

Hodge Genera of Algebraic Varieties I

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Abstract

The aim of this paper is to study the behavior of Hodge-theoretic (intersection homology) genera and their associated characteristic classes under proper morphisms of complex algebraic varieties. We obtain formulae that relate (parametrized families of) global invariants of a complex algebraic variety X to such invariants of singularities of proper algebraic maps defined on X . Such formulae severely constrain, both topologically and analytically, the singularities of complex maps, even between smooth varieties. Similar results were announced by the first and third author in [13, 32]. © 2007 Wiley Periodicals, Inc.

1 Introduction

This paper and its sequels study the behavior of Hodge-theoretic (intersection homology) genera and their associated characteristic classes under proper morphisms of complex algebraic varieties. The formulae obtained in the present paper relate global invariants to singularities of complex algebraic maps. They thus shed some light on the mysterious formulae announced some years ago by the first and third author in [13, 32].

These formulae can be viewed as, on the one hand, yielding powerful methods of inductively calculating (even parametrized families of) characteristic classes of algebraic varieties (e.g., by applying them to resolutions of singularities). On the other hand, they can be viewed as yielding powerful topological and analytic constraints on the singularities of any proper algebraic morphism (e.g., even between smooth varieties), expressed in terms of (even parametrized families of) their characteristic classes. (Both these perspectives will be more fully developed following our subsequent studies of the contributions of monodromy; e.g., see our forthcoming paper [10].) Among these severe parametrized constraints on singularities of maps obtained here in complex settings, only at one special value do these formulae have full analogues for noncomplex maps (at $y = 1$, where they yield topological

constraints on the signature and associated L -polynomials of Pontrjagin classes); for that constraint on topological maps, see Cappell and Shaneson [12] (which employed very different methods) and a comparison in Remark 2.11 below.

The main instrument used in proving our results is the functorial calculus of Grothendieck groups of algebraic mixed Hodge modules. Originally, some of these results were proven by using Hodge-theoretic aspects of a deep theorem of Bernstein, Beilinson, Deligne, and Gabber, namely, the decomposition theorem for the pushforward of an intersection cohomology complex under a proper algebraic morphism [5, 14, 15]. The functorial approach employed here was suggested to us by the referee. However, the core calculations used in proving the results in this paper are modeled on our original considerations based on BBDG. We assume the reader's familiarity with the language of sheaves and derived categories, as well as that of intersection homology, perverse sheaves, and Deligne's mixed Hodge structures. But we do not assume any knowledge about Saito's mixed Hodge modules, except maybe the now-classical notion of an admissible variation of mixed Hodge structures.

We now briefly outline the content of each section and summarize our main results. The paper is divided into three main sections: in Section 2 we discuss genera of complex algebraic varieties; in Section 3 we describe the functorial calculus of Grothendieck groups of algebraic mixed Hodge modules, while Section 4 deals with characteristic classes yielding the Hodge genera considered in Section 2.

In Section 2.1, we first recall the definition of Hirzebruch's χ_y -genus for a smooth, complex, projective variety [22] and explain some of its Hodge-theoretic extensions to genera of possibly singular and/or noncompact complex algebraic varieties, defined by means of Deligne's mixed Hodge structures. Then we study the behavior of these genera under proper algebraic maps $f : X \rightarrow Y$ that are locally trivial topological fibrations over a compact, connected, smooth base Y . If F denotes the general fiber of f , then, under the assumption of trivial monodromy, χ_y is multiplicative, i.e., $\chi_y(X) = \chi_y(Y) \cdot \chi_y(F)$.

In Section 2.2, we consider arbitrary proper algebraic morphisms $f : X \rightarrow Y$ of complex algebraic varieties and discuss generalizations of the above multiplicativity property to this general setting. By taking advantage of the mixed Hodge structure on the intersection cohomology groups of a possibly singular complex algebraic variety X [14, 27, 29, 31], we define intersection homology genera, $I\chi_y(X)$, that encode the intersection homology Hodge numbers and provide yet another extension of Hirzebruch's genera to the singular case. For example, $I\chi_{-1}(X)$ is the intersection homology Euler characteristic of X , and if X is projective, then $I\chi_1(X) = \sigma(X)$ is the Goresky-MacPherson signature [20] of the intersection form on the middle-dimensional intersection homology of X . $I\chi_0(X)$ can be regarded as an extension to singular varieties of the arithmetic genus. The main results of this section are Theorems 2.5 and 2.9, in which the (intersection homology) genera $\chi_y(X)$ and $I\chi_y(X)$, respectively, of the domain are expressed in terms

of the singularities of the map. More precisely, we first fix an algebraic stratification of the proper morphism f ; that is, we choose complex algebraic Whitney stratifications of X and Y so that f becomes a stratified submersion. In particular, the strata satisfy the frontier condition: $V \cap \bar{W} \neq \emptyset$ implies $V \subset \bar{W}$. Then the finite set $\mathcal{V} := \{V\}$ of all strata of Y is partially ordered by “ $V \leq W$ if and only if $V \subset \bar{W}$.” If, moreover, we assume that Y is irreducible, then there is exactly one top-dimensional stratum S of Y , with $\dim S = \dim Y$, and S is Zariski-open and dense in Y , with $V \leq S$ for all $V \in \mathcal{V}$. Let F denote the (generic) fiber of f above S . Then we have:

THEOREM 1.1 *Let $f : X \rightarrow Y$ be a proper algebraic map of complex algebraic varieties, with Y irreducible. Let \mathcal{V} be the set of components of strata of Y in an algebraic stratification of f , and assume $\pi_1(V) = 0$ for all $V \in \mathcal{V}$. For each $V \in \mathcal{V}$ with $\dim(V) < \dim(Y)$, define inductively*

$$\widehat{I\chi}_y(\bar{V}) = I\chi_y(\bar{V}) - \sum_{W < V} \widehat{I\chi}_y(\bar{W}) \cdot I\chi_y(c^\circ L_{W,V}),$$

where the sum is over all $W \in \mathcal{V}$ with $\bar{W} \subset \bar{V} \setminus V$, and $c^\circ L_{W,V}$ denotes the open cone on the link of W in \bar{V} . Then:

$$(1.1) \quad \begin{aligned} \chi_y(X) &= I\chi_y(Y) \cdot \chi_y(F) \\ &+ \sum_{V < S} \widehat{I\chi}_y(\bar{V}) \cdot (\chi_y(F_V) - \chi_y(F) \cdot I\chi_y(c^\circ L_{V,Y})), \end{aligned}$$

where F_V is the fiber of f above the stratum V . Assume, moreover, that X is pure dimensional. Then

$$(1.2) \quad \begin{aligned} I\chi_y(X) &= I\chi_y(Y) \cdot I\chi_y(F) \\ &+ \sum_{V < S} \widehat{I\chi}_y(\bar{V}) \cdot (I\chi_y(f^{-1}(c^\circ L_{V,Y})) - I\chi_y(F) \cdot I\chi_y(c^\circ L_{V,Y})). \end{aligned}$$

Let us explain in more detail the $I\chi_y$ -terms appearing in the above formulae. Since f is stratified, the restrictions $R^j f_*(\mathbb{Q}_X)|_W$ and $R^j f_*(IC'_X)|_W$, for all $j \in \mathbb{Z}$ and $W \in \mathcal{V}$, are admissible variations of mixed Hodge structures, where $IC'_X := IC_X[-\dim(X)]$ is the shifted intersection cohomology complex of X . In fact, the first is the classical example of a “geometric variation of mixed Hodge structures,” whereas the assertion for the second follows from Saito’s theory of algebraic mixed Hodge modules. Similarly, by Saito’s theory, the cohomology sheaves $\mathcal{H}^j(IC'_V)|_W$, for $j \in \mathbb{Z}$ and $W, V \in \mathcal{V}$, underlie admissible variations of mixed Hodge structures. And under the assumption that the fundamental group of W is trivial, these are trivial variations so that the isomorphism classes of the mixed Hodge structures

- (1) $R^j f_*(\mathbb{Q}_X)_w \simeq H^j(\{f = w\}; \mathbb{Q})$,
- (2) $R^j f_*(IC'_X)_w \simeq IH^j(\{f = w\}; \mathbb{Q})$ for $\dim(W) = \dim(Y)$ and $R^j f_*(IC'_X)_w \simeq IH^j(f^{-1}(c^\circ L_{W,Y}); \mathbb{Q})$ for $\dim(W) < \dim(Y)$,
- (3) $\mathcal{H}^j(IC'_V)_w \simeq IH^j(c^\circ L_{W,V}; \mathbb{Q})$,

do not depend on the choice of the point $w \in W$. The same is true for the corresponding I_{χ_y} -genera, even without the assumption $\pi_1(W) = 0$, because these genera depend only on the corresponding Hodge numbers, which are constant in a variation of mixed Hodge structures. Also, note that the inductive definition of $\widehat{IT}_y(\bar{V})$ depends only on the stratified space \bar{V} with its induced stratification. Finally, while the existence of the mixed Hodge structure on the global cohomology already follows from Deligne’s classical work, and the Hodge structure on the global intersection cohomology of compact algebraic varieties can be obtained by other methods as in [14, 15], the mixed Hodge structures on the intersection cohomology of the open links $c^\circ L_{V,Y}$ (or of their inverse images) use the stalk description above and essentially depend on Saito’s theory of mixed Hodge modules.

In Section 3, we explain in detail the functorial calculus of Grothendieck groups of algebraic mixed Hodge modules and prove in this language the main technical result of this paper. The proof of this result is exactly the same as that of the corresponding statement in the framework of Grothendieck groups of constructible sheaves used in our paper [11]. This is indeed the case because, by Saito’s work, the calculus of Grothendieck groups of constructible sheaves completely lifts to the context of Grothendieck groups of algebraic mixed Hodge modules. Theorem 1.1 above as well as the formulae for characteristic classes discussed below are direct consequences of this main theorem.

In Section 4, we outline the construction of a natural transformation, MHT_y , which, when evaluated at the intersection cohomology complex IC'_X of a variety X , yields a twisted homology class $IT_y(X)$, whose associated genus for X compact is $I_{\chi_y}(X)$. The definition uses Saito’s theory of mixed Hodge modules and is based on ideas of a recent paper of Brasselet, Schürmann, and Yokura [9]. If X is a nonsingular complex algebraic variety, then $IT_y(X)$ is the Poincaré dual of the modified Todd class $T_y^*(TX)$ that appears in the generalized Hirzebruch-Riemann-Roch theorem. For a proper algebraic map $f : X \rightarrow Y$ with X pure dimensional and Y irreducible, we prove a formula for the pushforward of the characteristic class $IT_y(X)$ in terms of characteristic classes of strata of f . The main result of this section can be stated as follows:

THEOREM 1.2 *With the notation and assumptions from the above theorem, for each $V \in \mathcal{V}$ define inductively*

$$\widehat{IT}_y(\bar{V}) = IT_y(\bar{V}) - \sum_{W < V} \widehat{IT}_y(\bar{W}) \cdot I_{\chi_y}(c^\circ L_{W,V}),$$

where all homology characteristic classes are regarded in the Borel-Moore homology of the ambient space Y (with coefficients in $\mathbb{Q}[y, y^{-1}, (1 + y)^{-1}]$). Then:

$$f_*IT_y(X) = IT_y(Y) \cdot I\chi_y(F) + \sum_{V < S} \widehat{IT}_y(\bar{V}) \cdot (I\chi_y(f^{-1}(c^\circ L_{V,Y})) - I\chi_y(F) \cdot I\chi_y(c^\circ L_{V,Y})),$$

where $L_{V,Y}$ is the link of V in Y .

Without the trivial monodromy assumption, the terms in the formulae of Theorems 1.1 and 1.2 must be written in terms of genera and characteristic classes, respectively, with coefficients in local systems (variations of Hodge structures) on open strata.

The paper ends by discussing immediate consequences of the pushforward formula of Theorem 1.2.

In a future paper, we will consider the behavior under proper algebraic maps of χ_y -genera that are defined by using the Hodge-Deligne numbers of (compactly supported) cohomology groups of a possibly singular algebraic variety and deal with nontrivial monodromy considerations (cf. [10]). We point out that preliminary results on the Euler characteristics χ_{-1} and $I\chi_{-1}$, and on the homology MacPherson-Chern classes [25] $T_{-1} = c_* \otimes \mathbb{Q}$ and $IT_{-1} = Ic_* \otimes \mathbb{Q}$ of complex algebraic (respectively, compact complex analytic) varieties, have been already obtained by the authors in [11].

2 Genera

In this section, we define Hodge-theoretic genera of complex algebraic varieties and study their behavior under proper algebraic morphisms.

2.1 Families over a Smooth Base

DEFINITION 2.1 For a smooth projective variety X , we define its Hirzebruch χ_y -genus [22] by the formula

$$(2.1) \quad \chi_y(X) = \sum_p \left(\sum_q (-1)^q h^{p,q}(X) \right) y^p,$$

where $h^{p,q}(X) = \dim_{\mathbb{C}} H^q(X; \Omega_X^p)$ are the Hodge numbers of X . Note that χ_{-1} is the usual Euler characteristic, χ_0 is the arithmetic genus, and χ_1 is the signature of X .

Various extensions of Hirzebruch’s genus to the singular and/or noncompact setting are explained below. First, note that the Grothendieck group $K_0(mHs^{(p)})$ of the abelian category of (graded polarizable) rational mixed Hodge structures is a ring with respect to the tensor product, with unit the pure Hodge structure \mathbb{Q} of weight 0. We can now make the following definition:

DEFINITION 2.2 The E -polynomial is the ring homomorphism

$$E : K_0(mHs^{(p)}) \rightarrow \mathbb{Z}[u, v, u^{-1}, v^{-1}]$$

defined by

$$(2.2) \quad (V, F^\bullet, W_\bullet) \mapsto E(V) := \sum_{p,q} \dim_{\mathbb{C}} (gr_F^p gr_{p+q}^W (V \otimes_{\mathbb{Q}} \mathbb{C})) \cdot u^p v^q.$$

This is well-defined since the functor $gr_F^p gr_{p+q}^W (- \otimes_{\mathbb{Q}} \mathbb{C})$ is exact on mixed Hodge structures. Specializing to $(u, v) = (-y, 1)$, we get the χ_y -genus

$$(2.3) \quad \begin{aligned} \chi_y : K_0(mHs^{(p)}) &\rightarrow \mathbb{Z}[y, y^{-1}], \\ [(V, F^\bullet, W_\bullet)] &\mapsto \sum_p \dim_{\mathbb{C}} (gr_F^p (V \otimes_{\mathbb{Q}} \mathbb{C})) \cdot (-y)^p. \end{aligned}$$

Using Deligne’s mixed Hodge structure on the cohomology (with compact support) $H_{(c)}^j(X; \mathbb{Q})$ of a complex algebraic variety X (cf. [17, 18]), we can define

$$(2.4) \quad [H_{(c)}^*(X; \mathbb{Q})] := \sum_j (-1)^j \cdot [H_{(c)}^j(X; \mathbb{Q})] \in K_0(mHs^p).$$

Then by applying one of the homomorphisms E or χ_y , we obtain $E(X)$, $E_c(X)$, $\chi_y(X)$, and $\chi_y^c(X)$, so that for X smooth and projective, this definition of $\chi_y(X) = \chi_y^c(X)$ agrees with the classical Hirzebruch genus of X from Definition 2.1.

We first show that if $f : X \rightarrow Y$ is a family of compact varieties (i.e., a locally trivial topological fibration in the complex topology) over a smooth, connected, compact variety Y , then under certain assumptions on monodromy, χ_y behaves multiplicatively. In the setting of algebraic geometry, this fact encodes as special cases the classical multiplicativity property of the Euler-Poincaré characteristic for a locally trivial topological fibration, and the Chern-Hirzebruch-Serre formula for the signature of fibre bundles with trivial monodromy.

PROPOSITION 2.3 *Let $f : X \rightarrow Y$ be a proper algebraic map of complex algebraic varieties, with Y compact, smooth, and connected. Suppose that all direct image sheaves $R^j f_* \mathbb{Q}_X$, $j \in \mathbb{Z}$, are locally constant; e.g., f is a locally trivial topological fibration. Assume $\pi_1(Y)$ acts trivially on the cohomology of the general fiber F of f (e.g., $\pi_1(Y) = 0$); i.e., all these local systems $R^j f_* \mathbb{Q}_X$ ($j \in \mathbb{Z}$) are constant. Then:*

$$(2.5) \quad \chi_y(X) = \chi_y(Y) \cdot \chi_y(F).$$

PROOF: The local systems $R^j f_* \mathbb{Q}_X$ ($j \in \mathbb{Z}$) define geometric variations of mixed Hodge structures, and thus admissible variations in the sense of Steenbrink-Zucker and Kashiwara (cf. [23, 36]). Since Y is a smooth, compact algebraic variety, the cohomology groups $H^i(Y; R^j f_* \mathbb{Q}_X)$ get an induced (polarizable) mixed

Hodge structure. So, for each $j \in \mathbb{Z}$ we can define

$$[H^*(Y; R^j f_* \mathbb{Q}_X)] := \sum_i (-1)^i \cdot [H^i(Y; R^j f_* \mathbb{Q}_X)] \in K_0(mHs^p).$$

Moreover, the following key equality holds:

$$(2.6) \quad [H^*(X; \mathbb{Q})] = \sum_j (-1)^j [H^*(Y; R^j f_* \mathbb{Q}_X)] \in K_0(mHs^p).$$

The proof of this equality will be given in Proposition 3.1 in terms of mixed Hodge modules. Finally, if the local system $R^j f_* \mathbb{Q}_X$ is constant, we have an isomorphism of mixed Hodge structures

$$H^i(Y; R^j f_* \mathbb{Q}_X) \cong H^i(Y; \mathbb{Q}) \otimes H^j(F; \mathbb{Q}).$$

Altogether, we get the equality

$$(2.7) \quad [H^*(X; \mathbb{Q})] = [H^*(Y; \mathbb{Q})] \cdot [H^*(F; \mathbb{Q})] \in K_0(mHs^p).$$

We obtain the claimed multiplicativity by applying the χ_y -genus homomorphism to the identity in formula (2.7). Note that by applying the E -polynomial to (2.7), we obtain a similar multiplicativity property for the E -polynomials. \square

2.2 Proper Maps of Complex Algebraic Varieties

Let $f : X \rightarrow Y$ be a proper map of complex algebraic varieties. Such a map can be stratified with subvarieties as strata. In particular, there is a filtration of Y by closed subvarieties, underlying a Whitney stratification \mathcal{V} , so that the restriction of f to the preimage of any component of a stratum in Y is a locally trivial map of Whitney stratified spaces; i.e., f becomes a stratified submersion.

In this paper all intersection cohomology complexes are those associated to the middle perversity. By convention, the restriction of the intersection cohomology complex IC_X to the dense open stratum of X is the constant sheaf shifted by the complex dimension of X . If X is a complex algebraic variety of pure dimension n , the intersection cohomology groups (with compact support) are defined by the rule $IH_{(c)}^k(X; \mathbb{Q}) := H_{(c)}^{k-n}(X; IC_X)$.

Another possible extension of Hirzebruch’s χ_y -genus to the singular setting is obtained by using intersection homology theory [20, 21] as follows:

DEFINITION 2.4 For a pure dimensional complex algebraic variety X we let

$$IC'_X := IC_X[-\dim(X)]$$

be the shifted intersection cohomology complex. Then, by Saito’s theory of mixed Hodge modules, the group $IH_{(c)}^k(X; \mathbb{Q}) := H_{(c)}^k(X; IC'_X)$ gets a (graded polarizable) mixed Hodge structure, which is pure of weight k if X is compact. So we can define

$$(2.8) \quad [IH_{(c)}^*(X; \mathbb{Q})] := \sum_j (-1)^j [IH_{(c)}^j(X; \mathbb{Q})] \in K_0(mHs^p)$$

and polynomials

$$(2.9) \quad IE_{(c)}(X) := E([IH_{(c)}^*(X; \mathbb{Q})])$$

and

$$(2.10) \quad I\chi_y^{(c)} := \chi_y([IH_{(c)}^*(X; \mathbb{Q})]).$$

As a natural extension of the multiplicativity property for χ_y -genera of families over a smooth base described in Proposition 2.3, we aim to find the deviation from multiplicativity of the χ_y - and $I\chi_y$ -genus, respectively, in the more general setting of an arbitrary proper algebraic map. The formulae proved in Theorem 2.5 and Theorem 2.9 below include correction terms corresponding, respectively, to genera of strata and of their normal slices and to those of fibers of f above each stratum in Y . The first main result of this section concerns the χ_y -genus:

THEOREM 2.5 *Let $f : X \rightarrow Y$ be a proper algebraic map of complex algebraic varieties, with Y irreducible. Let \mathcal{V} be the set of components of strata of Y in an algebraic stratification of f , and assume $\pi_1(V) = 0$ for all $V \in \mathcal{V}$.¹ For each $V \in \mathcal{V}$ with $\dim(V) < \dim(Y)$, define inductively*

$$\widehat{I\chi}_y(\bar{V}) = I\chi_y(\bar{V}) - \sum_{W < V} \widehat{I\chi}_y(\bar{W}) \cdot I\chi_y(c^\circ L_{W,V}),$$

where the sum is over all $W \in \mathcal{V}$ with $\bar{W} \subset \bar{V} \setminus V$, and $c^\circ L_{W,V}$ denotes the open cone on the link of W in \bar{V} . Then:

$$(2.11) \quad \begin{aligned} \chi_y(X) = & I\chi_y(Y) \cdot \chi_y(F) \\ & + \sum_{V < S} \widehat{I\chi}_y(\bar{V}) \cdot (\chi_y(F_V) - \chi_y(F) \cdot I\chi_y(c^\circ L_{V,Y})), \end{aligned}$$

where F is the (generic) fiber over the top-dimensional stratum S , and F_V is the fiber of f above the stratum $V \in \mathcal{V} \setminus \{S\}$.

The proof is a direct consequence of the calculus of Grothendieck groups of algebraic mixed Hodge modules and will be given in Proposition 3.5. We want to point out that similar formulae also hold for $\chi_y^c(X)$ and $E_{(c)}(X)$, which in the same way follow from the results of Section 3.3.

In the special case when f is the identity map, equation (2.11) measures the difference between the χ_y - and the $I\chi_y$ -genus of a complex algebraic variety (with no monodromy restrictions in the case of Euler characteristics, that is, for $y = -1$; see [11, cor. 3.5]):

¹Contributions of nontrivial monodromy to such formulae will be the subject of further studies; e.g., see our forthcoming paper [10]; see also [2, 3] for some results on monodromy contributions for signatures and related characteristic classes in topological settings.

COROLLARY 2.6 *Let Y be an irreducible complex algebraic variety, and assume there is a Whitney stratification \mathcal{V} of Y with all strata simply connected. Then, using the notation of the above theorem, we obtain*

$$(2.12) \quad \chi_y(Y) = I_{\chi_y}(Y) + \sum_{V < S} \widehat{I}_{\chi_y}(\bar{V}) \cdot (1 - I_{\chi_y}(c^\circ L_{V,Y})).$$

Example 2.7 (Smooth Blowup). Let Y be a smooth, compact, n -dimensional variety and $Z \subset Y$ a submanifold of pure codimension $r + 1$. Let X be the blowup of Y along Z , and $f : X \rightarrow Y$ be the blowup map. Then X is an n -dimensional smooth variety, and f is an isomorphism over $Y \setminus Z$ and a projective bundle (Zariski locally trivial with fibre $\mathbb{C}\mathbb{P}^r$) over Z , corresponding to the projectivization of the normal bundle of Z in Y of rank $r + 1$. As we later explain in Example 3.3, the formula (2.11) of Theorem 2.5 is also true in this context (without assuming that all strata V are simply connected), and it reduces to a more familiar one [9, example 3.3]:

$$(2.13) \quad \chi_y(X) = \chi_y(Y) + \chi_y(Z) \cdot (-y + \dots + (-y)^r).$$

In fact, formula (2.13) can be easily obtained just by using the (additivity and multiplicativity) properties of the χ_y^c -genera of complex algebraic varieties (cf. [16]), and it holds if one considers X to be the blowup of a complete variety Y along a *regularly* embedded subvariety Z of pure codimension $r + 1$. By the “weak factorization theorem” [1], any birational map $h : S \rightarrow T$ between complete, nonsingular, complex algebraic varieties can be decomposed as a finite sequence of projections from smooth spaces lying over T , which are obtained by blowing up or blowing down along smooth centers. Then (2.13) yields the birational invariance of the arithmetic genus χ_0 of nonsingular projective varieties (see the discussion in [9, example 3.3]).

Remark 2.8. Formula (2.11) yields calculations of classical topological and algebraic invariants of the complex algebraic variety X , e.g., the Euler characteristic, and if X is smooth and projective, the signature and arithmetic genus in terms of singularities of proper algebraic maps defined on X .

The second main result of this section asserts that the I_{χ_y} -genus defined in (2.10) satisfies the so-called “stratified multiplicative property” (cf. [13, 32]). More precisely, we have the following:

THEOREM 2.9 *Let $f : X \rightarrow Y$ be a proper algebraic map of complex algebraic varieties, with X pure dimensional and Y irreducible. Let \mathcal{V} be the set of components of strata of Y in an algebraic stratification of f , and assume $\pi_1(V) = 0$ for all $V \in \mathcal{V}$. Then in the notation of Theorem 2.5, we have*

$$(2.14) \quad \begin{aligned} I_{\chi_y}(X) &= I_{\chi_y}(Y) \cdot I_{\chi_y}(F) \\ &+ \sum_{V < S} \widehat{I}_{\chi_y}(\bar{V}) \cdot (I_{\chi_y}(f^{-1}(c^\circ L_{V,Y})) - I_{\chi_y}(F) \cdot I_{\chi_y}(c^\circ L_{V,Y})). \end{aligned}$$

Again, the proof will follow from the considerations of Proposition 3.5, which will also imply that similar formulae hold for $I\chi_y^c(X)$ and $IE_{(c)}(X)$. Let us only explain here the identification of stalks at a point w in a stratum W of Y that yield the $I\chi_y$ -terms in formula (2.14). Let $i_w : \{w\} \hookrightarrow Y$ be the inclusion map. Then:

LEMMA 2.10 *We have the following isomorphisms:*

- (i) $R^j f_*(IC'_X)_w = \mathcal{H}^j(i_w^* Rf_*(IC'_X)) \simeq IH^j(\{f = w\}; \mathbb{Q})$
 for $\dim(W) = \dim(Y)$, and
 $R^j f_*(IC'_X)_w = \mathcal{H}^j(i_w^* Rf_*(IC'_X)) \simeq IH^j(f^{-1}(c^\circ L_{W,Y}); \mathbb{Q})$
 for $\dim(W) < \dim(Y)$,
- (ii) $\mathcal{H}^j(IC'_{\bar{v}})_w = \mathcal{H}^j(i_w^* IC'_{\bar{v}}) \simeq IH^j(c^\circ L_{W,v}; \mathbb{Q})$,

which endow the intersection cohomology groups on the right-hand side of the above identities with canonical mixed Hodge structures.

PROOF: If $\dim(W) = \dim(Y)$, then $\{f = w\}$ is the generic fiber F of f ; thus it is locally normally nonsingular embedded in X . It follows from [21, sec. 5.4.1] that we have a quasi-isomorphism:

$$IC'_X|_F \simeq IC'_F.$$

Then by proper base change, we obtain that

$$\mathcal{H}^j(i_w^* Rf_*(IC'_X)) \simeq IH^j(\{f = w\}; \mathbb{Q}).$$

Assume now that $\dim(W) < \dim(Y)$. Let N be a normal slice to W at w in local analytic coordinates $(Y, w) \hookrightarrow (\mathbb{C}^n, w)$, that is, a germ of a complex manifold $(N, w) \hookrightarrow (\mathbb{C}^n, w)$, intersecting W transversally only at w , and with $\dim(W) + \dim(N) = n$. Recall that the link $L_{W,Y}$ of the stratum W in Y is defined as

$$L_{W,Y} := Y \cap N \cap \partial B_r(w),$$

where $B_r(w)$ is an open ball of (very small) radius r around w . Moreover, $Y \cap N \cap B_r(w)$ is isomorphic (in a stratified sense) to the open cone $c^\circ L_{W,Y}$ on the link [8, p. 44]. By factoring the inclusion map i_w as the composition

$$\{w\} \xrightarrow{\phi} Y \cap N \xrightarrow{\psi} Y$$

we can now write

$$\begin{aligned} \mathcal{H}^j(i_w^* Rf_* IC'_X) &\cong H^j(w, \phi^* \psi^* Rf_* IC'_X) \\ &\cong H^j(\psi^* Rf_* IC'_X)_w \\ &\cong H^j(c^\circ L_{W,Y}, Rf_* IC'_X) \\ &\cong H^j(f^{-1}(c^\circ L_{W,Y}), IC'_X) \end{aligned}$$

$$\begin{aligned}
& \stackrel{(*)}{\cong} H^j(f^{-1}(c^\circ L_{W,Y}), IC'_{f^{-1}(c^\circ L_{W,Y})}) \\
& \cong IH^j(f^{-1}(c^\circ L_{W,Y}); \mathbb{Q})
\end{aligned}$$

where in (*) we used the fact that the inverse image of a normal slice to a stratum of Y in a stratification of f is (locally) normally nonsingular embedded in X (this fact is a consequence of the first isotopy lemma).

The isomorphism $\mathcal{H}^j(i_w^* IC'_V) \simeq IH^j(c^\circ L_{W,V}; \mathbb{Q})$ follows, for example, from [8, prop. 4.2].

The statement about the existence of the canonical mixed Hodge structures is a consequence of Saito's theory of mixed Hodge modules (see Section 3.1). Indeed, since the complexes $Rf_* IC'_X$ and IC'_V , underlie complexes of mixed Hodge modules (cf. Section 3), their pullbacks over the point w become complexes of mixed Hodge structures, so their cohomologies are (graded polarizable) rational mixed Hodge structures. \square

Remark 2.11. For a projective algebraic variety X of pure dimension n , the value at $y = 1$ of the intersection homology genus $I\chi_y(X)$ is the Goresky-MacPherson signature $\sigma(X)$ of the intersection form in the middle-dimensional intersection cohomology $IH^n(X; \mathbb{Q})$ with middle perversity [20]. Therefore, under the trivial monodromy assumption, formula (2.14) calculates the signature of the domain of a proper map f in terms of singularities of the map. In [12], a different formula was given for the behavior of the signature (and associated L -class) under any stratified map. Those topological results were obtained by a very different sheaf-theoretic method, i.e., introducing a notion of cobordism of self-dual sheaves and showing sheaf decompositions up to such cobordism. By comparing the two formulae in the case of a proper map of algebraic varieties, one obtains interesting Hodge-theoretic interpretations of the normal data encoded in the topological formula for signature [12]. We exemplify this relation in a simple situation, namely that of blowing up a point: Let X be obtained from Y by blowing up a point y . Let L be the link of y in Y . Formula (2.14) becomes in this case

$$\sigma(X) = \sigma(Y) + I\chi_1(f^{-1}(c^\circ L)) - I\chi_1(c^\circ L).$$

On the other hand, the topological formula for signature in [12] yields

$$\sigma(X) = \sigma(Y) + \sigma(E_y),$$

where $E_y = f^{-1}(N)/f^{-1}(L)$ is the topological completion of $f^{-1}(\text{int } N)$ for N a piecewise linear neighborhood of y in Y with $\partial N = L$. By comparing the two formulae above, we obtain a Hodge-theoretic interpretation for the signature of the topological completion E_y , namely,

$$\sigma(E_y) = I\chi_1(f^{-1}(c^\circ L)) - I\chi_1(c^\circ L).$$

3 Grothendieck Groups of Algebraic Mixed Hodge Modules

In this section we explain the functorial calculus of Grothendieck groups of algebraic mixed Hodge modules and prove in this language the main technical result of this paper. All other results, in particular those stated in Section 2, are direct consequences of this main theorem.

3.1 Mixed Hodge Modules

For ease of reading, we begin by introducing a few notions that will be used throughout this paper. A quick introduction to Saito’s theory can be also found in [9, 26, 27]. Generic references are [28, 29], but see also [31].

Let X be an n -dimensional complex algebraic variety. To such an X one can associate an abelian category of algebraic mixed Hodge modules, $\text{MHM}(X)$, together with functorial pushdowns $f_!$ and f_* on the level of derived categories $D^b \text{MHM}(X)$ for any, not necessarily proper, map. If f is a proper map, then $f_* = f_!$. In fact, the derived category $D_c^b(X)$ of bounded (algebraically) constructible complexes of sheaves of \mathbb{Q} -vector spaces *underlies* the theory of mixed Hodge modules; i.e., there is a forgetful functor

$$\text{rat} : D^b \text{MHM}(X) \rightarrow D_c^b(X)$$

that associates their underlying \mathbb{Q} -complexes to complexes of mixed Hodge modules, so that

$$\text{rat}(\text{MHM}(X)) \subset \text{Perv}(\mathbb{Q}_X),$$

that is to say, that $\text{rat} \circ H = {}^p\mathcal{H} \circ \text{rat}$, where H stands for the cohomological functor in $D^b \text{MHM}(X)$ and ${}^p\mathcal{H}$ denotes the perverse cohomology. Then the functors f_* , $f_!$, f^* , $f^!$, \otimes , and \boxtimes on $D^b \text{MHM}(X)$ are “lifts” of the similar functors defined on $D_c^b(X)$, with (f^*, f_*) and $(f_!, f^!)$ also pairs of adjoint functors in the context of mixed Hodge modules.

The objects of the category $\text{MHM}(X)$ can be roughly described as follows: If X is *smooth*, then $\text{MHM}(X)$ is a full subcategory of the category of objects $((M, F), K, W)$ such that:

- (1) (M, F) is an algebraic holonomic filtered \mathcal{D} -module M on X , with an exhaustive, bounded-from-below, and increasing “Hodge” filtration F by algebraic \mathcal{O}_X -modules.
- (2) $K \in \text{Perv}(\mathbb{Q}_X)$ is the underlying rational sheaf complex, and there is a quasi-isomorphism $\alpha : DR(M) \simeq \mathbb{C} \otimes K$ in $\text{Perv}(\mathbb{C}_X)$, where DR is the de Rham functor shifted by the dimension of X .
- (3) W is a pair of filtrations on M and K compatible with α .

For a singular X , one works with suitable local embeddings into manifolds and corresponding filtered \mathcal{D} -modules with support on X . In addition, these objects have to satisfy a long list of very complicated properties, but the details of the full construction are not needed here. Instead, we will only use certain formal

properties that will be explained below. In this notation, the functor rat is defined by $\text{rat}((M, F), K, W) = K$.

It follows from the definition of mixed Hodge modules that every $M \in \text{MHM}(X)$ has a functorial increasing filtration W in $\text{MHM}(X)$, called the *weight filtration* of M , so that the functor $M \rightarrow Gr_k^W M$ is exact. We say that $M \in \text{MHM}(X)$ is *pure of weight k* if $Gr_i^W M = 0$ for all $i \neq k$. A complex $M^\bullet \in D^b \text{MHM}(X)$ is *mixed of weight $\leq k$ (respectively, $\geq k$)* if $Gr_i^W H^j M^\bullet = 0$ for all $i > j + k$ (respectively, $i < j + k$), and it is *pure of weight k* if $Gr_i^W H^j M^\bullet = 0$ for all $i \neq j + k$. If f is a map of algebraic varieties, then $f_!$ and f^* preserve weight $\leq k$, and f_* and $f^!$ preserve weight $\geq k$. If $M^\bullet \in D^b \text{MHM}(X)$ is of weight $\leq k$ (respectively, $\geq k$), then $H^j M^\bullet$ has weight $\leq j + k$ (respectively, $\geq j + k$). In particular, if $M \in D^b \text{MHM}(X)$ is pure and $f : X \rightarrow Y$ is proper, then $f_* M \in D^b \text{MHM}(Y)$ is pure.

If $j : U \hookrightarrow X$ is a Zariski-open dense subset in X , then the intermediate extension $j_{!*}$ (cf. [5]) preserves the weights.

We say that $M \in \text{MHM}(X)$ is supported on S if and only if $\text{rat}(M)$ is supported on S . Saito showed that the category of mixed Hodge modules supported on a point, $\text{MHM}(pt)$, coincides with the category mHs^p of (graded) polarizable, rational, mixed Hodge structures. Here one has to switch the increasing \mathcal{D} -module filtration F_* of the mixed Hodge module to the decreasing Hodge filtration of the mixed Hodge structure by $F^* := F_{-*}$, so that $gr_F^p \simeq gr_{-p}^F$. In this case, the functor rat associates to a mixed Hodge structure the underlying rational vector space. Following [29], there exists a unique object $\mathbb{Q}^H \in \text{MHM}(pt)$ such that $\text{rat}(\mathbb{Q}^H) = \mathbb{Q}$ and \mathbb{Q}^H is of type $(0, 0)$. In fact, $\mathbb{Q}^H = ((\mathbb{C}, F), \mathbb{Q}, W)$, with $gr_i^F = 0 = gr_i^W$ for all $i \neq 0$, and $\alpha : \mathbb{C} \rightarrow \mathbb{C} \otimes \mathbb{Q}$ the obvious isomorphism.

For a complex algebraic variety X , we define the complex of mixed Hodge modules

$$\mathbb{Q}_X^H := k^* \mathbb{Q}^H \in D^b \text{MHM}(X)$$

with $\text{rat}(\mathbb{Q}_X^H) = \mathbb{Q}_X$, where $k : X \rightarrow pt$ is the constant map to a point. If X is *smooth* and of dimension n , then $\mathbb{Q}_X[n] \in \text{Perv}(\mathbb{Q}_X)$, and $\mathbb{Q}_X^H[n] \in \text{MHM}(X)$ is a single mixed Hodge module (in degree 0) explicitly described by $\mathbb{Q}_X^H[n] = ((\mathcal{O}_X, F), \mathbb{Q}_X[n], W)$, where F and W are trivial filtrations so that $gr_i^F = 0 = gr_{i+n}^W$ for all $i \neq 0$. So if X is smooth of dimension n , then $\mathbb{Q}_X^H[n]$ is pure of weight n . By the stability of the intermediate extension functor, this shows that if X is an algebraic variety of pure dimension n and $j : U \hookrightarrow X$ is the inclusion of a smooth Zariski-open dense subset, then the intersection cohomology module $IC_X^H := j_{!*}(\mathbb{Q}_U^H[n])$ is pure of weight n , with underlying perverse sheaf $\text{rat}(IC_X^H) = IC_X$.

The pure objects in Saito's theory are the *polarized Hodge modules* (see [28], but here we work in the more restricted algebraic context as defined in [29]). The category $\text{MH}(X, k)^p$ of polarizable Hodge modules on X of weight k is a semisimple abelian category in the sense that every polarizable Hodge module on X can be

written in a unique way as a direct sum of polarizable Hodge modules with strict support in irreducible closed subvarieties of X . This is the so-called *decomposition by strict support* of a pure Hodge module. If $\text{MH}_Z(X, k)^p$ denotes the category of pure Hodge modules of weight k and strict support Z , then $\text{MH}_Z(X, k)^p$ depends only on Z , and any $M \in \text{MH}_Z(X, k)^p$ is *generically* a polarizable variation of Hodge structures \mathbb{V}_U on a Zariski open dense subset $U \subset Z$, with *quasi-unipotent monodromy at infinity*.

Conversely, any such polarizable variation of Hodge structures can be extended uniquely to a pure Hodge module. In other words, there is an equivalence of categories:

$$(3.1) \quad \text{MH}_Z(X, k)^p \simeq \text{VHS}_{\text{gen}}(Z, k - \dim(Z))^p,$$

where the right-hand side is the category of polarizable variations of Hodge structures of weight $k - \dim(Z)$ defined on nonempty smooth subvarieties of Z , whose local monodromies are quasi-unipotent. Note that, under this correspondence, if M is a pure Hodge module with strict support Z , then $\text{rat}(M) = IC_Z(\mathbb{V})$, where \mathbb{V} is the corresponding variation of Hodge structures.

If X is smooth of dimension n , an object $M \in \text{MHM}(X)$ is called *smooth* if and only if $\text{rat}(M)[-n]$ is a local system on X . Smooth mixed Hodge modules are (up to a shift) admissible (at infinity) variations of mixed Hodge structures. Conversely, an admissible variation of mixed Hodge structures \mathbb{V} on a smooth variety X of pure dimension n gives rise to a smooth mixed Hodge module (cf. [29]), i.e., to an element $\mathbb{V}^H[n] \in \text{MHM}(X)$ with $\text{rat}(\mathbb{V}^H[n]) = \mathbb{V}[n]$. A pure polarizable variation of weight k yields an element of $\text{MH}(X, k + n)^p$. By the stability by the intermediate extension functor, it follows that if X is an algebraic variety of pure dimension n and \mathbb{V} is an admissible variation of (pure) Hodge structures (of weight k) on a smooth Zariski-open dense subset $U \subset X$, then $IC_X^H(\mathbb{V})$ is an algebraic mixed Hodge module (pure of weight $k + n$), so that $\text{rat}(IC_X^H(\mathbb{V})|_U) = \mathbb{V}[n]$.

We conclude this section with a short explanation of the “rigidity” property for admissible variations of mixed Hodge structures, as this will be used later on. Assume X is smooth, connected, and of dimension n , with $M \in \text{MHM}(X)$ a smooth mixed Hodge module, so that the local system $\mathbb{V} := \text{rat}(M)[-n]$ has the property that the restriction map $r : H^0(X; \mathbb{V}) \rightarrow \mathbb{V}_x$ is an isomorphism for all $x \in X$. Then the (admissible) variation of mixed Hodge structures is a constant variation since r underlies the morphism of mixed Hodge structures (induced by the adjunction $id \rightarrow i_*i^*$):

$$H^0(k_*(M)[-n]) \rightarrow H^0(k_*i_*i^*(M)[-n])$$

with $k : X \rightarrow pt$ the constant map, and $i : \{x\} \hookrightarrow X$ the inclusion of the point. This implies

$$M[-n] = \mathbb{V}^H \simeq k^*i^*\mathbb{V}^H = k^*\mathbb{V}_x^H \in D^b \text{MHM}(X).$$

3.2 The Functorial Calculus of Grothendieck Groups

In this section, we describe the functorial calculus of Grothendieck groups of algebraic mixed Hodge modules. As a first application, we indicate the proof of the identity (2.6), which was used in Proposition 2.3.

Let X be a complex algebraic variety. By associating to (the class of) a complex the alternating sum of (the classes of) its cohomology objects, we obtain the following identification (e.g., compare [24, p. 77], [33, lemma 3.3.1]):

$$(3.2) \quad K_0(D^b \text{MHM}(X)) = K_0(\text{MHM}(X)).$$

In particular, if X is a point,

$$(3.3) \quad K_0(D^b \text{MHM}(pt)) = K_0(mHs^p),$$

and the latter is a commutative ring with respect to the tensor product, with unit \mathbb{Q}_{pt}^H . All functors f_* , $f_!$, f^* , $f^!$, \otimes , and \boxtimes induce corresponding functors on the groups $K_0(\text{MHM}(\cdot))$. Moreover, $K_0(\text{MHM}(X))$ becomes a $K_0(\text{MHM}(pt))$ -module, with the multiplication induced by the exact exterior product

$$\boxtimes : \text{MHM}(X) \times \text{MHM}(pt) \rightarrow \text{MHM}(X \times \{pt\}) \simeq \text{MHM}(X).$$

Also note that

$$M \otimes \mathbb{Q}_X^H \simeq M \boxtimes \mathbb{Q}_{pt}^H \simeq M$$

for all $M \in \text{MHM}(X)$. Therefore, $K_0(\text{MHM}(X))$ is a unitary $K_0(\text{MHM}(pt))$ -module. Finally, the functors f_* , $f_!$, f^* , and $f^!$ commute with exterior products (and f^* also commutes with the tensor product \otimes), so that the induced maps at the level of Grothendieck groups $K_0(\text{MHM}(\cdot))$ are $K_0(\text{MHM}(pt))$ -linear. Moreover, by the functor

$$\text{rat} : K_0(\text{MHM}(X)) \rightarrow K_0(D_c^b(X)) \simeq K_0(\text{Perv}(\mathbb{Q}_X)),$$

these maps lift the corresponding transformations from the (topological) level of Grothendieck groups of constructible (or perverse) sheaves.

As a first application, we can now explain the proof of Proposition 2.3 in the following, more general form:

PROPOSITION 3.1 *Let $f : X \rightarrow Y$ be a proper algebraic map, with Y a smooth, connected variety, of dimension n . Fix $M \in D^b \text{MHM}(X)$ so that all higher direct image sheaves $R^j f_* \text{rat}(M)$ ($j \in \mathbb{Z}$) are locally constant. Then we have*

$$(3.4) \quad [H^*(X; \text{rat}(M))] = \sum_j (-1)^j \cdot [H^*(Y; R^j f_* \text{rat}(M))] \in K_0(mHs^p).$$

If, moreover, the local systems $R^j f_ \text{rat}(M)$ ($j \in \mathbb{Z}$) are constant, then we obtain the following multiplicative relation in $K_0(mHs^p)$ (extending (2.7)):*

$$(3.5) \quad [H_{(c)}^*(X; \text{rat}(M))] = [H_{(c)}^*(Y; \mathbb{Q})] \cdot [H^*({f = y}; \text{rat}(M))].$$

PROOF: We first need to explain the various mixed Hodge structures appearing in the above formulae. Let $k : X \rightarrow pt$ be the constant map. Then the cohomology groups

$$H^j(X; \text{rat}(M)) = \text{rat}(H^j(k_*M)) \quad \text{and} \quad H_c^j(X; \text{rat}(M)) = \text{rat}(H^j(k_!M))$$

get induced (graded polarizable) mixed Hodge structures, so that we can define

$$[H_{(c)}^*(X; \text{rat}(M))] := k_{*(c)}[M] \in K_0(mHs^p).$$

By a deep theorem of Saito (cf. [30]), these mixed Hodge structures agree for $M = \mathbb{Q}_X^H$ with those of Deligne, so that in this case we get back our old notation from (2.4).

Since all direct image sheaves $R^j f_* \text{rat}(M)$ are locally constant ($j \in \mathbb{Z}$), the usual and perverse cohomology sheaves of $Rf_* \text{rat}(M)$ are the same up to a shift by n (since Y is smooth), so that for each $j \in \mathbb{Z}$, $H^j(f_*M)$ is a *smooth* mixed Hodge module with

$$\text{rat}(H^{j+n}(f_*M)) = R^j f_* \text{rat}(M)[n].$$

In particular, the local system $R^j f_* \text{rat}(M)$ underlies an admissible variation of mixed Hodge structures. By pushing down under the constant map $k' : Y \rightarrow pt$, we obtain

$$[H^*(Y; R^j f_* \text{rat}(M))] = (-1)^n \cdot k'_*([H^{j+n}(f_*M)]) \in K_0(mHs^p),$$

which, by construction, agrees in the case when $M = \mathbb{Q}_X^H$ and Y compact with the element of $K_0(mHs^p)$ appearing in equation (2.6). By taking the alternating sum of cohomology sheaves, we also have that

$$f_*[M] = [f_*M] = \sum_j (-1)^j \cdot [H^j(f_*M)] \in K_0(\text{MHM}(Y)),$$

and, by functoriality, we get the equality in the formula (3.4):

$$\begin{aligned} [H^*(X; \text{rat}(M))] &:= k_*[M] = k'_* f_*[M] \\ &= \sum_j (-1)^{j+n} \cdot k'_*[H^{j+n}(f_*M)] \\ &= \sum_j (-1)^j \cdot [H^*(Y; R^j f_* \text{rat}(M))] \in K_0(mHs^p). \end{aligned}$$

Note that in the case when $M = \mathbb{Q}_X^H$ and Y is compact, this implies the claimed identity (2.6) of Proposition 2.3.

If we assume in addition that the local system $R^j f_* \text{rat}(M)$ is constant with stalk the mixed Hodge structure M^j , then, by “rigidity,” when viewed as a mixed Hodge module it is isomorphic to $k'^*M^j \simeq \mathbb{Q}_Y^H \otimes k'^*M^j$. Therefore, by the $K_0(\text{MHM}(pt))$ -linearity of k'_* , this implies

$$[H^*(Y; R^j f_* \text{rat}(M))] = [H^*(Y; \text{rat}(\mathbb{Q}_Y^H))] \cdot [M^j] \in K_0(mHs^p).$$

Next note that for a fixed $y \in Y$,

$$\sum_j (-1)^j \cdot [M^j] = [H^*(\{f = y\}; \text{rat}(M))] \in K_0(mHs^p).$$

So, if all direct image sheaves $R^j f_* \text{rat}(M)$ ($j \in \mathbb{Z}$) are constant, then we obtain the multiplicative relation claimed in formula (3.5):

$$[H^*(X; \text{rat}(M))] = [H^*(Y; \mathbb{Q})] \cdot [H^*(\{f = y\}; \text{rat}(M))] \in K_0(mHs^p).$$

By using k'_1 instead of k'_* in the above arguments, we get a similar multiplicative relation for the cohomology with compact support. \square

3.3 Main Theorem and Immediate Consequences

We can now formulate the main technical result of this paper and explain, as an application, the proofs of Theorems 2.5 and 2.9.

Let Y be an irreducible complex algebraic variety endowed with a complex algebraic Whitney stratification \mathcal{V} so that the intersection cohomology complexes

$$IC'_{\bar{W}} := IC_{\bar{W}}[-\dim(W)]$$

are \mathcal{V} -constructible for all strata $W \in \mathcal{V}$. Denote by S the top-dimensional stratum, so S is Zariski-open and dense, and $V \leq S$ for all $V \in \mathcal{V}$. Let us fix for each $W \in \mathcal{V}$ a point $w \in W$ with inclusion $i_w : \{w\} \hookrightarrow Y$. Then

$$(3.6) \quad i_w^*[IC'^H_{\bar{W}}] = [i_w^* IC'^H_{\bar{W}}] = [\mathbb{Q}^H_{pt}] \in K_0(\text{MHM}(w)) = K_0(\text{MHM}(pt))$$

and $i_w^*[IC'^H_{\bar{V}}] \neq [0] \in K_0(\text{MHM}(pt))$ only if $W \leq V$. Moreover, for any $j \in \mathbb{Z}$, we have

$$(3.7) \quad \mathcal{H}^j(i_w^* IC'_{\bar{V}}) \simeq IH^j(c^\circ L_{W,V}),$$

with $c^\circ L_{W,V}$ the open cone on the link $L_{W,V}$ of W in \bar{V} for $W \leq V$ (cf. [8, prop. 4.2]). So

$$i_w^*[IC'^H_{\bar{V}}] = [IH^*(c^\circ L_{W,V})] \in K_0(\text{MHM}(pt)),$$

with the mixed Hodge structures on the right-hand side defined by the isomorphism (3.7).

The main technical result of this section is the following:

THEOREM 3.2 *For each stratum $V \in \mathcal{V} \setminus \{S\}$ define inductively*

$$(3.8) \quad \widehat{IC}^H(\bar{V}) := [IC'^H_{\bar{V}}] - \sum_{W < V} \widehat{IC}^H(\bar{W}) \cdot i_w^*[IC'^H_{\bar{V}}] \in K_0(D^b \text{MHM}(Y)).$$

As the notation suggests, $\widehat{IC}^H(\bar{V})$ depends only on the complex algebraic variety \bar{V} with its induced algebraic Whitney stratification. Assume $[M] \in K_0(D^b \text{MHM}(Y))$

is an element of the $K_0(\text{MHM}(pt))$ -submodule $\langle [IC_{\bar{V}}^{\prime H}] \rangle$ of $K_0(D^b \text{MHM}(Y))$ generated by the elements $[IC_{\bar{V}}^{\prime H}]$, $V \in \mathcal{V}$. Then we have the following equality in $K_0(D^b \text{MHM}(Y))$:

$$(3.9) \quad [M] = [IC_Y^{\prime H}] \cdot i_s^*[M] + \sum_{V < S} \widehat{IC}^H(\bar{V}) \cdot (i_v^*[M] - i_s^*[M] \cdot i_v^*[IC_Y^{\prime H}]).$$

PROOF: In order to prove formula (3.9), consider

$$(3.10) \quad [M] = \sum_{V \in \mathcal{V}} [IC_{\bar{V}}^{\prime H}] \cdot L(V)$$

for some $L(V) \in K_0(\text{MHM}(pt))$. The aim is to identify these coefficients $L(V)$. Since S is an open stratum, by applying i_s^* to (3.10) we obtain

$$i_s^*[M] = L(S) \in K_0(\text{MHM}(s)) = K_0(\text{MHM}(pt)).$$

Next fix a stratum $W \neq S$ and apply i_w^* to (3.10). Recall that $i_w^*[IC_{\bar{W}}^{\prime H}] = [\mathbb{Q}_{pt}^H] \in K_0(\text{MHM}(w)) = K_0(\text{MHM}(pt))$ and $i_w^*[IC_{\bar{V}}^{\prime H}] \neq [0] \in K_0(\text{MHM}(pt))$ only if $W \leq V$. We obtain

$$(3.11) \quad i_w^*[M] = L(W) + \sum_{W < V} i_w^*[IC_{\bar{V}}^{\prime H}] \cdot L(V) \in K_0(\text{MHM}(w)) = K_0(\text{MHM}(pt)).$$

Since S is dense, we have that $W < S$, so the stratum S appears in the summation on the right-hand side of (3.11). Therefore

$$(3.12) \quad i_w^*[M] - i_w^*[IC_Y^{\prime H}] \cdot i_s^*[M] = L(W) + \sum_{W < V < S} i_w^*[IC_{\bar{V}}^{\prime H}] \cdot L(V) \in K_0(\text{MHM}(pt)).$$

This implies that we can inductively calculate $L(V)$ in terms of

$$L'(W) := i_w^*[M] - i_w^*[IC_Y^{\prime H}] \cdot i_s^*[M].$$

Indeed, (3.12) can be rewritten as

$$(3.13) \quad L'(W) = \sum_{W \leq V < S} i_w^*[IC_{\bar{V}}^{\prime H}] \cdot L(V) \in K_0(\text{MHM}(pt)),$$

and the matrix $A = (a_{W,V})$ with $a_{W,V} := i_w^*[IC_{\bar{V}}^{\prime H}] \in K_0(\text{MHM}(pt))$ for $W, V \in \mathcal{V} \setminus \{S\}$ is upper-triangular with respect to \leq , with 1's on the diagonal. So A can be inverted. The nonzero coefficients of $A^{-1} = (a'_{W,V})$ can be calculated inductively (e.g., see [35, prop. 3.6.2]) by $a'_{V,V} = 1$ and

$$(3.14) \quad a'_{W,V} = - \sum_{W \leq T < V} a'_{W,T} \cdot a_{T,V}$$

for $W < V$. Then equation (3.10) becomes

$$\begin{aligned}
 [M] &= [IC'_Y{}^H] \cdot i_s^*[M] + \sum_{W < S} [IC'_W{}^H] \cdot L(W) \\
 (3.15) \quad &= [IC'_Y{}^H] \cdot i_s^*[M] + \sum_{W \leq V < S} [IC'_W{}^H] \cdot a'_{W,V} \cdot L'(V).
 \end{aligned}$$

The result follows by the inductive identification (for $V < S$ fixed)

$$\sum_{W \leq V} [IC'_W{}^H] \cdot a'_{W,V} = [IC'_V{}^H] - \sum_{W \leq T < V} [IC'_W{}^H] \cdot a'_{W,T} \cdot a_{T,V} = \widehat{IC}^H(\bar{V}).$$

□

Before stating immediate consequences of the above theorem, let us describe some cases when the technical hypothesis $[M] \in \langle [IC'_V{}^H] \rangle$ needed in the proof is satisfied for a fixed $M \in D^b(\text{MHM}(Y))$.

Example 3.3.

(1) Assume that all sheaf complexes IC'_V , $V \in \mathcal{V}$, not only are \mathcal{V} -constructible but satisfy the stronger property that they are “cohomologically \mathcal{V} -constant,” i.e., all cohomology sheaves $\mathcal{H}^j(IC'_V)|_W$ ($j \in \mathbb{Z}$) are constant for all $V, W \in \mathcal{V}$. Moreover, assume that either

- (1) $\text{rat}(M)$ is also cohomologically \mathcal{V} -constant, or
- (2) all perverse cohomology sheaves $\text{rat}(H^j(M))$ ($j \in \mathbb{Z}$) are cohomologically \mathcal{V} -constant; e.g., each $H^j(M)$ is a pure Hodge module with $\mathcal{H}^{-\dim(V)}(\text{rat}(H^j(M))|_V)$ constant for all $V \in \mathcal{V}$.

Then $[M] \in \langle [IC'_V{}^H] \rangle$. In particular, if all strata $V \in \mathcal{V}$ are simply connected, then

$$K_0(\text{MHM}(Y)) = \langle [IC'_V{}^H] \rangle,$$

so in this case we have that $[M] \in \langle [IC'_V{}^H] \rangle$ for all $M \in D^b \text{MHM}(X)$ with $\text{rat}(M)$ \mathcal{V} -constructible.

(2) *Toric varieties.* Another interesting example comes from a toric variety Y with its natural Whitney stratification \mathcal{V} by orbits; i.e., each stratum is of the form $V \simeq (\mathbb{C}^*)^{\dim(V)}$ (for details on toric varieties, the reader is advised to consult [19]). In this case not all the strata are simply connected, but nevertheless all intersection complexes IC'_V , $V \in \mathcal{V}$, are cohomologically \mathcal{V} -constant; e.g., see [6, lemma 15.15]. It follows that for any $M \in D^b \text{MHM}(Y)$ satisfying (a) or (b) above, we have $[M] \in \langle [IC'_V{}^H] \rangle$.

PROOF: It is clear that (2) follows from (1). Note that, by using the identity

$$[M] = \sum_i (-1)^i [H^i(M)] \in K_0(\text{MHM}(Y)),$$

it suffices to prove the claim (1) under the assumption (a). Moreover, by using the t -structure on $D^b \text{MHM}(Y)$ that corresponds to the usual t -structure on $D_c^b(Y)$

(and not to the perverse t -structure), cf. [29, remark 4.6(2)], we may assume that $\text{rat}(M)$ is a sheaf, and by (a) this is assumed to be cohomologically \mathcal{V} -constant (i.e., $\text{rat}(M)|_W$ underlies a constant variation of mixed Hodge structures for any $W \in \mathcal{V}$).

Under these considerations, we can now prove the claim by induction over the number of strata of the stratification. In the case when Y has only one stratum U , the variety $Y = U$ is nonsingular, connected, with $IC'_U{}^H = \mathbb{Q}_U^H$, and the claim follows trivially: indeed, under our assumptions, M is just a constant variation of mixed Hodge structures with stalk F , so $M \simeq \mathbb{Q}_Y^H \otimes k^*F$, for $k : Y \rightarrow pt$ the constant map; therefore $[M] = [IC'_U{}^H] \cdot [F]$. For the induction step, fix an open stratum $U \in \mathcal{V}$ with open inclusion $j : U \hookrightarrow Y$ and closed complement $i : Y' := Y \setminus U \rightarrow Y$. Then U is a smooth, connected variety as before, and Y' is an algebraic variety with a smaller number of strata, satisfying the same assumption on the intersection cohomology sheaves of the strata. The distinguished triangle (cf. [29, (4.4.1)])

$$j_!j^*M \rightarrow M \rightarrow i_*i^*M \rightarrow$$

implies that

$$[M] = [j_!j^*M] + [i_*i^*M] \in K_0(D^b \text{MHM}(Y)).$$

We can now apply the induction hypothesis to $[i^*M]$. Moreover, by assumption, $\text{rat}(j^*M)$ is a constant sheaf with stalk the (graded polarizable) mixed Hodge structure F , so that

$$j_![j^*M] = j_![j^*(IC'_U{}^H \otimes k^*F)] = [IC'_U{}^H] \cdot [F] - i_*[i^*(IC'_U{}^H \otimes k^*F)],$$

with $k : Y \rightarrow pt$ the constant map. But

$$i^*(IC'_U{}^H \otimes k^*F) \simeq i^*(IC'_U{}^H) \otimes i^*k^*F$$

also has a cohomologically \mathcal{V} -constant underlying complex with respect to the induced stratification on Y' . So our claim follows by induction. \square

In the following, we specialize to the relative context of a proper algebraic map $f : X \rightarrow Y$ of complex algebraic varieties, with Y irreducible, and indicate a proof of Theorem 2.5 and of Theorem 2.9. For a given $M \in D^b \text{MHM}(X)$, assume that $Rf_* \text{rat}(M)$ is constructible with respect to the given complex algebraic Whitney stratification \mathcal{V} of Y , with open dense stratum S . By proper base change, we get

$$i_v^*f_*[M] = [H^*(\{f = v\}, \text{rat}(M))] \in K_0(\text{MHM}(pt)).$$

So under the assumption $f_*[M] \in \langle [IC'_{\bar{V}}{}^H] \rangle$, Theorem 3.2 yields the following identity in $K_0(\text{MHM}(Y))$:

COROLLARY 3.4

$$\begin{aligned} f_*[M] &= [IC'_Y{}^H] \cdot [H^*(F; \text{rat}(M))] \\ &+ \sum_{V < S} \widehat{IC}^H(\bar{V}) \cdot ([H^*(F_V; \text{rat}(M))] - [H^*(F; \text{rat}(M))] \cdot [IH^*(c^\circ L_{V,Y})]), \end{aligned}$$

where F is the (generic) fiber over the top-dimensional stratum S , and F_V is the fiber over a stratum $V \in \mathcal{V} \setminus \{S\}$.

Note that the corresponding classes $[H^*(F; \text{rat}(M))]$ and $[H^*(F_V; \text{rat}(M))]$ may depend on the choice of fibers of f , but the above formula holds for any such choice. If all strata $V \in \mathcal{V}$ are simply connected, then these classes are independent of the choices made. By pushing the identity in Corollary 3.4 down to a point via k'_* , for $k' : Y \rightarrow pt$ the constant map, and using the fact that k'_* is $K_0(\text{MHM}(pt))$ -linear, an application of the χ_y -genus (which is a ring homomorphism) yields the following:

PROPOSITION 3.5 *Under the above notation and assumptions, the following identity holds in $\mathbb{Z}[y, y^{-1}]$:*

$$\begin{aligned} \chi_y([H^*(X; \text{rat}(M))]) &= I_\chi(Y) \cdot \chi_y([H^*(F; \text{rat}(M))]) \\ &+ \sum_{V < S} \widehat{I}_{\chi_y}(\bar{V}) \cdot (\chi_y([H^*(F_V; \text{rat}(M))]) - \chi_y([H^*(F; \text{rat}(M))]) \cdot I_{\chi_y}(c^\circ L_{V,Y})). \end{aligned}$$

Remark 3.6. Note that Theorem 2.5 follows from Proposition 3.5 above if we take $M = \mathbb{Q}_X^H$. Similarly, for X pure dimensional, Theorem 2.9 follows if we let $M = IC_X^{c^H}$ (together with the stalk identifications of Lemma 2.10).

Remark 3.7. Working with k'_* in place of k'_* , a similar argument yields the corresponding results for $\chi_y^c(\cdot)$ and $I_{\chi_y^c}(\cdot)$, respectively. Moreover, an application of the E -polynomials yields similar formulae for $IE(\cdot)$ and $IE_c(\cdot)$.

4 Characteristic Classes

Here we construct a natural characteristic class transformation, MHT_y , which for an algebraic variety X yields a twisted homology class $IT_y(X)$, whose associated genus for X compact is $I_{\chi_y}(X)$. The main result of this section is a formula for the proper pushforward of such a class and is a direct consequence of Theorem 3.2 and Corollary 3.4. The construction of MHT_y follows closely ideas of a recent paper of Brasselet, Schürmann, and Yokura ([9, remarks 5.3, 5.4], but see also Totaro’s paper [37, sec. 7]), and is based on Saito’s theory of mixed Hodge modules (cf. Section 3.1).

4.1 Construction of the Transformation MHT_y

For any $p \in \mathbb{Z}$, Saito constructed a functor of triangulated categories

$$gr_p^F DR : D^b \text{MHM}(X) \rightarrow D_{\text{coh}}^b(X)$$

commuting with proper pushdown. Here $D_{\text{coh}}^b(X)$ is the bounded derived category of sheaves of \mathcal{O}_X -modules with coherent cohomology sheaves. Moreover, $gr_p^F DR(M) = 0$ for almost all p and $M \in D^b \text{MHM}(X)$ fixed. If we let $\mathbb{Q}_X^H \in D^b \text{MHM}(X)$ be the constant Hodge module on X , and if X is smooth and pure dimensional, then $gr_{-p}^F DR(\mathbb{Q}_X^H) \simeq \Omega_X^p[-p]$.

The transformations $gr_p^F DR(M)$ are functors of triangulated categories, so they induce functors on the level of Grothendieck groups. Therefore, if $K_0(D_{\text{coh}}^b(X)) \simeq G_0(X)$ denotes the Grothendieck group of coherent sheaves on X , we obtain the following group homomorphism commuting with proper pushdown:

$$(4.1) \quad \begin{aligned} MHC_* : K_0(\text{MHM}(X)) &\rightarrow G_0(X) \otimes \mathbb{Z}[y, y^{-1}], \\ [M] &\mapsto \sum_p \left(\sum_i (-1)^i \mathcal{H}^i(gr_{-p}^F DR(M)) \right) \cdot (-y)^p. \end{aligned}$$

Recall that G_0 is a covariant functor with respect to the proper pushdown f_* , defined as follows: if $f : X \rightarrow Y$ is an algebraic map, then $f_* : G_0(X) \rightarrow G_0(Y)$ is given by $f_*([\mathcal{F}]) := \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}]$ for $R^i f_* \mathcal{F}$, the higher direct image sheaf of \mathcal{F} .

Note also that by the work of Yokura (cf. [9] and references therein) one can define the following generalization of the Baum-Fulton-MacPherson transformation for the Todd class:

$$(4.2) \quad \begin{aligned} td_{(1+y)} : G_0(X) \otimes \mathbb{Z}[y, y^{-1}] &\rightarrow H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}], \\ [\mathcal{F}] &\mapsto \sum_{k \geq 0} td_k([\mathcal{F}]) \cdot (1+y)^{-k}, \end{aligned}$$

with td_k the degree- k component of the Todd class transformation td_* of Baum-Fulton-MacPherson [4], which is linearly extended over $\mathbb{Z}[y, y^{-1}]$. Since td_* is degree preserving, this new transformation also commutes with proper pushdown (which is defined by linear extension over $\mathbb{Z}[y, y^{-1}]$).

We can now make the following definition (cf. [9, remark 5.3]):

DEFINITION 4.1 The transformation MHT_y is defined by the composition

$$\begin{aligned} MHT_y : K_0(\text{MHM}(X)) &\rightarrow H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}], \\ MHT_y &:= td_{(1+y)} \circ MHC_*. \end{aligned}$$

By the above discussion, MHT_y commutes with proper pushforward.

Example 4.2. Let $\mathbb{V} = ((V_{\mathbb{C}}, F), V_{\mathbb{Q}}, K) \in \text{MHM}(pt) = mHs^p$. Then:

$$(4.3) \quad \begin{aligned} MHT_y([\mathbb{V}]) &= \sum_p td_0([gr_F^p V_{\mathbb{C}}]) \cdot (-y)^p \\ &= \sum_p \dim_{\mathbb{C}}(gr_F^p V_{\mathbb{C}}) \cdot (-y)^p \\ &= \chi_y([\mathbb{V}]). \end{aligned}$$

Also, since over a point the twisted Todd transformation

$$\begin{aligned} td_{(1+y)} : K_0(pt) &= \mathbb{Z}[y, y^{-1}] \\ &\rightarrow \mathbb{Q}[y, y^{-1}, (1+y)^{-1}] = H_{2*}^{\text{BM}}(pt; \mathbb{Q})[y, y^{-1}, (1+y)^{-1}] \end{aligned}$$

is just the identity transformation, we get $MHC_* = MHT_y = \chi_y$ on $K_0(mHs^p)$.

DEFINITION 4.3 For a pure n -dimensional complex algebraic variety X , we define

$$(4.4) \quad IT_y(X) := MHT_y(IC_X'^H), \quad IC_*(X) := MHC_*(IC_X'^H).$$

Remark 4.4 (Normalization). If X is smooth and pure-dimensional, then $IC_X'^H \simeq \mathbb{Q}_X^H$ in $D^b \text{MHM}(X)$, and

$$MHC_*(IC_X'^H) = \sum_p [\Omega_X^p] \cdot y^p =: \lambda_y(T_X^*)$$

is the total λ -class of the cotangent bundle of X . Therefore, by [9, lemma 3.1],

$$IT_y(X) = td_{(1+y)} \left(\sum_p [\Omega_X^p] \cdot y^p \right) = T_y^*(TX) \cap [X] =: T_y(X),$$

where $T_y^*(TX)$ is the modified Todd class that appears in the generalized Hirzebruch-Riemann-Roch theorem, i.e., the cohomology class associated to the normalized power series defined by

$$Q_y(\alpha) := \frac{\alpha(1+y)}{1 - e^{-\alpha(1+y)}} - \alpha y.$$

If X is compact, the genus associated to $T_y^*(TX)$, that is, the degree of the zero-dimensional part of $T_y(X)$, is exactly the Hirzebruch χ_y -genus of Definition 2.1.

Remark 4.5. It is conjectured in [9] that for a complex projective variety X , the homology class $IT_1(X)$ is exactly the L -class $L_*(X)$ of Goresky-MacPherson. At least the equality of their degree follows from Saito's work. In general, for any compact, complex algebraic variety X , one has that the degree of $IT_y(X)$ is exactly $I\chi_y(X)$, i.e.,

$$I\chi_y(X) = \int_X IT_y(X)$$

(e.g., see Corollary 4.8 below).

Remark 4.6. For a possibly singular algebraic variety X , the twisted homology class $MHT_y(\mathbb{Q}_X^H)$ is the motivic Hirzebruch class $T_y(X)$ constructed in [9, 34], which for a complete variety X has as associated genus the generalized Hirzebruch χ_y -characteristic from Definition 2.2 (cf. [9, sec. 5]). In particular, if X is a rational homology manifold, then $\mathbb{Q}_X^H \cong IC_X'^H$ in $D^b \text{MHM}(X)$ and $IT_y(X) = T_y(X)$. Similarly, for any complex algebraic variety X , $MHC_*(\mathbb{Q}_X^H)$ is the motivic Chern class from [9].

4.2 Formula for Proper Pushforward

We begin this section with the following simple observation:² If $f : X \rightarrow Y$ is a proper algebraic map between irreducible n -dimensional complex algebraic varieties so that f is *homologically small of degree 1* in the sense of [21, sec. 6.2], then

$$f_*IT_y(X) = IT_y(Y).$$

Indeed, for such a map we have that $f_*IC_X \simeq {}^p\mathcal{H}^0(f_*IC_X) \simeq IC_Y$ in $D_c^b(Y)$ [21, theorem 6.2]. Moreover, since $\text{rat} : \text{MHM}(Y) \rightarrow \text{Perv}(\mathbb{Q}_Y)$ is a faithful functor, this isomorphism can be lifted to the level of mixed Hodge modules (cf. [27, theorem 1.12]). Then, since MHT_y commutes with proper pushdown and $[IC_X'^H] = (-1)^n[IC_X^H]$ in $K_0(\text{MHM}(X))$, we obtain

$$\begin{aligned} f_*IT_y(X) &= f_*MHT_y([IC_X'^H]) \\ &= (-1)^n MHT_y(f_*[IC_X^H]) \\ &= (-1)^n MHT_y([IC_Y^H]) \\ &= MHT_y([IC_Y'^H]) \\ &= IT_y(Y). \end{aligned}$$

In particular, if $f : X \rightarrow Y$ is a *small resolution*, that is, a resolution of singularities that is small in the sense of [21], then

$$(4.5) \quad IT_y(Y) = f_*T_y(X).$$

One might wish to take formula (4.5) as a definition of the class $IT_y(Y)$. Unfortunately, small resolutions do not always exist.

The main result of this section is the following:

THEOREM 4.7 *Let $f : X \rightarrow Y$ be a proper morphism of complex algebraic varieties, with Y irreducible. Let \mathcal{V} be the set of components of strata of Y in a stratification of f , with S the top-dimensional stratum (which is Zariski-open and dense in Y), and assume $\pi_1(V) = 0$ for all $V \in \mathcal{V}$. For each $V \in \mathcal{V} \setminus \{S\}$, define inductively*

$$\widehat{IT}_y(\bar{V}) := IT_y(\bar{V}) - \sum_{W < V} \widehat{IT}_y(\bar{W}) \cdot I\chi_y(c^\circ L_{W,V}),$$

where $c^\circ L_{W,V}$ denotes the open cone on the link of W in \bar{V} , and all homology characteristic classes are regarded in the Borel-Moore homology of the ambient variety Y (with coefficients in $\mathbb{Q}[y, y^{-1}, (1+y)^{-1}]$). Then:

$$(4.6) \quad \begin{aligned} f_*T_y(X) &= IT_y(Y) \cdot \chi_y(F) \\ &+ \sum_{V < S} \widehat{IT}_y(\bar{V}) \cdot (\chi_y(F_V) - \chi_y(F) \cdot I\chi_y(c^\circ L_{V,Y})), \end{aligned}$$

²Finding numerical invariants of complex varieties—more precisely, Chern numbers that are invariant under small resolutions—was Totaro’s guiding principle in his paper [37].

where F is the generic fiber of f , and F_V denotes the fiber over a stratum $V \in \mathcal{V} \setminus \{S\}$.

If, moreover, X is pure dimensional, then:

$$(4.7) \quad \begin{aligned} f_*IT_y(X) &= IT_y(Y) \cdot I\chi_y(F) \\ &+ \sum_{V \in \mathcal{S}} \widehat{IT}_y(\bar{V}) \cdot (I\chi_y(f^{-1}(c^\circ L_{V,Y})) - I\chi_y(F) \cdot I\chi_y(c^\circ L_{V,Y})). \end{aligned}$$

PROOF: Equations (4.6) and (4.7) follow directly by applying the transformation MHT_y to the identity of Corollary 3.4 for $M = \mathbb{Q}_X^H$ and for $M = IC_X^H$, respectively, and by using the fact that MHT_y commutes with the exterior product

$$K_0(\text{MHM}(Y)) \times K_0(\text{MHM}(pt)) \rightarrow K_0(\text{MHM}(Y \times \{pt\})) \simeq K_0(\text{MHM}(Y)).$$

More precisely, MHT_y commutes with the first exterior product and with the last isomorphism induced by the proper pushdown p_* for the isomorphism

$$p : Y \times \{pt\} \xrightarrow{\simeq} Y.$$

If i is the inverse to p and $k : Y \rightarrow pt$ is the constant map, then for $[M] \in K_0(\text{MHM}(Y))$ and $[M'] \in K_0(\text{MHM}(pt))$ we get

$$[M] \cdot [M'] = [M \otimes k^*M'] = [i^*(M \boxtimes M')] = [p_*(M \boxtimes M')].$$

Thus

$$\begin{aligned} MHT_y([M] \cdot [M']) &= p_*(MHT_y([M]) \boxtimes MHT_y([M'])) \\ &= MHT_y([M]) \cdot \chi_y([M']). \end{aligned}$$

□

COROLLARY 4.8 For any compact, pure-dimensional complex algebraic variety X , the degree of $IT_y(X)$ is the intersection homology genus $I\chi_y(X)$, i.e.,

$$I\chi_y(X) = \int_X IT_y(X).$$

PROOF: Apply Theorem 4.7 to the constant map $f : X \rightarrow pt$, which is proper since X is compact. □

Remark 4.9. Similar formulae can be obtained by applying the transformation MHC_* to the identity in Corollary 3.4, and even for a general mixed Hodge module M so that $f_*[M] \in \langle IC_{\bar{V}}^H \rangle$, e.g., if all strata $V \in \mathcal{V}$ are simply connected.

Example 4.10 (Smooth Blowup). Let Y be a smooth n -dimensional variety and $Z \xrightarrow{i} Y$ a submanifold of pure codimension $r + 1$. Let X be the blowup of Y along Z , and $f : X \rightarrow Y$ be the blowup map. Then $\mathcal{V} := \{Y \setminus Z, Z\}$ is a Whitney

stratification of Y with IC'_Y , IC'_Z , and $Rf_*(\mathbb{Q}_X)$ all cohomologically \mathcal{V} -constant. Then, as in Example 2.7, we have that (compare with [9, example 3.3(3)]):

$$(4.8) \quad f_*T_y(X) = T_y(Y) + T_y(Z) \cdot (-y + \cdots + (-y)^r).$$

In particular, for $y = 0$, this yields the well-known formula³:

$$f_*td_*(X) = td_*(Y).$$

As a special case, we also obtain the following generalization of some well-known facts concerning multiplicative properties of characteristic classes (e.g., see [22, sec. 23.6] for a discussion on Todd classes or [12] for a more general formula for L -classes):

COROLLARY 4.11 *Let $f : X \rightarrow Y$ be a proper algebraic map of complex algebraic varieties, with Y smooth and connected, so that all direct image sheaves $R^j f_*\mathbb{Q}_X$, or $R^j f_*IC'_X$ for X also pure dimensional, are locally constant (e.g., f is a locally trivial topological fibration). Let F be the general fiber of f , and assume that $\pi_1(Y)$ acts trivially on the (intersection-) cohomology of F (e.g., $\pi_1(Y) = 0$); i.e., all these $R^j f_*\mathbb{Q}_X$ or $R^j f_*IC'_X$ are constant. Then:*

$$(4.9) \quad f_*T_y(X) = \chi_y(F) \cdot T_y(Y), \quad f_*IT_y(X) = I\chi_y(F) \cdot T_y(Y).$$

We end by pointing out that the assumption of trivial monodromy is closely related but different from the situation of “algebraic piecewise trivial” maps coming up in the motivic context (e.g., see [9]). For example, the first formula in Corollary 4.11 is true for a Zariski locally trivial fibration of possibly singular complex algebraic varieties (see [9, example 3.3]).

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³This fact can be seen directly as follows: If $f : X \rightarrow Y$ is a blowup map along a smooth center, then $f_*([\mathcal{O}_X]) = [\mathcal{O}_Y]$ for $f_* : G_0(X) \rightarrow G_0(Y)$, the proper pushforward on the Grothendieck groups of coherent sheaves (e.g., see [7, lemma 2.2]). The claim follows since $td_*(X) := Td([\mathcal{O}_X])$, where $Td : G_0(X) \rightarrow H_{2*}^{BM}(X; \mathbb{Q})$ is the Baum-Fulton-MacPherson Todd transformation [4], and the latter commutes with proper pushforward.

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