Characteristic numbers of singular varieties

LAURENTIU MAXIM
University of Wisconsin-Madison

Heidelberg Colloquium, July 2012

1. Genera and characteristic numbers of manifolds

2. Genera and Characteristic numbers of singular varieties
   - Invariants of singular varieties of the MMP
   - Hodge-type invariants and functorial characteristic classes
   - Computational aspects

Genera of manifolds

Definition

A multiplicative genus ϕ is a ring homomorphism

$$\phi : \Omega_*^G \to \Lambda,$$

where

- Ω_*^G=cobordism ring of closed ($G = O$) and oriented ($G = SO$) or stably almost complex manifolds ($G = U$).
- Λ=commutative, unital $\mathbb{Q}$-algebra.

Hirzebruch: There is a one-to-one correspondence between:

- normalized power series f (i.e. $f(0) = 1$) in the variable $z = p^1$ or c^1;
- normalized and multiplicative cohomology characteristic classes cl_f^* over a finite-dim. base space X, i.e.

$$cl_f^* : (K(X), \oplus) \to (H^*(X; \Lambda), \cup)$$

(with $K(X)$ the Grothendieck group of $\mathbb{R}$-/$\mathbb{C}$- v.b. on X), s.t.

$$cl_f^*(L) = \begin{cases} f(p^1(L)), & L \text{ a real plane bundle} \\ f(c^1(L)), & L \text{ a complex line bundle} \end{cases}$$

- genera $\phi_f : \Omega_*^G \to \Lambda$, for $G = SO$ or $G = U$.

- Given f a normalized power series as above, with corresponding class cl_f^*, the associated genus ϕ_f is defined by:

$$\phi_f(X) = deg(cl_f^*(X)) := \langle cl_f^*(TX), [X] \rangle$$

Example: Signature of manifolds

Let $\sigma(X)$ be the signature of a closed oriented manifold X.

- Thom: the signature defines a genus $\sigma : \Omega_*^{SO} \otimes \mathbb{Q} \to \mathbb{Q}$
- σ comes from the power series $f(z) = \frac{z}{\tanh z}$ in $z = c^1$.
- associated characteristic class: Hirzebruch-Thom L-class
- correspondence: Hirzebruch signature theorem:

$$\sigma(X) = \langle L^*(TX), [X] \rangle$$

Example: Hirzebruch's χ_y-genus

Hirzebruch χ_y-genus of a complex manifold X:

$$\chi_y(X) := \sum_j \chi(X, \Omega_X^j) y^j := \sum_{i,j} (-1)^i \dim_{\mathbb{C}} H^i(X, \Omega_X^j) \cdot y^j .$$

- genus $\chi_y : \Omega_*^U \to \mathbb{Q}[y]$
- χ_y comes from the power series in $z = c^1$:

$$f_y(z) = \frac{z(1+y)}{1 - e^{-z(1+y)}} - zy \in \mathbb{Q}[y][[z]]$$

- associated characteristic class: Hirzebruch class T_y^*
- correspondence: generalized Hirzebruch-Riemann-Roch:

$$\chi_y(X) = \langle T_y^*(TX), [X] \rangle$$

Example: Elliptic genus

$$ell(-; y, q) : \Omega_*^U \otimes \mathbb{Q} \to \mathbb{Q}[[q]][y^{\pm 1}]$$

- power series:

$$f(x) = \frac{x}{2\pi i} \cdot \frac{\theta(\frac{x}{2\pi i} - z, \tau)\theta'(0, \tau)}{\theta(\frac{x}{2\pi i}, \tau)\theta(-z, \tau)}$$

  where $q = e^{2\pi i \tau}$, $y = e^{2\pi i z}$, for $z \in \mathbb{C}$, $\tau \in \mathbb{H}$, and θ the Jacobi theta function.
- Witten: $ell(X)" = "\chi_y(\Omega X; S^1)$, with ΩX the free loop space.
- $\lim_{q \to 0} ell(X; y, q) = y^{-dim(X)/2} \chi_{-y}(X)$

Definition

The value $\phi(X)$ of a genus $\phi : \Omega_*^G \to \mathbb{Q}$ on a closed manifold X is called a characteristic number of X.

Characteristic numbers are used to classify manifolds up to cobordism, e.g.,

- Milnor-Novikov: Two closed stably almost complex manifolds are cobordant $\iff$ all their Chern numbers are the same.

Invariants of singular varieties

Question (**Goresky-MacPherson**)

Which characteristic numbers can be defined for compact complex algebraic varieties with singularities ?

Motivation

- Goresky-MacPherson use intersection homology to define the signature $\sigma(X)$ of a singular compact complex variety.
- if $\widetilde{X} \xrightarrow{f} X$ is a small resolution of X, i.e.,

$$\textit{for all } i \geq 1, \; \textit{codim}_{\mathbb{C}}\{x \in X : \textit{dim}_{\mathbb{C}} f^{-1}(x) = i\} \geq 2i,$$

then $\sigma(X) = \sigma(\widetilde{X})$.

- **Question:** *Which characteristic numbers (linear combinations of Chern numbers)* $\phi : \Omega_*^U \to \mathbb{Q}$ *can be defined for a singular compact complex variety* X *so that* $\phi(X) = \phi(\widetilde{X})$ *for any small resolution* $f : \widetilde{X} \to X$ *?*

- Totaro: Any characteristic number which can be extended from smooth to singular complex varieties, *compatibly with small resolutions*, must be a specialization of the elliptic genus.
- **Question**: *How to define elliptic genus for singular varieties ?*
- Borisov-Libgober: Let X be a projective $\mathbb{Q}$-Gorenstein variety with log-terminal singularities. Then the elliptic genus $ell(X)$ can be defined so that for any crepant resolution $\widetilde{X} \to X$ of X one has $ell(X) = ell(\widetilde{X})$.
- Totaro: "Small" $\Longrightarrow$ "crepant".
- "Crepant" is the good notion of *minimality* for resolutions.
- $ell(X)$ is defined in terms of *any* resolution, then shown to be independent of the choice of resolution by using the *weak factorization theorem*.

Invariants of log-terminal varieties

Definition (Calabi-Yau manifolds; K-equivalence)

Let X be a complex algebraic manifold of dimension d.

- canonical divisor K_X: divisor of zeros and poles of a differential d-form on X.
- X is a Calabi-Yau manifold if K_X is trivial, i.e., if X admits a nowhere vanishing regular differential d-form.
- Complex algebraic manifolds X and Y are K-equivalent if there is a complex algebraic manifold Z and proper birational morphisms $h_X : Z \to X$ and $h_Y : Z \to Y$ s.t. $h_X^* K_X = h_Y^* K_Y$.

Example

Birational equivalent Calabi-Yau manifolds are K-equivalent.

Definition (Gorenstein varieties)

- A complex algebraic variety X of dimension d is Gorenstein if:
  - X is irreducible (or pure dimensional),
  - X is normal (so a canonical divisor K_X is well-defined),
  - K_X is a *Cartier divisor*, e.g., if X is smooth.
- X as above is $\mathbb{Q}$-Gorenstein if K_X is $\mathbb{Q}$-Cartier, i.e., if $r \cdot K_X$ (for some $r \in \mathbb{N}$) is a Cartier divisor.

Definition (Log-terminal varieties)

Let X be a $\mathbb{Q}$-Gorenstein variety, and $\pi : \widetilde{X} \to X$ a log resolution of singularities, i.e., the exceptional locus $E = \cup_i E_i$ of π is a simple normal crossing divisor (E_i's are smooth and meet transversally).

- The relative canonical divisor of π is the $\mathbb{Q}$-Cartier divisor:

$$K_\pi = K_{\widetilde{X}} - \pi^* K_X$$

- $Supp(K_\pi) \subset E$, so $K_\pi = \sum_i a_i \cdot E_i$, with discrepancies $a_i \in \mathbb{Q}$.
- X is log-terminal if $a_i > -1$, for all i (if this holds for one resolution, then it holds for any other resolution of this type).
- $\widetilde{X}$ is a crepant resolution of X if all $a_i = 0$, or $K_{\widetilde{X}} = \pi^* K_X$.

Invariants of log-terminal varieties

- Let H_*, H^* be covariant / contravariant theories with values in abelian groups or unitary rings.
- Let X be a log-terminal variety, $\pi : \widetilde{X} \to X$ a log resolution with exceptional components E_i and discrepancies a_i.
- Let $\phi(-)$ be an invariant of smooth varieties, and define:

$$\phi(X) := \pi_* \left(\phi(\widetilde{X}) \cdot J(E_i, a_i) \right) \in H_*(X)$$

where $J(E_i, a_i) \in H^*(X)$ is a correction term (Jacobian factor) depending on the resolution π.

Example: Elliptic genus (Borisov-Libgober)

The elliptic genus of a log-terminal variety X is defined by:

$$ell(X; y, q) :=$$

$$\langle \prod_l \frac{\xi_l}{2\pi i} \cdot \frac{\theta(\frac{\xi_l}{2\pi i} - z, \tau)\theta'(0, \tau)}{\theta(\frac{\xi_l}{2\pi i}, \tau)\theta(-z, \tau)} \cdot \prod_k \frac{\theta(\frac{e_k}{2\pi i} - (a_k + 1)z, \tau)\theta(-z, \tau)}{\theta(\frac{e_k}{2\pi i} - z, \tau)\theta(-(a_k + 1)z, \tau)}, [\widetilde{X}]\rangle \tag{1}$$

where $\{\xi_l\}_l$ are the Chern roots of the tangent bundle $T\widetilde{X}$ of the log resolution, $\{a_k\}_k$ are the discrepancies, $e_k := c_1(E_k)$, and θ is the Jacobi theta function.

- **Thm:** Such $\phi(X)$ is independent of the choice of resolution.
- If $\pi : \widetilde{X} \to X$ is crepant, then $K_\pi = 0$ and $J(E_i, a_i) = 1$, so:

$$\phi(X) = \pi_*(\phi(\widetilde{X}))$$

  and the right hand side does not depend on the choice of crepant resolution.
- In particular, if $H_*(X) = H^*(X) = R$, ring with unit, with $\pi_* = id_R$, and if $\widetilde{X}_1$ and $\widetilde{X}_2$ are crepant resolutions of X, then:

$$\phi(\widetilde{X}_1) = \phi(\widetilde{X}_2).$$

- **Corollary:** Hodge numbers of two crepant resolutions of a compact log-terminal variety coincide.

K-equivalence invariance

- Let X_1, X_2 be K-equivalent log-terminal varieties, with $\pi_i : Y \to X_i$ $(i = 1, 2)$ log resolutions so that $K_{\pi_1} = K_{\pi_2}$.
- (After taking another resolution Z of Y,) can assume that the exceptional loci of both π_1 and π_2 are contained in the same s.n.c.d., hence the Jacobian factor $J(E_i, a_i)$ is the same for both $\phi(X_1)$ and $\phi(X_2)$.
- In particular, if $H_*(-) = H^*(-) = R$, ring with unit, with $\pi_* = id_R$, then $\phi(X_1) = \phi(X_2)$.
- **Corollary** (Borisov-Libgober, Kontsevich): K-equivalent compact complex algebraic manifolds (e.g., birational C-Y's) have the same elliptic genera and the same Hodge numbers.

Birational Novikov conjecture

- $\sigma(X) = \langle L(TX), [X] \rangle \implies$ R.H.S. is a homotopy invariant.
- Kahn: No other combinations of Pontrjagin classes integrate to a homotopy invariant.
- Novikov's idea: exploit $\pi_1(X)$ to get more homotopy invariants.
- **Conjecture:** If $f : X \to B\pi_1(X)$ and $\alpha \in H^*(B\pi_1(X); \mathbb{Q})$, then

$$\sigma_\alpha(X) := \langle f^*(\alpha) \cup L(TX), [X] \rangle$$

is an oriented homotopy invariant.

If X is a compact complex algebraic manifold, then:

$$\chi_a(X) := \sum_i (-1)^i h^i(X; \mathcal{O}_X) \overset{(RR)}{=} td(X) := \langle Td(TX), [X] \rangle$$

- Hartog: $\chi_a(X)$ is a birational invariant, so $td(X)$ is so.
- **Conjecture** (Rosenberg): If $f : X \to B\pi_1(X)$ and $\alpha \in H^*(B\pi_1(X); \mathbb{Q})$, then the *higher Todd genus*

  $$td_\alpha(X) := \langle f^*(\alpha) \cup Td(TX), [X] \rangle$$

  is a birational invariant.
- proved by Block-Weinberger and Borisov-Libgober.
- **Theorem** (Borisov-Libgober): The higher elliptic genera

  $$ell_\alpha(X) := \langle f^*(\alpha) \cup Ell^*(TX), [X] \rangle$$

  are invariants under K-equivalence. Moreover, $ell_\alpha(X)$ (hence also $\sigma_\alpha(X)$) is invariant under crepant resolutions, and $td_\alpha(X)$ is a birational invariant.

Hodge-type invariants and functorial characteristic classes

Different extensions of genera and characteristic numbers to the singular setting for any type of singularities are derived from:

(a) Deligne's mixed Hodge structures on cohomology.

(b) functorial characteristic class theories.

(a) Hodge-type invariants of complex algebraic varieties

Let X be a complex algebraic variety, $dim_{\mathbb{C}}(X) = d$.

- The Hodge-Deligne polynomial of X is:

$$E_{(c)}(X; u, v) = \sum_{p,q=0}^{d} \left(\sum_{i=0}^{2d} (-1)^i dim_{\mathbb{C}} Gr_F^p Gr_{p+q}^W H_{(c)}^i(X; \mathbb{C}) \right) u^p v^q$$

- The Hirzebruch polynomial of X is:

$$\chi_y^{(c)}(X) = \sum_{i,p} (-1)^i dim_{\mathbb{C}} Gr_F^p H_{(c)}^i(X; \mathbb{C}) \cdot (-y)^p = E(X; -y, 1)$$

- $\chi_{-1}^{(c)}(X) = \chi(X) =$ Euler characteristic of X.
- if X smooth & compact: $\chi_y(X) =$ Hirzebruch χ_y-genus of X.

- Similar invariants $I\chi_y(X)$, $IE(X; u, v)$ etc. can be defined by using Saito's mixed Hodge structures on $IH^*_{(c)}(X)$.
- **Corrolary**: If X is compact, two small resolutions of X have the same Hodge-Deligne polynomials, hence the same Hodge numbers and Betti numbers.
- Hodge Index Theorem (Saito): If X is a complex projective variety, the Goresky-MacPherson signature is computed as:

$$\sigma(X) = I\chi_1(X) = \sum_{p,q} (-1)^q Ih^{p,q}(X),$$

with $Ih^{p,q}(X)$ the intersection homology Hodge numbers.

(b) Invariants derived from functorial characteristic classes

- A functorial characteristic class theory of singular spaces is a covariant transformation

$$cl_* : A(-) \to H_*(-) \otimes R,$$

with $A(-)$ a covariant theory depending on cl_*.

- For any X, there is a distinguished element $\alpha_X \in A(X)$.
- the characteristic class of the singular space X is:

$$cl_*(X) := cl_*(\alpha_X)$$

- cl_* satisfies the normalization property: if X is smooth and $cl^*(TX)$ is the corresponding cohomology class of X, then:

$$cl_*(\alpha_X) = cl^*(TX) \cap [X] \in H_*(X) \otimes R$$

Definition

A characteristic number of a *compact* singular space X is defined by:

$$\#(X) := deg(cl_*(\alpha_X)) := const_*(cl_*(\alpha_X))$$

for $const : X \to point$ the constant map.

Remark

If X is smooth, get by normalization:

$$\#(X) = \langle cl^*(TX), [X] \rangle,$$

so $\#(X)$ is a singular extension of the notion of characteristic numbers of manifolds.

Example (Topological Euler characteristic)

$$\chi(X) := \sum_i (-1)^i b_i(X) = deg(c_*(1_X)),$$

for

$$c_* : F(X) \to H_*(X)$$

the MacPherson-Chern class transformation on X.

Example (Hodge polynomial)

$$\chi_y(X) := \sum_{i,p} (-1)^i dim_{\mathbb{C}} Gr_F^p H^i(X;\mathbb{C}) \cdot (-y)^p = deg(T_{y*}([id_X])),$$

for

$$T_{y*} : K_0(var/X) \to H_*(X) \otimes \mathbb{Q}[y]$$

the Brasselet-Schürmann-Yokura Hirzebruch class transformation.

- **Question**: *How to compute invariants* $\phi(X)$ *of a given singular complex algebraic variety* X *?*

(1) Invariants of orbifolds.

Principle: If G is a finite group acting algebraically on a complex algebraic variety X, invariants of X/G are computed by averaging equivariant invariants of X, e.g.,

$$\chi_y(X/G) = \frac{1}{|G|} \sum_{g \in G} \chi_y(X; g),$$

where

$$\chi_y(X; g) := \sum_{i,p} (-1)^i trace\left(g | Gr_F^p H^i(X; \mathbb{C})\right) \cdot (-y)^p$$

- If X is smooth, Atiyah-Singer holomorphic Lefschetz formula:

$$\chi_y(X;g) = \langle T_y^*(X;g), [X^g] \rangle,$$

with $T_y^*(X;g) \in H^*(X^g) \otimes \mathbb{C}[y]$.

- Cappell, M., Schürmann, Shaneson: define Atiyah-Singer classes

$$T_{y*}(X;g) \in H_*^{BM}(X^g) \otimes \mathbb{C}[y]$$

for *singular* varieties, and prove a singular version of the Atiyah-Singer holomorphic Lefschetz formula for compact varieties:

$$\chi_y(X;g) = deg\left(T_{y*}(X;g)\right)$$

- in particular, $\chi_y(X;g)$ is a characteristic number (of X^g).

(2) Invariants of spaces built out of X.

- Let X be a *singular* quasi-projective variety. Consider spaces built out of X, and compute their invariants in terms of invariants of X:
  - symmetric products: $X^{(n)} := X^n/\Sigma_n$.
  - configuration spaces: $X^{\{n\}} := (X^n - \Delta)/\Sigma_n$.
  - Hilbert schemes of n points on X: $X^{[n]}$.
- Standard approach: for any invariant $\mathcal{I}(-)$ of algebraic varieties, consider the generating series

$$S_{\mathcal{I}}(X) := \sum_{n \geq 0} \mathcal{I}(X^{(n)}) \cdot t^n.$$

- Goal: calculate $S_{\mathcal{I}}(X)$ only in terms of invariants of X.
- Then $\mathcal{I}(X^{(n)})$ is equal to the coefficient of t^n in the resulting expression in invariants of X.

- **Sample result**: (M.-Schürmann)
  For any quasi-projective variety X,

$$\sum_{n\geq 0}\chi_{-y}(X^{(n)})\cdot t^n = exp\left(\sum_{r\geq 1}\chi_{-y^r}(X)\cdot\frac{t^r}{r}\right)$$

  - $y=1$: Macdonald's formula for χ.
  - $y=-1$ and X smooth and projective: Hirzebruch-Zagier formula for σ.

THANK YOU !!!