# HOMEWORK #3

**1.** Let $\pi : E \to B$ be a principal $S^1$-bundle, with $\pi_1(B) = 0$. Consider the cohomology Serre spectral sequence associated to $\pi$, and let $a \in H^1(S^1)$ be a generator. Show that the first Chern class of $\pi$ can be computed by $c_1(\pi) = d_2(a)$, where $d_2$ is the differential on the $E_2$-page of the spectral sequence.

**2.** Compute the cohomology ring $H^*(BO(n); \mathbb{Z}/2)$.

**3.** Show that if $f : M^m \to N^{n+k}$ is a map of differentiable manifolds of the respective dimensions $m$ and $m + k$, which is homotopic to an immersion or an embedding, then there is a rank $k$ real vector bundle $\nu$ so that $f^*TN = TM \oplus \nu$.

**4.** Show that if $M^{4n}$ is a connected manifold which is the boundary of a compact oriented $(4n + 1)$-dimensional manifold $W$, then the signature of $M$ is zero.

**5.** A differentiable $n$-dimensional manifold $M$ is orientable if its tangent bundle $\pi : TM \to M$ is a $SO(n)$-bundle. Show that $M$ is orientable if and only if its first Stiefel-Whitney class $w_1(M)$ is zero.

**6.** Find characteristic class obstructions to the existence of a complex structure on an even dimensional manifold. More precisely, prove the following statement: If $M$ is a $2n$-dimensional manifold which underlies a complex $n$-dimensional manifold, then for each $1 \leq i \leq n$, we have: $w_{2i-1}(M) = 0$ and $w_{2i}(M)$ is the reduction of an integral cohomology class of $M$ (which one?).

**7.** Show that $\mathbb{CP}^4$ cannot be smoothly embedded in $\mathbb{R}^n$ with $n \leq 11$.