HIRZEBRUCH-MILNOR CLASSES OF COMPLETE INTERSECTIONS

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To the memory of Friedrich Hirzebruch

ABSTRACT. We prove a new formula for the Hirzebruch-Milnor classes of global complete intersections with arbitrary singularities describing the difference between the Hirzebruch classes and the virtual ones. This generalizes a formula for the Chern-Milnor classes in the hypersurface case that was conjectured by S. Yokura and was proved by A. Parusinski and P. Pragacz. It also generalizes a formula of J. Seade and T. Suwa for the Chern-Milnor classes of complete intersections with isolated singularities.

Introduction

For the proof of his Riemann-Roch theorem [Hi], F. Hirzebruch defined the χ_y-genus of a compact complex manifold X by

$$\chi_y(X) := \sum_p \chi(\Omega_X^p)\, y^p \in \mathbf{Z}[y].$$

This specializes respectively to the Euler characteristic, the arithmetic genus, and the signature of X at $y = -1, 0, 1$. It is the highest degree part of the cohomology Hirzebruch characteristic class $T_y^*(TX)$ of the tangent bundle TX. Using the Chern roots $\{\alpha_i\}$ for TX, this cohomology Hirzebruch class is defined by

$$T_y^*(TX) := \prod_{i=1}^{\dim X} Q_y(\alpha_i) \in \mathbf{H}^{\bullet}(X)[y],$$

$$\text{with} \quad Q_y(\alpha) := \alpha(1+y)/\left(1 - e^{-\alpha(1+y)}\right) - \alpha y \in \mathbf{Q}[y][[\alpha]],$$

where $Q_y(\alpha)$ is as in [Hi], 1.8 (see also (1.1) below), and $\mathbf{H}^k(X) = H^{2k}(X, \mathbf{Q})$ in this paper. (It is known that the theory works also for the Chow cohomology groups as defined in [Fu1], [Fu2].) Substituting $y = -1,\ 0,\ 1$, we see that $Q_y(\alpha) \in \mathbf{Q}[y][[\alpha]]$ becomes respectively

$$1 + \alpha, \quad \alpha/(1 - e^{-\alpha}), \quad \alpha/\tanh\alpha,$$

and hence $T_y^*(TX)$ specializes respectively to the Chern class $c^*(TX)$, the Todd class $td^*(TX)$, and the Thom-Hirzebruch L-class $L^*(TX)$, see [HBJ], Sect. 5.4. In the smooth case, this cohomology class $T_y^*(TX)$ is identified by Poincaré duality with the (Borel-Moore) homology class $T_y^*(TX) \cap [X]$ which will be denoted by $T_{y*}(X)$.

A generalization of $T_{y*}(X)$ to the singular case was given by [BSY] using the Du Bois complex in [DB] or $\mathbf{Q}_{h,X} \in D^b\mathrm{MHM}(X)$, the bounded complex of mixed Hodge modules on X whose underlying $\mathbf{Q}$-complex is the constant sheaf $\mathbf{Q}_X$, where $\mathrm{MHM}(X)$ is the category of mixed Hodge modules [Sa2]. The Hirzebruch class $T_{y*}(X)$ is actually a special case of $T_{y*}(\mathcal{M}^{\bullet})$ defined for any $\mathcal{M}^{\bullet} \in D^b\mathrm{MHM}(X)$ in [BSY], see (1.2) below.

Hirzebruch [Hi] also introduced the notion of virtual χ_y-genus (or χ_y-characteristic) which is the χ_y-genus of smooth complete intersections X in smooth projective varieties Y. The *virtual Hirzebruch characteristic class* $T_{y*}^{\mathrm{vir}}(X)$ can be defined like the virtual genus (even

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in the singular case), see (1.3) below. In this paper, we show that the difference between the Hirzebruch class and the virtual one is given by the *Hirzebruch-Milnor class* supported on the singular locus of X, and prove an *inductive* formula to *calculate* it explicitly in the case of global complete intersections with arbitrary singularities as follows.

Let $Y^{(0)}$ be a smooth projective variety, and L be a very ample line bundle on $Y^{(0)}$. We have the graded ring

$$R_L := \bigoplus_{k \geq 0} R_{L,k} \quad \text{with} \quad R_{L,k} := \Gamma(Y^{(0)}, L^{\otimes k}).$$

Let $s_j \in R_{L,a_j}$ with $\{a_j\}$ a decreasing sequence of positive integers. Let r be a positive integer strictly smaller than $\dim Y^{(0)}$. For $j \in [1, r]$, set

$$Y^{(j)} := \bigcap_{k \in [1,j]} s_k^{-1}(0) \subset Y^{(0)},$$

and assume $\operatorname{codim}_{Y^{(0)}} Y^{(j)} = j$. Define

$$X := Y^{(r)}, \quad Y := Y^{(r-1)}.$$

Note that Y is not necessarily smooth. It is a complete intersection of codimension $r-1$ in a smooth projective variety $Y^{(0)}$. The sections s_j ($j \in [1, r]$) generate a graded ideal

$$I_X = \bigoplus_{k \geq 0} I_{X,k} \subset R_L.$$

Set

$$\mathbf{P}(R_{L,k}) := (R_{L,k} \setminus \{0\})/\mathbf{C}^*, \quad \mathbf{P}(I_{X,k}) := (I_{X,k} \setminus \{0\})/\mathbf{C}^*.$$

Replacing the s_j without changing I_X, we may assume by Proposition (3.2) below

$$\operatorname{Sing} Y^{(j)} \subset Y^{(r)} = X \quad (j \in [1, r]).$$

This condition is satisfied (and the s_j generate the ideal I_X) if the s_j are sufficiently general, i.e. if $([s_j])$ belongs to a sufficiently small Zariski-open subset

$$\mathcal{U}_I \subset \prod_{j=1}^{r} \mathbf{P}(I_{X,a_j}),$$

where $[s_j]$ denotes the image of s_j in $\mathbf{P}(I_{X,a_j})$.

Take a sufficiently general $s_r' \in R_{L,a_r}$, and set

$$X' := s_r'^{-1}(0) \cap Y.$$

This is viewed as a complete intersection of codimension $r-1$ in the *smooth* hypersurface section $s_r'^{-1}(0)$ of $Y^{(0)}$. Here "sufficiently general" means that $[s_r']$ belongs to a sufficiently small Zariski-open subset U_r of $\mathbf{P}(R_{L,a_r})$. It satisfies at least the following two conditions:

(1) The hypersurface section $s_r'^{-1}(0)$ transversally intersects all the strata of an algebraic Whitney stratification of Y such that $Y \setminus X$ is a stratum.

(2) For an embedded resolution of $X \subset Y$, the pullback of X' is smooth and transversally intersects any intersections of the irreducible components of the pullback of X.

Conditions (1) and (2) respectively define a non-empty Zariski-open subset of $\mathbf{P}(R_{L,a_r})$ for each choice of an algebraic Whitney stratification of Y or an embedded resolution of (Y, X). Note that the intersection of finitely many non-empty Zariski-open subsets of R_{L,a_r} is non-empty, and this is useful when we have to show the independence of the choices of the s_j, s_j' later. Note also that the Thom a_f-condition follows from condition (1) (see [BMM], [Pa]), and condition (2) follows from (1) if there is an algebraic Whitney stratification of the embedded resolution such that the desingularization is a stratified morphism, i.e. any stratum of the resolution is smooth over a stratum of Y.

We have

$$\Sigma := \operatorname{Sing} X \supset \operatorname{Sing} Y \supset \Sigma' := \operatorname{Sing} X' = \operatorname{Sing} Y \cap X'.$$

Let $i_{A,B}$ denote the inclusion morphism for $A \subset B$ in general. Set $f := (s_r/s'_r)|_{Y \setminus X'}$.

Theorem 1. *With the above notation and assumptions, there is $M_y(X) \in \mathbf{H}_\bullet(\Sigma)[y]$, called the Hirzebruch-Milnor class, where $\mathbf{H}_\bullet(\Sigma) := H^{BM}_{2\bullet}(\Sigma, \mathbf{Q})$, and satisfying*

(0.1) $$T^{vir}_{y*}(X) - T_{y*}(X) = (i_{\Sigma,X})_* M_y(X),$$

(0.2) $$M_y(X) = T_{y*}\big((i_{\Sigma \setminus X',\Sigma})_! \, \varphi_f \mathbf{Q}_{h,Y}\big) + (i_{\Sigma',\Sigma})_* M_y(X').$$

More precisely, choosing sufficiently general $s'_j \in R_{L,a_j}$ $(j \in [1,r])$ inductively, we have a mixed Hodge module $\mathcal{M}(s'_1, \ldots, s'_r)$ on Σ such that

$$M_y(X) = (-1)^{\dim X} T_{y*}\big(\mathcal{M}(s'_1, \ldots, s'_r)\big),$$

and $M_y(X)$ is independent of the choice of the s'_j.

Here $\varphi_f \mathbf{Q}_{h,Y}$ is viewed as a mixed Hodge module on $\Sigma \setminus X'$ up to a shift. (Note that $\mathbf{Q}_Y[\dim Y]$ is a perverse sheaf for any local complete intersection Y.) The monodromy around X' of the local systems $H^j \varphi_f \mathbf{Q}_{h,Y}|_S$ on a stratum S of a Whitney stratification coincides with the Milnor monodromy, since $f = (s_r/s'_r)|_{Y \setminus X'}$ by definition. So it is rare that the monodromy is trivial, and hence the situation is quite different from the one in [CMSS]. By Theorem 1, we have $T^{vir}_{y,k}(X) = T_{y,k}(X)$ for $k > \dim \Sigma$ where $T^{vir}_{y,k}(X) \in \mathbf{H}_k(X)[y]$ is the degree k part of $T^{vir}_{y*}(X)$. This generalizes [CMSS], Cor. 3.3 in the hypersurface case. Theorem 1 was proved in [CMSS], (1.17) in the case of hypersurfaces with isolated singularities.

The above $M_y(X)$ may depend on the choice of the s_j. However, it is well-defined if $([s_j])$ belongs to a sufficiently small non-empty Zariski-open subset $\mathcal{U}_I$ of $\prod^r_{j=1} \mathbf{P}(I_{X,a_j})$ (although it is not necessarily easy to write down explicitly $\mathcal{U}_I$), see Proposition (4.4) below. Note that $M_y(X)$ coincides with this canonical one if it is invariant by any small perturbation of the s_j, i.e. if it is constant for any element in a sufficiently small open neighborhood of $([s_j]) \in \prod^r_{j=1} \mathbf{P}(I_{X,a_j})$ in *classical* topology.

We briefly explain the construction of $M_y(X)$. There is a flat family $\mathcal{Z}_T$ over a Zariski-open subset T of $\mathbf{C}^r$ defined by

$$\mathcal{Z}_T := \mathcal{Z} \cap (Y^{(0)} \times T) \quad \text{with} \quad \mathcal{Z} := \bigcap^r_{i=1}\{s_i = t_i s'_i\} \subset Y^{(0)} \times \mathbf{C}^r.$$

Here $t_1, \ldots, t_r$ are the coordinates of $\mathbf{C}^r$, and $T \subset \mathbf{C}^r$ is defined by the following condition: $t \in T \iff \dim \mathcal{Z}_t = \dim Y^{(0)} - r$ with $\mathcal{Z}_t := \mathcal{Z} \cap (Y^{(0)} \times \{t\})$ for $t \in \mathbf{C}^r$. (Note that $\mathcal{Z}_T$ is a complete intersection in $Y^{(0)} \times T$, and is flat over T, i.e. the $t_i - c_i$ for $i \in [1,r]$ form a regular sequence in $\mathcal{O}_{\mathcal{Z}_T,(y,c)}$ for any $(y,c) \in \mathcal{Z}_T$. This follows from a well-known theory of commutative algebra about regular sequences and flat morphisms, see e.g. [Ei].)

We have $\mathcal{Z}_0 = Y^{(r)} = X$ by definition. Applying the iterated nearby cycle functors, we define a mixed Hodge module on X by

$$\mathcal{M}'(s'_1, \ldots, s'_r) := \psi_{t_r} \cdots \psi_{t_1} \mathbf{Q}_{h,\mathcal{Z}_T}[\dim X],$$

which satisfies

$$T^{vir}_{y*}(X) = (-1)^{\dim X} T_{y*}\big(\mathcal{M}'(s'_1, \ldots, s'_r)\big).$$

Indeed, the nearby cycle functor of mixed Hodge modules corresponds to the Gysin morphism of Borel-Moore homology via the transformation DR_y in (1.2.1), see Proposition (3.3) below. (Note that the latter transformation is denoted by mC_y in [Sch2].) We can moreover prove the injectivity of the canonical morphism of mixed Hodge modules

$$\mathbf{Q}_{h,X}[\dim X] \hookrightarrow \mathcal{M}'(s'_1, \ldots, s'_r),$$

together with the equality (0.2) at the level of the Grothendieck group of mixed Hodge modules by increasing induction on r. Then $\mathcal{M}(s'_1, \ldots, s'_r)$ is defined to be the cokernel of the above injection. It may depend on the s'_i $(i \in [1, r])$ although its image by T_{y*} (and hence $M_y(X)$) does not by using a one-parameter family together with Lemma (2.6) below, see also [Sch3], Cor. 3.7.

By [BSY] and (1.3.4) below, $T_{y*}(X)$ and $T^{\mathrm{vir}}_{y*}(X)$ respectively specialize for $y = -1$ to the MacPherson-Chern class $c(X)$ (see [Ma]) and the virtual Chern class $c^{\mathrm{vir}}(X)$ (called the Fulton or Fulton-Johnson class, see [Fu2], [FJ]) with rational coefficients. Specializing Theorem 1 and using [Sch1], Prop. 5.21, we then get the following.

Corollary 1. *With the notation and the assumptions of Theorem* 1, *there is* $M(X) \in \mathbf{H}_\bullet(\Sigma)$, *called the Milnor class, and satisfying*

$$c^{\mathrm{vir}}(X) - c(X) = (i_{\Sigma, X})_* M(X),$$

$$M(X) = c\big((i_{\Sigma \setminus X', \Sigma})_! \varphi_f \mathbf{Q}_Y\big) + (i_{\Sigma', \Sigma})_* M(X').$$

Here $c\big((i_{\Sigma \setminus X', \Sigma})_! \varphi_f \mathbf{Q}_Y\big)$ is the MacPherson-Chern class [Ma] with $\mathbf{Q}$-coefficients of the associated constructible function. To get Corollary 1 with $\mathbf{Z}$-coefficients (i.e. in $\mathbf{H}_\bullet(\Sigma, \mathbf{Z})$), we have to prove some assertions in this paper (e.g. Proposition (4.1)) with $\mathbf{Q}$ replaced by $\mathbf{Z}$, and use [Ve2] (and [Ke1], [Ke2] if $\mathbf{H}_\bullet(X) = \mathrm{CH}_\bullet(X)$).

As special cases of Theorem 1, we also get the following.

Corollary 2. *With the notation and the assumptions of Theorem* 1, *assume furthermore* $r = 1$, *i.e.* Y *is smooth, or* $\dim \Sigma = 0$, *i.e.* X *has only isolated singularities. Then the second term in the right-hand side of* (0.2) *vanishes so that*

$$(0.3) \qquad\qquad M_y(X) = T_{y*}\big((i_{\Sigma \setminus X', \Sigma})_! \varphi_f \mathbf{Q}_{h, Y}\big).$$

If $\dim \Sigma = 0$, *then we can omit* $(i_{\Sigma \setminus X', \Sigma})_!$ *(since* $\Sigma \cap X' = \emptyset$*), and*

$$(0.4) \qquad\qquad M_y(X) = \bigoplus_{x \in \mathrm{Sing}\, X} \chi_y(\widetilde{H}^\bullet(F_x)).$$

Here $\widetilde{H}^\bullet(F_x)$ *denotes the reduced Milnor cohomology of* $f : (Y, x) \to (\mathbf{C}, 0)$ *endowed with a canonical mixed Hodge structure, and* χ_y *is a polynomial defined by using its Hodge numbers, see* (1.2.7) *below.*

Note that $\chi_y(\widetilde{H}^\bullet(F_x))$ is essentially the Steenbrink spectrum [St3] with information on the action of the monodromy forgotten (see [CMSS], Remark 3.7 for a more precise statement). This spectrum coincides with the spectrum of the complete intersection $(Y^{(0)}, X)$ at a general point of $C_X Y^{(0)}$ over $x \in \mathrm{Sing}\, X$ defined in [DMS], if $s_1, \ldots, s_{r-1}$ and $s'_1, \ldots, s'_r$ are sufficiently general. One can calculate $\chi_y(\widetilde{H}^\bullet(F_x))$ by taking an embedded resolution of singularities and using the theory of motivic Milnor fibers [DL] (which is explained in a slightly different situation in Proposition (5.7) below). We can also apply a construction of Steenbrink [St2] in the isolated singularity case or the theory of mixed Hodge modules as in [BS], Th. 4.3.

By using [Sch1], Prop. 5.21, we see that Corollary 2 specializes for $y = -1$ to a formula for the Chern classes in the hypersurface case which was conjectured by S. Yokura [Yo2] (see also [Yo3], [Yo4]), and proved by A. Parusiński and P. Pragacz [PP]. Note that the formulation itself of Theorem 0.2 in loc. cit. cannot be generalized to the case of Hirzebruch classes because of some monodromy problem. The equality (0.4) specializes for $y = -1$ to a formula of J. Seade and T. Suwa for the Chern classes in the case of complete intersections with isolated singularities (see [SeSu], [Su]).

Note finally that the Hirzebruch-Milnor class $M_y(X)$ can be *calculated explicitly* by using (0.2), see Section 5 below. This is quite important form the view point of applications of Theorem 1.

In Section 1 we review some basics of the Hirzebruch characteristic classes. In Section 2 we prove some properties of the Gysin morphisms which are used in later sections. In Section 3 we show some assertions in the global complete intersection case, e.g. Proposition (3.4). In Section 4 we prove Theorem 1 after showing Proposition (4.1). In Section 5 we show how the Hirzebruch-Milnor classes can be calculated explicitly.

Conventions.

1. A variety means a complex algebraic variety which is always assumed reduced and irreducible except for the case of subvarieties (e.g. hypersurfaces) which may be non-reduced or reducible. We assume a variety is quasi-projective when we use [Fu1].

2. Cohomology and homology groups are always with **Q**-coefficients unless the coefficients are explicitly stated.

3. In this paper, $\mathbf{H}_k(X)$, $\mathbf{H}^k(X)$ are respectively $H_{2k}^{\mathrm{BM}}(X, \mathbf{Q})$ and $H^{2k}(X, \mathbf{Q})$ in order to simplify the explanations. We have similar assertions for $\mathbf{H}_k(X) := \mathrm{CH}_k(X)_{\mathbf{Q}}$ where $\mathbf{H}^k(X)$ is either the Chow cohomology in [Fu1] or Fulton-MacPherson's operational Chow cohomology in [FM], [Fu2] (see [To] for the relation with $H^{2k}(X, \mathbf{Q})$).

4. The nearby and vanishing cycle functors ψ_f, φ_f for $D^b\mathrm{MHM}(X)$ are not shifted by -1 as in [Sa1], [Sa2]. They are compatible with the corresponding functors for the underlying **Q**-complexes without a shift of complexes, but do not preserve mixed Hodge modules.

5. We use *left* $\mathcal{D}$-modules, although right $\mathcal{D}$-modules are mainly used in [Sa1], [Sa2]. The transformation between filtered left and right $\mathcal{D}$-modules is given by

$$(M, F) \mapsto (\Omega_X^{\dim X}, F) \otimes_{\mathcal{O}_X} (M, F),$$

where (M, F) are left $\mathcal{D}_X$-modules, and $\mathrm{Gr}_p^F \Omega_X^{\dim X} = 0$ for $p \neq -\dim X$.

1. Hirzebruch characteristic classes

In this section we review some basics of the Hirzebruch characteristic classes.

1.1. Cohomology Hirzebruch classes. Let X be a smooth complex algebraic variety of dimension n. The cohomology Hirzebruch characteristic class $T_y^*(TX)$ of the tangent bundle TX is defined by

(1.1.1) $$T_y^*(TX) := \prod_{i=1}^n Q_y(\alpha_i) \in \mathbf{H}^\bullet(X)[y],$$

where $Q_y(\alpha)$ is explained as below and the $\{\alpha_i\}$ are the (formal) Chern roots of TX, i.e.

$$\prod_{i=1}^n (1 + \alpha_i t) = \sum_{j=0}^n c_j(TX) t^j.$$

We have normalized and unnormalized power series (see [Hi], 1.8 and [HBJ], 5.4.):

(1.1.2) $$Q_y(\alpha) := \frac{\alpha(1+y)}{1 - e^{-\alpha(1+y)}} - \alpha y, \ \ \widetilde{Q}_y(\alpha) := \frac{\alpha(1 + ye^{-\alpha})}{1 - e^{-\alpha}} \in \mathbf{Q}[y][[\alpha]].$$

Their initial terms are as follows:

(1.1.3) $Q_y(0) = 1, \quad \widetilde{Q}_y(0) = 1 + y.$

These two power series have the following relation

(1.1.4) $Q_y(\alpha) = (1 + y)^{-1} \widetilde{Q}_y(\alpha(1 + y)).$

1.2. Homology Hirzebruch classes. Let X be a complex algebraic variety. Let MHM(X) be the category of mixed Hodge modules on X. For $\mathcal{M}^{\bullet} \in D^b\text{MHM}(X)$, its homology Hirzebruch characteristic class is defined by

(1.2.1)

$$T_{y*}(\mathcal{M}^{\bullet}) := td_{(1+y)*}\big(\text{DR}_y[\mathcal{M}^{\bullet}]\big) \in \mathbf{H}_{\bullet}(X)\big[y, \tfrac{1}{y(y+1)}\big] \quad \text{with}$$

$$\text{DR}_y[\mathcal{M}^{\bullet}] := \textstyle\sum_{i,p} (-1)^i \big[\mathcal{H}^i\text{Gr}_F^p\text{DR}(\mathcal{M}^{\bullet})\big] (-y)^p \in K_0(X)[y, y^{-1}],$$

setting $F^p := F_{-p}$. Here we define

$$td_{(1+y)*} : K_0(X)[y, y^{-1}] \to \mathbf{H}_{\bullet}(X)\big[y, \tfrac{1}{y(y+1)}\big]$$

to be the scalar extension of the Todd class transformation

$$td_* : K_0(X) \to \mathbf{H}_{\bullet}(X)$$

(denoted by τ in [BFM]) together with the multiplication by $(1 + y)^{-k}$ on the degree k part (see [BSY]). The last multiplication is closely related to the identity (1.1.4), and we have actually by [Sch1], Prop. 5.21

$$T_{y*}(\mathcal{M}^{\bullet}) \in \mathbf{H}_{\bullet}(X)[y, y^{-1}].$$

In this paper, we set $\mathbf{H}_k(X) := H_{2k}^{\text{BM}}(X, \mathbf{Q})$ to simplify the arguments, see Convention 3. In some other papers (as [BSY], [CMSS], etc.), $\text{DR}_y[\mathcal{M}^{\bullet}]$ is denoted by $mC_y(\mathcal{M}^{\bullet})$. (Note also that it is nontrivial to show that the $\mathcal{H}^i\text{Gr}_F^p\text{DR}(\mathcal{M}^{\bullet})$ are $\mathcal{O}_X$-modules in the singular case, see [Sa1], Lemma 3.2.6.)

By definition we have for $k \in \mathbf{Z}$

(1.2.2) $\text{DR}_y[\mathcal{M}^{\bullet}(k)] = \text{DR}_y[\mathcal{M}^{\bullet}] (-y)^{-k}, \quad T_{y*}(\mathcal{M}^{\bullet}(k)) = T_{y*}(\mathcal{M}^{\bullet}) (-y)^{-k},$

where the Tate twist (k) on the filtered $\mathcal{D}$-module part is given by the shift of the filtration $[k]$ which is defined by

(1.2.3) $(F[k])^p := F^{p+k}, \quad (F[k])_p := F_{p-k}.$

The *homology Hirzebruch characteristic class* $T_{y*}(X)$ is defined by applying the above definition to the case $\mathcal{M}^{\bullet} = \mathbf{Q}_{h,X}$ (see [BSY]), i.e.

$$T_{y*}(X) := T_{y*}(\mathbf{Q}_{h,X}) = td_{(1+y)*}\text{DR}_y[X] \in \mathbf{H}_{\bullet}(X)[y],$$

$$\text{with} \quad \text{DR}_y[X] := \text{DR}_y[\mathbf{Q}_{h,X}].$$

This coincides with the definition using the Du Bois complex [DB] by [Sa3], and it is known that $T_{y*}(X)$ belongs to $\mathbf{H}_{\bullet}(X)[y]$, see [BSY]. In case X is smooth, we have

(1.2.4) $\text{DR}_y[X] = \Lambda_y[T^*X],$

where $\Lambda_y[V] := \sum_p [\Lambda^p V] y^p$ for a vector bundle V. Indeed, we have

(1.2.5) $\text{DR}(\mathbf{Q}_{h,X}) = \text{DR}(\mathcal{O}_X)[-n] = \Omega_X^{\bullet} \quad \text{with} \ n := \dim X,$

and the Hodge filtration F^p on $\Omega_X^{\bullet}$ is given by the truncation $\sigma_{\geq p}$ in [D1].

To show the coincidence with the above definition of $T_{y*}(X)$ in the smooth case, we need the relation (1.1.4) together with some calculation about Hirzebruch's power series $Q_y(\alpha)$ as in [HBJ], Sect. 5.4 or in the proof of [Yo1], Lemma 2.3.7, which is closely related

to the generalized Hirzebruch-Riemann-Roch theorem in [Hi], Th. 21.3.1. This coincidence is part of the characterization of the Hirzebruch characteristic classes for singular varieties, see [BSY], Th. 3.1.

By a generalization of the Riemann-Roch theorem [BFM], td_* commutes with the pushforward under proper morphisms, and so does T_{y*}. If we apply this to $a_X : X \to pt$ with X compact, then the pushforward for $\mathbf{H_\bullet}$ is identified with the degree map or the trace morphism, and we have

$$(1.2.6) \qquad K_0(pt) = \mathbf{Z}, \quad \mathbf{H_\bullet}(pt) = \mathbf{Q}, \quad \mathrm{MHM}(pt) = \mathrm{MHS},$$

where MHS is the category of graded-polarizable mixed $\mathbf{Q}$-Hodge structures in [D1]. By definition we have for $H^\bullet \in D^b\mathrm{MHS}$

$$(1.2.7) \qquad T_{y*}(H^\bullet) = \chi_y(H^\bullet) := \sum_{j,p} (-1)^j \dim_\mathbf{C} \mathrm{Gr}_F^p H_\mathbf{C}^j (-y)^p.$$

So the degree 0 part of $T_{y*}(X)$ is identified with $\chi_y\big(H^\bullet(X)\big)$ if X is compact and connected. Here the cohomology groups $H^\bullet(X)$ (with the zero differential) is defined in $D^b\mathrm{MHS}$ by [D1], and this is compatible with [Sa2] by [Sa3] (using [Ca]).

1.3. Virtual Hirzebruch classes. Hirzebruch also introduced the notion of virtual χ_y-genus (or χ_y-characteristic) which is the χ_y-genus of general complete intersections X in smooth projective varieties Y. Let X be a complete intersection in a smooth projective variety Y. The virtual Hirzebruch characteristic class $T_{y*}^{\mathrm{vir}}(X)$ can be defined like the virtual genus by

$$(1.3.1) \qquad T_{y*}^{\mathrm{vir}}(X) := td_{(1+y)*}\mathrm{DR}_y^{\mathrm{vir}}[X] \in \mathbf{H_\bullet}(X)[y],$$

with $\mathrm{DR}_y^{\mathrm{vir}}[X]$ the image in $K_0(X)[[y]]$ of

$$(1.3.2) \qquad \Lambda_y(T_{\mathrm{vir}}^*X) = \Lambda_y[T^*Y|_X]/\Lambda_y[N_{X/Y}^*] \in K^0(X)[[y]],$$

and $\mathrm{DR}_y^{\mathrm{vir}}[X]$ belongs to $K_0(X)[y]$ by Proposition (3.4) below. Here $K^0(X)$, $K_0(X)$ are respectively the Grothendieck group of locally free sheaves of finite length and that of coherent sheaves. We denote respectively by T^*Y and $N_{X/Y}^*$ the cotangent and conormal bundles, and the virtual cotangent bundle is defined by

$$T_{\mathrm{vir}}^*X := [T^*Y|_X] - [N_{X/Y}^*] \in K^0(X).$$

More precisely, $N_{X/Y}^*$ in the non-reduced case is defined by the locally free sheaf $\mathcal{I}_X/\mathcal{I}_X^2$ on X where $\mathcal{I}_X \subset \mathcal{O}_Y$ is the ideal sheaf of the subvariety X of Y. Here we set for a virtual vector bundle V on X in general

$$(1.3.3) \qquad \Lambda_y V := \sum_{p \geq 0} \Lambda^p V\, y^p \in K^0(X)[[y]].$$

We can also define $T_{y*}^{\mathrm{vir}}(X)$ by using the virtual tangent bundle

$$T_{\mathrm{vir}}X := [TY|_X] - [N_{X/Y}] \in K^0(X),$$

together with the above cohomological transformation T_y^* as in [CMSS] (in the hypersurface case) so that

$$(1.3.4) \qquad T_{y*}^{\mathrm{vir}}(X) = T_y^*(T_{\mathrm{vir}}X) \cap [X] \quad \text{in } \mathbf{H_\bullet}(X)[y],$$

see Proposition (1.4) below. Here $[X] := \sum_j m_j[X_j]$ is the fundamental class of X with m_j the multiplicities along the irreducible components X_j of X. We have the equality $T_{y*}(X) = T_{y*}^{\mathrm{vir}}(X)$ if X is smooth. The problem is then how to describe the difference in the singular case, which is called the *Hirzebruch-Milnor class*. (For the degree-zero part, i.e. on the level of Hodge polynomials, see also [LM].)

1.4. Proposition. *The equality (1.3.4) holds in* $\mathbf{H}_\bullet(X)[[y]]$ *under the assumption that* X *is a locally complete intersection in a smooth variety* Y *of pure codimension* r.

Proof. By the compatibility of td_* with the cap product [BFM], we have

$$td_*\big(\mathrm{DR}_y^{\mathrm{vir}}[X]\big) = ch(\Lambda_y T_{\mathrm{vir}}^* X) \cap td_*(X) \quad \text{in } \mathbf{H}_\bullet(X)[[y]],$$

where $\mathrm{DR}_y^{\mathrm{vir}}[X]$ is as in (1.3.1), and ch denotes also the scalar extension of the Chern character $ch : K^0(X) \to \mathbf{H}^\bullet(X)$ under $\mathbf{Q} \hookrightarrow \mathbf{Q}[[y]]$. By [Ve1], Th. 7.1, we have

$$td_*(X) := td_*([\mathcal{O}_X]) = td_*(i^*[\mathcal{O}_Y]) = i^! td_*(Y) \cap td^*(N_{X/Y})^{-1},$$

and

$$td_*(Y) = td^*(TY) \cap [Y].$$

Using the relation of $i^!$, i^* with the cap product (see (2.3.4) below), we then get

$$td_*\big(\mathrm{DR}_y^{\mathrm{vir}}[X]\big) = ch(\Lambda_y T_{\mathrm{vir}}^* X) \cdot td^*(T_{\mathrm{vir}} X) \cap [X].$$

Let α_i and β_j be respectively the (formal) Chern roots of $TY|_X$ and $N_{X/Y}$. Then

$$ch(\Lambda_y T_{\mathrm{vir}}^* X) \cdot td^*(T_{\mathrm{vir}} X) = \frac{\prod_{i=1}^m (1 + ye^{-\alpha_i}) \cdot \alpha_i/(1 - e^{-\alpha_i})}{\prod_{j=1}^r (1 + ye^{-\beta_i}) \cdot \beta_j/(1 - e^{-\beta_j})},$$

where $m = \dim Y$ and $r = m - n$ with $n = \dim X$. Indeed, if $\ell(\gamma_j)$ denotes a (formal) line bundle class with the first Chern class γ_j in some ring extension of $K^0(X)$, then

$$ch\big(\ell(\gamma_1) \cdots \ell(\gamma_k)\big) = ch\big(\ell(\gamma_1 + \cdots + \gamma_k)\big) = e^{\gamma_1 + \cdots + \gamma_k},$$

and this implies (see also [Hi])

$$ch(\Lambda_y T_{\mathrm{vir}}^* X) = \prod_{i=1}^m (1 + ye^{-\alpha_i}) / \prod_{j=1}^r (1 + ye^{-\beta_i}).$$

Since $td_{(1+y)*}$ is the composition of td_* with the multiplication by $(1+y)^{-k}$ on $\mathbf{H}_k(X)$, and the last multiplication corresponds to the multiplication by $(1+y)^k$ on $\mathbf{H}^k(X)$ under the cap product, we then get the desired equality as follows:

$$\begin{aligned}
&td_{(1+y)*}\big(\mathrm{DR}_y^{\mathrm{vir}}[X]\big) \\
&= \frac{\prod_{i=1}^m \big(1 + ye^{-\alpha_i(1+y)}\big) \cdot \alpha_i(1+y)/(1 - e^{-\alpha_i(1+y)})}{\prod_{j=1}^r \big(1 + ye^{-\beta_i(1+y)}\big) \cdot \beta_j(1+y)/(1 - e^{-\beta_j(1+y)})} \cap (1+y)^{-n}[X] \\
&= \frac{\prod_{i=1}^m \big(\widetilde{Q}_y(\alpha_i(1+y))/(1+y)\big)}{\prod_{j=1}^r \big(\widetilde{Q}_y(\beta_j(1+y))/(1+y)\big)} \cap [X] \\
&= T_y^*\big([TY|_X] - [N_{X/Y}]\big) \cap [X] \\
&= T_y^*(T_{\mathrm{vir}} X) \cap [X],
\end{aligned}$$

where the relation (1.1.4) between the power series Q_y and $\widetilde{Q}_y$ is used, see also [BSY], (1.1) and [Yo1], Lemma 2.3.7. This finishes the proof of Proposition (1.4).

2. Gysin morphisms

In this section we prove some properties of the Gysin morphisms which are used in later sections.

2.1. Construction. Let $i : X \hookrightarrow Y$ be a locally complete intersection morphism of pure codimension r. By [Fu2], [Ve1] there are Gysin morphisms

$$(2.1.1) \qquad i^! : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k-r}(X), \quad i^! : H_{2k}^{\mathrm{BM}}(Y) \to H_{2k-2r}^{\mathrm{BM}}(X)(r),$$

in a compatible way with the cycle map. These are constructed by using the deformation to the normal cone $q : Z \to S := \mathbf{C}$ such that $q^{-1}(0) = N_{X/Y}$ and $q^{-1}(S') = Y \times S'$ where $S' := \mathbf{C}^*$. More precisely, Z is the complement of the proper transform of $Y \times \{0\}$ in the blow-up of $Y \times S$ along $X \times \{0\}$, see loc. cit.

We recall here the definition of the Gysin morphism for Borel-Moore homology (see [Ve1]). We have the isomorphisms

$$(R^k q_! \mathbf{Q}_Z)_0 = H_c^k(N_{X/Y}), \quad R^k q_! \mathbf{Q}_Z|_{S'} = a_{S'}^{-1} H_c^k(Y),$$

where $a_{S'} : S' \to pt$ is the natural morphism. So we get the cospecialization morphism

(2.1.3) $$\mathrm{sp}^* : H_c^k(N_{X/Y}) \to H_c^k(Y).$$

Taking the dual, we get the Gysin morphism

(2.1.4) $$i^! : H_k^{\mathrm{BM}}(Y) \xrightarrow{\mathrm{sp}_*} H_k^{\mathrm{BM}}(N_{X/Y}) \xleftarrow{\sim} H_{k-2r}^{\mathrm{BM}}(X)(r),$$

where the last isomorphism is induced by the pullback under the canonical morphism $\pi_N : N_{X/Y} \to X$.

The above definition implies the compatibility with the pushforward under a proper morphism $\rho : \widetilde{Y} \to Y$ such that $\widetilde{X} = X \times_Y \widetilde{Y}$ has codimension r everywhere in $\widetilde{Y}$. In this case we have the cartesian diagram

$$
\begin{array}{ccc}
\widetilde{X} & \overset{\tilde{i}}{\hookrightarrow} & \widetilde{Y} \\
\downarrow{\rho'} & & \downarrow{\rho} \\
X & \overset{i}{\hookrightarrow} & Y
\end{array}
$$

and

(2.1.5) $$i^! \circ \rho_* = \rho'_* \circ \tilde{i}^!.$$

Indeed, the specialization morphism sp_* in (2.1.4) is compatible with the pushforward by proper morphisms, and $N_{\widetilde{X}/\widetilde{Y}} \to \widetilde{X}$ is the base change of $N_{X/Y} \to X$ by the hypothesis.

2.2. Sheaf-theoretic description [Ve1]. With the above notation, set $Z_s := q^{-1}(s)$, and let $i_s : Z_s \hookrightarrow Z$ denote the inclusion for $s \in S$. We have

(2.2.1) $$Z_0 = N_{X/Y}, \quad Z_s = Y \ (s \neq 0).$$

Let $p : Z \to Y$ be the canonical projection. Set $p_s := p \circ i_s : Z_s \to Y$.

For $s \in S$ we have the canonical flasque resolutions

$$\mathbf{Q}_Z \xrightarrow{\mathrm{qi}} \mathcal{I}_Z^\bullet, \quad \mathbf{Q}_{Z_s} \xrightarrow{\mathrm{qi}} \mathcal{I}_{Z_s}^\bullet,$$

together with the canonical morphisms

(2.2.2) $$\mathcal{I}_Z^\bullet \to (i_s)_* \mathcal{I}_{Z_s}^\bullet,$$

which are compatible with $\mathbf{Q}_Z \to (i_s)_* \mathbf{Q}_{Z_s}$. Indeed, $(i_s)_* \mathcal{I}_{Z_s}^\bullet$ is naturally isomorphic to the canonical flasque resolution of $(i_s)_* \mathbf{Q}_{Z_s}$, and this resolution is functorial for morphisms of sheaves. The dualizing complex $\mathbf{D}_Z^\bullet$ is defined by

$$\mathbf{D}_Z^k(V) := \mathrm{Hom}(\Gamma_c(V, \mathcal{I}_Z^{-k}), \mathbf{Q}) \quad \text{for open } V \subset S,$$

and similarly for $\mathbf{D}_{Z_s}^\bullet$. Then we have by (2.2.2) the canonical morphisms

$$(i_s)_* \mathbf{D}_{Z_s}^\bullet \to \mathbf{D}_Z^\bullet.$$

They induce the canonical morphisms

(2.2.3) $$\Gamma(Z_s, \mathbf{D}_{Z_s}^\bullet) \to \Gamma_c(S, q_* \mathbf{D}_Z^\bullet),$$

and this is a quasi-isomorphism for $s = 0$. We thus get the canonical morphisms of complexes

$$(2.2.4) \qquad \Gamma(Y, \mathbf{D}_Y^\bullet) = \Gamma(Z_1, \mathbf{D}_{Z_1}^\bullet) \to \Gamma_c(S, q_*\mathbf{D}_Z^\bullet)] \xleftarrow{\mathrm{qi}} \Gamma(Z_0, \mathbf{D}_{Z_0}^\bullet) = \Gamma(N_{X/Y}, \mathbf{D}_{N_{X/Y}}^\bullet),$$

where qi means an quasi-isomorphism.

For any open subset $U \subset Y$, set

$$X_U := X \cap U, \quad Z_U := p^{-1}(U), \quad Z_{U,s} := Z_U \cap Z_s, \quad q_U := q|_{Z_U} : Z_U \to S.$$

Then

$$q_U^{-1}(0) = N_{X_U/U}, \quad q_U^{-1}(s) = U \ \ (s \neq 0),$$

and (2.2.4) implies the canonical morphisms of complexes

$$(2.2.5) \qquad \Gamma(U, \mathbf{D}_U^\bullet) \to \Gamma_c(S, (q_U)_*\mathbf{D}_{Z_U}^\bullet) \xleftarrow{\mathrm{qi}} \Gamma(N_{X_U/U}, \mathbf{D}_{N_{X_U/U}}^\bullet).$$

Here we have presheaves defined by associating to U respectively

$$\Gamma_c(S, (q_U)_*\mathbf{D}_{Z_U}^k), \quad \Gamma(N_{X_U/U}, \mathbf{D}_{N_{X_U/U}}^k),$$

and these are sheaves. So we get the canonical morphisms of sheaf complexes on Y, and this gives the morphism in the derived category $D_c^b(Y, \mathbf{Q}_Y)$

$$(2.2.6) \qquad \mathbf{D}_Y^\bullet \to (i \circ \pi_N)_*\mathbf{D}_{N_{X/Y}}^\bullet \xleftarrow{\sim} i_*\mathbf{D}_X^\bullet(r)[2r],$$

where $\pi_N : N_{X/Y} \to X$ denotes the canonical projection and the last quasi-isomorphism is given by π_N^*. So we get the sheaf-theoretic description of the Gysin morphism in (2.1). (This is a detailed version of a slight modification of a sketch of an argument in [Ve1], Sect. 8 where a point of $S' = \mathbf{C} \setminus \{0\}$ was not chosen as in the above argument. We may assume that this point is 1 by using an equivariant $\mathbf{C}^*$-action on $q : Z \to S$.)

2.3. Compatibility with the cap product. Let $\zeta \in H^j(Y, \mathbf{Q})$. By the canonical isomorphism

$$(2.3.1) \qquad H^j(Y, \mathbf{Q}) = \mathrm{Hom}(\mathbf{Q}_Y, \mathbf{Q}_Y[j]),$$

together with $\mathbf{D}_Y = \mathbf{Q}_Y \otimes \mathbf{D}_Y$, we get the morphism

$$(2.3.2) \qquad \zeta \cap : H_k^{\mathrm{BM}}(Y, \mathbf{Q}) \to H_{k-j}^{\mathrm{BM}}(Y, \mathbf{Q}).$$

So the above sheaf-theoretic construction of the Gysin morphism implies the compatibility with the cap product

$$(2.3.3) \qquad \begin{array}{ccc} H_k^{\mathrm{BM}}(Y, \mathbf{Q}) & \xrightarrow{i^!} & H_{k-2r}^{\mathrm{BM}}(X, \mathbf{Q})(r) \\ \downarrow{\zeta \cap} & & \downarrow{i^*\zeta \cap} \\ H_{k-j}^{\mathrm{BM}}(Y) & \xrightarrow{i^!} & H_{k-j-2r}^{\mathrm{BM}}(X)(r) \end{array}$$

i.e.

$$(2.3.4) \qquad i^!(\zeta \cap \xi) = i^*\zeta \cap i^!\xi \quad \text{for } \zeta \in H^j(Y, \mathbf{Q}), \ \xi \in H_k^{\mathrm{BM}}(Y).$$

Since $i^![Y] = [X]$, this implies the commutative diagram

$$(2.3.5) \qquad \begin{array}{ccc} H^{2n-k}(Y)(n) & \xrightarrow{i^*} & H^{2n-k}(X)(n) \\ \downarrow{\cap[Y]} & & \downarrow{\cap[X]} \\ H_k^{\mathrm{BM}}(Y) & \xrightarrow{i^!} & H_{k-2r}^{\mathrm{BM}}(X)(r) \end{array}$$

This is used in the proof of [CMSS], Lemma 3.1 in the hypersurface case (i.e. $r = 1$).

2.4. Principal divisor case. Assume X is a globally principal divisor on Y, i.e. there is a function on Y with $X = f^{-1}(0)$ and $r = 1$. Then the Gysin morphism induces a 'well-defined' morphism

$$(2.4.1) \qquad i^! : \mathrm{CH}_k(Y \setminus X) \to \mathrm{CH}_{k-1}(X),$$

using the exact sequence

$$\mathrm{CH}_k(X) \xrightarrow{i_*} \mathrm{CH}_k(Y) \xrightarrow{j^*} \mathrm{CH}_k(Y \setminus X) \to 0,$$

where 'well-defined' means that $i^! : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k-1}(X)$ factors through the surjection j^*. Indeed, the composition $i^! \circ i_*$ vanishes by the triviality of the normal bundle.

By [Ve1], Th. 7.1, we have

$$(2.4.2) \qquad td_* \circ i^* = i^! \circ td_*.$$

where $i^* : K_0(Y) \to K_0(X)$ is the pull-back of the Grothendieck groups of coherent sheaves defined by the mapping cone of the multiplication by the global defining function f of X. Using the exact sequence

$$K_0(X) \xrightarrow{i_*} K_0(Y) \xrightarrow{j^*} K_0(Y \setminus X) \to 0,$$

the last definition implies that the composition $i^* \circ i_*$ vanishes so that i^* induces also a 'well-defined' morphism

$$(2.4.3) \qquad i^* : K_0(Y \setminus X) \to K_0(X).$$

We have a similar property for the nearby cycle functor ψ_f with f a global defining function of X, and we have the 'well-defined' functor

$$(2.4.4) \qquad \psi_f[-1] : \mathrm{MHM}(Y \setminus X) \to \mathrm{MHM}(X).$$

Indeed, the restriction gives the surjection $\mathrm{MHM}(Y) \to \mathrm{MHM}(Y \setminus X)$ by extendability of mixed Hodge modules in the algebraic case, and $\psi_f \mathcal{M}'[-1]$ is independent of the extension $\mathcal{M}'$ of $\mathcal{M}$ to Y by using the functorial morphism $id \to j_* j^*$, since $\psi_f M''[-1]$ vanishes if $\mathrm{supp}\, M'' \subset X$.

As for Borel-Moore cohomology, we have the long exact sequence

$$(2.4.5) \qquad \cdots \to H_k^{\mathrm{BM}}(X) \xrightarrow{i_*} H_k^{\mathrm{BM}}(Y) \xrightarrow{j^*} H_k^{\mathrm{BM}}(Y \setminus X) \to H_{k-1}^{\mathrm{BM}}(X) \to \cdots,$$

and the next proposition shows the vanishing of the composition

$$(2.4.6) \qquad H_k^{\mathrm{BM}}(X) \xrightarrow{i_*} H_k^{\mathrm{BM}}(Y) \xrightarrow{i^!} H_{k-2}^{\mathrm{BM}}(X)(1).$$

2.5. Proposition. *With the above notation and the assumptions, the composition of the two morphisms in* (2.4.6) *vanishes.*

Proof. By the construction in (2.1), the restriction of sp_* to the image of i_* is essentially the identity. More precisely, we have

$$(2.5.1) \qquad \mathrm{sp}_* \circ i_* = (s_0)_* : H_k^{\mathrm{BM}}(X) \to H_k^{\mathrm{BM}}(N_{X/Y}),$$

where $s_0 : X \hookrightarrow N_{X/Y}$ is the zero-section. We have moreover

$$(2.5.2) \qquad s_0^! = (\pi_N^*)^{-1} : H_k^{\mathrm{BM}}(N_{X/Y}) \xrightarrow{\sim} H_{k-2}^{\mathrm{BM}}(X)(1),$$

where $\pi_N : N_{X/Y} \to X$ is the projection. Indeed, we have

$$(2.5.3) \qquad s_0^! \circ \pi_N^* = id : H_k^{\mathrm{BM}}(X) \to H_k^{\mathrm{BM}}(X),$$

since the deformation to the normal cone is identified with a trivial deformation.

So the assertion is reduced to

$$(2.5.4) \qquad s_0^! \circ (s_0)_* = 0.$$

Since the normal bundle $N_{X/Y}$ is trivial, we have a section s_1 whose image is disjoint from that of s_0, and $s_1^! \circ \pi_N^* = id$ by the same argument as above. So the assertion is further reduced to

$$(2.5.5) \qquad s_1^! \circ (s_0)_* = 0.$$

This is more or less trivial by the construction of $s_1^!$ since the images of s_0 and s_1 are disjoint. (The reader can also use (2.1.5) together with the fact that the codimension of an empty set can be any number.) This finishes the proof of Proposition (2.5).

By a similar argument, we have the following.

2.6. Lemma. *Let X be a complex algebraic variety. Let $i_t : X = X \times \{t\} \hookrightarrow X \times \mathbf{C}$ denote the inclusion for $t \in \mathbf{C}$. Then $i_t^! : \mathbf{H}_\bullet(X \times \mathbf{C}) \to \mathbf{H}_{\bullet-1}(X)$ is independent of $t \in \mathbf{C}$.*

Proof. Let $Y := X \times \mathbf{C}$ so that $Y = N_{X/Y}$ where $X = X \times \{t\} \subset Y$. Then the assertion follows from (2.5.2). $\square$

3. Global complete intersections

In this section we show some assertions in the global complete intersection case, e.g. Proposition (3.4). We first give proofs of the following lemma and proposition for the convenience of the reader. These are related to the independence of the choice of the sections s_i. They should be well-known to specialists.

3.1. Lemma. *Let L be a line bundle on a smooth variety X, and $s_i \in \Gamma(X, L)$ for $i \in [1, m]$. Assume $\bigcap_{i \in [1,m]} s_i^{-1}(0) = \emptyset$. Then there is a non-empty Zariski-open subset U of $\mathbf{P}^{m-1}$ such that the zero locus of $\sum_{i \in [1,m]} t_i s_i$ in X is smooth for any $t = (t_1, \ldots, t_m) \in U$.*

Proof. Consider

$$Z := \big\{ (y, t) \in X \times \mathbf{P}^{m-1} \,\big|\, \textstyle\sum_{i \in [1,m]} t_i s_i(y) = 0 \big\}.$$

By the Sard-type theorem, it is enough to show that Z is smooth. Set

$$U_j := \{t_j \neq 0\} \subset \mathbf{P}^{m-1}, \quad X_i := X \setminus s_i^{-1}(0).$$

Then $X = \bigcup_i X_i$, and $Z \cap (X_i \times \mathbf{P}^{m-1})$ is covered by $X_i \times U_j$ with $i \neq j$ by the definition of Z. Moreover, $Z \cap (X_i \times U_j)$ is defined by

$$t_i / t_j = -\textstyle\sum_{k \neq i} (t_k/t_j)(s_k(y)/s_i(y)) \quad \text{in } X_i \times U_j,$$

where the $t_k/t_j \ (k \neq j)$ are the affine coordinates of U_j. So the assertion follows. $\square$

3.2. Proposition. *Let L be a very ample line bundle on a projective variety Y. Let $s_i \in \Gamma(Y, L^{\otimes a_i}) \ (i = 1, \ldots, r)$ with a_i a decreasing sequence of positive integers where $r \geq 2$. Set $Y^{(j)} := \bigcap_{i \in [1,j]} s_i^{-1}(0) \ (j = 1, \ldots, r)$. Assume $Y \setminus Y^{(r)}$ is smooth and $\mathrm{codim}_Y Y^{(r)} = r$. Then, the $Y^{(j)} \setminus Y^{(r)}$ are smooth for any $j \in [1, r-1]$ by replacing s_i with $s_i + \sum_{i' > i} s'_{i,i'} s_{i'}$ if we choose the $s'_{i,i'} \in \Gamma(Y, L^{\otimes(a_i - a_{i'})})$ generically.*

Proof. We proceed by increasing induction on $r \geq 2$. Consider the embedding $Y \hookrightarrow \mathbf{P}^N$ by the line bundle L. Let $\{\tau_k^{(i)}\}_{k \in [1,\nu_i]}$ be a basis of $\Gamma(Y, L^{\otimes i})$. We apply Lemma (3.1) to the case where $X = Y \setminus Y^{(r)}$, $m = \sum_{i=1}^r \nu_{a_1 - a_i}$, and the $s_j \ (j \in [1, m])$ in Lemma (3.1) are given by the restrictions of

$$\tau_k^{(a_1 - a_i)} s_i \quad (k \in [1, \nu_{a_1 - a_i}], \ i \in [1, r]).$$

Note that the intersection of the zero loci of $\tau_k^{(a_1-a_i)} s_i$ ($k \in [1, \nu_{a_1-a_i}]$) coincides with $s_i^{-1}(0)$ for each $i \in [1, r]$. We may assume that the coefficient of s_1 does not vanish since this condition defines a non-empty Zariski-open subset of $\mathbf{P}^{m-1}$ (and the intersection of any two non-empty Zariski-open subsets of $\mathbf{P}^{m-1}$ is non-empty). So we get the assertion for $j = 1$, and hence for $r = 2$. Moreover, the assertion for $r \geq 3, j \geq 2$ is reduced to the assertion with Y replaced by $Y^{(1)}$, and (r, j) by $(r - 1, j - 1)$. Thus we can proceed by induction on r. This finishes the proof of Proposition (3.2).

The following was shown in [Sch2] without assuming the condition that the variety Y is embeddable into a smooth variety. We give here a simplified proof assuming this condition. This proposition will be used in the proof of Proposition (3.4) below.

3.3. Proposition ([Sch2]). *Let $f : Y \to \mathbf{C}$ be a non-constant function on a complex algebraic variety. Assume Y is embeddable into a smooth variety. Set $Y_0 := f^{-1}(0)$ with the inclusion $i_0 : Y_0 \hookrightarrow Y$. Then, for any bounded complex of mixed Hodge modules $\mathcal{M}^\bullet$ on Y, we have*

(3.3.1)
$$i_0^* \mathrm{DR}_y[\mathcal{M}^\bullet] = \mathrm{DR}_y[\psi_f \mathcal{M}^\bullet](1 + y) \quad in \ K_0(Y_0)[y, y^{-1}],$$
$$i_0^! T_{y*}(\mathcal{M}^\bullet) = T_{y*}(\psi_f \mathcal{M}^\bullet) \quad in \ \mathbf{H}_\bullet(Y_0)[y, y^{-1}],$$

where $\psi_f \mathcal{M}^\bullet$ is viewed as a complex of mixed Hodge modules on Y_0.

Proof. It is enough to show the first equality since the second equality follows from it using [Ve1], Th. 7.1. (Here the term $(1 + y)$ disappears since $i_0^!$ send $\mathbf{H}_k(Y)$ to $\mathbf{H}_{k-1}(Y_0)$.) We may assume that $\mathcal{M}^\bullet$ is a mixed Hodge module $\mathcal{M}$ on Y. By assumption there is a closed embedding $Y \hookrightarrow Y'$ with Y' smooth. Set

$$Z := Y \times \mathbf{C} \subset Z' := Y' \times \mathbf{C}.$$

Let $i_f : Y \hookrightarrow Z$ be the graph embedding, and (M, F) be the underlying filtered left $\mathcal{D}_{Z'}$-module of the direct image of $\mathcal{M}$ by i_f, see [Sa1]. It has an increasing filtration V of Kashiwara and Malgrange such that $\partial_t t - a$ is nilpotent on $\mathrm{Gr}_V^a M$. (In this paper we use left $\mathcal{D}$-modules, and V^a corresponds to V_{-a} in loc. cit., see Convention 5.)

Setting $Z'^* := Y' \times \mathbf{C}^*$, we have $V^a M|_{Z'^*} = M|_{Z'^*}$ for any $a \in \mathbf{Q}$. (Indeed, for any $m \in M$, we have $t^i m \in V^a M$ for $i \gg a$ and t is invertible on Z'^*). Using the well-definedness of (2.4.3), the left-hand side of (3.3.1) can be given by applying i_0^* to the mapping cone

(3.3.2)
$$C\big(\mathrm{Gr}_F^\bullet \mathrm{DR}_{Z'/\mathbf{C}}(V^a M) \xrightarrow{\mathrm{Gr}\,\partial_t} \mathrm{Gr}_F^{\bullet-1} \mathrm{DR}_{Z'/\mathbf{C}}(V^b M)\big),$$

and considering it in $K_0(Y_0)[y, y^{-1}]$ (using [Sa1], Lemma 3.2.6). Here a, b can be any rational numbers satisfying

(3.3.3)
$$a - 1 \geq b > 0.$$

(This implies that [Sa1], (3.2.1.3) is not needed in [Sch2].) By [Sa1], (3.2.1.2), we have the isomorphisms

(3.3.4)
$$t : V^a M \xrightarrow{\sim} V^{a+1} M \quad for \ any \ a > 0.$$

So the left-hand side of (3.3.1) is given by

(3.3.5)
$$-\mathrm{DR}_y[(V^a/V^{a+1})(M, F)] + \mathrm{DR}_y[(V^b/V^{b+1})(M, F[-1])],$$

where $(V^a/V^{a+1})(M, F)$ and $(V^b/V^{b+1})(M, F)$ are viewed as filtered left $\mathcal{D}_{Y'}$-module. (For the shift of filtration $[-1]$, see (1.2.3).)

As for the right-hand side of (3.3.1), we have by definition

(3.3.6)
$$\psi_t(M, F) = \bigoplus_{a \in (0,1]} \mathrm{Gr}_V^a(M, F)[1],$$

where the shift of complex by 1 comes from Convention 4. So we get (3.3.1) by using (1.2.2) and (3.3.4). This finishes the proof of Proposition (3.3).

3.4. Proposition. *Let $\mathcal{M}'(s'_1, \ldots, s'_r)$ and $\Lambda_y(T^*_{\mathrm{vir}}X)$ be as in the introduction and (1.3) respectively. Then*

$$\mathrm{DR}_y[\mathcal{M}'(s'_1, \ldots, s'_r)] = (-1)^{\dim X}\Lambda_y(T^*_{\mathrm{vir}}X) \quad in \ K_0(X)[[y]][y^{-1}].$$

Hence $\Lambda_y(T^*_{\mathrm{vir}}X) \in K_0(X)[y] \left(= K_0(X)[y, y^{-1}] \cap K_0(X)[[y]]\right)$, and

$$T_{y*}\big(\mathcal{M}'(s'_1, \ldots, s'_r)\big) = (-1)^{\dim X}T^{\mathrm{vir}}_{y*}(X).$$

Proof. It is enough to show the first equality, since the last assertion follows from it by [Ve1], Th. 7.1 applied to the inclusion $X \hookrightarrow \mathcal{Z}_T$. In the notation of the introduction, set

$$T_j := \{t_k = 0 \ (k \le j)\} \subset T, \quad \mathcal{Z}_j := \mathcal{Z}_T \times_T T_j,$$

with the inclusions $i_j : \mathcal{Z}_j \hookrightarrow \mathcal{Z}_{j-1}$. Let $i_X : X \hookrightarrow \mathcal{Z}_T$ denote the composition of the i_j for $j \in [1, r]$. Set $n := \dim X$. Applying Proposition (3.3) to the i_j, we get

(3.4.1) $$(-1)^n\mathrm{DR}_y[\mathcal{M}'(s'_1, \ldots, s'_r)] = i_X^*\mathrm{DR}_y[\mathbf{Q}_{h,\mathcal{Z}_T}](1 + y)^{-r}.$$

Let U be a non-empty Zariski-open subset of T such that $\mathcal{Z}_U := \mathcal{Z}_T \times_T U$ is smooth over U. Let $T^*(\mathcal{Z}_U/U)$ denote the relative cotangent bundle. Since T^*U is trivial, we have

(3.4.2)
$$\mathrm{DR}_y[\mathbf{Q}_{h,\mathcal{Z}_U}](1 + y)^{-r} = \Lambda_y[T^*\mathcal{Z}_U](1 + y)^{-r}$$
$$= \Lambda_y\big[T^*(\mathcal{Z}_U/U)\big] = i_{\mathcal{Z}_U}^* \circ pr_1^*\big(\Lambda_y[T^*Y^{(0)}]/\prod_{j=1}^r \Lambda_y[L_j^*]\big),$$

where $i_{\mathcal{Z}_U} : \mathcal{Z}_U \hookrightarrow Y^{(0)} \times U$ and $pr_1 : Y^{(0)} \times U \to Y^{(0)}$ are natural morphisms, and L_j^* is the dual of a very ample line bundle L_j such that $s_j \in \Gamma(Y^{(0)}, L_j)$. (Here it is not necessary to assume that $L_j = L^{\otimes a_j}$.)

In order to apply inductively (2.4.3) and (2.4.4), we have to take Zariski-open subsets U'_j, U_j of T_j ($j \in [0, r-1]$) satisfying the two conditions

(3.4.3) $$U'_j \setminus T_{j+1} = U_j \setminus T_{j+1}, \quad U'_{j+1} := U_j \cap T_{j+1} \ne \emptyset,$$

by increasing induction on $j \in [0, r-1]$, where $U'_r = T_r = \{0\}$, and (2.4.3), (2.4.4) will be applied to the base changes of the inclusions

$$U'_{j+1} = U_j \cap T_{j+1} \hookrightarrow U_j.$$

If $j = 0$, we set $U'_0 := U$. In general, if U'_j is given, then U_j can be any Zariski-open subset of T_j satisfying the two conditions in (3.4.3). Here a canonical choice would be the maximal one satisfying the two conditions, i.e.

$$U_j := T_j \setminus \overline{(T_j \setminus (U'_j \cup T_{j+1}))}.$$

This means that $T_j \setminus U_j$ is the union of the irreducible components of $T_j \setminus U'_j$ which are not contained in T_{j+1}.

Consider now the inclusions

(3.4.4) $$\mathcal{Z}_{U'_{j+1}} \hookrightarrow \mathcal{Z}_{U_j},$$

which are obtained by the base change of the inclusions $U'_{j+1} \hookrightarrow U_j \hookrightarrow U$ by $\mathcal{Z}_U \to U$ for $j \in [0, r-1]$. The assertion then follows from the 'well-definedness' of (2.4.3) and (2.4.4) (as is explained in (2.4)) which we apply to the inclusions (3.4.4) inductively. This finishes the proof of Proposition (3.4).

4. Proof of the main theorem

In this section we prove Theorem 1 after showing Proposition (4.1) below. A proof of the latter has been presented in the proof of [PP], Prop. 5.1 in case Y is smooth, where they demonstrated the contractibility of the Milnor fiber using the integration of a controlled vector field as in [Mi]. We give a sheaf-theoretic proof using an embedded resolution.

4.1. Proposition. *Let X be a hypersurface of a projective variety Y such that X is a very ample divisor, and $Y \setminus X$ is smooth. Let X' be a sufficiently general member of the linear system associated with X so that we have a one-parameter family $\mathcal{Y} = \coprod_{c \in \mathbf{P}^1} Y_c$ with $Y_0 = X$, $Y_\infty = X'$. Here $\mathbf{P}^1$ is identified with $\mathbf{C} \cup \{\infty\}$ by the inhomogeneous coordinate t of $\mathbf{P}^1$. Let $\pi : \mathcal{Y} \to \mathbf{P}^1$ be the projection. Set $X'' := X \cap X'$. Then*

$$\varphi_{\pi^*t}\mathbf{Q}_{\mathcal{Y}}|_{X''} = 0, \quad i.e. \quad \psi_{\pi^*t}\mathbf{Q}_{\mathcal{Y}}|_{X''} = \mathbf{Q}_{X''}.$$

Proof. We first treat the following simple case.

(a) Normal crossing case. Assume that Y, X' are smooth, X is a divisor with simple normal crossings, and X' transversally intersects any intersections of irreducible components of X. (Here it is not necessary to assume that X is a very ample divisor.) Let g be a local equation of X around $x \in X'' = X \cap X'$. Let $y_1, \ldots, y_n$ be local coordinates of Y such that locally

$$g = \prod_{i=1}^r y_i^{m_i}, \quad X' = \{y_n = 0\},$$

where $r < n := \dim Y$ and $m_i \geq 1 \, (i \in [1, r])$. Then we have locally

$$\mathcal{Y} = \{g = y_n t\} \subset Y \times \mathbf{C},$$

where t is identified with the affine coordinate of $\mathbf{C} \subset \mathbf{P}^1$. So the Milnor fiber of $\pi : \mathcal{Y} \to \mathbf{P}^1$ around $x \in X \cap X' \subset \pi^{-1}(0)$ is given by

$$\left\{ (y_1, \ldots, y_n) \in \mathbf{C}^n \,\middle|\, g = y_n t, \, \sum_{i=1}^n |y_i|^2 < \varepsilon^2 \right\}.$$

where $0 < |t| \ll \varepsilon \ll 1$. This is identified with

$$\left\{ (y_1, \ldots, y_{n-1}) \in \mathbf{C}^{n-1} \,\middle|\, \sum_{i=1}^{n-1} |y_i|^2 + |t|^{-2} \prod_{i=1}^r |y_i|^{2m_i} < \varepsilon^2 \right\}.$$

This set is contractible by using the natural action of $\lambda \in [0, 1]$ defined by

$$\lambda(y_1, \ldots, y_{n-1}) = (\lambda y_1, \ldots, \lambda y_{n-1}).$$

So the assertion in the normal crossing case follows.

(b) General case. Let $\sigma : (\widetilde{Y}, \widetilde{X}) \to (Y, X)$ be an embedded resolution of singularities such that $\widetilde{X} := \sigma^{-1}(X)$ is a divisor with simple normal crossings. By assumption, $\widetilde{X}' := \sigma^{-1}(X')$ is smooth and transversally intersects any intersections of irreducible components of $\widetilde{X}$. So $\widetilde{X}'$ is the total transform of X', and we get the one-parameter family

$$\widetilde{\mathcal{Y}} := \coprod_{c \in \mathbf{P}^1} \widetilde{Y}_c \quad \text{with} \quad \widetilde{Y}_0 = \widetilde{X}, \, \widetilde{Y}_\infty = \widetilde{X}'.$$

Set

$$\mathcal{Y}_U := \coprod_{c \in U} Y_c, \quad \widetilde{\mathcal{Y}}_U := \coprod_{c \in U} \widetilde{Y}_c \quad \text{with} \quad U := \mathbf{P}^1 \setminus \{0\}.$$

Let $\pi_U : \mathcal{Y}_U \to \mathbf{P}^1$, $\widetilde{\pi}_U : \widetilde{\mathcal{Y}}_U \to \mathbf{P}^1$ denote the canonical morphisms. Then σ induces

$$\widetilde{\sigma}_U : \widetilde{\mathcal{Y}}_U \to \mathcal{Y}_U \quad \text{with} \quad \widetilde{\pi}_U = \pi_U \circ \widetilde{\sigma}_U : \widetilde{\mathcal{Y}}_U \to \mathcal{Y}_U \to U.$$

Let $\sigma_c : \widetilde{Y}_c \to Y_c \, (c \in U)$ be morphisms induced by σ. Set

$$\widetilde{X}'' := \widetilde{X} \cap \widetilde{X}', \quad X'' := X \cap X' \quad \text{with} \quad \sigma'' := \sigma|_{\widetilde{X}''} : \widetilde{X}'' \to X''.$$

By the definition of $\widetilde{\mathcal{Y}}_U$, $\mathcal{Y}_U$, we have the inclusions

$$\widetilde{i}''_U : \widetilde{X}'' \times U \hookrightarrow \widetilde{\mathcal{Y}}_U, \quad i''_U : X'' \times U \hookrightarrow \mathcal{Y}_U \quad \text{over } U,$$

and $\widetilde{\sigma}$ induces an isomorphism

$$(4.1.1) \qquad \widetilde{\mathcal{Y}}_U \setminus \widetilde{i}''_U(\widetilde{X}'' \times U) \xrightarrow{\sim} \mathcal{Y}_U \setminus i''_U(X'' \times U).$$

Consider now the distinguished triangle

$$(4.1.2) \qquad \mathbf{Q}_{\mathcal{Y}_U} \xrightarrow{\widetilde{\sigma}^\#_U} \mathbf{R}(\widetilde{\sigma}_U)_* \mathbf{Q}_{\widetilde{\mathcal{Y}}_U} \to \operatorname{Cone}\widetilde{\sigma}^\#_U \xrightarrow{+1} .$$

By the definition of $\widetilde{\sigma}_U$ and using (4.1.1), we have

$$(4.1.3) \qquad \operatorname{Cone}\widetilde{\sigma}^\#_U \cong (i''_U)_* pr_1^* K'' \quad \text{with} \quad K'' := \operatorname{Cone}\big(\mathbf{Q}_{X''} \to \mathbf{R}\sigma''_* \mathbf{Q}_{\widetilde{X}''}\big),$$

where $pr_1 : X'' \times U \to X''$ is the first projection.

Let t be the affine coordinate of $\mathbf{C} \subset \mathbf{P}^1$. Since the nearby cycle functor commutes with the direct image by a proper morphism, we have a canonical isomorphism

$$\mathbf{R}(\sigma_0)_* \psi_{\widetilde{\pi}^* t} \mathbf{Q}_{\widetilde{\mathcal{Y}}_U} = \psi_{\pi^* t} \mathbf{R}\widetilde{\sigma}_* \mathbf{Q}_{\widetilde{\mathcal{Y}}_U}.$$

By the assertion in the normal crossing case together with the proper base change theorem, we then get

$$(4.1.4) \qquad \begin{aligned} \mathbf{R}\sigma''_* \mathbf{Q}_{\widetilde{X}''} &\xrightarrow{\sim} \mathbf{R}\sigma''_* \big(\psi_{\widetilde{\pi}^* t} \mathbf{Q}_{\widetilde{\mathcal{Y}}_U}\big|_{\widetilde{X}''}\big) \\ &= \mathbf{R}(\sigma_0)_* \psi_{\widetilde{\pi}^* t} \mathbf{Q}_{\widetilde{\mathcal{Y}}_U}\big|_{X''} = \psi_{\pi^* t} \mathbf{R}\widetilde{\sigma}_* \mathbf{Q}_{\widetilde{\mathcal{Y}}_U}\big|_{X''}. \end{aligned}$$

Let $i_0 : Y_0 \hookrightarrow \mathcal{Y}$ denote the inclusion. Apply the functorial morphism $i_0^* \to \psi_{\pi^* t}$ to the distinguished triangle

$$\mathbf{Q}_{\mathcal{Y}} \xrightarrow{\widetilde{\sigma}^\#} \mathbf{R}(\widetilde{\sigma})_* \mathbf{Q}_{\widetilde{\mathcal{Y}}} \to \operatorname{Cone}\widetilde{\sigma}^\# \xrightarrow{+1},$$

which is the extension of (4.1.2) over $\mathbf{P}^1$. Restricting these over $X'' \subset X = Y_0$, and using the 'well-definedness' of the nearby cycle functor as in (2.4.4), we then get a morphism of distinguished triangles

$$(4.1.5) \qquad \begin{array}{ccccccc} \mathbf{Q}_{X''} & \to & \mathbf{R}\sigma''_* \mathbf{Q}_{\widetilde{X}''} & \to & K'' & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \| & & \\ \psi_{\pi^* t} \mathbf{Q}_{\mathcal{Y}_U}\big|_{X''} & \to & \psi_{\pi^* t} \mathbf{R}\widetilde{\sigma}_* \mathbf{Q}_{\widetilde{\mathcal{Y}}_U}\big|_{X''} & \to & K'' & \xrightarrow{+1} & \end{array}$$

where the right vertical isomorphism follows from (4.1.3). Moreover, the middle vertical morphism is an isomorphism by (4.1.4). So the left vertical morphism is an isomorphism. This finishes the proof of Proposition (4.1).

4.2. Proof of Theorem 1. We proceed by increasing induction on r. We fix $s_1, \ldots, s_r$ such that

$$\Sigma := \operatorname{Sing} X \supset \operatorname{Sing} Y \supset \Sigma' := \operatorname{Sing} X' = \operatorname{Sing} Y \cap X',$$

in the notation of the introduction, e.g. $Y := Y^{(r-1)}$, $X := Y^{(r)}$, $X' = Y \cap s_r'^{-1}(0)$.

Let $\mathcal{Y}$ be the blow-up of Y along $X'' := X \cap X'$ so that

$$\mathcal{Y} = \coprod_{c \in \mathbf{P}^1} Y_c \quad \text{with} \quad Y_0 = X, \ Y_\infty = X'.$$

Here we use c to denote a point of $\mathbf{P}^1$, and the coordinate t_r will be used to define the nearby and vanishing cycle functors.

Set $T_r := \{t_i = 0 \ (1 \le i < r)\} \subset T$. Then

$$\mathcal{Y} \setminus Y_\infty \cong \mathcal{Z}_T \cap (Y^{(0)} \times T_r).$$

Set $\mathcal{Y}_U := \mathcal{Y} \times_{T_r} U$ with U a sufficiently small non-empty Zariski-open subset of T_r such that the hypersurface $\{s_r = c\, s_r'\} \subset Y^{(0)}$ is smooth and intersects transversally any strata of Whitney stratifications of $Y^{(k)}$ $(0 < k < r)$ for any $c \in U$. Set $U' := U \cup \{\infty\} \subset \mathbf{P}^1$. Let $i_c : Y_c \hookrightarrow \mathcal{Y}_{U'}$ denote the inclusion for $c \in U'$. Set

$$\mathcal{M}'_{U'} := \mathcal{M}'(s_1', \ldots, s_{r-1}')|_{\mathcal{Y}_{U'}}, \quad \mathcal{M}'_c := i_c^* \mathcal{M}'_{U'}[-1] \ (c \in U').$$

These are mixed Hodge modules on $\mathcal{Y}_{U'}$ and Y_c $(c \in U')$ respectively, and moreover Y_c transversally intersects any strata of a Whitney stratification of $\mathcal{Y}_U$ compatible with $\mathcal{M}'_U$ by shrinking U if necessary. (Here U' may contain $\infty \in \mathbf{P}^1$, since $X' = \{s_r' = 0\} \subset Y$ is assumed to be sufficiently general in the one-parameter family $Y_c = \{s_r = c\, s_r'\} \subset Y$ $(c \in U)$.)

Set $n = \dim X$. By inductive hypothesis applied to the smooth projective variety $\{s_r = c\, s_r'\} \subset Y^{(0)}$ for $c \in U'$, we have short exact sequences of mixed Hodge modules on Y_c $(c \in U')$

(4.2.1) $$0 \to \mathbf{Q}_{h,Y_c}[n] \to \mathcal{M}'_c \to \mathcal{M}_c \to 0,$$

where the last term is defined by the cokernel of the injection. (Indeed, the restriction by the inclusion $\{s_r = c\, s_r'\} \hookrightarrow Y^{(0)}$ commutes with the nearby cycle functors by [DMST].) They imply a short exact sequence of mixed Hodge modules on $\mathcal{Y}_{U'}$

(4.2.2) $$0 \to \mathbf{Q}_{h,\mathcal{Y}_{U'}}[n+1] \to \mathcal{M}'_{U'} \to \mathcal{M}_{U'} \to 0,$$

together with the isomorphism

$$\mathcal{M}_c = i_c^* \mathcal{M}_{U'}[-1] \ (c \in U').$$

Applying the nearby cycle functor ψ_{t_r} to (4.2.2), we get an exact sequence of mixed Hodge modules on X

(4.2.3) $$0 \to \psi_{t_r} \mathbf{Q}_{h,\mathcal{Y}_{U'}}[n] \to \mathcal{M}'(s_1', \ldots, s_r') \to \psi_{t_r} \mathcal{M}_{U'}[-1] \to 0.$$

Here we use the well-definedness of (2.4.4) which implies that ψ_{t_r} can be defined for mixed Hodge modules defined on the complement of $\{t_r = 0\}$. We have moreover the short exact sequence

(4.2.4) $$0 \to \mathbf{Q}_{h,X}[n] \to \psi_{t_r} \mathbf{Q}_{h,\mathcal{Y}_{U'}}[n] \to \varphi_{t_r} \mathbf{Q}_{h,\mathcal{Y}}[n] \to 0,$$

since X is a complete intersection so that $\mathbf{Q}_X[n]$ is a perverse sheaf. These exact sequences imply the injectivity of the natural morphism

$$\mathbf{Q}_{h,X}[n] \to \mathcal{M}'(s_1', \ldots, s_r').$$

Let $\mathcal{M}(s_1', \ldots, s_r')$ be its cokernel. Then we have a short exact sequence of mixed Hodge modules

(4.2.5) $$0 \to \varphi_{t_r} \mathbf{Q}_{h,\mathcal{Y}}[n] \to \mathcal{M}(s_1', \ldots, s_r') \to \psi_{t_r} \mathcal{M}_{U'}[-1] \to 0.$$

Set $f := (s_r/s_r')|_{Y \setminus X'}$ as in the introduction. Since $\Sigma := \mathrm{Sing}\, X \supset \mathrm{Sing}\, Y$, the support of $\varphi_f \mathbf{Q}_{h,Y}$ is contained in $\Sigma \setminus X'$, and $\varphi_f \mathbf{Q}_{h,Y}[n]$ can be viewed as a mixed Hodge module on $\Sigma \setminus X'$. By Proposition (4.1) we get

(4.2.6) $$\varphi_{t_r} \mathbf{Q}_{h,\mathcal{Y}}[n] = (i_{\Sigma \setminus X', \Sigma})_! \, \varphi_f \mathbf{Q}_{h,Y}[n],$$

where $i_{X \setminus X', X} : X \setminus X' \hookrightarrow X$ is the inclusion, and the left-hand side is viewed as a mixed Hodge module on Σ. So the proof of Theorem 1 is reduced to Proposition (4.3) below, except for the independence of the s_j'. To show the latter, we consider the one-parameter families

$$s'_{j,\lambda} := (1-\lambda)s_j' + \lambda s_j'' \ (\lambda \in \mathbf{C}) \quad \text{for } j \in [1,r],$$

if we are given sufficiently general s'_j and s''_j for $j \in [1, r]$, where λ is independent of $j \in [1, r]$. Then the assertion follows from Lemma (2.6). (See also the proof of Proposition (4.4) below.)

4.3. Proposition. *With the above notation, $\psi_{t_r} \mathcal{M}_{U'}[-1]$ is supported in Σ', and*

$$M_y(X') = (-1)^n T_{y*} \big(\psi_{t_r} \mathcal{M}_{U'}[-1] \big).$$

Proof. By inductive hypothesis, we have

$$\operatorname{supp} \mathcal{M}_c \subset \operatorname{Sing} Y_c = \Sigma' \quad (c \in U'),$$

where the last equality holds since s'_r is assumed to be sufficiently general. So $\mathcal{M}_{U'}$ is viewed as a mixed Hodge module on $\Sigma \times U'$, and this can be extended to a mixed Hodge module on $\Sigma \times \mathbf{C}$ by extendability of mixed Hodge modules in the algebraic case, where $\mathbf{C}$ is the complement of some point of $\mathbf{P}^1$. By Proposition (3.3) together with (2.4.2) and Lemma (2.6), we then get the assertion. (Note that the non-characteristic restriction i_c^* for any $c \in U'$ trivially commutes with DR_y up to the factor as in Proposition (3.3).) This finishes the proofs of Proposition (4.3) and Theorem 1.

4.4. Proposition. *In Theorem 1, $M_y(X)$ is independent of the choice of the s_j if $([s_j])$ belongs to a sufficiently small non-empty Zariski-open subset $\mathcal{U}_I$ of $\prod_{j=1}^r \mathbf{P}(I_{X,a_j})$.*

Proof. Since $\mathcal{M}(s'_1, \ldots, s'_r)$ in the introduction may depend also on $s_1, \ldots, s_r$, we denote it by $\mathcal{M}(\mathbf{s}, \mathbf{s}')$ with $\mathbf{s} := (s_1, \ldots, s_r)$, $\mathbf{s}' := (s'_1, \ldots, s'_r)$. By the construction of $\mathcal{M}(\mathbf{s}, \mathbf{s}')$ in the introduction, there is a non-empty Zariski-open conical subset $\mathcal{U}_{I,R}$ of

$$\prod_{j=1}^r I_{X,a_j} \times \prod_{j=1}^r R_{L,a_j},$$

together with a mixed Hodge module $\mathcal{M}_{\mathcal{U}_{I,R}}$ on $\Sigma \times \mathcal{U}_{I,R}$ such that $\mathcal{M}(\mathbf{s}, \mathbf{s}')$ is the restriction of $\mathcal{M}_{\mathcal{U}_{I,R}}$ by the inclusion $\Sigma \times \{\mathbf{s}, \mathbf{s}'\} \hookrightarrow \Sigma \times \mathcal{U}_{I,R}$, and moreover this inclusion is strictly non-characteristic for the underlying perverse sheaf $\mathcal{F}_{\mathcal{U}_{I,R}}$ of $\mathcal{M}_{\mathcal{U}_{I,R}}$, i.e. $\mathcal{F}_{\mathcal{U}_{I,R}}$ is a topologically locally constant family of perverse sheaves parametrized by $\mathcal{U}_{I,R}$. (This can be shown for instance by using [BMM] together with a Whitney stratification of the total space, since its restriction to a sufficiently general fiber is a Whitney stratification.)

We then get the independence of $M_y(X)$ by the choices of $\mathbf{s}, \mathbf{s}'$ using Lemma (2.6) together with a one-parameter family as in the last part of the proof of Theorem 1 in (4.2). Since the independence of $\mathbf{s}'$ is already shown, the assertion follows.

4.5. Remarks. (i) Let $\mathcal{Z}_T \to T$ be as in the introduction. If $X = \mathcal{Z}_0$ has an isolated singular point x, then we have a mixed Hodge structure on the vanishing cohomology. This can be defined by restricting the morphism $\mathcal{Z}_T \to T$ over a generic smooth curve on T passing through 0 and using an embedded resolution of singularities as in [St2]. (This can be calculated also by using mixed Hodge modules as in [BS], Th. 4.3.) The associated mixed Hodge numbers are independent of the choice of the generic smooth curve by the theory of spectra for arbitrary varieties which uses the deformation to the normal cone (see e.g. [DMS], Remark (1.3)(i)). By a similar argument these mixed Hodge numbers coincide with those given by Corollary 2 assuming that $s_1, \ldots, s_{r-1}$ and $s'_1, \ldots, s'_r$ are sufficiently general (by using Proposition (3.2)).

(ii) For a germ of a complete intersection with an isolated singularity $(X, 0)$, there is a versal flat deformation $\rho : (\mathcal{X}, 0) \to (S, 0)$ in the following sense: any flat deformation $(\mathcal{X}', 0) \to (S', 0)$ of $(X, 0)$ is analytically isomorphic to the pull-back of ρ by some complex analytic morphism $(S', 0) \to (S, 0)$, see [KS], [Tju]. Choosing a generic smooth curve on S passing through 0, we also get a mixed Hodge structure on the vanishing cohomology, and the associated mixed Hodge numbers are independent of the choice of the generic curve, see

also [ES]. Moreover these numbers coincide with those given in Remark (i) above by using [DMS], Cor. 3.4.

5. Explicit calculations of the Hirzebruch-Milnor classes

In this section we show how the Hirzebruch-Milnor classes can be calculated explicitly.

5.1. Stratification. The first term of the right-hand side of (0.2) in Theorem 1 can be described explicitly as follows. Let $\mathcal{S}$ be a complex algebraic stratification of $\Sigma \setminus X'$ such that the $\mathcal{H}^j \varphi_f \mathbf{Q}_Y|_S$ are local systems for any $j \in \mathbf{N}$ and $S \in \mathcal{S}$. Then $\mathcal{H}^j \varphi_f \mathbf{Q}_Y|_S$ underlies the variation of mixed Hodge structure $\mathcal{H}^j i^*_{S,\Sigma \setminus X'} \varphi_f \mathbf{Q}_{h,Y}$. Here it is enough to assume that each stratum $S \in \mathcal{S}$ is a locally closed smooth subvariety and the Whitney conditions are not needed as long as we have local systems on each stratum. We have the following.

5.2. Proposition. *With the above notation and assumption, we have*

$$(5.2.1) \qquad T_{y*}\big((i_{\Sigma \setminus X',\Sigma})_! \varphi_f \mathbf{Q}_{h,Y}\big) = \sum_{j \in \mathbf{N}, S \in \mathcal{S}} (-1)^j \, T_{y*}\big((i_{S,\Sigma})_!(\mathcal{H}^j i^*_{S,\Sigma \setminus X'} \varphi_f \mathbf{Q}_{h,Y})\big).$$

Indeed, this follows from the following.

5.3. Proposition. *Let X be a complex algebraic variety. Let $\mathcal{M}^\bullet \in D^b\mathrm{MHM}(X)$, and $K^\bullet$ its underlying $\mathbf{Q}$-complex. Let $\mathcal{S} = \{S\}$ be a complex algebraic stratification of X such that for any $S \in \mathcal{S}$, S is smooth, $\bar{S} \setminus S$ is a union of strata, and the $\mathcal{H}^i K^\bullet|_S$ are local systems on S for any i. Let $j_S : S \hookrightarrow X$ denote the inclusion. Then*

$$(5.3.1) \qquad T_{y*}(\mathcal{M}^\bullet) = \sum_{S,i} (-1)^i \, T_{y*}\big((j_S)_! \mathcal{H}^i(j_S)^* \mathcal{M}^\bullet\big),$$

where $\mathcal{H}^i$ for $(j_S)^*\mathcal{M}^\bullet$ is associated to the classical t-structure as in [Sa2], 4.6.2 (which coincides with the usual t-structure on the derived category of mixed Hodge modules up to a shift if it is restricted to a stratum of the stratification) so that $\mathcal{H}^i K^\bullet|_S$ underlies variations of mixed Hodge structures $\mathcal{H}^i(j_S)^*\mathcal{M}^\bullet$.

Proof. Set $X_k := \bigcup_{\dim S \le k} S \subset X$. We have the canonical inclusions

$$j_k : X_k \setminus X_{k-1} \hookrightarrow X_k, \quad i_k : X_k \hookrightarrow X,$$

together with the distinguished triangles

$$(5.3.2) \qquad (i_k)_*(j_k)_! j_k^* i_k^* \mathcal{M}^\bullet \to (i_k)_* i_k^* \mathcal{M}^\bullet \to (i_{k-1})_* i_{k-1}^* \mathcal{M}^\bullet \xrightarrow{+1}.$$

These imply the following identity in the Grothendieck group of $D^b\mathrm{MHM}(X)$

$$(5.3.3) \qquad [\mathcal{M}^\bullet] = \sum_k \big[(i_k)_*(j_k)_! j_k^* i_k^* \mathcal{M}^\bullet\big].$$

Moreover, j_S is the composition of $j'_S : S \hookrightarrow X_k$ with i_k where $k = \dim S$. So the assertion follows.

5.4. Logarithmic forms. The right-hand side of (5.2.1) can be described as follows. Let $\mathcal{M}$ be an admissible variation of mixed Hodge structure on a stratum S more generally, where we may assume that $\mathcal{M}$ is a polarizable variation of Hodge structure by taking the graded pieces of the weight filtration W. Let M be the underlying $\mathcal{O}_S$-module with the Hodge filtration F of the variation of mixed Hodge structure. Take a smooth partial compactification $i_{S,Z} : S \hookrightarrow Z$ such that $D := Z \setminus S$ is a divisor with simple normal crossings and moreover $i_{S,\Sigma} = \pi_Z \circ i_{S,Z}$ for a proper morphism $\pi_Z : Z \to \Sigma$. Here D cannot be empty, since we take a stratification of $\Sigma \setminus X'$ and X' is a hyperplane section. Let $M_Z^{\ge 0}$ be the Deligne extension with eigenvalues of the residues of the logarithmic connection contained in $(0,1]$. Then we have the following.

5.5. Proposition. *With the above notation, we have*

(5.5.1) $T_{y*}\big((i_{S,\Sigma})_!\mathcal{M}\big) = \sum_{p,q} (-1)^q (\pi_Z)_* td_{(1+y)*} \big[\mathrm{Gr}_F^p M_Z^{\geq 0} \otimes \Omega_Z^q (\log D)\big](-y)^{p+q}.$

Proof. Let (M_Z, F) be the underlying filtered $\mathcal{D}_Z$-module of $(i_{S,Z})_!\mathcal{M}$. Then we have a canonical inclusion

(5.5.2) $(M_Z^{\geq 0} \otimes \Omega_Z^\bullet (\log D), F)[\dim Z] \hookrightarrow \mathrm{DR}_Z(M_Z, F),$

where the filtration F on $(M_Z^{\geq 0} \otimes \Omega_Z^p (\log D)$ is shifted by $-p$ as usual. Moreover, it is a filtered quasi-isomorphism by [Sa2], Prop. 3.11, see also [CMSS], 3.3. So the assertion follows from the commutativity of td_* with the pushforward by proper morphisms [BFM], see also [CMSS], 3.3.

5.6. Motivic Milnor fibers. It is also possible to use the theory of motivic nearby fibers [DL] (see also [St1]) to calculate the first term of the right-hand side of (0.2). Let $\sigma : (\widetilde{Y}, \widetilde{X}) \to (Y, X)$ be an embedded resolution inducing the isomorphism outside the singular locus of X_{red} and such that $\widetilde{X} := \sigma^{-1}(X)$ and $\sigma^{-1}(\Sigma)$ are divisors with simple normal crossings, where simple means that the irreducible components E_i of $\widetilde{X}$ are smooth. (Such a condition for $\sigma^{-1}(\Sigma)$ is satisfied if it is a divisor and the other conditions are satisfied.) Let m_i be the multiplicity of $\widetilde{X}$ along E_i $(i = 1, \ldots, s)$. Put $\widetilde{X}' := \sigma^{-1}(X')$. For $I \subset \{1, \ldots, s\}$, set

$$E_I := \bigcap_{i \in I} E_i, \quad E_I' := E_I \setminus \widetilde{X}', \quad E_I'^{o} := E_I' \setminus \big(\bigcup_{i \notin I} E_i\big).$$

Let $\tilde{f}$ be the pull-back of f to $\widetilde{Y} \setminus \widetilde{X}'$. There are smooth varieties $\widetilde{E}_I'^{o}$ together with finite étale morphisms $\gamma_I : \widetilde{E}_I'^{o} \to E_I'^{o}$ such that $(\gamma_I)_* \mathbf{Q}_{\widetilde{E}_I'^{o}} = \mathcal{H}^0 \psi_{\tilde{f}} \mathbf{Q}_{\widetilde{Y}}$, see [DL]. Note that we can get these varieties by taking the normalization of the base change of $\tilde{f}$ by a ramified m-fold covering of an open disk where $m = \mathrm{LCM}(m_i)$, see [St1]. They may be called the Stein factorization of the Milnor fibers, see [Lo]. Let $\tilde{j}_I : \widetilde{E}_I'^{o} \hookrightarrow \widetilde{E}_I$ be a smooth compactification such that γ_I is extended to $\widetilde{\gamma}_I : \widetilde{E}_I \to E_I$. We have canonical morphisms

$$i_{I,J} : \widetilde{E}_{I,J} \hookrightarrow \widetilde{E}_I, \quad \sigma_\Sigma := \sigma|_{\sigma^{-1}(\Sigma)} : \sigma^{-1}(\Sigma) \to X.$$

Set $\Lambda' := \{I \subset \{1, \ldots, r\} \mid E_I \subset \sigma^{-1}(\Sigma)\}$. Then we have the following.

5.7. Proposition. *With the above notation and assumption, we have*

(5.7.1) $\begin{aligned} T_{y*}\big((i_{\Sigma \setminus X', \Sigma})_! \varphi_f \mathbf{Q}_{h,Y}\big) &= \sum_{I \in \Lambda'} (\sigma_\Sigma \circ \widetilde{\gamma}_I)_* T_{y*}\big((\tilde{j}_I)_! \mathbf{Q}_{h, \widetilde{E}_I'^{o}}\big)(1+y)^{|I|-1} \\ &\quad - T_{y*}(\Sigma) + (i_{\Sigma \cap X', \Sigma})_* T_{y*}(\Sigma \cap X'). \end{aligned}$

Proof. Let $h : \mathcal{Z}' \to \mathbf{C}$ be the normalization of the base change of $\tilde{f} : \widetilde{Y}' := \widetilde{Y} \setminus \widetilde{X}' \to \mathbf{C}$ by the m-fold ramified covering $\mathbf{C} \to \mathbf{C}$ defined by $t \mapsto t^m$ where $m = \mathrm{LCM}(m_i)$ with m_i the multiplicity of $\widetilde{X}$ along E_i. Set $\mathcal{Z}_0' := h^{-1}(0)$. There is a canonical morphism $\gamma : \mathcal{Z}_0' \to \widetilde{X} \setminus \widetilde{X}'$. Set

$$\widetilde{E}_I'^{o} := \gamma^{-1}(E_I'^{o}), \quad m_I := \mathrm{GCD}\big(m_i \, (i \in I)\big).$$

Then the induced morphism $\gamma_I : \widetilde{E}_I'^{o} \to E_I'^{o}$ is finite étale of degree m_I. By [St1] we have

$$\mathcal{H}^i \psi_h \mathbf{Q}_{\mathcal{Z}'}|_{\widetilde{E}_I'^{o}} = \overset{\nu(I,i)}{\bigoplus} \mathbf{Q}_{\widetilde{E}_I'^{o}}(-i) \quad \text{with} \quad \nu(I,i) = \binom{|I|-1}{i},$$

since $\mathcal{Z}_0'$ is a divisor with V-normal crossings on a V-manifold $\mathcal{Z}'$. So we get

$$\mathcal{H}^i \psi_{\tilde{f}} \mathbf{Q}_{\widetilde{Y}'}|_{E_I'^{o}} = \overset{\nu(I,i)}{\bigoplus} (\gamma_I)_* \mathbf{Q}_{\widetilde{E}_I'^{o}}(-i).$$

This is an isomorphism as $\mathbf{Q}$-complexes. However, it holds as variations of Hodge structures of type (i, i) since the Hodge filtration is trivial.

Let $j : X \setminus X' \hookrightarrow X$ denote the inclusion. By Proposition (4.1), we have

$$\varphi_{\pi^* t} \mathbf{Q}_{h, \mathcal{Y}} = j_! \varphi_f \mathbf{Q}_{h, Y \setminus X'} = (i_\Sigma)_* i_\Sigma^* j_! \big(C(\mathbf{Q}_{h, X \setminus X'} \to \psi_f \mathbf{Q}_{h, Y \setminus X'}) \big).$$

Here the second isomorphism follows from the distinguished triangle

$$\mathbf{Q}_{h, X \setminus X'} \to \psi_f \mathbf{Q}_{h, Y \setminus X'} \to \varphi_f \mathbf{Q}_{h, Y \setminus X'} \xrightarrow{+1},$$

together with the fact that $\varphi_f \mathbf{Q}_{h, Y \setminus X'}$ is supported on $\Sigma \setminus X'$.

By the commutativity of the nearby cycle functor with the direct image by a proper morphism, we have

$$\psi_f \mathbf{Q}_{h, Y \setminus X'} = (\sigma_0')_* \psi_{\tilde{f}} \mathbf{Q}_{h, \widetilde{Y} \setminus \widetilde{X}'},$$

where $\sigma_0' : \widetilde{X} \setminus \widetilde{X}' \to X \setminus X'$ is the restriction of σ. So the assertion follows from Proposition (5.3). This finishes the proof of Proposition (5.7).

5.8. Remarks. (i) The term $(1 + y)^{|I| - 1}$ in (5.7.1) comes from the mixed Hodge structure on the cohomology of the irreducible components of the Milnor fiber which is homeomorphic to $(\mathbf{C}^*)^{|I| - 1}$ in the normal crossing case and corresponds to $(1 - \mathbf{L})^{|I| - 1}$ in [DL]. (Note that the stalks of the nearby cycle sheaves are given by the cohomology of the Milnor fibers.) Here we may assume that $\widetilde{E}_I \setminus \widetilde{E}_I'^\circ$ is a divisor with simple normal crossings replacing $\widetilde{E}_I$ if necessary. In this case, let $\widetilde{E}_{I,j}$ be the irreducible components of $\widetilde{E}_I \setminus \widetilde{E}_I'^\circ$ which are assumed smooth. Set $\widetilde{E}_{I,J} := \bigcap_{j \in J} \widetilde{E}_{I,j}$ where $\widetilde{E}_{I,\emptyset} = \widetilde{E}_I$ if $J = \emptyset$. Then the usual resolution argument implies

(5.8.1) $$T_{y*}\big((\widetilde{j}_I)_! \mathbf{Q}_{h, \widetilde{E}_I'^\circ}\big) = \sum_J (-1)^{|J|} (i_{I,J})_* T_{y*}(\widetilde{E}_{I,J}).$$

(ii) In Theorem 1 and its corollaries and also in Propositions of this section, most of the formulas hold also on the level of the Grothendieck group of mixed Hodge modules and hence on that of coherent sheaves by using DR_y (except for (5.5.1) which is also valid, or rather meaningful, only on the level of the Grothendieck group of coherent sheaves).

REFERENCES

[BFM]   Baum, P., Fulton, W. and MacPherson, R., Riemann-Roch for singular varieties, Inst. Hautes Etudes Sci. Publ. Math. 45 (1975), 101–145.

[BBD]   Beilinson, A., Bernstein, J. and Deligne, P., Faisceaux pervers, Astérisque, 100, Soc. Math. France, Paris, 1982.

[Bi]   Bittner, F., On motivic zeta functions and motivic nearby cycles, Math. Z. 249 (2005), 63–83.

[BSY]   Brasselet, J.-P., Schürmann, J. and Yokura, S., Hirzebruch classes and motivic Chern classes of singular spaces, Journal of Topology and Analysis 2, (2010), 1–55.

[BMM]   Briançon, J., Maisonobe, P. and Merle, M., Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom, Invent. Math. 117 (1994), 531–550.

[BS]   Budur, N. and Saito, M., Multiplier ideals, V-filtration, and spectrum, J. Alg. Geom. 14 (2005), 269–282.

[CMSS]   Cappell, S.E., Maxim, L., Schürmann, J. and Shaneson, J.L., Characteristic classes of complex hypersurfaces, Adv. Math. 225 (2010), 2616–2647.

[Ca]   Carlson, J., Extensions of mixed Hodge structures, in Journées de Géométrie Algébrique d'Angers 1979, Sijthoff-Noordhoff Alphen a/d Rijn, 1980, 107–128.

[D1]   Deligne, P., Théorie de Hodge, II, Inst. Hautes Etudes Sci. Publ. Math. 40 (1971), 5–57.

[D2]   Deligne, P., Le formalisme des cycles évanescents, in SGA7 XIII and XIV, Lect. Notes in Math. 340, Springer, Berlin, 1973, 82–115 and 116–164.

[D3]   Deligne, P., Théorie de Hodge, III, Inst. Hautes Etudes Sci. Publ. Math. 44 (1974), 5–77.

[DL]   Denef, J. and Loeser, F., Motivic Igusa zeta functions, J. Alg. Geom. 7 (1998), 505–537.

[DMS]   Dimca, A., Maisonobe, Ph. and Saito, M., Spectrum and multiplier ideals of arbitrary subvarieties, Ann. Inst. Fourier 61 (2011), 1633–1653.

[DMST]   Dimca, A., Maisonobe, Ph., Saito, M. and Torrelli, T., Multiplier ideals, V-filtrations and transversal sections, Math. Ann. 336 (2006), 901–924.

[DB]   Du Bois, Ph., Complexe de De Rham filtré d'une variété singulière, Bull. Soc. Math. France 109 (1981) 41–81.

[Ei]   Eisenbud, D., Commutative algebra with a view toward algebraic geometry, Springer, 1994.

[ES]   Ebeling, W. and Steenbrink, J.H.M., Spectral pairs for isolated complete intersection singularities, J. Algebraic Geom. 7 (1998), 55–76.

[Fu1]   Fulton, W., Rational equivalence on singular varieties, Inst. Hautes Etudes Sci. Publ. Math. 45 (1975), 147–167.

[Fu2]   Fulton, W., Intersection theory, Springer, Berlin, 1984.

[FJ]   Fulton, W. and Johnson, K., Canonical classes on singular varieties, Manuscripta Math. 32 (1980), 381–389.

[FM]   Fulton, W. and MacPherson, R., Categorical framework for the study of singular spaces, Mem. Amer. Math. Soc. 243 (1981).

[Ha]   Hamm, H., Lokale topologische Eigenschaften komplexer Räume, Math. Ann. 191 (1971), 235–252.

[Hi]   Hirzebruch, F., Topological methods in algebraic geometry, Springer-Verlag, Berlin, 1966.

[HBJ]   Hirzebruch, F., Berger, T. and Jung, R., Manifolds and modular forms, Aspects of Mathematics, E20, Friedr. Vieweg & Sohn, Braunschweig, 1992.

[KS]   Kas, A. and Schlessinger, M., On the versal deformation of a complex space with an isolated singularity, Math. Ann. 196 (1972), 23–29.

[Ke1]   Kennedy, G., MacPherson's Chern classes of singular algebraic varieties, Comm. Algebra 18 (1990), 2821–2839.

[Ke2]   Kennedy, G., Specialization of MacPherson's Chern classes, Math. Scand. 66 (1990), 12–16.

[LM]   Libgober, A. and Maxim, L., Hodge polynomials of singular hypersurfaces, Michigan Math. J. 60 (2011), 661–673.

[Lo]   Looijenga, E., Motivic measures, in Séminaire Bourbaki 1999/2000, Astérisque 276 (2002), 267–297.

[Ma]   MacPherson, R.D., Chern classes for singular algebraic varieties, Ann. of Math. (2) 100 (1974), 423–432.

[Mi]   Milnor, J., Singular points of complex hypersurfaces, Ann. Math. Stud. vol. 61, Princeton Univ. Press, 1968.

[Pa]   Parusiński A., Limits of tangent spaces to fibres and the w_f condition, Duke Math. J. 72 (1993), 99–108.

[PP]   Parusiński A. and Pragacz, P., Characteristic classes of hypersurfaces and characteristic cycles, J. Alg. Geom. 10 (2001), 63–79.

[Sa1]   Saito, M., Modules de Hodge polarisables, Publ. RIMS, Kyoto Univ. 24 (1988), 849–995.

[Sa2]   Saito, M., Mixed Hodge modules, Publ. RIMS, Kyoto Univ. 26 (1990), 221–333.

[Sa3]   Saito, M., Mixed Hodge complexes on algebraic varieties, Math. Ann. 316 (2000), 283–331.

[Sch1]   Schürmann, J., Characteristic classes of mixed Hodge modules, in Topology of Stratified Spaces, MSRI Publications Vol. 58, Cambridge University Press (2011), 419–471.

[Sch2]   Schürmann, J., Specialization of motivic Hodge-Chern classes, arXiv:0909.3478.

[Sch3]   Schürmann, J., Nearby cycles and characteristic classes of singular spaces, IRMA Lectures in Mathematics and Theoretical Physics, European Math. Soc., Vol. 20 (2012), 181–205. [arXiv:1003.2343]

[SeSu]   Seade, J. and Suwa,T., An adjunction formula for local complete intersections, Int. J. Math. 9. (1998), 759–768

[St1]   Steenbrink, J.H.M., Mixed Hodge structure on the vanishing cohomology, in Real and complex singularities (Proc. Ninth Nordic Summer School, Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, 525–563.

[St2]   Steenbrink, J.H.M., Mixed Hodge structures associated with isolated singularities, Proc. Sympos. Pure Math., 40, A.M.S., Providence, RI, 1983, 513–536.

[St3]   Steenbrink, J.H.M., The spectrum of hypersurface singularities, Astérisque 179-180 (1989), 163–184.

[Su]   Suwa, T., Classes de Chern des intersections complètes locales, C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), 67–70.

[To]   Totaro, B., Chow groups, Chow cohomology, and linear varieties, preprint, to appear in J. Alg. Geom.

[Tju]   Tjurina, G.N., Locally semi-universal flat deformations of isolated singularities of complex spaces, Izv. Akad. Nauk SSSR, Ser. Mat. 33 (1969), 1026–1058.

[Ve1]   Verdier, J.-L., Le théorème de Riemann-Roch pour les intersections complètes, Astérisque 36–37 (1976), 189–228.

[Ve2]   Verdier, J.-L., Spécialisation des classes de Chern, Astérisque 82–83 (1981), 149–159.

[Yo1]   Yokura, S., A generalized Grothendieck-Riemann-Roch theorem for Hirzebruch's χ_y-characteristic and T_y-characteristic, Publ. RIMS, Kyoto Univ. 30 (1994), 603–610.

[Yo2]   Yokura, S., On a Milnor class, unpublished preprint, 1997.

[Yo3]   Yokura, S., On characteristic classes of complete intersections, in Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), Contemp. Math., 241, Amer. Math. Soc., Providence, RI, 1999, 349–369.

[Yo4]   Yokura, S., Motivic Milnor classes, Journal of Singularities, 1 (2010), 39–59.

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