HOMEWORK #1

1. Use homotopy groups in order to show that there is no retraction $\mathbb{RP}^n \to \mathbb{RP}^k$ if $n > k > 0$.

2. Show that an n-connected, n-dimensional CW complex is contractible.

3. (*Extension Lemma*)
Given a CW pair (X, A) and a map $f : A \to Y$ with Y path-connected, show that f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n.

4. Show that a CW complex retracts onto any contractible subcomplex. (Hint: Use the above extension lemma.)

5. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic. Conclude that S^∞ is contractible, and more generally, this is true for the infinite suspension $\Sigma^\infty(X) := \bigcup_{n \geq 0} \Sigma^n(X)$ of any CW complex X.

6. Use cellular approximation to show that the n-skeletons of homotopy equivalent CW complexes without cells of dimension $n+1$ are also homotopy equivalent.

7. Show that a closed simply-connected 3-manifold is homotopy equivalent to S^3. (Hint: Use Poincaré Duality, and also the fact that closed manifolds are homotopy equivalent to CW complexes.)

8. Suppose X is a CW complex with $\tilde{H}_i(X; \mathbb{Z}) = 0$ for all $i \geq 0$. Show that the suspension of X is contractible.

9. Show that a map $f : X \to Y$ of connected CW complexes is a homotopy equivalenc eif it induces an isomorphism on π_1 and if a lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ to the universal covers induces an isomorphism on homology.

10. Show that $\pi_7(S^4)$ is non-trivial. [Hint: It contains a $\mathbb{Z}$-summand.]

11. Prove that the space $SO(3)$ of orthogonal 3×3 matrices with determinant 1 is homeomorphic to $\mathbb{RP}^3$.

12.

(a) Show that if $S^k \to S^m \to S^n$ is a fiber bundle, then $k = n-1$ and $m = 2n-1$.
(b) Show that if there were fiber bundles $S^{n-1} \to S^{2n-1} \to S^n$ for *all* n, then the groups $\pi_i(S^n)$ would be finitely generated free abelian groups computable by induction, and non-zero if $i \geq n \geq 2$.