}, \tilde{x}) \\ & \nearrow \tilde{f} & \downarrow p \\ (S^n, s_0) & \xrightarrow{f} & (X, x) \end{array}$$

Next, we show that  $p_*$  is injective. Suppose  $[\tilde{f}] \in \ker p_*$ . So  $p_*([\tilde{f}]) = [p \circ \tilde{f}] = 0$ . Let  $p \circ \tilde{f} = f$ . Then  $f \simeq c_x$  via some homotopy  $\phi_t : (S^n, s_0) \rightarrow (X, x_0)$  with  $\phi_1 = f$  and  $\phi_0 = c_x$ .

Again, by the lifting criterion, there is a unique  $\tilde{\phi}_t : (S^n, s_0) \rightarrow (\tilde{X}, \tilde{x})$  with  $p \circ \tilde{\phi}_t = \phi_t$ .

$$\begin{array}{ccc} & & (\tilde{X}, \tilde{x}) \\ & \nearrow \tilde{\phi}_t & \downarrow p \\ (S^n, s_0) & \xrightarrow{\phi_t} & (X, x) \end{array}$$

Then we have  $p \circ \tilde{\phi}_1 = \phi_1 = f$  and  $p \circ \tilde{\phi}_0 = \phi_0 = c_x$ , so by the uniqueness of lifts, we must have  $\tilde{\phi}_1 = \tilde{f}$  and  $\tilde{\phi}_0 = c_{\tilde{x}}$ . Then  $\tilde{\phi}_t$  is a homotopy between  $\tilde{f}$  and  $c_{\tilde{x}}$ . So  $[\tilde{f}] = 0$ . Thus  $p_*$  is injective.  $\square$

**Example 1.14.** Consider  $S^1$  with its universal covering map  $p : \mathbb{R} \rightarrow S^1$  given by  $p(t) = e^{2\pi it}$ . We already know that  $\pi_1(S^1) = \mathbb{Z}$ . If  $n \geq 2$ , Proposition 1.13 yields that  $\pi_n(S^1) = \pi_n(\mathbb{R}) = 0$ .

**Example 1.15.** Consider  $T^n = S^1 \times S^1 \times \cdots \times S^1$ , the  $n$ -torus. We have  $\pi_1(T^n) = \mathbb{Z}^n$ . By using the universal covering map  $p : \mathbb{R}^n \rightarrow T^n$ , we have by Proposition 1.13 that  $\pi_i(T^n) = \pi_i(\mathbb{R}^n) = 0$  for  $i \geq 2$ .

**Definition 1.16.** If  $\pi_n(X) = 0$  for all  $n \geq 2$ , the space  $X$  is called aspherical.

**Remark 1.17.** As a side remark, the celebrated *Singer-Hopf conjecture* asserts that if  $X$  is a smooth closed aspherical manifold of dimension  $2k$ , then  $(-1)^k \cdot \chi(X) \geq 0$ , where  $\chi$  denotes the Euler characteristic.

**Proposition 1.18.** Let  $\{X_\alpha\}_\alpha$  be a collection of path-connected spaces. Then

$$\pi_n \left( \prod_\alpha X_\alpha \right) \cong \prod_\alpha \pi_n(X_\alpha)$$

for all  $n$ .

*Proof.* First note that a map  $f : Y \rightarrow \prod_\alpha X_\alpha$  is a collection of maps  $f_\alpha : Y \rightarrow X_\alpha$ . For elements of  $\pi_n$ , take  $Y = S^n$  (note that since all spaces are path-connected, we may drop the reference to base points). For homotopies, take  $Y = S^n \times I$ .  $\square$

**Example 1.19.** A natural question to ask is if there exist spaces  $X$  and  $Y$  such that  $\pi_n(X) \cong \pi_n(Y)$  for all  $n$ , but with  $X$  and  $Y$  not homotopy equivalent. Whitehead's Theorem (to be discussed later on) states that if a map of CW complexes  $f : X \rightarrow Y$  induces isomorphisms on all  $\pi_n$ , then  $f$  is a homotopy equivalence. So for the above question to have a positive answer, we must find  $X$  and  $Y$  so that there is no continuous map  $f : X \rightarrow Y$  inducing the isomorphisms on  $\pi_n$ 's. Consider

$$X = S^2 \times \mathbb{R}P^3 \quad \text{and} \quad Y = \mathbb{R}P^2 \times S^3.$$

Then  $\pi_n(X) = \pi_n(S^2 \times \mathbb{R}P^3) = \pi_n(S^2) \times \pi_n(\mathbb{R}P^3)$ . Since  $S^3$  is a covering of  $\mathbb{R}P^3$ , for all  $n \geq 2$  we have that  $\pi_n(X) = \pi_n(S^2) \times \pi_n(S^3)$ . We also have  $\pi_1(X) = \pi_1(S^2) \times \pi_1(\mathbb{R}P^3) = \mathbb{Z}/2$ . Similarly, we have  $\pi_n(Y) = \pi_n(\mathbb{R}P^2 \times S^3) = \pi_n(\mathbb{R}P^2) \times \pi_n(S^3)$ . And since  $S^2$  is a covering of  $\mathbb{R}P^2$ , for  $n \geq 2$  we have that  $\pi_n(Y) = \pi_n(S^2) \times \pi_n(S^3)$ . Finally,  $\pi_1(Y) = \pi_1(\mathbb{R}P^2) \times \pi_1(S^3) = \mathbb{Z}/2$ . So

$$\pi_n(X) = \pi_n(Y) \text{ for all } n.$$

By considering homology groups, however, we see that  $X$  and  $Y$  are not homotopy equivalent. Indeed, by the Künneth formula, we get that  $H_5(X) = \mathbb{Z}$  while  $H_5(Y) = 0$  (since  $\mathbb{R}P^3$  is oriented while  $\mathbb{R}P^2$  is not).

Just like there is a homomorphism  $\pi_1(X) \longrightarrow H_1(X)$ , we can also construct homomorphisms

$$\pi_n(X) \longrightarrow H_n(X)$$

defined by

$$[f : S^n \rightarrow X] \mapsto f_*[S^n],$$

where  $[S^n]$  is the fundamental class of  $S^n$ . A very important result in homotopy theory is the following:

**Theorem 1.20.** (*Hurewicz*)

*If  $n \geq 2$  and  $\pi_i(X) = 0$  for all  $i < n$ , then  $H_i(X) = 0$  for  $i < n$  and  $\pi_n(X) \cong H_n(X)$ .*

Moreover, there is also a relative version of the Hurewicz theorem (see the next section for a definition of the relative homotopy groups), which can be used to prove the following:

**Corollary 1.21.** *If  $X$  and  $Y$  are CW complexes with  $\pi_1(X) = \pi_1(Y) = 0$ , and a map  $f : X \rightarrow Y$  induces isomorphisms on all integral homology groups  $H_n$ , then  $f$  is a homotopy equivalence.*

We'll discuss all of these in the subsequent sections.

## 2 Relative Homotopy Groups

Given a triple  $(X, A, x_0)$  where  $x_0 \in A \subseteq X$ , we define relative homotopy groups as follows:

**Definition 2.1.** *Let  $X$  be a space and let  $A \subseteq X$  and  $x_0 \in A$ . Let*

$$I^{n-1} = \{(s_1, \dots, s_n) \in I^n \mid s_n = 0\}$$

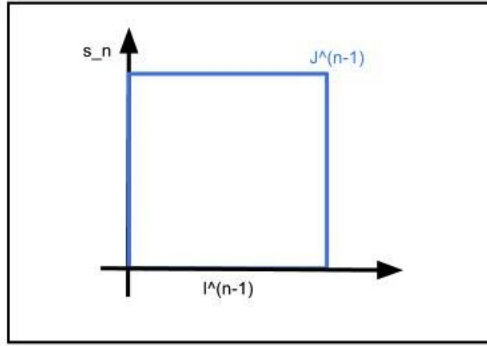
*and set*

$$J^{n-1} = \overline{\partial I^n \setminus I^{n-1}}.$$

*Then define the  $n$ -th homotopy group of the pair  $(X, A)$  with basepoint  $x_0$  as:*

$$\pi_n(X, A, x_0) = \{f : (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)\} / \sim$$

*where, as before,  $\sim$  is the homotopy equivalence relation.*

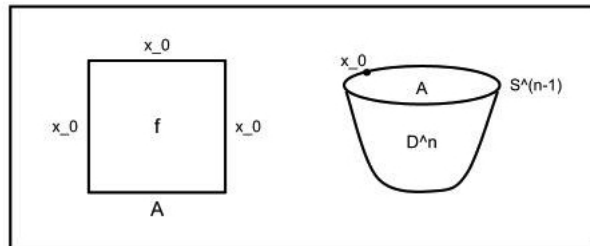


Alternatively, by collapsing  $J^{n-1}$  to a point, we obtain a commutative diagram

$$\begin{array}{ccc}
 (I^n, \partial I^n, J^{n-1}) & \xrightarrow{f} & (X, A, x_0) \\
 \searrow g & & \nearrow \\
 & (D^n, S^{n-1}, s_0) &
 \end{array}$$

where  $g$  is obtained by collapsing  $J^{n-1}$ . So we can take

$$\pi_n(X, A, x_0) = \{g : (D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)\} / \sim .$$



A sum operation is defined on  $\pi_n(X, A, x_0)$  by the same formulas as for  $\pi_n(X, x_0)$ , except that the coordinate  $s_n$  now plays a special role and is no longer available for the sum operation. Thus, we have:

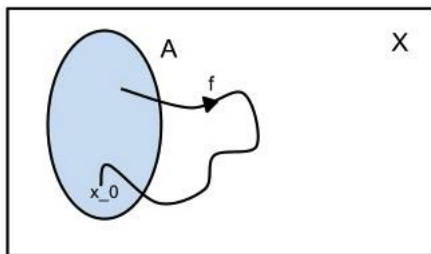
**Proposition 2.2.** *If  $n \geq 2$ , then  $\pi_n(X, A, x_0)$  forms a group under the usual sum operation. Further, if  $n \geq 3$ , then  $\pi_n(X, A, x_0)$  is abelian.*

**Remark 2.3.** Note that the proposition fails in the case  $n = 1$ . Indeed, we have that

$$\pi_1(X, A, x_0) = \{f : (I, \{0, 1\}, \{1\}) \rightarrow (X, A, x_0)\} / \sim .$$

Then  $\pi_1(X, A, x_0)$  consists of homotopy classes of paths starting anywhere  $A$  and ending at  $x_0$ , so we cannot always concatenate two paths.





Just as in the absolute case, a map of pairs  $\phi : (X, A, x_0) \rightarrow (Y, B, y_0)$  induces homomorphisms  $\phi_* : \pi_n(X, A, x_0) \rightarrow \pi_n(Y, B, y_0)$  for all  $n \geq 2$ .

A very important feature of the relative homotopy groups is the following:

**Proposition 2.4.** *The relative homotopy groups of  $(X, A, x_0)$  fit into a long exact sequence*

$$\cdots \rightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \rightarrow \cdots \rightarrow \pi_0(X, x_0) \rightarrow 0,$$

where the map  $\partial_n$  is defined by  $\partial_n[f] = [f|_{I^{n-1}}]$  and all others are induced by inclusions.

**Remark 2.5.** Near the end of the above sequence, where group structures are not defined, exactness still makes sense: the image of one map is the kernel of the next, which consists of those elements mapping to the homotopy class of the constant map.

**Example 2.6.** Let  $X$  be a path-connected space, and

$$CX := X \times [0, 1] / X \times \{0\}$$

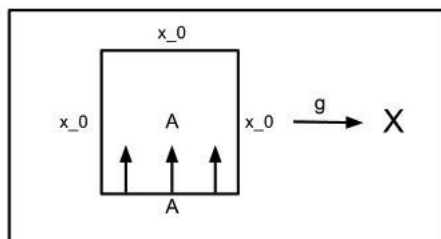
be the cone on  $X$ . We can regard  $X$  as a subspace of  $CX$  via  $X \times \{1\} \subset CX$ . Since  $CX$  is contractible, the long exact sequence of homotopy groups gives isomorphisms

$$\pi_n(CX, X, x_0) \cong \pi_{n-1}(X, x_0).$$

In what follows, it will be important to have a good description of the zero element  $0 \in \pi_n(X, A, x_0)$ .

**Lemma 2.7.** *Let  $[f] \in \pi_n(X, A, x_0)$ . Then  $[f] = 0$  if, and only if,  $f \simeq g$  for some map  $g$  with image contained in  $A$ .*

*Proof.* ( $\Leftarrow$ ) Suppose  $f \simeq g$  for some  $g$  with  $\text{Image } g \subset A$ .



Then we can deform  $I^n$  to  $J^{n-1}$  as indicated in the above picture, and so  $g \simeq c_{x_0}$ . Since homotopy is a transitive relation, we then get that  $f \simeq c_{x_0}$ .

( $\Rightarrow$ ) Suppose  $[f] = 0$  in  $\pi_n(X, A, x_0)$ . So  $f \simeq c_{x_0}$ . Take  $g = c_{x_0}$ . □

Recall that if  $X$  is path-connected, then  $\pi_n(X, x_0)$  is independent of our choice of base point, and  $\pi_1(X)$  acts on  $\pi_n(X)$  for all  $n \geq 1$ . In the relative case, we have:

**Lemma 2.8.** *If  $A$  is path-connected, then  $\beta_\gamma : \pi_n(X, A, x_1) \rightarrow \pi_n(X, A, x_0)$  is an isomorphism, where  $\gamma$  is a path in  $A$  from  $x_1$  to  $x_0$ .*

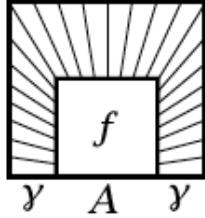


Figure 5: relative  $\beta_\gamma$

**Remark 2.9.** In particular, if  $x_0 = x_1$ , we get an action of  $\pi_1(A)$  on  $\pi_n(X, A)$ .

It is easy to see that the following three conditions are equivalent:

1. every map  $S^i \rightarrow X$  is homotopic to a constant map,
2. every map  $S^i \rightarrow X$  extends to a map  $D^{i+1} \rightarrow X$ , with  $S^i = \partial D^{i+1}$ ,
3.  $\pi_i(X, x_0) = 0$  for all  $x_0 \in X$ .

In the relative setting, the following are equivalent for any  $i > 0$ :

1. every map  $(D^i, \partial D^i) \rightarrow (X, A)$  is homotopic rel.  $\partial D^i$  to a map  $D^i \rightarrow A$ ,
2. every map  $(D^i, \partial D^i) \rightarrow (X, A)$  is homotopic through such maps to a map  $D^i \rightarrow A$ ,
3. every map  $(D^i, \partial D^i) \rightarrow (X, A)$  is homotopic through such maps to a constant map  $D^i \rightarrow A$ ,
4.  $\pi_i(X, A, x_0) = 0$  for all  $x_0 \in A$ .

**Remark 2.10.** As seen above, if  $\alpha : S^n = \partial e^{n+1} \rightarrow X$  represents an element  $[\alpha] \in \pi_n(X, x_0)$ , then  $[\alpha] = 0$  if and only if  $\alpha$  extends to a map  $e^{n+1} \rightarrow X$ . Thus if we enlarge  $X$  to a space  $X' = X \cup_\alpha e^{n+1}$  by adjoining an  $(n+1)$ -cell  $e^{n+1}$  with  $\alpha$  as attaching map, then the inclusion  $j : X \hookrightarrow X'$  induces a homomorphism  $j_* : \pi_n(X, x_0) \rightarrow \pi_n(X', x_0)$  with  $j_*[\alpha] = 0$ . We say that  $[\alpha]$  “has been killed”.

The following is left as an exercise:

**Lemma 2.11.** *Let  $(X, x_0)$  be a space with a basepoint, and let  $X' = X \cup_{\alpha} e^{n+1}$  be obtained from  $X$  by adjoining an  $(n+1)$ -cell. Then the inclusion  $j : X \hookrightarrow X'$  induces a homomorphism  $j_* : \pi_i(X, x_0) \rightarrow \pi_i(X', x_0)$ , which is an isomorphism for  $i < n$  and surjective for  $i = n$ .*

**Definition 2.12.** *We say that the pair  $(X, A)$  is  $n$ -connected if  $\pi_i(X, A) = 0$  for  $i \leq n$  and  $X$  is  $n$ -connected if  $\pi_i(X) = 0$  for  $i \leq n$ .*

In particular,  $X$  is 0-connected if and only if  $X$  is connected. Moreover,  $X$  is 1-connected if and only if  $X$  is simply-connected.

### 3 Homotopy Extension Property

**Definition 3.1** (Homotopy Extension Property). *Given a pair  $(X, A)$ , a map  $F_0 : X \rightarrow Y$ , and a homotopy  $f_t : A \rightarrow Y$  such that  $f_0 = F_0|_A$ , we say that  $(X, A)$  satisfies the homotopy extension property (HEP) if there is a homotopy  $F_t : X \rightarrow Y$  extending  $f_t$  and  $F_0$ . In other words,  $(X, A)$  has homotopy extension property if any map  $X \times \{0\} \cup A \times I \rightarrow Y$  extends to a map  $X \times I \rightarrow Y$ .*

**Proposition 3.2.** *Any CW pair has the homotopy extension property. In fact, for every CW pair  $(X, A)$ , there is a deformation retract  $r : X \times I \rightarrow X \times \{0\} \cup A \times I$ , hence  $X \times I \rightarrow Y$  can be defined by the composition  $X \times I \xrightarrow{r} X \times \{0\} \cup A \times I \rightarrow Y$ .*

*Proof.* We have an obvious deformation retract  $D^n \times I \xrightarrow{r} D^n \times \{0\} \cup S^{n-1} \times I$ . For every  $n$ , consider the pair  $(X_n, A_n \cup X_{n-1})$ , where  $X_n$  denotes the  $n$ -skeleton of  $X$ . Then

$$X_n \times I = [X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I] \cup D^n \times I,$$

where the cylinders  $D^n \times I$  corresponding to  $n$ -cells  $D^n$  in  $X \setminus A$  are glued along  $D^n \times \{0\} \cup S^{n-1} \times I$  to the pieces  $X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I$ . By deforming these cylinders  $D^n \times I$  we get a deformation retraction

$$r_n : X_n \times I \rightarrow X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I.$$

Concatenating these deformation retractions by performing  $r_n$  over  $[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}]$ , we get a deformation retraction of  $X \times I$  onto  $X \times \{0\} \cup A \times I$ . Continuity follows since CW complexes have the weak topology with respect to their skeleta, so a map of CW complexes is continuous if and only if its restriction to each skeleton is continuous.  $\square$

### 4 Cellular Approximation

All maps are assumed to be continuous.

**Definition 4.1.** Let  $X$  and  $Y$  be CW-complexes. A map  $f : X \rightarrow Y$  is called cellular if  $f(X_n) \subset Y_n$  for all  $n$ , where  $X_n$  denotes the  $n$ -skeleton of  $X$  and similarly for  $Y$ .

**Definition 4.2.** Let  $f : X \rightarrow Y$  be a map of CW complexes. A map  $f' : X \rightarrow Y$  is a cellular approximation of  $f$  if  $f'$  is cellular and  $f$  is homotopic to  $f'$ .

**Theorem 4.3** (Cellular Approximation Theorem). Any map  $f : X \rightarrow Y$  between CW-complexes has a cellular approximation  $f' : X \rightarrow Y$ . Moreover, if  $f$  is already cellular on a subcomplex  $A \subseteq X$ , we can take  $f'|_A = f|_A$ .

The proof of Theorem 4.3 uses the following key technical result.

**Lemma 4.4.** Let  $f : X \cup e^n \rightarrow Y \cup e^k$  be a map of CW complexes, with  $e^n, e^k$  denoting an  $n$ -cell and, resp.,  $k$ -cell attached to  $X$  and, resp.,  $Y$ . Assume that  $f(X) \subseteq Y$ ,  $f|_X$  is cellular, and  $n < k$ . Then  $f \stackrel{h.e.}{\simeq} f'$  (rel.  $X$ ), with  $\text{Image}(f') \subseteq Y$ .

**Remark 4.5.** If in the statement of Lemma 4.4 we assume that  $X$  and  $Y$  are points, then we get that the inclusion  $S^n \hookrightarrow S^k$  ( $n < k$ ) is homotopic to the constant map  $S^n \rightarrow \{s_0\}$  for some point  $s_0 \in S^k$ .

Lemma 4.4 is used along with induction on skeleta to prove the cellular approximation theorem as follows.

*Proof of Theorem 4.3.* Suppose  $f|_{X_n}$  is cellular, and let  $e^n$  be an (open)  $n$ -cell of  $X$ . Since  $\bar{e}^n$  is compact,  $f(\bar{e}^n)$  (hence also  $f(e^n)$ ) meets only finitely many open cells of  $Y$ . Let  $e^k$  be an open cell of maximal dimension in  $Y$  which meets  $f(e^n)$ . If  $k \leq n$ ,  $f$  is already cellular on  $e^n$ . If  $n < k$ , Lemma 4.4 can be used to homotop  $f|_{X_{n-1} \cup e^n}$  (rel.  $X_{n-1}$ ) to a map whose image on  $e^n$  misses  $e^k$ . By finitely many iterations of this process, we eventually homotop  $f|_{X_{n-1} \cup e^n}$  (rel.  $X_{n-1}$ ) to a map  $f' : X_{n-1} \cup e^n \rightarrow Y_n$ , i.e., whose image on  $e^n$  misses all cells in  $Y$  of dimension  $> n$ . Doing this for all  $n$ -cells of  $X$ , staying fixed on  $n$ -cells in  $A$  where  $f$  is already cellular, we obtain a homotopy of  $f|_{X_n}$  (rel.  $X_{n-1} \cup A_n$ ) to a cellular map. By the homotopy extension property 3.2, we can extend this homotopy (together with the constant homotopy on  $A$ ) to a homotopy defined on all of  $X$ . This completes the induction step.

For varying  $n \rightarrow \infty$ , we concatenate the above homotopies to define a homotopy from  $f$  to a cellular map  $f'$  (rel.  $A$ ) by performing the above construction (i.e., the  $n$ -th homotopy) on the  $t$ -interval  $[1 - 1/2^n, 1 - 1/2^{n+1}]$ .  $\square$

We also have the following relative version of Theorem 4.3:

**Theorem 4.6** (Relative cellular approximation). Any map  $f : (X, A) \rightarrow (Y, B)$  of CW pairs has a cellular approximation by a homotopy through such maps of pairs.

*Proof.* First we use the cellular approximation for  $f|_A : A \rightarrow B$ . Let  $f' : A \rightarrow B$  be a cellular map, homotopic to  $f|_A$  via a homotopy  $H$ . By the Homotopy Extension Property 3.2, we can regard  $H$  as a homotopy on all of  $X$ , so we get a map  $f' : X \rightarrow Y$  such that  $f'|_A$  is a cellular map. By the second part of the cellular approximation theorem 4.3,  $f' \stackrel{h.e.}{\simeq} f''$ , with  $f'' : X \rightarrow Y$  a cellular map satisfying  $f''|_A = f'|_A$ . The map  $f''$  provides the required cellular approximation of  $f$ .  $\square$

**Corollary 4.7.** *Let  $A \subset X$  be CW complexes and suppose that all cells of  $X \setminus A$  have dimension  $> n$ . Then  $\pi_i(X, A) = 0$  for  $i \leq n$ .*

*Proof.* Let  $[f] \in \pi_i(X, A)$ . By the relative version of the cellular approximation, the map of pairs  $f : (D^i, S^{i-1}) \rightarrow (X, A)$  is homotopic to a map  $g$  with  $g(D^i) \subset X_i$ . But for  $i \leq n$ , we have that  $X_i \subset A$ , so  $\text{Image } g \subset A$ . Therefore, by Lemma 2.7,  $[f] = [g] = 0$ .  $\square$

**Corollary 4.8.** *If  $X$  is a CW complex, then  $\pi_i(X, X_n) = 0$  for all  $i \leq n$ .*

Therefore, the long exact sequence for the homotopy groups of the pair  $(X, X_n)$  yields the following:

**Corollary 4.9.** *Let  $X$  be a CW complex. For  $i < n$ , we have  $\pi_i(X) \cong \pi_i(X_n)$ .*

## 5 Excision for homotopy groups. The Suspension Theorem

We state here the following useful result without proof:

**Theorem 5.1** (Excision). *Let  $X$  be a CW complex which is a union of subcomplexes  $A$  and  $B$ , such that  $C = A \cap B$  is path connected. Assume that  $(A, C)$  is  $m$ -connected and  $(B, C)$  is  $n$ -connected, with  $m, n \geq 1$ . Then the map  $\pi_i(A, C) \rightarrow \pi_i(X, B)$  induced by inclusion is an isomorphism if  $i < m + n$  and a surjection for  $i = m + n$ .*

The following consequence is very useful for iterating homotopy groups of spheres:

**Theorem 5.2** (Freudenthal Suspension Theorem). *Let  $X$  be an  $(n - 1)$ -connected CW complex. For any map  $f : S^i \rightarrow X$ , consider its suspension,*

$$\Sigma f : \Sigma S^i = S^{i+1} \rightarrow \Sigma X.$$

*The assignment*

$$[f] \in \pi_i(X) \mapsto [\Sigma f] \in \pi_{i+1}(\Sigma X)$$

*defines a homomorphism  $\pi_i(X) \rightarrow \pi_{i+1}(\Sigma X)$ , which is an isomorphism for  $i < 2n - 1$  and a surjection for  $i = 2n - 1$ .*

*Proof.* Decompose the suspension  $\Sigma X$  as the union of two cones  $C_+X$  and  $C_-X$  intersecting in a copy of  $X$ . By using long exact sequences of pairs and the fact that the cones  $C_+X$  and  $C_-X$  are contractible, the suspension map can be written as a composition:

$$\pi_i(X) \cong \pi_{i+1}(C_+, X) \longrightarrow \pi_{i+1}(\Sigma X, C_-X) \cong \pi_{i+1}(\Sigma X),$$

with the middle map induced by inclusion.

Since  $X$  is  $(n - 1)$ -connected, from the long exact sequence of  $(C_\pm X, X)$ , we see that the pairs  $(C_\pm X, X)$  are  $n$ -connected. Therefore, the Excision Theorem 5.1 yields that  $\pi_{i+1}(C_+, X) \rightarrow \pi_{i+1}(\Sigma X, C_-X)$  is an isomorphism for  $i + 1 < 2n$  and it is surjective for  $i + 1 = 2n$ .  $\square$

## 6 Homotopy Groups of Spheres

We now turn our attention to computing (some of) the homotopy groups  $\pi_i(S^n)$ . For  $i \leq n, i = n + 1, n + 2, n + 3$  and a few more cases, these homotopy groups are known (and we will work them out later on). In general, however, this is a very difficult problem. For  $i = n$ , we would expect to have  $\pi_n(S^n) = \mathbb{Z}$  by associating to each (homotopy class of a) map  $f : S^n \rightarrow S^n$  its degree. For  $i < n$ , we will show that  $\pi_i(S^n) = 0$ . Note that if  $f : S^i \rightarrow S^n$  is not surjective, i.e., there is  $y \in S^n \setminus f(S^i)$ , then  $f$  factors through  $\mathbb{R}^n$ , which is contractible. By composing  $f$  with the retraction  $\mathbb{R}^n \rightarrow x_0$  we get that  $f \simeq c_{x_0}$ . However, there are surjective maps  $S^i \rightarrow S^n$  for  $i < n$ , in which case the above “proof” fails. To make things work, we “alter”  $f$  to make it cellular, so the following holds.

**Proposition 6.1.** *For  $i < n$ , we have  $\pi_i(S^n) = 0$ .*

*Proof.* Choose the standard CW-structure on  $S^i$  and  $S^n$ . For  $[f] \in \pi_i(S^n)$ , we may assume by Theorem 4.3 that  $f : S^i \rightarrow S^n$  is cellular. Then  $f(S^i) \subset (S^n)_i$ . But  $(S^n)_i$  is a point, so  $f$  is a constant map.  $\square$

Recall now the following special case of the Suspension Theorem 5.2 for  $X = S^n$ :

**Theorem 6.2.** *Let  $f : S^i \rightarrow S^n$  be a map, and consider its suspension,*

$$\Sigma f : \Sigma S^i = S^{i+1} \rightarrow \Sigma S^n = S^{n+1}.$$

*The assignment*

$$[f] \in \pi_i(S^n) \mapsto [\Sigma f] \in \pi_{i+1}(S^{n+1})$$

*defines a homomorphism  $\pi_i(S^n) \rightarrow \pi_{i+1}(S^{n+1})$ , which is an isomorphism  $\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$  for  $i < 2n - 1$  and a surjection for  $i = 2n - 1$ .*

**Corollary 6.3.**  *$\pi_n(S^n)$  is either  $\mathbb{Z}$  or a finite quotient of  $\mathbb{Z}$  (for  $n \geq 2$ ), generated by the degree map.*

*Proof.* By the Suspension Theorem 6.2, we have the following:

$$\mathbb{Z} \cong \pi_1(S^1) \twoheadrightarrow \pi_2(S^2) \cong \pi_3(S^3) \cong \pi_4(S^4) \cong \dots$$

$\square$

To show that  $\pi_1(S^1) \cong \pi_2(S^2)$ , we can use *the long exact sequence for the homotopy groups of a fibration*, see Theorem 11.8 below. For any fibration (e.g., a covering map)

$$\begin{array}{ccc} F & & \\ & \searrow & \\ & & E \\ & & \downarrow \\ & & B \end{array}$$

there is a long exact sequence

$$\cdots \longrightarrow \pi_i(F) \longrightarrow \pi_i(E) \longrightarrow \pi_i(B) \longrightarrow \pi_{i-1}(F) \longrightarrow \cdots \quad (6.1)$$

Applying the above long exact sequence to the Hopf fibration  $S^1 \hookrightarrow S^3 \rightarrow S^2$ , we obtain:

$$\cdots \longrightarrow \pi_2(S^1) \longrightarrow \pi_2(S^3) \longrightarrow \pi_2(S^2) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(S^3) \longrightarrow \cdots$$

Using the fact that  $\pi_2(S^3) = 0$  and  $\pi_1(S^3) = 0$ , we therefore have an isomorphism:

$$\pi_2(S^2) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Note that by using the vanishing of the higher homotopy groups of  $S^1$ , the long exact sequence (11.8) also yields that

$$\pi_3(S^2) \cong \pi_2(S^2) \cong \mathbb{Z}.$$

**Remark 6.4.** Unlike the homology and cohomology groups, the homotopy groups of a finite CW-complex can be infinitely generated. This fact is discussed in the next example.

**Example 6.5.** For  $n \geq 2$ , consider the finite CW complex  $S^1 \vee S^n$ . We then have that

$$\pi_n(S^1 \vee S^n) = \pi_n(\widetilde{S^1 \vee S^n}),$$

where  $\widetilde{S^1 \vee S^n}$  is the universal cover of  $S^1 \vee S^n$ , depicted below. By contracting the segments

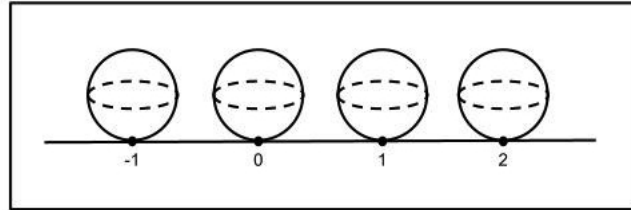


Figure 6: universal cover of  $S^1 \vee S^n$

between consecutive integers, we have that

$$\widetilde{S^1 \vee S^n} \simeq \bigvee_{k \in \mathbb{Z}} S_k^n,$$

with  $S_k^n$  denoting the  $n$ -sphere corresponding to the integer  $k$ . So for any  $n \geq 2$ , we have:

$$\pi_n(S^1 \vee S^n) = \pi_n\left(\bigvee_{k \in \mathbb{Z}} S_k^n\right),$$

which is the free abelian group generated by the inclusions  $S_k^n \hookrightarrow \bigvee_{k \in \mathbb{Z}} S_k^n$ . Indeed, we have the following:

**Lemma 6.6.**  $\pi_n(\bigvee_{\alpha} S_{\alpha}^n)$  is free abelian and generated by the inclusions of factors.

*Proof.* Suppose first that there are only finitely many  $S_{\alpha}^n$ 's in the wedge  $\bigvee_{\alpha} S_{\alpha}^n$ . Then we can regard  $\bigvee_{\alpha} S_{\alpha}^n$  as the  $n$ -skeleton of  $\prod_{\alpha} S_{\alpha}^n$ . The cell structure of a particular  $S_{\alpha}^n$  consists of a single 0-cell  $e_{\alpha}^0$  and a single  $n$ -cell,  $e_{\alpha}^n$ . Thus, in the product  $\prod_{\alpha} S_{\alpha}^n$  there is one 0-cell  $e^0 = \prod_{\alpha} e_{\alpha}^0$ , which, together with the  $n$ -cells

$$\bigcup_{\alpha} \left( \prod_{\beta \neq \alpha} e_{\beta}^0 \right) \times e_{\alpha}^n,$$

form the  $n$ -skeleton  $\bigvee_{\alpha} S_{\alpha}^n$ . Hence  $\prod_{\alpha} S_{\alpha}^n \setminus \bigvee_{\alpha} S_{\alpha}^n$  has only cells of dimension at least  $2n$ , which by Corollary 4.8 yields that the pair  $(\prod_{\alpha} S_{\alpha}^n, \bigvee_{\alpha} S_{\alpha}^n)$  is  $(2n - 1)$ -connected. In particular, as  $n \geq 2$ , we get:

$$\pi_n\left(\bigvee_{\alpha} S_{\alpha}^n\right) \cong \pi_n\left(\prod_{\alpha} S_{\alpha}^n\right) \cong \prod_{\alpha} \pi_n(S_{\alpha}^n) = \bigoplus_{\alpha} \pi_n(S_{\alpha}^n) = \bigoplus_{\alpha} \mathbb{Z}.$$

To reduce the case of infinitely many summands  $S_{\alpha}^n$  to the finite case, consider the homomorphism  $\Phi : \bigoplus_{\alpha} \pi_n(S_{\alpha}^n) \rightarrow \pi_n(\bigvee_{\alpha} S_{\alpha}^n)$  induced by the inclusions  $S_{\alpha}^n \hookrightarrow \bigvee_{\alpha} S_{\alpha}^n$ . Then  $\Phi$  is onto since any map  $f : S^n \rightarrow \bigvee_{\alpha} S_{\alpha}^n$  has compact image contained in the wedge sum of finitely many  $S_{\alpha}^n$ 's, so by the above finite case,  $[f]$  is in the image of  $\Phi$ . Moreover, a nullhomotopy of  $f$  has compact image contained in the wedge sum of finitely many  $S_{\alpha}^n$ 's, so by the above finite case we have that  $\Phi$  is also injective.  $\square$

To conclude our example, we showed that  $\pi_n(S^1 \vee S^n) \cong \pi_n(\bigvee_{k \in \mathbb{Z}} S_k^n)$ , and  $\pi_n(\bigvee_{k \in \mathbb{Z}} S_k^n)$  is free abelian generated by the inclusion of each of the infinite number of  $n$ -spheres. Therefore,  $\pi_n(S^1 \vee S^n)$  is infinitely generated.

**Remark 6.7.** Under the action of  $\pi_1$  on  $\pi_n$ , we can regard  $\pi_n$  as a  $\mathbb{Z}[\pi_1]$ -module. Here  $\mathbb{Z}[\pi_1]$  is the group ring of  $\pi_1$  with  $\mathbb{Z}$ -coefficients, whose elements are of the form  $\sum_{\alpha} n_{\alpha} \gamma_{\alpha}$ , with  $n_{\alpha} \in \mathbb{Z}$  and only finitely many non-zero, and  $\gamma_{\alpha} \in \pi_1$ . Since all the  $n$ -spheres  $S_k^n$  in the universal cover  $\bigvee_{k \in \mathbb{Z}} S_k^n$  are identified under the  $\pi_1$ -action,  $\pi_n$  is a free  $\mathbb{Z}[\pi_1]$ -module of rank 1, i.e.,

$$\begin{aligned} \pi_n &\cong \mathbb{Z}[\pi_1] \cong \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}], \\ &1 \mapsto t \\ &-1 \mapsto t^{-1} \\ &n \mapsto t^n, \end{aligned}$$

which is infinitely generated (by the powers of  $t$ ) over  $\mathbb{Z}$  (i.e., as an abelian group).

**Remark 6.8.** If we consider the class of spaces for which  $\pi_1$  acts trivially on all of  $\pi_n$ 's, a result of Serre asserts that the homotopy groups of such spaces are finitely generated if and only if homology groups are finitely generated.



## 7 Whitehead's Theorem

**Definition 7.1.** A map  $f : X \rightarrow Y$  is a weak homotopy equivalence if it induces isomorphisms on all homotopy groups  $\pi_n$ .

Notice that a homotopy equivalence is a weak homotopy equivalence. The following important result provides a converse to this fact in the world of CW complexes.

**Theorem 7.2** (Whitehead). *If  $X$  and  $Y$  are CW complexes and  $f : X \rightarrow Y$  is a weak homotopy equivalence, then  $f$  is a homotopy equivalence. Moreover, if  $X$  is a subcomplex of  $Y$ , and  $f$  is the inclusion map, then  $X$  is a deformation retract of  $Y$ .*

The following consequence is very useful in practice:

**Corollary 7.3.** *If  $X$  and  $Y$  are CW complexes with  $\pi_1(X) = \pi_1(Y) = 0$ , and  $f : X \rightarrow Y$  induces isomorphisms on homology groups  $H_n$  for all  $n$ , then  $f$  is a homotopy equivalence.*

The above corollary follows from Whitehead's theorem and the following relative version of the Hurewicz Theorem 10.1 (to be discussed later on):

**Theorem 7.4** (Hurewicz). *If  $n \geq 2$ , and  $\pi_i(X, A) = 0$  for  $i < n$ , with  $A$  simply-connected and non-empty, then  $H_i(X, A) = 0$  for  $i < n$  and  $\pi_n(X, A) \cong H_n(X, A)$ .*

Before discussing the proof of Whitehead's theorem, let us give an example that shows that having induced isomorphisms on all homology groups is not sufficient for having a homotopy equivalence (so the simply-connectedness assumption in Corollary 7.3 cannot be dropped):

**Example 7.5.** Let

$$f : X = S^1 \hookrightarrow (S^1 \vee S^n) \cup e^{n+1} = Y \quad (n \geq 2)$$

be the inclusion map, with the attaching map for the  $(n+1)$ -cell of  $Y$  described below. We know from Example 6.5 that  $\pi_n(S^1 \vee S^n) \cong \mathbb{Z}[t, t^{-1}]$ . We define  $Y$  by attaching the  $(n+1)$ -cell  $e^{n+1}$  to  $S^1 \vee S^n$  by a map  $g : S^n = \partial e^{n+1} \rightarrow S^1 \vee S^n$  so that  $[g] \in \pi_n(S^1 \vee S^n)$  corresponds to the element  $2t - 1 \in \mathbb{Z}[t, t^{-1}]$ . We then see that

$$\pi_n(Y) = \mathbb{Z}[t, t^{-1}]/(2t - 1) \neq 0 = \pi_n(X),$$

since by setting  $t = \frac{1}{2}$  we get that  $\mathbb{Z}[t, t^{-1}]/(2t - 1) \cong \mathbb{Z}[\frac{1}{2}] = \{\frac{a}{2^k} \mid k \in \mathbb{Z}_{\geq 0}\} \subset \mathbb{Q}$ . In particular,  $f$  is not a homotopy equivalence. Moreover, from the long exact sequence of homotopy groups for the  $(n-1)$ -connected pair  $(Y, X)$ , the inclusion  $X \hookrightarrow Y$  induces an isomorphism on homotopy groups  $\pi_i$  for  $i < n$ . Finally, this inclusion map also induces isomorphisms on all homology groups,  $H_n(X) \cong H_n(Y)$  for all  $n$ , as can be seen from cellular homology. Indeed, the cellular boundary map

$$H_{n+1}(Y_{n+1}, Y_n) \rightarrow H_n(Y_n, Y_{n-1})$$

is an isomorphism since the degree of the composition of the attaching map  $S^n \rightarrow S^1 \vee S^n$  of  $e^{n+1}$  with the collapse map  $S^1 \vee S^n \rightarrow S^n$  is  $2 - 1 = 1$ .

Let us now get back to the proof of Whitehead's Theorem 7.2. To prove Whitehead's theorem, we will use the following:

**Lemma 7.6** (Compression Lemma). *Let  $(X, A)$  be a CW pair, and  $(Y, B)$  be a pair with  $Y$  path-connected and  $B \neq \emptyset$ . Suppose that for each  $n > 0$  for which  $X \setminus A$  has cells of dimension  $n$ ,  $\pi_n(Y, B, b_0) = 0$  for all  $b_0 \in B$ . Then any map  $f : (X, A) \rightarrow (Y, B)$  is homotopic to some map  $f' : X \rightarrow B$  fixing  $A$  (i.e., with  $f'|_A = f|_A$ ).*

*Proof.* Assume inductively that  $f(X_{k-1} \cup A) \subseteq B$ . Let  $e^k$  be a  $k$ -cell in  $X \setminus A$ , with characteristic map  $\alpha : (D^k, S^{k-1}) \rightarrow X$ . Ignoring basepoints, we regard  $\alpha$  as an element  $[\alpha] \in \pi_k(X, X_{k-1} \cup A)$ . Then  $f_*[\alpha] = [f \circ \alpha] \in \pi_k(Y, B) = 0$  by our hypothesis, since  $e^k$  is a  $k$ -cell in  $X \setminus A$ . By Lemma 2.7, there is a homotopy  $H : (D^k, S^{k-1}) \times I \rightarrow (Y, B)$  such that  $H_0 = f \circ \alpha$  and  $\text{Image}(H_1) \subseteq B$ .

Performing this process for all  $k$ -cells in  $X \setminus A$  simultaneously, we get a homotopy from  $f$  to  $f'$  such that  $f'(X_k \cup A) \subseteq B$ . Using the homotopy extension property 3.2, we can regard this as a homotopy on all of  $X$ , i.e.,  $f \simeq f'$  as maps  $X \rightarrow Y$ , so the induction step is completed.

Finitely many applications of the induction step finish the proof if the cells of  $X \setminus A$  are of bounded dimension. In general, we have

$$\begin{aligned} f &\underset{H_1}{\simeq} f_1, \text{ with } f_1(X_1 \cup A) \subseteq B, \\ f_1 &\underset{H_2}{\simeq} f_2, \text{ with } f_2(X_2 \cup A) \subseteq B, \\ &\dots \\ f_{n-1} &\underset{H_n}{\simeq} f_n, \text{ with } f_n(X_n \cup A) \subseteq B, \end{aligned}$$

and so on. Any finite skeleton is eventually fixed under these homotopies.

Define a homotopy  $H : X \times I \rightarrow Y$  as

$$H = H_i \text{ on } \left[1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}\right].$$

Note that  $H$  is continuous by CW topology, so it gives the required homotopy.  $\square$

*Proof of Whitehead's theorem.* We can split the proof of Theorem 7.2 into two cases:

Case 1: If  $f$  is an inclusion  $X \hookrightarrow Y$ , since  $\pi_n(X) = \pi_n(Y)$  for all  $n$ , we get by the long exact sequence for the homotopy groups of the pair  $(Y, X)$  that  $\pi_n(Y, X) = 0$  for all  $n$ . Applying the above compression lemma 7.6 to the identity map  $id : (Y, X) \rightarrow (Y, X)$  yields a deformation retraction  $r : Y \rightarrow X$  of  $Y$  onto  $X$ .

Case 2: The general case of a map  $f : X \rightarrow Y$  can be reduced to the above case of an inclusion by using the *mapping cylinder* of  $f$ , i.e.,

$$M_f := (X \times I) \sqcup Y / (x, 1) \sim f(x).$$

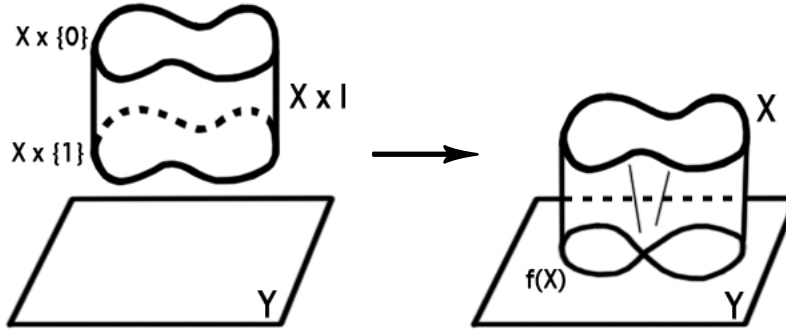


Figure 7: The mapping cylinder  $M_f$  of  $f$

Note that  $M_f$  contains both  $X = X \times \{0\}$  and  $Y$  as subspaces, and  $M_f$  deformation retracts onto  $Y$ . Moreover, the map  $f$  can be written as the composition of the inclusion  $i$  of  $X$  into  $M_f$ , and the retraction  $r$  from  $M_f$  to  $Y$ :

$$f : X = X \times \{0\} \xrightarrow{i} M_f \xrightarrow{r} Y.$$

Since  $M_f$  is homotopy equivalent to  $Y$  via  $r$ , it suffices to show that  $M_f$  deformation retracts onto  $X$ , so we can replace  $f$  with the inclusion map  $i$ . If  $f$  is a cellular map, then  $M_f$  is a CW complex having  $X$  as a subcomplex. So we can apply Case 1. If  $f$  is not cellular, then  $f$  is homotopic to some cellular map  $g$ , so we may work with  $g$  and the mapping cylinder  $M_g$  and again reduce to Case 1.  $\square$

We can now prove Corollary 7.3:

*Proof.* After replacing  $Y$  by the mapping cylinder  $M_f$ , we may assume that  $f$  is an inclusion  $X \hookrightarrow Y$ . As  $H_n(X) \cong H_n(Y)$  for all  $n$ , we have by the long exact sequence for the homology groups of the pair  $(Y, X)$  that  $H_n(Y, X) = 0$  for all  $n$ .

Since  $X$  and  $Y$  are simply-connected, we have  $\pi_1(Y, X) = 0$ . So by the relative Hurewicz Theorem 10.1, the first non-zero  $\pi_n(Y, X)$  is isomorphic to the first non-zero  $H_n(Y, X)$ . So  $\pi_n(Y, X) = 0$  for all  $n$ . Then, by the homotopy long exact sequence for the pair  $(Y, X)$ , we get that

$$\pi_n(X) \cong \pi_n(Y)$$

for all  $n$ , with isomorphisms induced by the inclusion map  $f$ . Finally, Whitehead's theorem 7.2 yields that  $f$  is a homotopy equivalence.  $\square$

**Example 7.7.** Let  $X = \mathbb{R}P^2$  and  $Y = S^2 \times \mathbb{R}P^\infty$ . First note that  $\pi_1(X) = \pi_1(Y) \cong \mathbb{Z}/2$ . Also, since  $S^2$  is a covering of  $\mathbb{R}P^2$ , we have that

$$\pi_i(X) \cong \pi_i(S^2), \quad i \geq 2.$$

Moreover,  $\pi_i(Y) \cong \pi_i(S^2) \times \pi_i(\mathbb{R}P^\infty)$ , and as  $\mathbb{R}P^\infty$  is covered by  $S^\infty = \bigcup_{n \geq 0} S^n$ , we get that

$$\pi_i(Y) \cong \pi_i(S^2) \times \pi_i(S^\infty), \quad i \geq 2.$$

To calculate  $\pi_i(S^\infty)$ , we use cellular approximation. More precisely, we can approximate any  $f : S^i \rightarrow S^\infty$  by a cellular map  $g$  so that  $\text{Image } g \subset S^n$  for  $i \ll n$ . Thus,  $[f] = [g] \in \pi_i(S^n) = 0$ , and we see that

$$\pi_i(X) \cong \pi_i(S^2) \cong \pi_i(Y), \quad i \geq 2.$$

Altogether, we have shown that  $X$  and  $Y$  have the same homotopy groups. However, as can be easily seen by considering homology groups,  $X$  and  $Y$  are not homotopy equivalent. In particular, by Whitehead's theorem, there cannot exist a map  $f : \mathbb{R}P^2 \rightarrow S^2 \times \mathbb{R}P^\infty$  inducing isomorphisms on  $\pi_n$  for all  $n$ . (If such a map existed, it would have to be a homotopy equivalence.)

**Example 7.8.** As we will see later on, the CW complexes  $S^2$  and  $S^3 \times \mathbb{C}P^\infty$  have isomorphic homotopy groups, but they are not homotopy equivalent.

## 8 CW approximation

Recall that map  $f : X \rightarrow Y$  is a *weak homotopy equivalence* if it induces isomorphisms on all homotopy groups  $\pi_n$ . As seen in Theorem 10.3, a weak homotopy equivalence induces isomorphisms on all homology and cohomology groups. Furthermore, Whitehead's Theorem 7.2 shows that a weak homotopy equivalence of CW complexes is a homotopy equivalence.

In this section we show that given any space  $X$ , there exists a (unique up to homotopy) CW complex  $Z$  and a weak homotopy equivalence  $f : Z \rightarrow X$ . Such a map  $f : Z \rightarrow X$  is called a *CW approximation* of  $X$ .

**Definition 8.1.** *Given a pair  $(X, A)$ , with  $\emptyset \neq A$  a CW complex, an  $n$ -connected CW model of  $(X, A)$  is an  $n$ -connected CW pair  $(Z, A)$ , together with a map  $f : Z \rightarrow X$  with  $f|_A = id_A$ , so that  $f_* : \pi_i(Z) \rightarrow \pi_i(X)$  is an isomorphism for  $i > n$  and an injection for  $i = n$  (for any choice of basepoint).*

**Remark 8.2.** If such models exist, by letting  $A$  consist of one point in each path-component of  $X$  and  $n = 0$ , we get a CW approximation  $Z$  of  $X$ .

**Theorem 8.3.** *For any pair  $(X, A)$  with  $A$  a nonempty CW complex such  $n$ -connected models  $(Z, A)$  exist. Moreover,  $Z$  can be built from  $A$  by attaching cells of dimension greater than  $n$ . (Note that by cellular approximation this implies that  $\pi_i(Z, A) = 0$  for  $i \leq n$ ).*

We will prove this theorem after discussing the following consequences:

**Corollary 8.4.** *Any pair of spaces  $(X, X_0)$  has a CW approximation  $(Z, Z_0)$ .*

*Proof.* Let  $f_0 : Z_0 \rightarrow X_0$  be a CW approximation of  $X_0$ , and consider the map  $g : Z_0 \rightarrow X$  defined by the composition of  $f_0$  and the inclusion map  $X_0 \hookrightarrow X$ . Let  $M_g$  be the mapping cylinder of  $g$ . Hence we get the sequence of maps  $Z_0 \hookrightarrow M_g \rightarrow X$ , where the map  $r : M_g \rightarrow X$  is a deformation retract.

Now, let  $(Z, Z_0)$  be a 0-connected CW model of  $(M_g, Z_0)$ . Consider the composition:

$$(f, f_0) : (Z, Z_0) \longrightarrow (M_g, Z_0) \xrightarrow{(r, f_0)} (X, X_0)$$

So the map  $f : Z \rightarrow X$  is obtained by composing the weak homotopy equivalence  $Z \rightarrow M_g$  and the deformation retract (hence homotopy equivalence)  $M_g \rightarrow X$ . In other words,  $f$  is a weak homotopy equivalence and  $f|_{Z_0} = f_0$ , thus proving the result.  $\square$

**Corollary 8.5.** *For each  $n$ -connected CW pair  $(X, A)$  there is a CW pair  $(Z, A)$  that is homotopy equivalent to  $(X, A)$  relative to  $A$ , and such that  $Z$  is built from  $A$  by attaching cells of dimension  $> n$ .*

*Proof.* Let  $(Z, A)$  be an  $n$ -connected CW model of  $(X, A)$ . By Theorem 8.3,  $Z$  is built from  $A$  by attaching cells of dimension  $> n$ . We claim that  $Z \xrightarrow{h.e.} X$  (rel.  $A$ ). First, by definition, the map  $f : Z \rightarrow X$  has the property that  $f_*$  is an isomorphism on  $\pi_i$  for  $i > n$  and an injection on  $\pi_n$ . For  $i < n$ , by the  $n$ -connectedness of the given model,  $\pi_i(X) \cong \pi_i(A) \cong \pi_i(Z)$  where the isomorphisms are induced by  $f$  since the following diagram commutes,

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ A & \xrightarrow{id} & A \end{array}$$

(with  $A \hookrightarrow Z$  and  $A \hookrightarrow X$  the inclusion maps.) For  $i = n$ , by  $n$ -connectedness of  $(X, A)$  the composition

$$\pi_n(A) \rightarrow \pi_n(Z) \rightarrow \pi_n(X)$$

is onto. So the induced map  $f_* : \pi_n(Z) \rightarrow \pi_n(X)$  is surjective. Altogether,  $f_*$  induces isomorphisms on all  $\pi_i$ , so by Whitehead's Theorem we conclude that  $f : Z \rightarrow X$  is a homotopy equivalence.

We make  $f$  stationary on  $A$  as follows. Define the quotient space

$$W_f := M_f / \{ \{a\} \times I \sim \text{pt}, \forall a \in A \}$$

of the mapping cylinder  $M_f$  obtained by collapsing each segment  $\{a\} \times I$  to a point, for any  $a \in A$ . Assuming  $f$  has been made cellular,  $W_f$  is a CW complex containing  $X$  and  $Z$  as subcomplexes, and  $W_f$  deformation retracts onto  $X$  just as  $M_f$  does.

Consider the map  $h : Z \rightarrow X$  given by the composition  $Z \hookrightarrow W_f \rightarrow X$ , where  $W_f \rightarrow X$  is the deformation retract. We claim that  $Z$  is a deformation retract of  $W_f$ , thus giving us that  $h$  is a homotopy equivalence relative to  $A$ . Indeed,  $\pi_i(W_f) \cong \pi_i(X)$  (since  $W_f$  is a deformation retract of  $X$ ) and  $\pi_i(X) \cong \pi_i(Z)$  since  $X$  is homotopy equivalent to  $Z$ . Using Whitehead's theorem, we conclude that  $Z$  is a deformation retract of  $W_f$ .  $\square$

*Proof of Theorem 8.3.* We will construct  $Z$  as a union of subcomplexes

$$A = Z_n \subseteq Z_{n+1} \subseteq \cdots$$

such that for each  $k \geq n + 1$ ,  $Z_k$  is obtained from  $Z_{k-1}$  by attaching  $k$ -cells.

We will show by induction that we can construct  $Z_k$  together with a map  $f_k : Z_k \rightarrow X$  such that  $f_k|_A = id_A$  and  $f_{k*}$  is injective on  $\pi_i$  for  $n \leq i < k$  and onto on  $\pi_i$  for  $n < i \leq k$ . We start the induction at  $k = n$ , with  $Z_n = A$ , in which case the conditions on  $\pi_i$  are void.

For the induction step,  $k \rightarrow k + 1$ , consider the set  $\{\phi_\alpha\}_\alpha$  of generators  $\phi_\alpha : S^k \rightarrow Z_k$  of  $\ker(f_{k*} : \pi_k(Z_k) \rightarrow \pi_k(X))$ . Define

$$Y_{k+1} := Z_k \cup_\alpha \cup_{\phi_\alpha} e_\alpha^{k+1},$$

where  $e_\alpha^{k+1}$  is a  $(k + 1)$ -cell attached to  $Z_k$  along  $\phi_\alpha$ .

Then  $f_k : Z_k \rightarrow X$  extends to  $Y_{k+1}$ . Indeed,  $f_k \circ \phi_\alpha : S^k \rightarrow Z_k \rightarrow X$  is nullhomotopic, since  $[f_k \circ \phi_\alpha] = f_{k*}[\phi_\alpha] = 0$ . So we get a map  $g : Y_{k+1} \rightarrow X$ . It is easy to check that the  $g_*$  is injective on  $\pi_i$  for  $n \leq i \leq k$ , and onto on  $\pi_k$ . In fact, since we extend  $f_k$  on  $(k + 1)$ -cells, we only need to check the effect on  $\pi_k$ . The elements of  $\ker(g_*)$  on  $\pi_k$  are represented by nullhomotopic maps (by construction)  $S^k \rightarrow Z_k \subset Y_{k+1} \rightarrow X$ . So  $g_*$  is one-to-one on  $\pi_k$ . Moreover,  $g_*$  is onto on  $\pi_k$  since, by hypothesis, the composition  $\pi_k(Z_k) \rightarrow \pi_k(Y_{k+1}) \rightarrow \pi_k(X)$  is onto.

Let  $\{\phi_\beta : S^{k+1} \rightarrow X\}$  be a set of generators of  $\pi_{k+1}(X, x_0)$  and let  $Z_{k+1} = Y_{k+1} \vee_\beta S_\beta^{k+1}$ . We extend  $g$  to a map  $f_{k+1} : Z_{k+1} \rightarrow X$  by defining  $f_{k+1}|_{S_\beta^{k+1}} = \phi_\beta$ . This implies that  $f_{k+1}$  induces an epimorphism on  $\pi_{k+1}$ . The remaining conditions on homotopy groups are easy to check.  $\square$

**Remark 8.6.** If  $X$  is path-connected and  $A$  is a point, the construction of a CW model for  $(X, A)$  gives a CW approximation of  $X$  with a single 0-cell. In particular, by Whitehead's Theorem 7.2, *any connected CW complex is homotopy equivalent to a CW complex with a single 0-cell.*

**Proposition 8.7.** *Let  $g : (X, A) \rightarrow (X', A')$  be a map of pairs, where  $A, A'$  are nonempty CW complexes. Let  $(Z, A)$  be an  $n$ -connected CW model of  $(X, A)$  with associated map  $f : (Z, A) \rightarrow (X, A)$ , and let  $(Z', A')$  be an  $n'$ -connected model of  $(X', A')$  with associated map  $f' : (Z', A') \rightarrow (X', A')$ . Assume that  $n \geq n'$ . Then there exists a map  $h : Z \rightarrow Z'$ , unique up to homotopy, such that  $h|_A = g|_A$  and,*

$$\begin{array}{ccc} (Z, A) & \xrightarrow{f} & (X, A) \\ h \downarrow & & g \downarrow \\ (Z', A') & \xrightarrow{f'} & (X', A') \end{array}$$

*commutes up to homotopy.*

*Proof.* The proof is a standard induction on skeleta. We begin with the map  $g : A \rightarrow A' \subseteq Z'$ , and recall that  $Z$  is obtained from  $A$  by attaching cells of dimension  $> n$ . Let  $k$  be the smallest dimension of such a cell, thus  $(A \cup Z_k, A)$  has a  $k$ -connected model,  $f_k : (Z^k, A) \rightarrow (A \cup Z_k, A)$  such that  $f_k|_A = id_A$ . Composing this new map with  $g$  allows us to consider  $g$  as having been extended to the  $k$  skeleton of  $Z$ . Iterating this process produces our map.  $\square$

**Corollary 8.8.** *CW-approximations are unique up to homotopy equivalence. More generally,  $n$ -connected models of a pair  $(X, A)$  are unique up to homotopy relative to  $A$ .*

*Proof.* Assume that  $f : (Z, A) \rightarrow (X, A)$  and  $f' : (Z', A) \rightarrow (X, A)$  are two  $n$ -connected models of  $(X, A)$ . Then we may take  $(X, A) = (X', A')$  and  $g = id$  in the above lemma twice, and conclude that there are two maps  $h_0 : Z \rightarrow Z'$  and  $h_1 : Z' \rightarrow Z$ , such that  $f \circ h_1 \simeq f'$  (rel.  $A$ ) and  $f' \circ h_0 \simeq f$  (rel.  $A$ ). In particular,  $f \circ (h_1 \circ h_0) \simeq f$  (rel.  $A$ ) and  $f' \circ (h_0 \circ h_1) \simeq f'$  (rel.  $A$ ). The uniqueness in Proposition 8.7 then implies that  $h_1 \circ h_0$  and  $h_0 \circ h_1$  are homotopic to the respective identity maps (rel.  $A$ ).  $\square$

**Remark 8.9.** By taking  $n = n'$  in Proposition 8.7, we get a functoriality property for  $n$ -connected CW models. For example, a map  $X \rightarrow X'$  of spaces induces a map of CW approximations  $Z \rightarrow Z'$ .

**Remark 8.10.** By letting  $n$  vary, and by letting  $(Z^n, A)$  be an  $n$ -connected CW model for  $(X, A)$ , then Proposition 8.7 gives a *tower* of CW models

$$\begin{array}{ccccc}
 & & & & \downarrow \\
 & & & & Z^2 \\
 & & \nearrow & \downarrow & \searrow \\
 & & Z^1 & & \\
 & \nearrow & \downarrow & \searrow & \\
 A & \longrightarrow & Z^0 & \longrightarrow & X
 \end{array}$$

with commutative triangle on the left, and homotopy-commutative triangles on the right.

**Example 8.11** (Whitehead towers). Assume  $X$  is an arbitrary CW complex with  $A \subset X$  a point. Then the resulting tower of  $n$ -connected CW modules of  $(X, A)$  amounts to a sequence of maps

$$\dots \longrightarrow Z^2 \longrightarrow Z^1 \longrightarrow Z^0 \longrightarrow X$$

with  $Z^n$   $n$ -connected and the map  $Z^n \rightarrow X$  inducing isomorphisms on all homotopy groups  $\pi_i$  with  $i > n$ . The space  $Z^0$  is path-connected and homotopy equivalent to the component of  $X$  containing  $A$ , so one may assume that  $Z^0$  equals this component. The space  $Z^1$  is simply-connected, and the map  $Z^1 \rightarrow X$  has the homotopy properties of the universal cover of the component  $Z^0$  of  $X$ . In general, if  $X$  is connected the map  $Z^n \rightarrow X$  has the homotopy properties of an  $n$ -connected cover of  $X$ . An example of a 2-connected cover of  $S^2$  is the Hopf map  $S^3 \rightarrow S^2$ .

**Example 8.12** (Postnikov towers). If  $X$  is a connected CW complex, the tower of  $n$ -connected models for the pair  $(CX, X)$ , with  $CX$  the cone on  $X$ , is called the *Postnikov tower* of  $X$ . Relabeling  $Z^n$  as  $X^{n-1}$ , the Postnikov tower gives a commutative diagram

$$\begin{array}{ccc}
 & & \downarrow \\
 & & X^3 \\
 & \nearrow & \downarrow \\
 & & X^2 \\
 & \nearrow & \downarrow \\
 X & \longrightarrow & X^1
 \end{array}$$

where the induced homomorphism  $\pi_i(X) \rightarrow \pi_i(X^n)$  is an isomorphism for  $i \leq n$  and  $\pi_i(X^n) = 0$  if  $i > n$ . Indeed, by Definition 8.1 we get  $\pi_i(X^n) = \pi_i(Z^{n+1}) \cong \pi_i(CX) = 0$  for  $i \geq n + 1$ .

## 9 Eilenberg-MacLane spaces

**Definition 9.1.** A space  $X$  having just one nontrivial homotopy group  $\pi_n(X) = G$  is called an *Eilenberg-MacLane space*  $K(G, n)$ .

**Example 9.2.** We have already seen that  $S^1$  is a  $K(\mathbb{Z}, 1)$  space, and  $\mathbb{R}P^\infty$  is a  $K(\mathbb{Z}/2\mathbb{Z}, 1)$  space. The fact that  $\mathbb{C}P^\infty$  is a  $K(\mathbb{Z}, 2)$  space will be discussed in Example 11.16 by making use of fibrations and the associated long exact sequence of homotopy groups.

**Lemma 9.3.** If a CW-pair  $(X, A)$  is  $r$ -connected ( $r \geq 1$ ) and  $A$  is  $s$ -connected ( $s \geq 0$ ), then the map  $\pi_i(X, A) \rightarrow \pi_i(X/A)$  induced by the quotient map  $X \rightarrow X/A$  is an isomorphism if  $i \leq r + s$  and onto if  $i = r + s + 1$ .

*Proof.* Let  $CA$  be the cone on  $A$  and consider the complex

$$Y = X \cup_A CA$$

obtained from  $X$  by attaching the cone  $CA$  along  $A \subseteq X$ . Since  $CA$  is a contractible subcomplex of  $Y$ , the quotient map

$$q : Y \longrightarrow Y/CA = X/A$$

is obtained by deforming  $CA$  to the cone point inside  $Y$ , so it is a homotopy equivalence. So we have a sequence of homomorphisms

$$\pi_i(X, A) \longrightarrow \pi_i(Y, CA) \xleftarrow{\cong} \pi_i(Y) \xrightarrow{\cong} \pi_i(X/A),$$



where the first and second maps are induced by the inclusion of pairs, the second map is an isomorphism by the long exact sequence of the pair  $(Y, CA)$

$$0 = \pi_i(CA) \rightarrow \pi_i(Y) \rightarrow \pi_i(Y, CA) \rightarrow \pi_{i-1}(CA) = 0,$$

and the third map is the isomorphism  $q_*$ . It therefore remains to investigate the map  $\pi_i(X, A) \rightarrow \pi_i(Y, CA)$ . We know that  $(X, A)$  is  $r$ -connected and  $(CA, A)$  is  $(s + 1)$ -connected. The second fact once again follows from the long exact sequence of the pair and the fact that  $A$  is  $s$ -connected. Using the Excision Theorem 5.1, the desired result follows immediately.  $\square$

**Lemma 9.4.** *Assume  $n \geq 2$ . If  $X = (\bigvee_{\alpha} S_{\alpha}^n) \cup \bigcup_{\beta} e_{\beta}^{n+1}$  is obtained from  $\bigvee_{\alpha} S_{\alpha}^n$  by attaching  $(n + 1)$ -cells  $e_{\beta}^{n+1}$  via basepoint-preserving maps  $\phi_{\beta} : S_{\beta}^n \rightarrow \bigvee_{\alpha} S_{\alpha}^n$ , then*

$$\pi_n(X) = \pi_n(\bigvee_{\alpha} S_{\alpha}^n) / \langle \phi_{\beta} \rangle = (\bigoplus_{\alpha} \mathbb{Z}) / \langle \phi_{\beta} \rangle.$$

*Proof.* Consider the following portion of the long exact sequence for the homotopy groups of the  $n$ -connected pair  $(X, \bigvee_{\alpha} S_{\alpha}^n)$ :

$$\pi_{n+1}(X, \bigvee_{\alpha} S_{\alpha}^n) \xrightarrow{\partial} \pi_n(\bigvee_{\alpha} S_{\alpha}^n) \rightarrow \pi_n(X) \rightarrow \pi_n(X, \bigvee_{\alpha} S_{\alpha}^n) = 0,$$

where the fact that  $\pi_n(X, \bigvee_{\alpha} S_{\alpha}^n) = 0$  follows by Corollary 4.8 of the Cellular Approximation theorem. So  $\pi_n(X) \cong \pi_n(\bigvee_{\alpha} S_{\alpha}^n) / \text{Image}(\partial)$ .

We have the identification  $X / \bigvee_{\alpha} S_{\alpha}^n \simeq \bigvee_{\beta} S_{\beta}^{n+1}$ , so by Lemma 9.3 and Lemma 6.6 we get that  $\pi_{n+1}(X, \bigvee_{\alpha} S_{\alpha}^n) \cong \pi_{n+1}(\bigvee_{\beta} S_{\beta}^{n+1})$  is free with a basis consisting of the characteristic maps  $\Phi_{\beta}$  of the cells  $e_{\beta}^{n+1}$ . Since  $\partial([\Phi_{\beta}]) = [\phi_{\beta}]$ , the claim follows.  $\square$

**Example 9.5.** Any abelian group  $G$  can be realized as  $\pi_n(X)$  with  $n \geq 2$  for some space  $X$ . In fact, given a presentation  $G = \langle g_{\alpha} \mid r_{\beta} \rangle$ , we can take

$$X = \left( \bigvee_{\alpha} S_{\alpha}^n \right) \cup \bigcup_{\beta} e_{\beta}^{n+1},$$

with the  $S_{\alpha}^n$ 's corresponding to the generators of  $G$ , and with  $e_{\beta}^{n+1}$  attached to  $\bigvee_{\alpha} S_{\alpha}^n$  by a map  $f : S_{\beta}^n \rightarrow \bigvee_{\alpha} S_{\alpha}^n$  satisfying  $[f] = r_{\beta}$ . Note also that by cellular approximation,  $\pi_i(X) = 0$  for  $i < n$ , but nothing can be said about  $\pi_i(X)$  with  $i > n$ .

**Theorem 9.6.** *For any  $n \geq 1$  and any group  $G$  (which is assumed abelian if  $n \geq 2$ ) there exists an Eilenberg-MacLane space  $K(G, n)$ .*

*Proof.* Let  $X_{n+1} = (\bigvee_{\alpha} S_{\alpha}^n) \cup \bigcup_{\beta} e_{\beta}^{n+1}$  be the  $(n - 1)$ -connected CW complex of dimension  $n + 1$  with  $\pi_n(X_{n+1}) = G$ , as constructed in Example 9.5. Enlarge  $X_{n+1}$  to a CW complex  $X_{n+2}$  obtained from  $X_{n+1}$  by attaching  $(n + 2)$ -cells  $e_{\gamma}^{n+2}$  via maps representing some set of generators of  $\pi_{n+1}(X_{n+1})$ . Since  $(X_{n+2}, X_{n+1})$  is  $(n + 1)$ -connected (by Corollary 4.8), the

long exact sequence for the homotopy groups of the pair  $(X_{n+2}, X_{n+1})$  yields isomorphisms  $\pi_i(X_{n+2}) = \pi_i(X_{n+1})$  for  $i \leq n$ , together with the exact sequence

$$\cdots \rightarrow \pi_{n+2}(X_{n+2}, X_{n+1}) \xrightarrow{\partial} \pi_{n+1}(X_{n+1}) \rightarrow \pi_{n+1}(X_{n+2}) \rightarrow 0.$$

Next note that  $\partial$  is an isomorphism by construction and Lemma 9.3. Indeed, Lemma 9.3 yields that the quotient map  $X_{n+2} \rightarrow X_{n+2}/X_{n+1}$  induces an epimorphism

$$\pi_{n+2}(X_{n+2}, X_{n+1}) \rightarrow \pi_{n+2}(X_{n+2}/X_{n+1}) \cong \pi_{n+2}\left(\bigvee_{\gamma} S_{\gamma}^{n+2}\right),$$

which is an isomorphism for  $n \geq 2$ . Moreover, we also have an epimorphism  $\pi_{n+2}(\bigvee_{\gamma} S_{\gamma}^{n+2}) \rightarrow \pi_{n+1}(X_{n+1})$  by our construction of  $X_{n+2}$ . As  $\partial$  is onto, we then get that  $\pi_{n+1}(X_{n+2}) = 0$ .

Repeat this construction inductively, at the  $k$ -th stage attaching  $(n+k+1)$ -cells to  $X_{n+k}$  to create a CW complex  $X_{n+k+1}$  with vanishing  $\pi_{n+k}$  and without changing the lower homotopy groups. The union of this increasing sequence of CW complexes is then a  $K(G, n)$  space.  $\square$

**Corollary 9.7.** *For any sequence of groups  $\{G_n\}_{n \in \mathbb{N}}$ , with  $G_n$  abelian for  $n \geq 2$ , there exists a space  $X$  such that  $\pi_n(X) \cong G_n$  for any  $n$ .*

*Proof.* Call  $X^n = K(G_n, n)$ . Then  $X = \prod_n X^n$  has the desired prescribed homotopy groups.  $\square$

**Lemma 9.8.** *Let  $X$  be a CW complex of the form  $(\bigvee_{\alpha} S_{\alpha}^n) \cup \bigcup_{\beta} e_{\beta}^{n+1}$  for some  $n \geq 1$ . Then for every homomorphism  $\psi : \pi_n(X) \rightarrow \pi_n(Y)$  with  $Y$  a path-connected space, there exists a map  $f : X \rightarrow Y$  such that  $f_* = \psi$  on  $\pi_n$ .*

*Proof.* Recall from Lemma 9.4 that  $\pi_n(X)$  is generated by the inclusions  $i_{\alpha} : S_{\alpha}^n \hookrightarrow X$ . Let  $f$  send the wedge point of  $X$  to a basepoint of  $Y$ , and extend  $f$  onto  $S_{\alpha}^n$  by choosing a fixed representative for  $\psi([i_{\alpha}]) \in \pi_n(Y)$ . This then allows us to define  $f$  on the  $n$ -skeleton  $X_n = \bigvee_{\alpha} S_{\alpha}^n$  of  $X$ , and we notice that, by construction of  $f : X_n \rightarrow Y$ , we have that

$$f_*([i_{\alpha}]) = [f \circ i_{\alpha}] = [f|_{S_{\alpha}^n}] = \psi([i_{\alpha}]).$$

Because the  $i_{\alpha}$  generate  $\pi_n(X_n)$ , we then get that  $f_* = \psi$ .

To extend  $f$  over a cell  $e_{\beta}^{n+1}$ , we need to show that the composition of the attaching map  $\phi_{\beta} : S^n \rightarrow X_n$  for this cell with  $f$  is nullhomotopic in  $Y$ . We have  $[f \circ \phi_{\beta}] = f_*([\phi_{\beta}]) = \psi([\phi_{\beta}]) = 0$ , as the  $\phi_{\beta}$  are precisely the relators in  $\pi_n(X)$  by Example 9.5. Thus we obtain an extension  $f : X \rightarrow Y$ . Moreover,  $f_* = \psi$  since the elements  $[i_{\alpha}]$  generate  $\pi_n(X_n) = \pi_n(X)$ .  $\square$

**Proposition 9.9.** *The homotopy type of a CW complex  $K(G, n)$  is uniquely determined by  $G$  and  $n$ .*

*Proof.* Let  $K$  and  $K'$  be  $K(G, n)$  CW complexes, and assume without loss of generality (since homotopy equivalence is an equivalence relation) that  $K$  is the particular  $K(G, n)$  constructed in Theorem 9.6, i.e., built from a space  $X$  as in Lemma 9.8 by attaching cells of dimension  $n + 2$  and higher. Since  $X = K_{n+1}$ , we have that  $\pi_n(X) = \pi_n(K) = \pi_n(K')$ , and call the composition of these isomorphisms  $\psi : \pi_n(X) \rightarrow \pi_n(K')$ . By Lemma 9.8, there is a map  $f : X \rightarrow K'$  inducing  $\psi$  on  $\pi_n$ . To extend this map over  $K$ , we proceed inductively, first extending it over the  $(n + 2)$ -cells, then over the  $(n + 3)$ -cells, and so on.

Let  $e_\gamma^{n+2}$  be an  $(n + 2)$ -cell of  $K$ , with attaching map  $\phi_\gamma : S^{n+1} \rightarrow X$ . Then  $f \circ \phi_\gamma : S^{n+1} \rightarrow K'$  is nullhomotopic since  $\pi_{n+1}(K') = 0$ . Therefore,  $f$  extends over  $e_\gamma^{n+2}$ . Proceed similarly for higher dimensional cells of  $K$  to get a map  $f : K \rightarrow K'$  which is a weak homotopy equivalence. By Whitehead's Theorem 7.2, we conclude that  $f$  is a homotopy equivalence.  $\square$

## 10 Hurewicz Theorem

**Theorem 10.1** (Hurewicz). *If a space  $X$  is  $(n - 1)$ -connected and  $n \geq 2$ , then  $\tilde{H}_i(X) = 0$  for  $i < n$  and  $\pi_n(X) \cong H_n(X)$ . Moreover, if a pair  $(X, A)$  is  $(n - 1)$ -connected with  $n \geq 2$ , and  $\pi_1(A) = 0$ , then  $H_i(X, A) = 0$  for all  $i < n$  and  $\pi_n(X, A) \cong H_n(X, A)$ .*

*Proof.* First, since all hypotheses and assertions in the statement deal with homology and homotopy groups, if we prove the statement for a CW approximation of  $X$  (or  $(X, A)$ ) then the results will also hold for the original space (or pair). Hence, we assume without loss of generality that  $X$  is a CW complex and  $(X, A)$  is a CW-pair.

Secondly, the relative case can be reduced to the absolute case. Indeed, since  $(X, A)$  is  $(n - 1)$ -connected and that  $A$  is 1-connected, Lemma 9.3 implies that  $\pi_i(X, A) = \pi_i(X/A)$  for  $i \leq n$ , while  $H_i(X, A) = \tilde{H}_i(X/A)$  always holds for CW-pairs.

In order to prove the absolute case of the theorem, let  $x_0$  be a 0-cell in  $X$ . Since  $X$ , hence also  $(X, x_0)$ , is  $(n - 1)$ -connected, Corollary 8.5 tells us that we can replace  $X$  by a homotopy equivalent CW complex with  $(n - 1)$ -skeleton a point, i.e.,  $X_{n-1} = x_0$ . In particular,  $\tilde{H}_i(X) = 0$  for  $i < n$ . For showing that  $\pi_n(X) \cong H_n(X)$ , we may disregard any cells of dimension greater than  $n + 1$  since these have no effect on  $\pi_n$  or  $H_n$ . Thus we may assume that  $X$  has the form  $(\bigvee_\alpha S_\alpha^n) \cup \bigcup_\beta e_\beta^{n+1}$ . By Lemma 9.4, we then have that  $\pi_n(X) \cong (\bigoplus_\alpha \mathbb{Z}) / \langle \phi_\beta \rangle$ . On the other hand, cellular homology yields the same calculation for  $H_n(X)$ , so we are done.  $\square$

**Remark 10.2.** One cannot expect any sort of relationship between  $\pi_i(X)$  and  $H_i(X)$  beyond  $n$ . For example,  $S^n$  has trivial homology in degrees  $> n$ , but many nontrivial homotopy groups in this range, if  $n \geq 2$ . On the other hand,  $\mathbb{C}P^\infty$  has trivial higher homotopy groups in the range  $> 2$  (as a  $K(\mathbb{Z}, 2)$  space), but many nontrivial homology groups in this range.

Recall the Hurewicz Theorem has been already used for proving the important Corollary 7.3. Here we give another important application of Theorem 10.1:

**Theorem 10.3.** *If  $f : X \rightarrow Y$  induces isomorphisms on homotopy groups  $\pi_n$  for all  $n$ , then it induces isomorphisms on homology and cohomology groups with  $G$  coefficients, for any group  $G$ .*

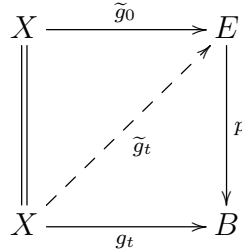
*Proof.* By the universal coefficient theorems, it suffices to show that  $f$  induces isomorphisms on integral homology groups  $H_*(-; \mathbb{Z})$ .

We only prove here the assertion under the extra condition that  $X$  is simply connected (the general case follows easily from spectral sequence theory, and it will be dealt with later on). As before, after replacing  $Y$  with the homotopy equivalent space defined by the mapping cylinder  $M_f$  of  $f$ , we can assume that  $f$  is an inclusion. Since by the hypothesis,  $\pi_n(X) \cong \pi_n(Y)$  for all  $n$ , with isomorphisms induced by the inclusion  $f$ , the homotopy long exact sequence of the pair  $(Y, X)$  yields that  $\pi_n(Y, X) = 0$  for all  $n$ . By the relative Hurewicz theorem (as  $\pi_1(X) = 0$ ), this gives that  $H_n(Y, X) = 0$  for all  $n$ . Hence, by the long exact sequence for homology,  $H_n(X) \cong H_n(Y)$  for all  $n$ , and the proof is complete.  $\square$

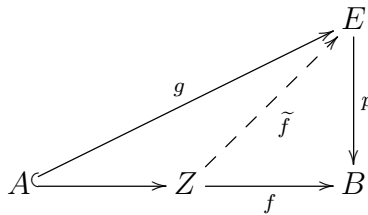
**Example 10.4.** Take  $X = \mathbb{R}P^2 \times S^3$  and  $Y = S^2 \times \mathbb{R}P^3$ . As seen in Example 1.19,  $X$  and  $Y$  have isomorphic homotopy groups  $\pi_n$  for all  $n$ , but  $H_5(X) \not\cong H_5(Y)$ . So there cannot exist a map  $f : X \rightarrow Y$  inducing the isomorphisms on the  $\pi_n$ .

## 11 Fibrations. Fiber bundles

**Definition 11.1** (Homotopy Lifting Property). *A map  $p : E \rightarrow B$  has the homotopy lifting property (HLP) with respect to a space  $X$  if, given a homotopy  $g_t : X \rightarrow B$ , and a lift  $\tilde{g}_0 : X \rightarrow E$  of  $g_0$ , there exists a homotopy  $\tilde{g}_t : X \rightarrow E$  lifting  $g_t$  and extending  $\tilde{g}_0$ .*



**Definition 11.2** (Lift Extension Property). *A map  $p : E \rightarrow B$  has the lift extension property (LEP) with respect to a pair  $(Z, A)$  if for all maps  $f : Z \rightarrow B$  and  $g : A \rightarrow E$ , there exists a lift  $\tilde{f} : Z \rightarrow E$  of  $f$  extending  $g$ .*



**Remark 11.3.** (HLP) is a special case of (LEP), with  $Z = X \times [0, 1]$ , and  $A = X \times \{0\}$ .

**Definition 11.4.** A fibration  $p : E \rightarrow B$  is a map having the homotopy lifting property with respect to all spaces  $X$ .

**Definition 11.5** (Homotopy Lifting Property with respect to a pair). A map  $p : E \rightarrow B$  has the homotopy lifting property with respect to a pair  $(X, A)$  if each homotopy  $g_t : X \rightarrow B$  lifts to a homotopy  $\tilde{g}_t : X \rightarrow E$  starting with a given lift  $\tilde{g}_0$  and extending a given lift  $\tilde{g}_t : A \rightarrow E$ .

**Remark 11.6.** The homotopy lifting property with respect to the pair  $(X, A)$  is the lift extension property for  $(X \times I, X \times \{0\} \cup A \times I)$ .

**Remark 11.7.** The homotopy lifting property with respect to a disk  $D^n$  is equivalent to the homotopy lifting property with respect to the pair  $(D^n, \partial D^n)$ , since the pairs  $(D^n \times I, D^n \times \{0\})$  and  $(D^n \times I, D^n \times \{0\} \cup \partial D^n \times I)$  are homeomorphic. This implies that a fibration has the homotopy lifting property with respect to all CW pairs  $(X, A)$ . Indeed, the homotopy lifting property for disks is in fact equivalent to the homotopy lifting property with respect to all CW pairs  $(X, A)$ . This can be easily seen by induction over the skeleta of  $X$ , so it suffices to construct a lifting  $\tilde{g}_t$  one cell of  $X \setminus A$  at a time. Composing with the characteristic map  $D^n \rightarrow X$  of a cell then gives the reduction to the case  $(X, A) = (D^n, \partial D^n)$ .

**Theorem 11.8** (Long exact sequence for homotopy groups of a fibration). Given a fibration  $p : E \rightarrow B$ , points  $b \in B$  and  $e \in F := p^{-1}(b)$ , there is an isomorphism  $p_* : \pi_n(E, F, e) \xrightarrow{\cong} \pi_n(B, b)$  for all  $n \geq 1$ . Hence, if  $B$  is path-connected, there is a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_n(F, e) \longrightarrow \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \longrightarrow \pi_{n-1}(F, e) \longrightarrow \cdots \longrightarrow \pi_0(E, e) \longrightarrow 0$$

*Proof.* To show that  $p_*$  is onto, represent an element of  $\pi_n(B, b)$  by a map  $f : (I^n, \partial I^n) \rightarrow (B, b)$ , and note that the constant map to  $e$  is a lift of  $f$  to  $E$  over  $J^{n-1} \subset I^n$ . The homotopy lifting property for the pair  $(I^{n-1}, \partial I^{n-1})$  extends this to a lift  $\tilde{f} : I^n \rightarrow E$ . This lift satisfies  $\tilde{f}(\partial I^n) \subset F$  since  $f(\partial I^n) = b$ . So  $\tilde{f}$  represents an element of  $\pi_n(E, F, e)$  with  $p_*([\tilde{f}]) = [f]$  since  $p\tilde{f} = f$ .

To show the injectivity of  $p_*$ , let  $\tilde{f}_0, \tilde{f}_1 : (I^n, \partial I^n, J^{n-1}) \rightarrow (E, F, e)$  be so that  $p_*(\tilde{f}_0) = p_*(\tilde{f}_1)$ . Let  $H : (I^n \times I, \partial I^n \times I) \rightarrow (B, b)$  be a homotopy from  $p\tilde{f}_0$  to  $p\tilde{f}_1$ . We have a partial lift given by  $\tilde{f}_0$  on  $I^n \times \{0\}$ ,  $\tilde{f}_1$  on  $I^n \times \{1\}$  and the constant map to  $e$  on  $J^{n-1} \times I$ . The homotopy lifting property for CW pairs extends this to a lift  $\tilde{H} : I^n \times I \rightarrow E$  giving a homotopy  $\tilde{f}_t : (I^n, \partial I^n, J^{n-1}) \rightarrow (E, F, e)$  from  $\tilde{f}_0$  to  $\tilde{f}_1$ .

Finally, the long exact sequence of the fibration follows by plugging  $\pi_n(B, b)$  in for  $\pi_n(E, F, e)$  in the long exact sequence for the pair  $(E, F)$ . The map  $\pi_n(E, e) \rightarrow \pi_n(E, F, e)$  in the latter sequence becomes the composition  $\pi_n(E, e) \rightarrow \pi_n(E, F, e) \xrightarrow{p_*} \pi_n(B, b)$ , which is exactly  $p_* : \pi_n(E, e) \rightarrow \pi_n(B, b)$ . The surjectivity of  $\pi_0(F, e) \rightarrow \pi_0(E, e)$  follows from the path-connectedness of  $B$ , since a path in  $E$  from an arbitrary point  $x \in E$  to  $F$  can be obtained by lifting a path in  $B$  from  $p(x)$  to  $b$ .  $\square$

**Definition 11.9.** Given two fibrations  $p_i : E_i \rightarrow B$ ,  $i = 1, 2$ , a map  $f : E_1 \rightarrow E_2$  is fiber-preserving if the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ p_1 \searrow & & \swarrow p_2 \\ & B & \end{array}$$

commutes. Such a map  $f$  is called a fiber homotopy equivalence if  $f$  is both fiber-preserving and a homotopy equivalence, i.e., there is a map  $g : E_2 \rightarrow E_1$  such that  $f$  and  $g$  are fiber-preserving and  $f \circ g$  and  $g \circ f$  are homotopic to the identity maps by fiber-preserving maps.

**Definition 11.10** (Fiber Bundle). A map  $p : E \rightarrow B$  is a fiber bundle with fiber  $F$  if, for any point  $b \in B$ , there exists a neighborhood  $U_b$  of  $b$  with a homeomorphism  $h : p^{-1}(U_b) \rightarrow U_b \times F$  so that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U_b) & \xrightarrow{h} & U_b \times F \\ p \searrow & & \swarrow pr \\ & U_b & \end{array}$$

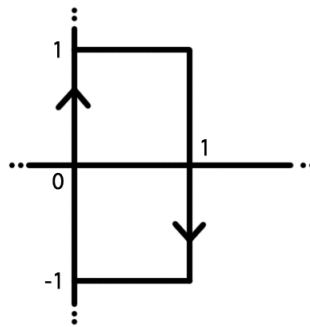
**Remark 11.11.** Fibers of fibrations are homotopy equivalent, while fibers of fiber bundles are homeomorphic.

**Theorem 11.12** (Hurewicz). Fiber bundles over paracompact spaces are fibrations.

Here are some easy examples of fiber bundles.

**Example 11.13.** If  $F$  is discrete, a fiber bundle with fiber  $F$  is a covering map. Moreover, the long exact sequence for the homotopy groups yields that  $p_* : \pi_i(E) \rightarrow \pi_i(B)$  is an isomorphism if  $i \geq 2$  and a monomorphism for  $i = 1$ .

**Example 11.14.** The Möbius band  $I \times [-1, 1] / (0, y) \sim (1, -y) \rightarrow S^1$  is a fiber bundle with fiber  $[-1, 1]$ , induced from the projection map  $I \times [-1, 1] \rightarrow I$ .



**Example 11.15.** By glueing the unlabeled edges of a Möbius band, we get  $K \rightarrow S^1$  (where  $K$  is the Klein bottle), a fiber bundle with fiber  $S^1$ .

**Example 11.16.** The following is a fiber bundle with fiber  $S^1$ :

$$\begin{aligned} S^1 \hookrightarrow S^{2n+1}(\subset \mathbb{C}^{n+1}) &\longrightarrow \mathbb{C}P^n \\ (z_0, \dots, z_n) &\mapsto [z_0 : \dots : z_n] = [\underline{z}] \end{aligned}$$

For  $[\underline{z}] \in \mathbb{C}P^n$ , there is an  $i$  such that  $z_i \neq 0$ . Then we have a neighborhood

$$U_{[\underline{z}]} = \{[z_0 : \dots : 1 : \dots : z_n]\} \cong \mathbb{C}^n$$

(with the entry 1 in place of the  $i$ th coordinate) of  $[\underline{z}]$ , with a homeomorphism

$$\begin{aligned} p^{-1}(U_{[\underline{z}]}) &\longrightarrow U_{[\underline{z}]} \times S^1 \\ (z_0, \dots, z_n) &\mapsto ([z_0 : \dots : z_n], z_i/|z_i|). \end{aligned}$$

By letting  $n$  go to infinity, we get a diagram of fibrations

$$\begin{array}{ccccccc} S^1 & = & S^1 & = & \dots & = & S^1 \\ \searrow & & \searrow & & & & \searrow \\ & S^{2n+1} & \subset & S^{2n+3} & \subset & \dots & \subset & S^\infty \simeq \{\text{pt}\} \\ & \downarrow & & \downarrow & & & & \downarrow \\ & \mathbb{C}P^n & \subset & \mathbb{C}P^{n+1} & \subset & \dots & \subset & \mathbb{C}P^\infty \end{array}$$

In particular, from the long exact sequence of the fibration

$$S^1 \hookrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$$

with  $S^\infty$  contractible, we obtain that

$$\pi_i(\mathbb{C}P^\infty) \cong \pi_{i-1}(S^1) = \begin{cases} \mathbb{Z} & i = 2 \\ 0 & i \neq 2 \end{cases}$$

i.e.,

$$\mathbb{C}P^\infty = K(\mathbb{Z}, 2),$$

as already mentioned in our discussion about Eilenberg-MacLane spaces.

**Remark 11.17.** As we will see later on, for any topological group  $G$  there exists a “universal fiber bundle”  $G \hookrightarrow EG \xrightarrow{\pi_G} BG$  with  $EG$  contractible, classifying the space of (principal)

$G$ -bundles. That is, any  $G$ -bundle  $\pi : E \rightarrow B$  over a space  $B$  is determined by (the homotopy class of) a classifying map  $f : B \rightarrow BG$  by pull-back:  $\pi \cong f^*\pi_G$ :

$$\begin{array}{ccc}
 G & & G \\
 \searrow & & \searrow \\
 E & & EG \\
 \downarrow \pi & & \downarrow \pi_G \\
 B & \xrightarrow{f} & BG
 \end{array}
 \quad \simeq \{pt\}$$

From this point of view,  $\mathbb{C}P^\infty$  can be identified with the classifying space  $BS^1$  of (principal)  $S^1$ -bundles.

**Example 11.18.** By letting  $n = 1$  in the fibration of Example 11.16, the corresponding bundle

$$S^1 \hookrightarrow S^3 \longrightarrow \mathbb{C}P^1 \cong S^2 \tag{11.1}$$

is called the *Hopf fibration*. The long exact sequence of homotopy group for the Hopf fibration gives:  $\pi_2(S^2) \cong \pi_1(S^1)$  and  $\pi_n(S^3) \cong \pi_n(S^2)$  for all  $n \geq 3$ . Together with the fact that  $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ , this shows that  $S^2$  and  $S^3 \times \mathbb{C}P^\infty$  are simply-connected CW complexes with isomorphic homotopy groups, though they are not homotopy equivalent as can be easily seen from cellular homology.

**Example 11.19.** A fiber bundle similar to that of Example 11.16 can be obtained by replacing  $\mathbb{C}$  with the quaternions  $\mathbb{H}$ , namely:

$$S^3 \hookrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n.$$

(Note that  $S^{4n+3}$  can be identified with the unit sphere in  $\mathbb{H}^{n+1}$ .) In particular, by letting  $n = 1$  we get a second Hopf fiber bundle

$$S^3 \hookrightarrow S^7 \longrightarrow \mathbb{H}P^1 \cong S^4. \tag{11.2}$$

A third example of a Hopf bundle

$$S^7 \hookrightarrow S^{15} \longrightarrow S^8 \tag{11.3}$$

can be constructed by using the nonassociative 8-dimensional algebra  $\mathbb{O}$  of Cayley octonions, whose elements are pair of quaternions  $(a_1, a_2)$  with multiplication defined by  $(a_1, a_2) \cdot (b_1, b_2) = (a_1b_1 - \bar{b}_2a_2, a_2\bar{b}_1 + b_2a_1)$ . Here we regard  $S^{15}$  as the unit sphere in the 16-dimensional vector space  $\mathbb{O}^2$ , and the projection map  $S^{15} \rightarrow S^8 = \mathbb{O} \cup \{\infty\}$  is  $(z_0, z_1) \mapsto z_0z_1^{-1}$  (just like for the other Hopf bundles). There are no fiber bundles with fiber, total space and base spheres, other than those provided by the Hopf bundles of (11.1), (11.2) and (11.3). Finally, note that there is an ‘‘octonion projective plane’’  $\mathbb{O}P^2$  obtained by glueing a cell  $e^{16}$  to  $S^8$  via the Hopf map  $S^{15} \rightarrow S^8$ ; however, there is no octonion analogue of  $\mathbb{R}P^n$ ,  $\mathbb{C}P^n$  or  $\mathbb{H}P^n$  for higher  $n$ , since the associativity of multiplication is needed for the relation  $(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$  to be an equivalence relation.



**Example 11.20.** Other examples of fiber bundles are provided by the orthogonal and unitary groups:

$$\begin{aligned} O(n-1) &\hookrightarrow O(n) \rightarrow S^{n-1} \\ A &\mapsto Ax, \end{aligned}$$

where  $x$  is a fixed unit vector in  $\mathbb{R}^n$ . Similarly, there is a fibration

$$\begin{aligned} U(n-1) &\hookrightarrow U(n) \rightarrow S^{2n-1} \\ A &\mapsto Ax, \end{aligned}$$

with  $x$  a fixed unit vector in  $\mathbb{C}^n$ . These examples will be discussed in some detail in the next section.

## 12 More examples of fiber bundles

**Definition 12.1.** For  $n \leq k$ , the  $n$ -th Stiefel manifold associated to  $\mathbb{R}^k$  is defined as

$$V_n(\mathbb{R}^k) := \{n\text{-frames in } \mathbb{R}^k\},$$

where an  $n$ -frame in  $\mathbb{R}^k$  is an  $n$ -tuple  $\{v_1, \dots, v_n\}$  of orthonormal vectors in  $\mathbb{R}^k$ , i.e.,  $v_1, \dots, v_n$  are pairwise orthonormal:  $\langle v_i, v_j \rangle = \delta_{ij}$ .

We assign  $V_n(\mathbb{R}^k)$  the subspace topology induced from

$$V_n(\mathbb{R}^k) \subset \underbrace{S^{k-1} \times \dots \times S^{k-1}}_{n \text{ times}},$$

where  $S^{k-1} \times \dots \times S^{k-1}$  has the usual product topology.

**Example 12.2.**  $V_1(\mathbb{R}^k) = S^{k-1}$ .

**Example 12.3.**  $V_n(\mathbb{R}^n) \cong O(n)$ .

**Definition 12.4.** The  $n$ -th Grassmann manifold associated to  $\mathbb{R}^k$  is defined as:

$$G_n(\mathbb{R}^k) := \{n\text{-dimensional vector subspaces in } \mathbb{R}^k\}.$$

**Example 12.5.**  $G_1(\mathbb{R}^k) = \mathbb{R}P^{k-1}$

There is a natural surjection

$$p : V_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k)$$

given by

$$\{v_1, \dots, v_n\} \mapsto \text{span}\{v_1, \dots, v_n\}.$$

The fact that  $p$  is onto follows by the Gram-Schmidt procedure. So  $G_n(\mathbb{R}^k)$  is endowed with the quotient topology via  $p$ .

**Lemma 12.6.** *The projection  $p$  is a fiber bundle with fiber  $V_n(\mathbb{R}^n) = O(n)$ .*

*Proof.* Let  $V \in G_n(\mathbb{R}^k)$  be fixed. The fiber  $p^{-1}(V)$  consists on  $n$ -frames in  $V \cong \mathbb{R}^n$ , so it is homeomorphic to  $V_n(\mathbb{R}^n)$ . Let us now choose an orthonormal frame on  $V$ . By projection and Gram-Schmidt, we get orthonormal frames on all “nearby” (in some neighborhood  $U$  of  $V$ ) vector subspaces  $V'$ . Indeed, by projecting the frame of  $V$  orthogonally onto  $V'$  we get a (non-orthonormal) basis for  $V'$ , then apply the Gram-Schmidt process to this basis to make it orthonormal. This is a continuous process. The existence of such frames on all  $n$ -planes in  $U$  allows us to identify them with  $\mathbb{R}^n$ , so  $p^{-1}(U)$  is identified with  $U \times V_n(\mathbb{R}^n)$ .  $\square$

To conclude this discussion, we have shown that for  $k > n$ , there are fiber bundles:

$$O(n) \hookrightarrow V_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k) \quad (12.1)$$

A similar method gives the following fiber bundle for all triples  $m < n \leq k$ :

$$\begin{aligned} V_{n-m}(\mathbb{R}^{k-m}) \hookrightarrow V_n(\mathbb{R}^k) &\xrightarrow{p} V_m(\mathbb{R}^k) \\ \{v_1, \dots, v_n\} &\longmapsto \{v_1, \dots, v_m\} \end{aligned} \quad (12.2)$$

Here, the projection  $p$  sends an  $n$ -frame onto the  $m$ -frame formed by its first  $m$  vectors, so the fiber consists of  $(n - m)$ -frames in the  $(k - m)$ -plane orthogonal to the given frame.

**Example 12.7.** If  $k = n$  in the bundle (12.2), we get the fiber bundle

$$O(n - m) \hookrightarrow O(n) \longrightarrow V_m(\mathbb{R}^n). \quad (12.3)$$

Here,  $O(n - m)$  is regarded as the subgroup of  $O(n)$  fixing the first  $m$  standard basis vectors. So  $V_m(\mathbb{R}^n)$  is identifiable with the coset space  $O(n)/O(n - m)$ , or the orbit space of the free action of  $O(n - m)$  on  $O(n)$  by right multiplication. Similarly,

$$G_m(\mathbb{R}^n) \cong O(n)/O(m) \times O(n - m),$$

where  $O(m) \times O(n - m)$  consists of the orthogonal transformations of  $\mathbb{R}^n$  taking the  $m$ -plane spanned by the first  $m$  standard basis vectors to itself.

If, moreover, we take  $m = 1$  in (12.3), we get the fiber bundle

$$\begin{aligned} O(n - 1) \hookrightarrow O(n) &\longrightarrow S^{n-1} \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \\ B &\longmapsto Bu \end{aligned} \quad (12.4)$$

with  $u \in S^{n-1}$  some fixed unit vector. In particular, this identifies  $S^{n-1}$  as an orbit (or homogeneous) space:

$$S^{n-1} \cong O(n)/O(n - 1).$$

**Example 12.8.** If  $m = 1$  in the bundle (12.2), we get the fiber bundle

$$V_{n-1}(\mathbb{R}^{k-1}) \hookrightarrow V_n(\mathbb{R}^k) \longrightarrow S^{k-1}. \quad (12.5)$$

By using the long exact sequence for bundle (12.5) and induction on  $n$ , it follows readily that  $V_n(\mathbb{R}^k)$  is  $(k - n - 1)$ -connected.

**Remark 12.9.** The long exact sequence of homotopy groups for the bundle (12.4) shows that  $\pi_i(O(n))$  is independent of  $n$  for  $n$  large. We call this the stable homotopy group  $\pi_i(O)$ . Bott Periodicity shows that  $\pi_i(O)$  is periodic in  $i$  with period 8. Its values are:

$i$	1	2	3	4	5	6	7	8
$\pi_i(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

**Definition 12.10.**

$$V_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} V_n(\mathbb{R}^k) \qquad G_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} G_n(\mathbb{R}^k)$$

The infinite grassmanian  $G_n(\mathbb{R}^\infty)$  carries a lot of topological information. As we will see later on, the space  $G_n(\mathbb{R}^\infty)$  is the classifying space for rank- $n$  real vector bundles. In fact, we get a “limit” fiber bundle:

$$O(n) \hookrightarrow V_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty). \quad (12.6)$$

Moreover, we have the following:

**Proposition 12.11.**  $V_n(\mathbb{R}^\infty)$  is contractible.

*Proof.* By using the bundle (12.5) for  $k \rightarrow \infty$ , we see that  $\pi_i(V_n(\mathbb{R}^\infty)) = 0$  for all  $i$ . Using the CW structure and Whitehead’s Theorem 7.2 shows that  $V_n(\mathbb{R}^\infty)$  is contractible.

Alternatively, we can define an explicit homotopy  $h_t : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$  by

$$h_t(x_1, x_2, \dots) := (1 - t)(x_1, x_2, \dots) + t(0, x_1, x_2, \dots).$$

Then  $h_t$  is linear for each  $t$  with  $\ker h_t = \{0\}$ . So  $h_t$  preserves independence of vectors. Applying  $h_t$  to an  $n$ -frame we get an  $n$ -tuple of independent vectors, which can be made orthonormal by the Gram-Schmidt process. We then get a deformation retraction of  $V_n(\mathbb{R}^\infty)$  onto the subspace of  $n$ -frames with first coordinate zero. Repeating this procedure  $n$  times, we get a deformation of  $V_n(\mathbb{R}^\infty)$  to the subspace of  $n$ -frames with first  $n$  coordinates zero.

Let  $\{e_1, \dots, e_n\}$  be the standard  $n$ -frame in  $\mathbb{R}^\infty$ . For an  $n$ -frame  $\{v_1, \dots, v_n\}$  of vectors with first  $n$  coordinates zero, define a homotopy  $k_t : V_n(\mathbb{R}^\infty) \rightarrow V_n(\mathbb{R}^\infty)$  by

$$k_t(\{v_1, \dots, v_n\}) := [(1 - t)\{v_1, \dots, v_n\} + t\{e_1, \dots, e_n\}] \circ (\text{Gram - Schmidt}).$$

Then  $k_t$  preserves linear independence and orthonormality by Gram-Schmidt.

Composing  $h_t$  and  $k_t$ , any  $n$ -frame is moved continuously to  $\{e_1, \dots, e_n\}$ , the standard  $n$ -frame. Thus  $k_t \circ h_t$  is a contraction of  $V_n(\mathbb{R}^\infty)$ .  $\square$

Similar considerations apply if we use  $\mathbb{C}$  or  $\mathbb{H}$  instead of  $\mathbb{R}$ , so we can define complex or quaternionic Stiefel and Grassman manifolds, by using the usual hermitian inner products in  $\mathbb{C}^k$  and  $\mathbb{H}^k$ , respectively. In particular,  $O(n)$  gets replaced by  $U(n)$  if  $\mathbb{C}$  is used, and  $Sp(n)$  is the quaternionic analog of this. Then similar fiber bundles can be constructed in the complex and quaternionic setting. For example, over  $\mathbb{C}$  we get fiber bundles

$$U(n) \hookrightarrow V_n(\mathbb{C}^k) \xrightarrow{p} G_n(\mathbb{C}^k), \quad (12.7)$$

with  $V_n(\mathbb{C}^k)$  a  $(2k - 2n)$ -connected space. As  $k \rightarrow \infty$ , we get a fiber bundle

$$U(n) \hookrightarrow V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty), \quad (12.8)$$

with  $V_n(\mathbb{C}^\infty)$  contractible. As we will see later on, this means that  $V_n(\mathbb{C}^\infty)$  is the classifying space for rank- $n$  complex vector bundles. We also have a fiber bundle similar to (12.4)

$$U(n-1) \hookrightarrow U(n) \longrightarrow S^{2n-1}, \quad (12.9)$$

whose long exact sequence of homotopy groups then shows that  $\pi_i(U(n))$  is stable for large  $n$ . Bott periodicity shows that this stable group  $\pi_i(U)$  repeats itself with period 2: the relevant groups are 0 for  $i$  even, and  $\mathbb{Z}$  for  $i$  odd. Note that by (12.9), odd-dimensional spheres can be realized as complex homogeneous spaces via

$$S^{2n-1} \cong U(n)/U(n-1).$$

Many of these fiber bundles will become essential tools in the next chapter for computing (co)homology of matrix groups, with a view towards *classifying spaces* and *characteristic classes* of manifolds.

## 13 Turning maps into fibration

In this section, we show that *any map is homotopic to a fibration*.

Given a map  $f : A \rightarrow B$ , define

$$E_f := \{(a, \gamma) \mid a \in A, \gamma : [0, 1] \rightarrow B \text{ with } \gamma(0) = f(a)\}.$$

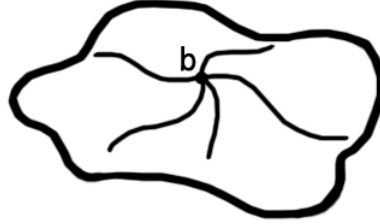
$E_f$  is a topological space with respect to the compact-open topology. Then  $A$  can be regarded as a subset of  $E_f$ , by mapping  $a \in A$  to  $(a, c_{f(a)})$ , where  $c_{f(a)}$  is the constant path based at the image of  $a$  under  $f$ . Define

$$\begin{aligned} E_f &\xrightarrow{p} B \\ (a, \gamma) &\mapsto \gamma(1) \end{aligned}$$

Then  $p|_A = f$ , so  $f = p \circ i$  where  $i$  is the inclusion of  $A$  in  $E_f$ . Moreover,  $i : A \rightarrow E_f$  is a homotopy equivalence, and  $p : E_f \rightarrow B$  is a fibration with fiber  $A$ . So  $f$  can be factored as a composition of a homotopy equivalence and a fibration:

$$\begin{array}{ccccc} A & \xhookrightarrow{i} & E_f & \xrightarrow[p]{\text{fibration}} & B \\ & \searrow & \text{f} & \nearrow & \\ & & & & \end{array}$$

**Example 13.1.** If  $A = \{b\} \hookrightarrow B$  and  $f$  is the inclusion of  $b$  in  $B$ , then  $E_f =: P_B$  is the contractible space of paths in  $B$  starting at  $b$  (called the *path-space* of  $B$ ):



In this case, the above construction yields the *path fibration*

$$\Omega B = p^{-1}(b) \hookrightarrow P_B \longrightarrow B,$$

where  $\Omega B$  is the space of all loops in  $B$  based at  $b$ , and  $P_B \longrightarrow B$  is given by  $\gamma \mapsto \gamma(1)$ . Since  $P_B$  is contractible, the associated long exact sequence of the fibration yields that

$$\pi_i(B) \cong \pi_{i-1}(\Omega B) \tag{13.1}$$

for all  $i$ .

The isomorphism (13.1) suggests that the Hurewicz Theorem 10.1 can also be proved by induction on the degree of connectivity. Indeed, if  $B$  is  $n$ -connected then  $\Omega B$  is  $(n - 1)$ -connected. We'll give the details of such an approach by using spectral sequences.

The following result is useful for computations:

**Proposition 13.2** (Puppé sequence). *Given a fibration  $F \hookrightarrow E \rightarrow B$ , there is a sequence of maps*

$$\dots \longrightarrow \Omega^2 B \longrightarrow \Omega F \longrightarrow \Omega E \longrightarrow \Omega B \longrightarrow F \longrightarrow E \longrightarrow B$$

*with any two consecutive maps forming a fibration.*

## Exercises

1. Let  $f : X \rightarrow Y$  be a homotopy equivalence. Let  $Z$  be any other space. Show that  $f$  induces bijections:

$$f_* : [Z, X] \rightarrow [Z, Y] \quad \text{and} \quad f^* : [Y, Z] \rightarrow [X, Z],$$

where  $[A, B]$  denotes the set of homotopy classes of maps from the space  $A$  to  $B$ .

2. Find examples of spaces  $X$  and  $Y$  which have the same homology groups, cohomology groups, and cohomology rings, but with different homotopy groups.
3. Use homotopy groups in order to show that there is no retraction  $\mathbb{R}\mathbb{P}^n \rightarrow \mathbb{R}\mathbb{P}^k$  if  $n > k > 0$ .
4. Show that an  $n$ -connected,  $n$ -dimensional CW complex is contractible.
5. (*Extension Lemma*)  
Given a CW pair  $(X, A)$  and a map  $f : A \rightarrow Y$  with  $Y$  path-connected, show that  $f$  can be extended to a map  $X \rightarrow Y$  if  $\pi_{n-1}(Y) = 0$  for all  $n$  such that  $X \setminus A$  has cells of dimension  $n$ .
6. Show that a CW complex retracts onto any contractible subcomplex. (Hint: Use the above extension lemma.)
7. If  $p : (\tilde{X}, \tilde{A}, \tilde{x}_0) \rightarrow (X, A, x_0)$  is a covering space with  $\tilde{A} = p^{-1}(A)$ , show that the map  $p_* : \pi_n(\tilde{X}, \tilde{A}, \tilde{x}_0) \rightarrow \pi_n(X, A, x_0)$  is an isomorphism for all  $n > 1$ .
8. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes  $X_1 \subset X_2 \subset \dots$  such that each inclusion  $X_i \hookrightarrow X_{i+1}$  is nullhomotopic. Conclude that  $S^\infty$  is contractible, and more generally, this is true for the infinite suspension  $\Sigma^\infty(X) := \bigcup_{n \geq 0} \Sigma^n(X)$  of any CW complex  $X$ .
9. Use cellular approximation to show that the  $n$ -skeletons of homotopy equivalent CW complexes without cells of dimension  $n + 1$  are also homotopy equivalent.
10. Show that a closed simply-connected 3-manifold is homotopy equivalent to  $S^3$ . (Hint: Use Poincaré Duality, and also the fact that closed manifolds are homotopy equivalent to CW complexes.)
11. Show that a map  $f : X \rightarrow Y$  of connected CW complexes is a homotopy equivalence if it induces an isomorphism on  $\pi_1$  and if a lift  $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$  to the universal covers induces an isomorphism on homology.
12. Show that  $\pi_7(S^4)$  is non-trivial. [Hint: It contains a  $\mathbb{Z}$ -summand.]
13. Prove that the space  $SO(3)$  of orthogonal  $3 \times 3$  matrices with determinant 1 is homeomorphic to  $\mathbb{R}\mathbb{P}^3$ .
14. Show that if  $S^k \rightarrow S^m \rightarrow S^n$  is a fiber bundle, then  $k = n - 1$  and  $m = 2n - 1$ .
15. Show that if there were fiber bundles  $S^{n-1} \rightarrow S^{2n-1} \rightarrow S^n$  for all  $n$ , then the groups  $\pi_i(S^n)$  would be finitely generated free abelian groups computable by induction, and non-zero if  $i \geq n \geq 2$ .
16. Let  $U(n)$  be the unitary group. Find  $\pi_k(U(n))$  for  $k = 1, 2, 3$  and  $n \geq 2$ .
17. If  $p : E \rightarrow B$  is a fibration over a contractible space  $B$ , then  $p$  is fiber homotopy equivalent to the trivial fibration  $B \times F \rightarrow B$ .