Characteristic classes of Hilbert schemes of points via symmetric products

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Symmetric Products

- X – quasi-projective complex variety, $\dim_{\mathbb{C}} X = d$.
- $S^n X := X^{\times n}/S_n$ – n^{th} symmetric product of X.
- $S^n X$ parametrizes effective 0-cycles on X:

$$S^n X = \left\{ \sum_{i=1}^{\ell} n_i [x_i] \mid x_i \in X, n_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{\ell} n_i = n \right\}$$

- $S^n X$ has a stratification with strata in 1-1 correspondence with partitions of n: to a partition $\nu = (n_1, \cdots, n_\ell)$ associate:

$$S^n_\nu X := \left\{ \sum_{i=1}^{\ell} n_i [x_i] \mid x_i \neq x_j, \text{if } i \neq j \right\} \cong \left((\prod_{i=1}^{n} X^{k_i}) \setminus \Delta \right) / \prod_{i=1}^{n} S_{k_i}$$

where k_i is the number of occurrences of i among the n_j's and Δ is the large diagonal in $X^{\sum k_i}$.

Hilbert schemes of points on a quasi-projective manifold

- X – smooth quasi-projective variety, $\dim_{\mathbb{C}} X = d$.
- $Hilb_X^n$ = Hilbert scheme of 0-dim subschemes of length n on X
- $Hilb_{X,x}^n$ = punctual Hilbert scheme of length n at x.
- Hilbert-Chow morphism:

$$\pi_n : Hilb_X^n \to S^n X \ , \ Z \mapsto \sum_{x \in Z} \mathrm{length}(Z_x) \cdot [x].$$

- $\pi_n(Hilb_{X,x}^n) = n[x]$.
- if $d = 1$: $Hilb_X^n \cong S^n X$ is smooth.
- if $d = 2$: $Hilb_X^n$ is smooth and π_n is a crepant resolution.
- if $d \geq 3$: $Hilb_X^n$ is singular for $n \geq 4$ (little is known about its topology).

Computing invariants of Hilbert schemes

- Cheah: computed generating series for the Hodge-Deligne polynomials $e(Hilb^n_X)$ in terms of $e(X)$ and $e(Hilb^i_{\mathbb{C}^d,0})$.
- Gusein-Zade, Luengo, Melle-Hernandez: defined a power structure on $K_0(var/\mathbb{C})$ and proved:

$$1+\sum_{n\geq 1}[Hilb^n_X]\cdot t^n = \left(1+\sum_{i\geq 1}\left[\mathrm{Hilb}^i_{\mathbb{C}^d,0}\right]\cdot t^i\right)^{[X]} \in K_0(var/\mathbb{C})[[t]]$$

- Cheah's formula is obtained from this motivic identity by an application of the pre-lambda ring homomorphism:

$$e(-;u,v): K_0(var/\mathbb{C}) \to \mathbb{Z}[u,v];$$

$$e(X) := e([X];u,v) := \sum_{p,q}\left(\sum_i (-1)^i h^{p,q}(H^i_c(X;\mathbb{C}))\right)\cdot u^p v^q$$

Power structure on $K_0(var/\mathbb{C})$

- Power structure on a ring R is a map

$$(1 + tR[[t]]) \times R \to 1 + tR[[t]], \quad (A(t), m) \mapsto (A(t))^m$$

with the usual properties of powers.

- Gusein-Zade, Luengo, Melle-Hernandez: For $R = K_0(var/\mathbb{C})$:

$$\left(1 + \sum_{i=1}^{\infty}[A_i]t^i\right)^{[X]}$$

$$:= 1 + \sum_{n=1}^{\infty}\left\{\sum_{\underline{k}\ :\sum ik_i = n}\left((\prod_i X^{k_i}) \setminus \Delta\right) \times \prod_i A_i^{k_i} / \prod_i S_{k_i}\right\} \cdot t^n$$

Example (Kapranov zeta function)

$$(1-t)^{-[X]} := (1+t+t^2+\cdots)^{[X]} = 1 + \sum_{n=1}^{\infty} [S^n X] \cdot t^n$$

Example

$\mathrm{Conf}^n X := (X^n \setminus \Delta)/\Sigma_n =$ *configuration space of n unlabeled pts*

$$(1+t)^{[X]} = 1 + \sum_{n=1}^{\infty} [\mathrm{Conf}^n X] \cdot t^n$$

Theorem (Gusein-Zade, Luengo, Melle-Hernandez)

$$\left(1 + \sum_{i \geq 1} \left[\mathrm{Hilb}^i_{\mathbb{C}^d,0}\right] \cdot t^i\right)^{[X]} = 1 + \sum_{n \geq 1} [Hilb^n_X] \cdot t^n$$

Pre-lambda structure, power structure, Adams operation

Definition

- Pre-lambda structure on a comm. ring R is a group morphism

$$\lambda_t : (R,+) \to (1+tR[[t]],\cdot)$$

so that $\lambda_t(m) = 1 + mt \pmod{t^2}$.

- A pre-lambda ring homomorphism is a ring homomorphism commuting with the pre-lambda structures.

Remark

A pre-lambda structure is related to the Adams operations by:

$$(1-t)^{-(\cdot)} := \lambda_t(\cdot) = \exp\left(\sum_{r=1}^{\infty} \Psi_r(\cdot)\frac{t^r}{r}\right)$$

- A pre-lambda structure $\lambda_t(\cdot) =: (1-t)^{-(\cdot)}$ on a ring R determines *algebraically* a power structure $(A(t))^m$ on R via the Euler product decomposition (with exponents $b_k \in R$):

$$A(t) = 1 + \sum_{i=1}^{\infty} a_i t^i = \prod_{k=1}^{\infty} (1-t^k)^{-b_k} = \prod_{k=1}^{\infty} \left((1-t)^{-b_k}|_{t \mapsto t^k} \right)$$

- Define:

$$(A(t))^m = \prod_{k=1}^{\infty} (1-t^k)^{-m \cdot b_k}$$

- A pre-lambda ring homomorphism $\phi : R_1 \to R_2$ respects the corresponding power structures, i.e.,

$$\phi \left(A(t)^m \right) = \left(\phi(A(t)) \right)^{\phi(m)}$$

Example

- Pre-lambda structure on $K_0(var/\mathbb{C})$ is given by the Kapranov zeta function:

$$\lambda_t(X) := 1 + \sum_{n \geq 1} [S^n X] \cdot t^n = (1-t)^{-[X]}$$

- Pre-lambda structure on $\mathbb{Z}[u_1, \cdots, u_r]$ $(r \geq 1)$:

$$\lambda_t\Big(\sum_{\vec{k} \in \mathbb{Z}^r_{\geq 0}} a_{\vec{k}} \cdot \vec{u}^{\,\vec{k}} \Big) := \prod_{\vec{k} \in \mathbb{Z}^r_{\geq 0}} \Big(1 - \vec{u}^{\,\vec{k}} \cdot t\Big)^{-a_{\vec{k}}}$$

- The Hodge-Deligne homomorphism

$$e(-; u, v) : K_0(var/\mathbb{C}) \to \mathbb{Z}[u, v];$$

is a pre-lambda ring homomorphism.

Relative motivic Grothendieck group $K_0(var/X)$

- $K_0(var/X)$ = free abelian group of isomorphism classes $[Y \to X]$ modulo the *scissor relation*:

$$[Y \to X] = [Z \hookrightarrow Y \to X] + [Y \setminus Z \hookrightarrow Y \to X]$$

for $Z \subset Y$ a closed algebraic subvariety of Y.

- if $Z = Y_{red}$, get $[Y \to X] = [Y_{red} \to X]$.
- if $X = point$, get: $K_0(var/pt) = K_0(var/\mathbb{C})$.
- functorial push-forward: for $f : X' \to X$ get:

$$f_* : K_0(var/X') \to K_0(var/X) \ , \ [Z \stackrel{h}{\to} X'] \mapsto [Z \stackrel{f \circ h}{\to} X].$$

- exterior product:

$$K_0(var/X) \boxtimes K_0(var/X') \to K_0(var/X \times X')$$

$$[Z \to X] \times [Z' \to X'] = [Z \times Z' \to X \times X'].$$

(Motivic) Pontrjagin ring

- $F(X) = K_0(var/X)$ or $F(X) = H_*(X) := H^{BM}_{even}(X) \otimes \mathbb{Q}[y]$.
- Pontrjagin ring $(PF(X), \odot)$ is defined as:

$$PF(X) := \sum_{n=0}^{\infty} F(S^n X) \cdot t^n,$$

with product $\odot$ induced via

$$\odot : F(S^n X) \times F(S^m X) \stackrel{\boxtimes}{\to} F(S^n X \times S^m X) \stackrel{(-)_*}{\to} F(S^{n+m} X).$$

- A (proper) morphism $f : X \to Y$ induces a ring homomorphism

$$f_* = \{(S^n f)_*\}_n : PF(X) \to PF(Y)$$

Motivic exponentiation (Cappell-M.-Ohmoto-Schürmann-Yokura)

- For X *fixed* and $A(t) = 1 + \sum_i [A_i] t^i \in K_0(var/\mathbb{C})[[t]]$, define:

$$(A(t))^X \in PK_0(var/X) := \sum_{n \geq 0} K_0(var/S^n X) \cdot t^n$$

by the same formula as $(A(t))^{[X]} \in K_0(var/\mathbb{C})[[t]]$,
but keeping track of the strata of symmetric products corresponding to each partition.

- For $\nu = (n_1, \cdots, n_\ell)$ with k_i=# of i among n_j's, regard

$$\left((\prod_i X^{k_i}) \setminus \Delta \right) \times \prod_i A_i^{k_i} / \prod_i S_{k_i} \in K_0(var/S^n X)$$

via:

$$\left((\prod_i X^{k_i}) \setminus \Delta \right) \times \prod_i A_i^{k_i} / \prod_i S_{k_i} \overset{proj}{\to} (\prod_i X^{k_i}) \setminus \Delta / \prod_i S_{k_i} \cong S^n_\nu X \hookrightarrow S^n X$$

Properties of motivic exponentiation

- if $k : X \to pt$ is the constant map, then:

$$k_* \left((A(t))^X \right) = (A(t))^{[X]}$$

- if $Y \overset{i}{\hookrightarrow} X$ is a closed inclusion with $U := X \setminus Y \overset{j}{\hookrightarrow} X$, then

$$(A(t))^X = i_* (A(t))^Y \odot j_* (A(t))^U$$

- if $\pi : X' \times X \to X$ is the projection map, then

$$\pi_* \left((A(t))^{X' \times X} \right) = \left((A(t))^{[X']} \right)^X$$

- $(1 + t)^X = 1 + [id_X]t + \text{higher order terms}$

Example

$$(1-t)^{-X} := ((1-t)^{-1})^X = 1 + \sum_{n=1}^{\infty} [id_{S^n X}] \cdot t^n$$

$$(1+t)^X = 1 + \sum_{n=1}^{\infty} [\mathrm{Conf}^n X \overset{i_n}{\hookrightarrow} S^n X] \cdot t^n = 1 + \sum_{n=1}^{\infty} (i_n)_* [id_{\mathrm{Conf}^n X}] \cdot t^n$$

$$\left((1-t)^{-[X']}\right)^X = 1 + \sum_{n=1}^{\infty} [S^n(X' \times X) \to S^n X] \cdot t^n$$

Theorem (Behrend, Bryan, Szendröi)

$$\left(1 + \sum_{i \geq 1} \left[\mathrm{Hilb}^i_{\mathbb{C}^d,0}\right] \cdot t^i\right)^X = 1 + \sum_{n \geq 1} \left[Hilb^n_X \xrightarrow{\pi_n} S^n X\right] \cdot t^n$$

Motivic Hirzebruch classes

- Brasselet-Schürmann-Yokura:

$$T_{y*} : K_0(var/X) \to H_*(X) := H^{BM}_{even}(X) \otimes \mathbb{Q}[y]$$

- if $X = pt$: $T_{y*} = \chi_y := e(-; -y, 1) : K_0(var/\mathbb{C}) \to \mathbb{Z}[y]$
- Hirzebruch class of X: $T_{y*}(X) := T_{y*}([id_X])$.
- T_{y*} commutes with proper push-forward and $\boxtimes$
- T_{y*} extends to a *ring homomorphism*:

$$T_{y*} : PK_0(var/X) \to PH_*(X)$$

Main Result

Theorem (Cappell-M.-Ohmoto-Schürmann-Yokura)

For X a quasi-projective variety and $\alpha \in K_0(var/\mathbb{C})$ fixed, we have:

$$T_{(-y)_*}\left(\left((1-t)^{-\alpha}\right)^X\right) = (1 - t \cdot d_*)^{-\chi_{-y}(\alpha) \cdot T_{(-y)_*}(X)}$$

with

$$(1 - t \cdot d_*)^{-(\cdot)} := \exp\left(\sum_{r=1}^{\infty} \Psi_r d_*^r(\cdot)\frac{t^r}{r}\right) : H_*(X) \to PH_*(X),$$

and

- $d^r : X \overset{diag}{\to} X^r \overset{proj}{\to} X^{(r)}$
- Ψ_r *is the r-th homological Adams operation defined by*
  - $\cdot \frac{1}{r^k}$ *on* $H_{2k}^{BM}(-;\mathbb{Q})$
  - $y \mapsto y^r$.

Hirzebruch classes of symmetric products

Corollary (Cappell-M.-Schürmann-Shaneson-Yokura)

If X is a quasi-projective complex algebraic variety, then:

$$\begin{aligned}\sum_{n \geq 0} T_{(-y)_*}(S^n X) \cdot t^n &= T_{(-y)_*}((1-t)^{-X}) \\ &= (1 - t \cdot d_*)^{-T_{(-y)_*}(X)} \\ &= \exp\left(\sum_{r=1}^{\infty} \Psi_r d_*^r \left(T_{(-y)_*}(X) \right) \frac{t^r}{r} \right).\end{aligned}$$

Recall:

Theorem (Behrend, Bryan, Szendröi)

$$1 + \sum_{n \geq 1} \left[Hilb_X^n \xrightarrow{\pi_n} S^n X \right] \cdot t^n = \left(1 + \sum_{i \geq 1} \left[\mathrm{Hilb}^i_{\mathbb{C}^d, 0} \right] \cdot t^i \right)^X$$

Hirzebruch classes of Hilbert schemes

Theorem (Cappell-M.-Ohmoto-Schürmann-Yokura)

Let X be a smooth quasi-projective variety of dimension d.

$$\sum_{n=0}^{\infty} \pi_{n*} T_{(-y)_*}(Hilb_X^n) \cdot t^n = \left(1 + \sum_{n=1}^{\infty} \chi_{-y}(\mathrm{Hilb}^n_{\mathbb{C}^d,0}) \cdot t^n \cdot d_*^n\right)^{T_{(-y)_*}(X)}$$

$$:= \prod_{k=1}^{\infty} (1 - t^k \cdot d_*^k)^{-\chi_{-y}(\alpha_k) \cdot T_{(-y)_*}(X)},$$

for

$$(1 - t^k \cdot d_*^k)^{-(\cdot)} = \exp\left(\sum_{r=1}^{\infty} \Psi_r d_*^{rk}(\cdot) \frac{t^{rk}}{r}\right)$$

and $\alpha_k \in K_0(var/\mathbb{C})$ given by the Euler product decomposition:

$$1 + \sum_{n \geq 1} \left[\mathrm{Hilb}^n_{\mathbb{C}^d,0}\right] \cdot t^n = \prod_{k=1}^{\infty} (1 - t^k)^{-\alpha_k}.$$

MacPherson-Chern classes of Hilbert schemes

Theorem (CMOSY)

$$\sum_{n=0}^{\infty} \pi_{n*} c_*(Hilb^n_X) \cdot t^n = \left(1 + \sum_{n=1}^{\infty} \chi(\mathrm{Hilb}^n_{\mathbb{C}^d,0}) \cdot t^n \cdot d_*^n\right)^{c_*(X)}$$

$$:= \prod_{k=1}^{\infty} (1 - t^k \cdot d_*^k)^{-\chi(\alpha_k)\cdot c_*(X)}$$

$$\in \sum_{n=0}^{\infty} H^{BM}_{even}(X^{(n)}; \mathbb{Q}) \cdot t^n.$$

with

$$(1 - t \cdot d_*)^{-(\cdot)} := \exp\left(\sum_{r=1}^{\infty} d_*^r(\cdot)\frac{t^r}{r}\right)$$

Example

- If X is a smooth surface, then $\alpha_k = \mathbb{L}^{k-1}$, so:

$$\sum_{n=0}^{\infty} \pi_{n*} c_*(Hilb_X^n) \cdot t^n = \sum_n c_*^{orb}(S^n X) \cdot t^n = \prod_{k=1}^{\infty} (1 - t^k \cdot d_*^k)^{-c_*(X)}$$

- If X is a smooth 3-fold, then $\chi(\alpha_k) = k$ and:

$$\sum_{n=0}^{\infty} \pi_{n*} c_*(Hilb_X^n) \cdot t^n = \prod_{k=1}^{\infty} (1 - t^k \cdot d_*^k)^{-k \cdot c_*(X)}$$

(Relative) virtual motives

- $f : M \to \mathbb{C}$ be a regular function on a smooth q-proj variety.
- $Z = \{df = 0\} \subset M$.
- (relative) virtual motive of Z:

$$[Z]_{\mathrm{relvir}} = -\mathbb{L}^{-\frac{\dim M}{2}}[\varphi_f]_Z \in K_0(var/Z)[\mathbb{L}^{-1/2}],$$

$$[Z]_{\mathrm{vir}} = -\mathbb{L}^{-\frac{\dim M}{2}}[\varphi_f] \in K_0(var/\mathbb{C})[\mathbb{L}^{-1/2}]$$

with φ_f the motivic vanishing cycles of Denef-Loeser.

- this applies to $Hilb^n_{\mathbb{C}^3}$, so get $[Hilb^n_{\mathbb{C}^3}]_{\mathrm{relvir}}$, $[Hilb^n_{\mathbb{C}^3}]_{\mathrm{vir}}$ and $[\mathrm{Hilb}^n_{\mathbb{C}^3,0}]_{\mathrm{vir}}$
- For X a smooth q-proj 3-fold, Behrend-Bryan-Szendröi define

$$[Hilb^n_X]_{\mathrm{vir}} \in K_0(var/\mathbb{C})[\mathbb{L}^{-1/2}]$$

Theorem (Behrend-Bryan-Szendröi)

Let X be a smooth quasi-projective 3-fold. Then:

$$1+\sum_{n\geq 1}[Hilb_X^n]_{\mathrm{vir}}\cdot t^n = \left(1+\sum_{n\geq 1}\left[\mathrm{Hilb}^n_{\mathbb{C}^3,0}\right]_{\mathrm{vir}}\cdot t^n\right)^{[X]}$$

Remark

Relative virtual motives of $Hilb^n_X$ are not defined, but can define

$$\pi_{n*}[Hilb^n_X]_{\mathrm{relvir}} \in K_0(var/S^nX)[\mathbb{L}^{-1/2}]$$

by the generating series:

$$1+\sum_{n\geq 1} \pi_{n*}[Hilb^n_X]_{\mathrm{relvir}} \cdot t^n := \left(1+\sum_{n\geq 1}\left[\mathrm{Hilb}^n_{\mathbb{C}^3,0}\right]_{\mathrm{vir}} \cdot t^n\right)^X$$

in $PK_0(var/X)[\mathbb{L}^{-1/2}]$.

Theorem (CMOSY)

For any smooth quasi-projective 3-fold X we have:

$$\sum_{n=0}^{\infty} T_{(-y)_*}\big(\pi_{n*}[Hilb_X^n]_{\mathrm{relvir}}\big) \cdot (-t)^n = \prod_{k=1}^{\infty}(1 - t^k \cdot d_*^k)^{-\chi_{-y}(\alpha_k)\cdot T_{(-y)_*}(X)},$$

with coefficients $\alpha_k \in K_0(var/\mathbb{C})[\mathbb{L}^{-1/2}]$ *given by*

$$\alpha_k = \frac{(-\mathbb{L}^{1/2})^{-k} - (-\mathbb{L}^{1/2})^{k}}{\mathbb{L}(1-\mathbb{L})}.$$

Theorem (CMOSY)

For any smooth quasi-projective 3-fold X we have:

$$\sum_{n=0}^{\infty} \pi_{n*}(c_*^A(Hilb_X^n)) \cdot (-t)^n = \prod_{k=1}^{\infty} (1 - t^k \cdot d_*^k)^{-k \cdot c_*(X)}.$$

with

$$c_*^A(Hilb_X^n) := c_*(\nu_{Hilb_X^n})$$

the Aluffi class (= Chern class of the Behrend function $\nu_{Hilb_X^n}$.)

Remark

- *Comparing the Chern class formulae, obtain:*

$$\pi_{n*}(c_*^A(X^{[n]})) = (-1)^n \pi_{n*}(c_*(X^{[n]}))$$

- *For X a projective CY 3-fold, taking degrees in the last result yields the degree-zero MNOP conjecture.*

THANK YOU !!!

and

Feliz cumpleaños !!!