Singularities through the lens of characteristic classes

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Lecture plan

♣ **Part I**:
Introduction to characteristic classes for singular varieties

♣ **Part 2**:
Applications of characteristic classes to Ehrhart theory

Part I.

Introduction to characteristic classes for singular varieties

Motivation. Overview

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Manifolds: genera and characteristic classes

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♣ Here we focus on $G = U$.

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- genera $\phi_f : \Omega_*^U \to R$
- normalized and multiplicative cohomology characteristic classes cl_f^* over a finite-dim. base space X,

$$cl_f^* : (K^0(X), \oplus) \to (H^*(X) \otimes R, \cup)$$

(with $H^*(X) = H^{2*}(X; \mathbb{Z})$, and $K^0(X)$ the Grothendieck group of $\mathbb{C}$-vector bundles on X), s.t.

$cl_f^*(L) = f(c^1(L))$, if L is a complex line bundle

♣ Given f a normalized power series as above, with corresponding class cl_f^*, the associated genus ϕ_f is defined by:

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♣ Every multiplicative genus on Ω_*^U is completely determined by its values on all complex projective spaces, since (Milnor):

$$\Omega_*^U \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^1, \mathbb{CP}^2, \mathbb{CP}^3, \cdots]$$

Example: Hirzebruch's χ_y-genus

Hirzebruch χ_y-genus of a compact complex manifold X:

$$\chi_y(X) := \sum_{p \geq 0} \chi(X, \Omega_X^p) y^p$$

- genus $\chi_y : \Omega_*^U \to \mathbb{Q}[y]$, with $\chi_y(\mathbb{CP}^n) = \sum_{p=0}^n (-y)^p$.

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- χ_y comes from the power series in $z = c^1$:

$$f_y(z) = \frac{z(1+y)}{1 - e^{-z(1+y)}} - zy \in \mathbb{Q}[y][[z]]$$

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- $y = 0$: arithmetic genus, Todd class, and Riemann-Roch.

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Singular spaces do not usually have tangent bundles, so cohomology characteristic classes and genera cannot be defined as in the manifold case. Instead, one works with homology characteristic classes defined via suitable natural transformations.

Functorial characteristic classes for singular varieties

♣ A functorial characteristic class theory of singular complex algebraic varieties is a covariant transformation

$$cl_* : A(-) \to H_*(-) \otimes R,$$

with $A(-)$ a covariant theory depending on cl_*, and $H_*(-) = H_*^{BM}(-;\mathbb{Z})$.

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♣ cl_* satisfies the normalization property: if X is smooth and $cl^*(T_X)$ is the corresponding cohomology class of X, then:

$$cl_*(\alpha_X) = cl^*(T_X) \cap [X] \in H_*(X) \otimes R$$

♣ A characteristic number of a *compact* singular variety X is defined by:

$$\#(X) := deg(cl_*(\alpha_X)) := const_*(cl_*(\alpha_X))$$

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♣ If X is smooth, get by normalization that

$$\#(X) = \langle cl^*(T_X), [X] \rangle,$$

so $\#(X)$ is a singular extension of the notion of characteristic numbers of manifolds.

Example (Euler characteristic)

The topological Euler characteristic

$$\chi(X) := \sum_i (-1)^i b_i(X)$$

is a characteristic number via the singular version of Gauss-Bonnet-Chern theorem:

$$\chi(X) := deg(c_*^{SM}(X)),$$

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$$c_* : CF(X) \to H_*(X)$$

the MacPherson-Chern class transformation on X, defined on the group $CF(X)$ of constructible functions on X.

Example (Arithmetic genus)

The arithmetic genus of a compact complex algebraic variety,

$$\chi_a(X) := \chi(X, \mathcal{O}_X) = \sum_i (-1)^i \dim_{\mathbb{C}} H^i(X; \mathcal{O}_X)$$

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$$td_* : K_0(X) := K_0(Coh(X)) \to H_*(X) \otimes \mathbb{Q}$$

the Baum-Fulton-MacPherson Todd class transformation.

Example (Hirzebruch polynomial)

The Hirzebruch polynomial of a compact complex algebraic variety,

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is a characteristic number of X via the singular (g-HRR):

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$$T_{y*}, \widehat{T}_{y*} : K_0(\mathrm{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}]$$

the Brasselet-Schürmann-Yokura Hirzebruch class transformations, defined on the Grothendieck group of mixed Hodge modules on X.

Hirzebruch class transformations

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♣ If X is *singular*, use suitable local embeddings into manifolds and filtered $\mathcal{D}$-modules supported on X.

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♣ $\alpha : \mathrm{DR}(\mathcal{L})^{an} \simeq L[n]$, with shifted de Rham complex

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with $\mathcal{L}$ in degree $-n$.

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♣ This data defines a mixed Hodge module $L^H[n]$ on X.

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- $f : Y \to X$ a proper morphism from a complex algebraic manifold Y,
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- L a good variation of mixed Hodge structures on U.

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(This follows by induction from resolution of singularities using the standard attaching triangles for closed/open inclusions.)

Filtered de Rham complexes

Theorem (Saito)

For any variety X, there is a functor of triangulated categories

$$Gr_p^F DR : D^b MHM(X) \longrightarrow D^b_{\mathrm{coh}}(X)$$

commuting with proper pushforward, with $Gr_p^F DR(\mathscr{M}) = 0$ for almost all p and $\mathscr{M}$ fixed.

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(a) *If X is a (pure) n-dimensional complex algebraic manifold, and $\mathscr{M} \in MHM(X)$, then $Gr_p^F DR(\mathscr{M})$ is the complex associated to the de Rham complex of the underlying algebraic left $\mathcal{D}_X$-module $\mathcal{M}$ with its integrable connection ∇:*

$$DR(\mathcal{M}) = [\mathcal{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^n_X]$$

with $\mathcal{M}$ in degree $-n$, filtered by

$$F_p DR(\mathcal{M}) = [F_p\mathcal{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} F_{p+n}\mathcal{M} \otimes_{\mathcal{O}_X} \Omega^n_X]$$

Theorem (Filtered de Rham complexes, cont'd)

(b) $\bar{X}$ – *smooth partial compactification of the algebraic manifold* X *with complement* D *a sncd, and open inclusion* $j : X \hookrightarrow \bar{X}$.

Theorem (Filtered de Rham complexes, cont'd)

(b) $\bar{X}$ – *smooth partial compactification of the algebraic manifold X with complement D a sncd, and open inclusion $j : X \hookrightarrow \bar{X}$.*

For a good variation (L, F, W) of MHS on X, $(DR(j_* L^H), F)$ is filtered quasi-isomorphic to the log de Rham complex

$$DR_{\log}(\mathcal{L}) := [\overline{\mathcal{L}} \xrightarrow{\overline{\nabla}} \cdots \xrightarrow{\overline{\nabla}} \overline{\mathcal{L}} \otimes_{\mathcal{O}_{\bar{X}}} \Omega^n_{\bar{X}}(\log(D))]$$

with increasing filtration $F_{-p} := F^p$ given by

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where $\overline{\mathcal{L}}$ is the canonical Deligne extension of $\mathcal{L} := L \otimes_{\mathbb{Q}_X} \mathcal{O}_X$.

Theorem (Filtered de Rham complexes, cont'd)

(b) $\bar{X}$ – *smooth partial compactification of the algebraic manifold* X *with complement* D *a sncd, and open inclusion* $j : X \hookrightarrow \bar{X}$.

For a good variation (L, F, W) *of MHS on* X, $(DR(j_* L^H), F)$ *is filtered quasi-isomorphic to the log de Rham complex*

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In particular, $Gr^F_{-p} DR(j_* L^H) \overset{q.i}{\simeq} Gr^p_F DR_{log}(\mathcal{L})$.

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In particular, $Gr^F_{-p} DR(j_* L^H) \overset{q.i}{\simeq} Gr^p_F DR_{log}(\mathcal{L})$.

(c) *For $(DR(j_! L^H), F)$, consider instead the log de Rham complex associated to the Deligne extension $\overline{\mathcal{L}} \otimes \mathcal{O}_{\bar{X}}(-D)$ of $\mathcal{L}$.*

Hodge–Chern classes

The transformations $Gr_p^F \mathrm{DR}$ induce group homomorphisms

$$Gr_p^F \mathrm{DR} : K_0(\mathrm{MHM}(X)) \longrightarrow K_0(X) \simeq K_0(D^b_{\mathrm{coh}}(X))$$

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Definition (Brasselet–Schürmann–Yokura)

The *Hodge–Chern class transformation* of a variety X is:

$$\mathrm{DR}_y : K_0(\mathrm{MHM}(X)) \longrightarrow K_0(X) \otimes \mathbb{Z}[y^{\pm 1}]$$

$$\mathrm{DR}_y([\mathscr{M}]) := \sum_{i,p} (-1)^i \left[\mathcal{H}^i Gr^F_{-p} \mathrm{DR}(\mathscr{M})\right] \cdot (-y)^p$$

$$= \sum_{p} \left[Gr^F_{-p} \mathrm{DR}(\mathscr{M})\right] \cdot (-y)^p$$

Hirzebruch classes of mixed Hodge modules

Definition (Brasselet–Schürmann–Yokura)

♣ The *un-normalized Hirzebruch class transformation* is:

$$T_{y*} := td_* \circ \mathrm{DR}_y : K_0(\mathrm{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}]$$

with $td_* : K_0(X) \to H_*(X) \otimes \mathbb{Q}$ the *Todd class transformation* of the *singular (G-R-R) thm* of Baum-Fulton-MacPherson, linearly extended over $\mathbb{Z}[y^{\pm 1}]$, and $H_*(X) := H^{BM}_{2*}(X)$.

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where

$$td_{(1+y)*} : K_0(X) \otimes \mathbb{Z}[y^{\pm 1}] \to H_*(X) \otimes \mathbb{Q}\left[y, \tfrac{1}{y(y+1)}\right]$$

is the scalar extension of td_* together with the multiplication by $(1+y)^{-k}$ on the degree k component.

Homology Hirzebruch characteristic classes of varieties

Definition (Brasselet-Schürmann-Yokura)

Homology Hirzebruch characteristic classes of a complex algebraic variety X are defined by evaluating at the (class of the) constant Hodge module $\mathbb{Q}^H_X$:

$$T_{y*}(X) := T_{y*}([\mathbb{Q}^H_X]), \ \widehat{T}_{y*}(X) := \widehat{T}_{y*}([\mathbb{Q}^H_X]) \in H_*(X) \otimes \mathbb{Q}[y].$$

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♣ The classes $T_{y*}(X)$ and $\widehat{T}_{y*}(X)$ are "motivic", i.e., they are images of $[id_X]$ under natural transformations (motivic lifts):

$$T_{y*}, \widehat{T}_{y*} : K_0(var/X) \to H_*(X) \otimes \mathbb{Q}[y],$$

where $K_0(var/X)$ is generated by isomorphism classes $[f : Y \to X]$ and the scissor relation.

Properties

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and for $y = -1$:

$$\widehat{T}_{-1*}([\mathscr{M}]) = c_*([\mathrm{rat}(\mathscr{M})]) \in H_*(X) \otimes \mathbb{Q}$$

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♣ If X is *Du Bois* (e.g., a toric variety) then

$$T_{0*}(X) = \widehat{T}_{0*}(X) = td_*([\mathcal{O}_X]) =: td_*(X)$$

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♣ Degree: If X is compact, and $\mathscr{M} \in D^b\mathrm{MHM}(X)$:

$$\begin{aligned} deg(T_{y*}([\mathscr{M}])) = deg(\widehat{T}_{y*}([\mathscr{M}])) &= \sum_{j,p} (-1)^j \dim \mathrm{Gr}^p_F H^j(X; \mathscr{M}) \cdot (-y)^p \\ &=: \chi_y(X; \mathscr{M}) \end{aligned}$$

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$$\left(\mathrm{DR}(j_* L^H), F_{-\bullet}\right) \simeq \left(\mathrm{DR}_{log}(\mathcal{L}), F^{\bullet}\right)$$

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- For $j = id : X \to X$, get the *Atiyah-Meyer type formula*:
$$\mathrm{DR}_y([L^H]) = DR^y(L) \cap (\Lambda_y(T^*_X) \cap [\mathcal{O}_X]) \in K_0(X)[y^{\pm 1}]$$

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In particular,

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Corollary

$$DR_y(\left[(i_{S,\bar{S}})_! \mathbb{Q}^H_S\right]) = (\pi_Z)_* \left[\mathcal{O}_Z(-D) \otimes \Lambda_y \Omega^1_Z(\log(D)\right] \in K_0(\bar{S})[y]$$

Application: Hirzebruch classes of *toric varieties*

Theorem (M.-Schürmann)

Let X_Σ be the toric variety defined by the fan Σ. For any cone $\sigma \in \Sigma$, with orbit O_σ and inclusion $i_\sigma : O_\sigma \hookrightarrow \overline{O}_\sigma = V_\sigma$, have:

$$DR_y([(i_\sigma)_! \mathbb{Q}^H_{O_\sigma}]) = (1+y)^{\dim(O_\sigma)} \cdot [\omega_{V_\sigma}],$$

where ω_{V_σ} is the canonical sheaf on the toric variety V_σ.

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Let $X = X_\Sigma$ be the toric variety defined by the fan Σ. For each cone $\sigma \in \Sigma$, let V_σ be the closure of the orbit $O_\sigma \subset X$, and choose a point $x_\sigma \in O_\sigma$. Let $\mathscr{M} \in D^b\mathrm{MHM}(X)$ be a mixed Hodge module complex on $X = X_\Sigma$ with constant cohomology sheaves along the torus orbits O_σ, $\sigma \in \Sigma$ (e.g., $\mathbb{Q}^H_X$ or IC^H_X).

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♣ In Part 2, I will discuss applications of the above formula to (generalized) weighted Ehrhart theory for convex lattice polytopes.

Singularities through the lens of characteristic classes

LAURENTIU MAXIM
University of Wisconsin-Madison

Heidelberg, September 2024

Part 2.

Applications of characteristic classes to Ehrhart theory

Lattice polytopes, Fans and Toric Varieties

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$$\mathrm{Ehr}_P(\ell) = \chi(X_P, \mathcal{O}(\ell D_P)) \overset{(RR)}{=} \sum_{k \geq 0} \left(\frac{1}{k!} \int_{X_P} [D_P]^k \cap td_k(X_P) \right) \ell^k = \sum_{k \geq 0} a_k \ell^k,$$

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- $a_n = \mathrm{vol}(P)$, $a_{n-1} = \frac{1}{2}\mathrm{vol}(\partial P)$, $a_0 = \chi(P) = 1$.

Reciprocity

♣ Ehrhart reciprocity:

$$\mathrm{Ehr}_P(-\ell) = (-1)^n \cdot \#\left(\mathrm{Int}(\ell P) \cap M\right) = (-1)^n \cdot \mathrm{Ehr}_{\mathrm{Int}(P)}(\ell)$$

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Weighted Ehrhart theory

♣ Face decomposition for P:

$$\#(\ell P \cap M) = \sum_{Q \preceq P} \#(\mathrm{Relint}(\ell Q) \cap M),$$

with $\mathrm{Relint}(\ell Q)$ the relative interior of the face ℓQ of the dilated polytope ℓP.

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♣ Assign Laurent polynomial weights $f_Q(y) \in \mathbb{Z}[y^{\pm 1}]$ to each face $Q \preceq P$ of P, and define for any $\ell \in \mathbb{Z}_{>0}$ the weighted Ehrhart "polynomial" of P and $f = \{f_Q\}_{Q \preceq P}$ by

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♣ If $f = \mathbf{1} := \{1\}$, get for $y = 0$: $\mathrm{Ehr}_{P,\mathbf{1}}(\ell, 0) = \mathrm{Ehr}_P(\ell)$

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♣ If P is a *simple* polytope, the polar polytope P° is *simplicial*, so that $g_{Q^\circ}(-y) = 1$, for all faces Q of P.

Purity

Theorem (Beck-Gunnells-Materov)

For $f_Q(y) = g_{Q^\circ}(-y)$ the weight vector given by Stanley's g-polynomials for the faces of the polar polytope P° of P, the following purity property holds:

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- We use Hodge theory, and recover all properties of $\mathrm{Ehr}_{P,f}(\ell)$ from the calculus of characteristic classes of mixed Hodge modules on X_P (via a generalized Hirzebruch-Riemann-Roch).

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- There is a Hodge polynomial ring homomorphism

$$\chi_y : K_0(\mathrm{MHS}) \longrightarrow \mathbb{Z}[y^{\pm 1}]$$

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- For a complex projective algebraic variety X (e.g., X_P), can take:
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♣ M.-Saito-Schürmann: $I\chi_1(X) = \sigma(X)$ is the intersection cohomology signature of X (Goresky-MacPherson).

Lemma (M.-Schürmann)

For any weight vector $f = \{f_Q\}_{Q \preceq P}$ *on the faces of the lattice polytope* P*, there exists some* $\mathcal{M} \in D^b\mathrm{MHM}(X_P)$ *with constant cohomology sheaves along the torus orbits such that*

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Example (Fieseler, Denef-Loeser)

For a lattice polytope P with Stanley's g-polynomials $f_Q(y) = g_{Q^\circ}(-y)$, one can choose $\mathscr{M} = IC_{X_P}[-n]$.

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♣ For X projective,

$$\chi_y(X;\mathscr{M}) := \chi_y([H^\bullet(X;\mathscr{M})]) = \int_X T_{y*}([\mathscr{M}]),$$

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- Set $T_{y*}(X) := T_{y*}([\mathbb{Q}_X])$, $IT_{y*}(X) := T_{y*}([IC_X[-\dim(X)]])$.
- If X is a toric variety, then

$$T_{0*}(X) = td_*(X).$$

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$$T_{y*}([\mathscr{M}]) = \sum_{Q \preceq P} \chi_y(\mathcal{H}^\bullet(\mathscr{M})_{x_Q}) \cdot (1+y)^{\dim(Q)} \cdot td_*([\omega_{V_{\sigma_Q}}]),$$

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$$\chi_y(X_P; \mathscr{M}) = \sum_{Q \preceq P} \chi_y(\mathcal{H}^\bullet(\mathscr{M})_{x_Q}) \cdot (-1-y)^{\dim(Q)}.$$

For $\mathscr{M} = \mathbb{Q}_{X_P}$, get:

Corollary

(a) *The Hodge polynomial $\chi_y(X_P)$ is computed by:*

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(b) *The Euler characteristic* $e(X)$ *is computed by:*

$$e(X_P) = \text{number of vertices of } P.$$

Stanley's g-polynomials and intersection cohomology

Corollary

Assume $0 \in \mathrm{Int}(P)$, *and* $\mathscr{M} = IC_{X_P}[-n]$. *Then:*

$$I\chi_y(X_P) := \chi_y([IH^\bullet(X_P)]) = \sum_{Q \preceq P} g_{Q^\circ}(-y) \cdot (-1-y)^{\dim(Q)}.$$

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In particular, for $y = 1$, *the Goresky-MacPherson signature is*

$$\sigma(X_P) = \sum_{Q \preceq P} g_{Q^\circ}(-1) \cdot (-2)^{\dim(Q)}.$$

Weighted Ehrhart theory via generalized HRR for X_P

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Let $P \subset M_{\mathbb{R}} \cong \mathbb{R}^n$ be a full-dimensional lattice polytope with associated toric variety X_P and ample Cartier divisor D_P.

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$$\begin{aligned}\mathrm{Ehr}_{P,f}(\ell, y) &= \int_{X_P} e^{\ell[D_P]} \cap T_{y*}([\mathscr{M}]) \\ &= \sum_{k=0}^{n} \left(\frac{1}{k!} \int_X [D_P]^k \cap T_{y,k}([\mathscr{M}]) \right) \cdot \ell^k\end{aligned}$$

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with $\mathscr{M} \in D^b\mathrm{MHM}(X_P)$ a mixed Hodge module complex with constant cohomology sheaves along orbits chosen so that $f_Q(y) = \chi_y(\mathcal{H}^\bullet(\mathscr{M})_{x_Q})$ for some (any) $x_Q \in O_{\sigma_Q} \subset X_P$.

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- *Constant term:*

$$\mathrm{Ehr}_{P,f}(0, y) = \chi_y(X_P; \mathscr{M}) = \sum_{Q \preceq P} f_Q(y) \cdot (-1-y)^{\dim(Q)}$$

Reciprocity and Purity for arbitrary weight vectors

Theorem (M.-Schürmann)

For any $\mathscr{M} \in D^b\mathrm{MHM}(X_P)$ with constant cohomology sheaves along the torus orbits, we have the reciprocity property

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More generally, for any weight vector f on the faces of P, we have

$$\mathrm{Ehr}_{P,f}(-\ell, y) = \sum_{Q \preceq P} f_Q(y) \cdot (-1-y)^{\dim(Q)} \cdot \#(\ell Q \cap M).$$

Remark

The purity of Beck-Gunnells-Materov for $\mathrm{Ehr}_{P,f}(\ell, y)$*, with* f *given by Stanley's* g*-polynomials of faces of the polar polytope* P°*, follows for the special case of* IC_{X_P}*, which is self-dual pure Hodge module of weight* n *on* X_P*.*

Generalized Ehrhart theory

♣ Let $\varphi\colon M_{\mathbb{R}} \cong \mathbb{R}^n \to \mathbb{C}$ be a homogeneous polynomial function.

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Theorem (Brion-Vergne, 1997)

$\mathrm{Ehr}_P^{\varphi}(\ell)$ *is a polynomial in* ℓ *of degree* $\dim(P) + \deg(\varphi)$, *with constant term* $\varphi(0)$, *which satisfies the reciprocity law*

$$\mathrm{Ehr}_P^{\varphi}(-\ell) = (-1)^{\dim(P)+\deg(\varphi)} \sum_{m \in \mathrm{Int}(\ell P) \cap M} \varphi(m)$$
$$= (-1)^{\dim(P)+\deg(\varphi)} \cdot \mathrm{Ehr}_{\mathrm{Int}(P)}^{\varphi}(\ell)$$

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♣ The generalized weighted Ehrhart "polynomial" is defined by

$$\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y) := \sum_{Q \preceq P} f_Q(y) \cdot (1+y)^{\dim(Q)+\deg(\varphi)} \cdot \sum_{m \in \mathrm{Relint}(\ell Q) \cap M} \varphi(m)$$

with $\mathrm{Relint}(\ell Q)$ denoting the relative interior of the face ℓQ of the dilated polytope ℓP.

Properties of $\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y)$

The Brion-Vergne combinatorial approach to reciprocity can be linearly extended (over the faces of P) to this generalized weighted Ehrhart theory, so that $\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y)$ has the following properties:

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- $\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y)$ is a polynomial in ℓ.
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$$\mathrm{Ehr}^{\varphi}_{P,f}(0, y) = \sum_{Q \preceq P} f_Q(y) \cdot (-1 - y)^{\dim(Q) + \deg(\varphi)} \cdot \varphi(0),$$

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- (Reciprocity formula) For $\ell \in \mathbb{Z}_{>0}$,

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Theorem (Beck-Gunnells-Materov)

The following purity property holds:

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♣ Finally, the homogeneous polynomial φ defines a homomorphism

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♣ To explain geometrically the polynomial behavior in ℓ of $\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y)$ we work in equivariant homology, using equivariant localization at torus fixed points (a combinatorial proof can be given using work of Brion-Vergne).

Remark

If P is a simple lattice polytope (so X_P is an orbifold), $\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y)$ can be computed by Euler-Maclaurin type formulae, like in works of Beck-Gunnells-Materov (combinatorially) or Cappell-M.-Schürmann-Shaneson (via the equivariant Hirzebruch-Riemann-Roch formalism).

Happy Birthday, Jörg !!!