Euler Characteristics of Algebraic Varieties

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Abstract

This note studies the behavior of Euler characteristics and of intersection homology Euler characteristics under proper morphisms of algebraic (respectively, analytic) varieties. The methods also yield, for algebraic (respectively, analytic) varieties, formulae comparing these two kinds of Euler characteristics. The main results are direct consequences of the calculus of constructible functions and Grothendieck groups of constructible sheaves. Similar formulae for Hodge-theoretic invariants of algebraic varieties under morphisms were announced by the first and third authors in [5, 14]. © 2007 Wiley Periodicals, Inc.

1 Introduction

We study the behavior of (intersection homology) Euler characteristics under proper morphisms of complex algebraic (respectively, analytic) varieties. We begin by discussing simple formulae for the usual Euler-Poincaré characteristic, then show that similar formulae hold for the intersection homology Euler characteristic as well as for the corresponding Chern homology classes of MacPherson. The methods used in the present paper also yield formulae expressing the Euler characteristics of usual and intersection homology of an algebraic (respectively, analytic) variety in terms of each other and corresponding invariants of the subvarieties formed by the closures of its singular strata.

The main results of this note are direct applications of the standard calculus of constructible functions and Grothendieck groups of constructible sheaves. Some of the formulae on the intersection homology Euler characteristic were originally proven with the aid of the deep decomposition theorem of Bernstein, Beilinson, Deligne, and Gabber (in short, BBDG) for the pushforward of an intersection homology complex under a proper morphism (cf. [1, 6]). The functorial approach employed here was suggested by the referee. However, the core calculations used in proving these results are modeled on our original approach based on BBDG.

Communications on Pure and Applied Mathematics, Vol. LXI, 0409–0421 (2008)

This note is a first step in an ongoing project that deals with the study of genera of complex algebraic (respectively, analytic) varieties. In forthcoming papers [3, 4] we will discuss the behavior of Hodge-theoretic genera under proper morphisms and provide explicit formulae for the pushforward of various characteristic classes. The functorial approach and the language of Grothendieck groups of constructible sheaves used in this paper allow a simple translation of the underlying ideas to the forthcoming papers, where Grothendieck groups of Saito's algebraic mixed Hodge modules will be employed.

Unless otherwise specified, all homology and intersection homology groups in this paper are those with rational coefficients. We assume the reader's familiarity with intersection homology, and for some arguments also with (Grothendieck groups of) constructible sheaves and derived categories. However, our results are also explained in the simpler language of constructible functions, which relies only on Euler characteristic information.

2 Topological Euler-Poincaré Characteristic

For a complex algebraic variety X, let $\chi(X)$ denote its topological Euler characteristic. Then $\chi(X)$ equals the compactly supported Euler characteristic, $\chi_c(X)$ (cf. [8, p. 141], [12, §6.0.6]).[1] The additivity property for the Euler characteristic reads as follows: for Z a Zariski closed subset of X, the long exact sequence of the compactly supported cohomology

$$\cdots \to H^i_c(X \setminus Z) \to H^i_c(X) \to H^i_c(Z) \to H^{i+1}_c(X \setminus Z) \to \cdots$$

yields that $\chi_c(X) = \chi_c(Z) + \chi_c(X \setminus Z)$; therefore the same relation holds for χ. The multiplicative property for fibrations asserts that if $F \hookrightarrow E \to B$ is a locally trivial topological fibration such that the three Euler characteristics $\chi(B)$, $\chi(F)$, and $\chi(E)$ are defined, then $\chi(E) = \chi(B) \cdot \chi(F)$ (e.g., see [7, cor. 2.5.5]). In particular, if $f : X \to Y$ is a proper smooth submersion of smooth manifolds, with Y connected and generic fiber F, then

$$\chi(X) = \chi(Y) \cdot \chi(F). \tag{2.1}$$

Indeed, by Ehresmann's theorem, such a map is a locally trivial fibration in the complex topology.

In this note we generalize this multiplicative property of proper smooth submersions in two different directions: first, we study the behavior of the usual Euler characteristic under arbitrary proper maps of possibly singular varieties; second, we replace the usual cohomology by intersection cohomology when dealing with singular varieties and study the behavior of the intersection homology Euler characteristic under arbitrary proper morphisms. The formulae we obtain here are classically referred to as the *stratified multiplicative property* for Euler characteristics (cf. [5, 14]).

[1] This fact is not true outside the category of complex varieties (e.g., if X is an oriented n-dimensional topological manifold, then Poincaré duality yields that $\chi_c(X) = (-1)^n \chi(X)$).

Let Y be a topological space with a finite partition $\mathcal{V}$ into a disjoint union of finitely many connected subsets V satisfying the *frontier condition*: "$V \cap \bar{W} \neq \varnothing$ implies that $V \subset \bar{W}$." (The main examples of such spaces are complex algebraic or compact analytic varieties with a fixed Whitney stratification.) Then $\mathcal{V}$ is partially ordered by "$V \leq W$ if and only if $V \subset \bar{W}$." Let $F_{\mathcal{V}}(Y)$ be the abelian group of $\mathcal{V}$-constructible functions on Y, i.e., of functions $\alpha : Y \to \mathbb{Z}$ such that $\alpha|_V$ is constant for all $V \in \mathcal{V}$. This is a free abelian group with basis $\{1_V | V \in \mathcal{V}\}$, so that

$$\alpha = \sum_{V \in \mathcal{V}} \alpha(V) \cdot 1_V .$$

Note that $\{1_{\bar{V}} | V \in \mathcal{V}\}$ is another basis for $F_{\mathcal{V}}(Y)$, since

$$1_{\bar{V}} = \sum_{W \leq V} 1_W$$

and the matrix $A = (a_{W,V})$, with $a_{W,V} := 1$ for $W \leq V$ and 0 otherwise, is upper-triangular with respect to $\leq$, with all diagonal entries equal to 1. Thus A is invertible. The nonzero entries of $A^{-1} = (a'_{W,V})$ can inductively be calculated (e.g., see [15, prop. 3.6.2]) by $a'_{V,V} = 1$ and, for $W < V$,

$$a'_{W,V} = -\sum_{W \leq S < V} a'_{W,S} \cdot a_{S,V} .$$

This implies the following:

PROPOSITION 2.1 *For each $V \in \mathcal{V}$, define inductively $\hat{1}_{\bar{V}}$ by the formula*

$$\hat{1}_{\bar{V}} = 1_{\bar{V}} - \sum_{W < V} \hat{1}_{\bar{W}} .$$

Then, for any $\alpha \in F_{\mathcal{V}}(Y)$, one has the equality

$$\alpha = \sum_{V} \alpha(V) \cdot \hat{1}_{\bar{V}} . \tag{2.2}$$

PROOF: As the notation indicates, $\hat{1}_{\bar{V}}$ depends only on the space $\bar{V}$ with its induced partition. Then by the above considerations we have

$$\alpha = \sum_{V} \alpha(V) \cdot 1_V = \sum_{W \leq V} \alpha(V) \cdot a'_{W,V} \cdot 1_{\bar{W}} ,$$

and formula (2.2) follows from the inductive identification (for V fixed):

$$\sum_{W \leq V} a'_{W,V} \cdot 1_{\bar{W}} = 1_{\bar{V}} - \sum_{W \leq S < V} a'_{W,S} \cdot a_{S,V} \cdot 1_{\bar{W}} = \hat{1}_{\bar{V}} .$$

□

Remark 2.2.

(1) If there is a stratum $S \in \mathcal{V}$ that is dense in Y, i.e., $\bar{S} = Y$ or $V \leq S$ for all $V \in \mathcal{V}$, then formula (2.2) can be rewritten as

$$\alpha = \alpha(S) \cdot 1_Y + \sum_{V<S} (\alpha(V) - \alpha(S)) \cdot \hat{1}_{\bar{V}}. \tag{2.3}$$

(2) For a group homomorphism $\phi : F_{\mathcal{V}}(Y) \to G$ for some abelian group G, one obtains similar descriptions for $\phi(\alpha)$ in terms of

$$\hat{\phi}(\bar{V}) := \phi(\hat{1}_{\bar{V}}) = \phi(1_{\bar{V}}) - \sum_{W<V} \phi(\hat{1}_{\bar{W}}).$$

For the rest of this section we specialize to the complex algebraic (respectively, compact analytic) context, with Y a reduced complex algebraic variety (respectively, a reduced compact complex analytic space), and all $V \in \mathcal{V}$ locally closed constructible subsets. The group $F_c(Y)$ of all complex algebraically (respectively, analytically) constructible functions is defined as the direct limit of these $F_{\mathcal{V}}(Y)$. Then one has the following important group homomorphisms on $F_c(Y)$ (e.g., see [10, 11, 12, 13]):

(1) The Euler characteristic with compact support $\chi_c : F_c(Y) \to \mathbb{Z}$ characterized by $\chi_c(1_Z) = \chi_c(Z)$ for $Z \subset Y$ a locally closed constructible subset.
(2) The Euler characteristic $\chi : F_c(Y) \to \mathbb{Z}$ characterized by $\chi(1_Z) = \chi(Z)$ for $Z \subset Y$ a closed algebraic (respectively, analytic) subset.
(3) For $f : X \to Y$ a proper complex algebraic (respectively, analytic) map, the functorial pushdown $f_* : F_c(X) \to F_c(Y)$ is characterized by

$$f_*(1_Z)(y) = \chi(Z \cap \{f = y\})$$

for $Z \subset X$ a closed algebraic (respectively, analytic) subset.
(4) The Chern class transformation of MacPherson,

$$c_* : F_c(Y) \to H_{2*}^{\mathrm{BM}}(Y; \mathbb{Z}),$$

which commutes with proper pushdowns and is uniquely characterized by this property together with the normalization $c_*(1_M) = c^*(TM) \cap [M]$ for M a complex algebraic (respectively, analytic) manifold.

In fact, as already pointed out, in this context we have $\chi = \chi_c$. Moreover, $\chi \circ f_* = \chi$, and for Y compact one gets by functoriality $\chi(\alpha) = \deg(c_*(\alpha))$ for any $\alpha \in F_c(Y)$.

Now let $f : X \to Y$ be a proper complex algebraic (respectively, analytic) map, with Y as above. Assume $f_*(\alpha) \in F_{\mathcal{V}}(Y)$ for a given $\alpha \in F_c(X)$ (e.g., $\mathcal{V}'$ and $\mathcal{V}$ are complex Whitney stratifications of X and Y, respectively, such that f is a stratified submersion, and $\alpha \in F_{\mathcal{V}'}(X)$). Then

$$f_*(\alpha) = \sum_{V \in \mathcal{V}} f_*(\alpha)(V) \cdot 1_V$$

with

$$f_*(\alpha)(V) = \chi(\alpha|_{F_V})$$

for F_V the fiber of f over a point in V. Assume, moreover, that Y is irreducible, so there is a dense stratum $S \in \mathcal{V}$, with $F := F_S$ a general fiber of f. In terms of $f_*(\alpha)$, formula (2.3) yields the following:

$$f_*(\alpha) = \chi(\alpha|_F) \cdot 1_Y + \sum_{V<S} (\chi(\alpha|_{F_V}) - \chi(\alpha|_F)) \cdot \hat{1}_{\bar{V}}. \tag{2.4}$$

By applying the homomorphism χ and c_*, respectively, to equation (2.4), we obtain the following formulae:

COROLLARY 2.3

$$\chi(\alpha) = \chi(\alpha|_F) \cdot \chi(Y) + \sum_{V<S} (\chi(\alpha|_{F_V}) - \chi(\alpha|_F)) \cdot \hat{\chi}(\bar{V}), \tag{2.5}$$

$$f_*(c_*(\alpha)) = \chi(\alpha|_F) \cdot c_*(Y) + \sum_{V<S} (\chi(\alpha|_{F_V}) - \chi(\alpha|_F)) \cdot \hat{c}_*(\bar{V}), \tag{2.6}$$

where $c_*(Y) := c_*(1_Y)$, and similarly for $\hat{c}_*(\bar{V})$, which by the functoriality of c_* is regarded as a homology class in the Borel-Moore homology $H_{2*}^{\mathrm{BM}}(Y; \mathbb{Z})$.

By letting $\alpha = 1_X$ in the formulae (2.5) and (2.6) above, we obtained the stratified multiplicative property for the topological Euler-Poincaré characteristic and for the Chern-MacPherson class, respectively:

PROPOSITION 2.4 *Let $f : X \to Y$ be a proper complex algebraic* (*respectively, analytic*) *map, with Y irreducible* (*and compact in the analytic context*) *and endowed with a complex algebraic* (*respectively, analytic*) *Whitney stratification $\mathcal{V}$. Assume $f_*(1_X) \in F_{\mathcal{V}}(Y)$. Then*:

$$\chi(X) = \chi(F) \cdot \chi(Y) + \sum_{V<S} (\chi(F_V) - \chi(F)) \cdot \hat{\chi}(\bar{V}), \tag{2.7}$$

$$f_*(c_*(X)) = \chi(F) \cdot c_*(Y) + \sum_{V<S} (\chi(F_V) - \chi(F)) \cdot \hat{c}_*(\bar{V}). \tag{2.8}$$

3 Intersection Homology Euler Characteristics

Let Y be a topological pseudomanifold (or a locally conelike stratified space [12, p. 232]) with a stratification $\mathcal{V}$ by finitely many oriented strata of *even* dimension. By definition, strata of $\mathcal{V}$ satisfy the frontier condition, and $\mathcal{V}$ is locally topologically trivial along each stratum V, with fibers being the cone on a compact pseudomanifold $L_{V,Y}$, the "link" of V in Y. Note that each stratum V, its closure $\bar{V}$, and in general any locally closed union of strata gets an induced stratification of the same type. Examples are given by a complex algebraic (respectively, analytic) Whitney stratification of a reduced complex algebraic (respectively, compact complex analytic) variety.

Let $\mathrm{Sh}_{\mathcal{V}}(Y)$ be the category of $\mathcal{V}$-constructible sheaves of rational vector spaces, i.e., sheaves $\mathcal{F}$ with the property that for all $V \in \mathcal{V}$ the restriction $\mathcal{F}|_V$ is a locally constant sheaf of $\mathbb{Q}$-vector spaces, with finite-dimensional stalks. Denote by $D^b_{\mathcal{V}}(Y)$ the corresponding derived category of bounded complexes with $\mathcal{V}$-constructible cohomology sheaves (compare [2, 10, 12]). Then one has an equality of Grothendieck groups (e.g., compare [10, p. 77], [12, lemma 3.3.1])

$$K_0(\mathrm{Sh}_{\mathcal{V}}(Y)) = K_0(D^b_{\mathcal{V}}(Y))$$

obtained by identifying the class of a complex with the alternating sum of the classes of its cohomology sheaves.

Moreover, one has a canonical group epimorphism

$$\chi_Y : K_0(D^b_{\mathcal{V}}(Y)) \to F_{\mathcal{V}}(Y)$$

defined by taking stalkwise the Euler characteristic. Note that χ_Y is not injective in general (e.g., see [7, p. 98]), except for when all strata $V \in \mathcal{V}$ are simply connected, e.g., for $Y = \{pt\}$, in which case we use the shorter notion $K_0(pt)$. So $K_0(pt)$ is just the Grothendieck group of finite-dimensional $\mathbb{Q}$-vector spaces, and it is a commutative ring with respect to the tensor product with unit $\mathbb{Q}_{pt}$. Moreover, there is an isomorphism $K_0(pt) \cong \mathbb{Z}$ induced by the Euler characteristic homomorphism. $K_0(D^b_{\mathcal{V}}(Y))$ becomes a unitary $K_0(pt)$-module, with the multiplication defined by the exterior product

$$K_0(D^b_{\mathcal{V}}(Y)) \times K_0(pt) \to K_0(D^b_{\mathcal{V}\times\{pt\}}(Y \times \{pt\})) = K_0(D^b_{\mathcal{V}}(Y)),$$

and the Euler characteristic homomorphisms χ_Y and χ are compatible with this structure (more generally, χ_Y commutes with exterior products).

Important examples of $\mathcal{V}$-constructible complexes are provided by the intersection cohomology complexes $IC_{\bar{V}}$ of the closures of the strata $V \in \mathcal{V}$, extended by 0 to all of Y (cf. [1, 2, 9]). These are self-dual with respect to Verdier duality (and become important in the context of perverse sheaves and mixed Hodge modules, as in our forthcoming papers [3, 4]). The normalization axiom for $IC_{\bar{V}}$ (in the conventions of [1]) yields that $IC_{\bar{V}}|_V = \mathbb{Q}_V[\dim(V)]$, with $\dim(V) := \dim_{\mathbb{R}}(V)/2$ (the complex dimension in the complex algebraic/analytic context). Since we work in Grothendieck groups, in order to avoid signs in our calculations, we will use the normalization condition of [2], that is, we work with $IC'_{\bar{V}} := IC_{\bar{V}}[-\dim(V)]$, whose hypercohomology is exactly the intersection cohomology of $\bar{V}$.

Let us fix for each $W \in \mathcal{V}$ a point $w \in W$ with inclusion $i_w : \{w\} \hookrightarrow Y$. Then

$$i_w^*[IC'_{\bar{W}}] = [i_w^* IC'_{\bar{W}}] = [\mathbb{Q}_{pt}] \in K_0(w) = K_0(pt) \tag{3.1}$$

and $i_w^*[IC'_{\bar{V}}] \neq [0] \in K_0(pt)$ only if $W \leq V$. If we let

$$ic_{\bar{V}} := \chi_Y(IC'_{\bar{V}}) \in F_{\mathcal{V}}(Y) \tag{3.2}$$

be the corresponding constructible function, then

$$\mathrm{supp}(ic_{\bar{V}}) = \bar{V} \quad \text{and} \quad ic_{\bar{V}}|_V = 1_V. \tag{3.3}$$

Note that $ic_{\bar{V}}(w)$ does not depend on the choice of $w \in W$, and this is also the case for $i_w^*[IC_{\bar{V}}] \in K_0(pt)$. In fact, since for any $j \in \mathbb{Z}$,

$$\mathcal{H}^j(i_w^* IC'_{\bar{V}}) \simeq IH^j(c^\circ L_{W,V})$$

with $c^\circ L_{W,V}$, the open cone on the link $L_{W,V}$ of W in $\bar{V}$ for $W \leq V$ (cf. [2, prop. 4.2]), we have that

$$i_w^*[IC'_{\bar{V}}] = [IH^*(c^\circ L_{W,V})] \in K_0(pt).$$

In terms of constructible functions, this gives

$$ic_{\bar{V}}(w) = I\chi(c^\circ L_{W,V}) := \chi([IH^*(c^\circ L_{W,V})]). \tag{3.4}$$

In particular, $\{ic_{\bar{V}} | V \in \mathcal{V}\}$ is another distinguished basis of $F_{\mathcal{V}}(Y)$ since, by (3.3), the transition matrix to the basis $\{1_V\}$ is upper-triangular with respect to $\leq$, with all diagonal entries equal to 1. Moreover, by (3.1) the $K_0(pt)$-submodule $\langle [IC'_{\bar{V}}] \rangle$ of $K_0(D^b_{\mathcal{V}}(Y))$ generated by the elements $[IC'_{\bar{V}}]$ $(V \in \mathcal{V})$ is in fact freely generated by them, and the restriction

$$\chi_Y : \langle [IC'_{\bar{V}}] \rangle \to F_{\mathcal{V}}(Y) \tag{3.5}$$

is an isomorphism.

The main technical result of this section is the following:

THEOREM 3.1 *Assume Y has an open dense stratum $S \in \mathcal{V}$ so that $V \leq S$ for all V. For each $V \in \mathcal{V} \setminus \{S\}$ define inductively*

$$\widehat{IC}(\bar{V}) := [IC'_{\bar{V}}] - \sum_{W<V} \widehat{IC}(\bar{W}) \cdot i_w^*[IC'_{\bar{V}}] \in K_0(D^b_{\mathcal{V}}(Y)), \tag{3.6}$$

and similarly

$$\widehat{ic}(\bar{V}) := ic_{\bar{V}} - \sum_{W<V} \widehat{ic}(\bar{W}) \cdot I\chi(c^\circ L_{W,V}) \in F_{\mathcal{V}}(Y) \tag{3.7}$$

so that $\chi_Y(\widehat{IC}(\bar{V})) = \widehat{ic}(\bar{V})$. As the notation suggests, $\widehat{IC}(\bar{V})$ and $\widehat{ic}(\bar{V})$ depend only on the stratified space $\bar{V}$ with its induced stratification.

(i) *Assume that $[\mathcal{F}] \in K_0(D^b_{\mathcal{V}}(Y))$ is an element of the $K_0(pt)$-submodule $\langle [IC'_{\bar{V}}] \rangle$. Then*

$$[\mathcal{F}] = [IC'_Y] \cdot i_s^*[\mathcal{F}] + \sum_{V<S} \widehat{IC}(\bar{V}) \cdot (i_v^*[\mathcal{F}] - i_s^*[\mathcal{F}] \cdot i_v^*[IC'_Y]) \in K_0(D^b_{\mathcal{V}}(Y)). \tag{3.8}$$

(ii) *For any $\mathcal{V}$-constructible function $\alpha \in F_{\mathcal{V}}(Y)$, one has the equality*

$$\alpha = \alpha(s) \cdot ic_Y + \sum_{V<S} (\alpha(v) - \alpha(s) \cdot I\chi(c^\circ L_{V,Y})) \cdot \widehat{ic}(\bar{V}). \tag{3.9}$$

PROOF: Note that equation (3.9) of the second part of the theorem is a direct consequence of formula (3.8) from the first part. Indeed, by (3.5) we can first represent any $\alpha \in F_{\mathcal{V}}(Y)$ as $\alpha = \chi_Y([\mathcal{F}])$ for some $[\mathcal{F}] \in \langle [IC'_{\bar{V}}] \rangle$. Then, assuming (3.8) holds for this choice of $[\mathcal{F}]$, we can apply χ_Y to this equation and obtain (3.9).

In order to prove formula (3.8), consider

$$[\mathcal{F}] = \sum_V [IC'_{\bar{V}}] \cdot L(V) \tag{3.10}$$

for some $L(V) \in K_0(pt)$. The aim is to identify these coefficients $L(V)$. Since S is an open stratum, by applying i_s^* to (3.10) we obtain

$$i_s^*[\mathcal{F}] = L(S) \in K_0(s) = K_0(pt).$$

Next fix a stratum $W \neq S$ and apply i_w^* to (3.10). Recall that $i_w^*[IC'_{\bar{W}}] = [\mathbb{Q}_{pt}] \in K_0(w) = K_0(pt)$, and $i_w^*[IC'_{\bar{V}}] \neq [0] \in K_0(pt)$ only if $W \leq V$. We obtain

$$i_w^*[\mathcal{F}] = L(W) + \sum_{W<V} i_w^*[IC'_{\bar{V}}] \cdot L(V) \in K_0(w) = K_0(pt). \tag{3.11}$$

Since S is dense, we have that $W < S$, so the stratum S appears in the summation on the right-hand side of (3.11). Therefore

$$i_w^*[\mathcal{F}] - i_w^*[IC'_Y] \cdot i_s^*[\mathcal{F}] = L(W) + \sum_{W<V<S} i_w^*[IC'_{\bar{V}}] \cdot L(V) \in K_0(w) = K_0(pt). \tag{3.12}$$

This implies that we can inductively calculate $L(V)$ in terms of

$$L'(W) := i_w^*[\mathcal{F}] - i_w^*[IC'_Y] \cdot i_s^*[\mathcal{F}].$$

Indeed, (3.12) can be rewritten as

$$L'(W) = \sum_{W \leq V<S} i_w^*[IC'_{\bar{V}}] \cdot L(V) \in K_0(pt), \tag{3.13}$$

and the matrix $A = (a_{W,V})$, with $a_{W,V} := i_w^*[IC'_{\bar{V}}] \in K_0(pt)$ for $W, V \in \mathcal{V} \setminus \{S\}$, is upper-triangular with respect to $\leq$, with 1's on the diagonal. So A can be inverted. The nonzero coefficients of $A^{-1} = (a'_{W,V})$ can be inductively calculated by $a'_{V,V} = 1$ and

$$a'_{W,V} = -\sum_{W \leq T < V} a'_{W,T} \cdot a_{T,V} \tag{3.14}$$

for $W < V$. Then (3.10) becomes

$$\begin{aligned} [\mathcal{F}] &= [IC'_Y] \cdot i_s^*[\mathcal{F}] + \sum_{W<S} [IC'_{\bar{W}}] \cdot L(W) \\ &= [IC'_Y] \cdot i_s^*[\mathcal{F}] + \sum_{W \leq V<S} [IC'_{\bar{W}}] \cdot a'_{W,V} \cdot L'(V). \end{aligned} \tag{3.15}$$

The result follows by the inductive identification (for $V < S$ fixed)

$$\sum_{W \leq V} [IC'_{\bar{W}}] \cdot a'_{W,V} = [IC'_{\bar{V}}] - \sum_{W \leq T < V} [IC'_{\bar{W}}] \cdot a'_{W,T} \cdot a_{T,V} = \widehat{IC}(\bar{V}).$$

□

Remark 3.2. In this paper, we only make use of equation (3.9), and this could be proven directly by working in $F_{\mathcal{V}}(Y)$ by following the same arguments as above. However, the formula of equation (3.8) is particularly important since in the complex algebraic context it extends to the framework of Grothendieck groups of algebraic mixed Hodge modules that will be used in our forthcoming paper [4]. Of course, the technical condition used in proving formula (3.8) is not generally satisfied, but it holds under the assumption of *trivial monodromy along all strata* $V \in \mathcal{V}$ (e.g., if all strata V are simply connected). For more details, see [4].

For the remaining part of this section, we will specialize to the complex algebraic (respectively, compact complex analytic) context; that is, Y is a reduced complex algebraic variety (respectively, a reduced compact complex analytic space), with a complex algebraic (respectively, analytic) Whitney stratification $\mathcal{V}$. In this setting, let $f : X \to Y$ be a proper complex algebraic (respectively, analytic) map. Assume $f_*(\alpha) \in F_{\mathcal{V}}(Y)$ for a given $\alpha \in F_c(X)$; e.g., we choose $\mathcal{V}'$ and $\mathcal{V}$ complex Whitney stratifications of X and Y, respectively, such that f is a stratified submersion, and $\alpha \in F_{\mathcal{V}'}(X)$. Then

$$f_*(\alpha) = \sum_{V \in \mathcal{V}} f_*(\alpha)(V) \cdot 1_V,$$

with

$$f_*(\alpha)(V) = \chi(\alpha|_{F_V}),$$

for F_V the fiber of f over a point in V. Assume, moreover, that Y is irreducible so there is a dense stratum $S \in \mathcal{V}$ with $F := F_S$ a general fiber of f. In terms of $f_*(\alpha)$, the equation (3.9) of Theorem 3.1 becomes

$$f_*(\alpha) = \chi(\alpha|_F) \cdot ic_Y + \sum_{V<S} \big(\chi(\alpha|_{F_V}) - \chi(\alpha|_F) \cdot I\chi(c^\circ L_{V,Y})\big) \cdot \widehat{ic}(\bar{V}) \in F_{\mathcal{V}}(Y). \tag{3.16}$$

By applying the group homomorphism χ and c_* to equation (3.16), respectively, we obtain the following (recall $\chi \circ f_* = \chi$ and $c_* \circ f_* = f_* \circ c_*$ for f proper):

COROLLARY 3.3

$$\chi(\alpha) = \chi(\alpha|_F) \cdot I\chi(Y) + \sum_{V<S} \big(\chi(\alpha|_{F_V}) - \chi(\alpha|_F) \cdot I\chi(c^\circ L_{V,Y})\big) \cdot \widehat{I\chi}(\bar{V}), \tag{3.17}$$

$$f_*(c_*(\alpha)) = \chi(\alpha|_F) \cdot Ic_*(Y) + \sum_{V<S} \big(\chi(\alpha|_{F_V}) - \chi(\alpha|_F) \cdot I\chi(c^\circ L_{V,Y})\big) \cdot \widehat{Ic}_*(\bar{V}). \tag{3.18}$$

Here $I\chi(Y) := \chi(ic_Y) = \chi([IH^*(Y;\mathbb{Q})])$ *is the intersection homology Euler characteristic of* Y*, and similarly for* $\widehat{I\chi}(\bar{V})$*. Also,* $Ic_*(Y) := c_*(ic_Y)$*, and similarly for* $\widehat{Ic}_*(\bar{V})$*, which by functoriality is regarded as a homology class in* $H_{2*}^{\mathrm{BM}}(Y;\mathbb{Z})$*, the even degree Borel-Moore homology of* Y*.*

By letting $\alpha = 1_X$ in the formulae (3.17) and (3.18) above, respectively, we obtain:

PROPOSITION 3.4 *Let* $f : X \to Y$ *be a proper complex algebraic* (*respectively, analytic*) *map, with* Y *irreducible* (*and compact in the analytic context*) *and endowed with a complex algebraic* (*respectively, analytic*) *Whitney stratification* $\mathcal{V}$*. Assume* $f_*(1_X) \in F_{\mathcal{V}}(Y)$*. Then*:

$$\chi(X) = \chi(F) \cdot I\chi(Y) + \sum_{V<S} \big(\chi(F_V) - \chi(F) \cdot I\chi(c^\circ L_{V,Y})\big) \cdot \widehat{I\chi}(\bar{V}), \tag{3.19}$$

$$f_*(c_*(X)) = \chi(F) \cdot Ic_*(Y) + \sum_{V<S} \big(\chi(F_V) - \chi(F) \cdot I\chi(c^\circ L_{V,Y})\big) \cdot \widehat{Ic}_*(\bar{V}). \tag{3.20}$$

In the special case when f is the identity map, equation (3.19) yields a formula expressing the Euler characteristics of usual and intersection homology of an algebraic (respectively, analytic) variety in terms of each other and corresponding invariants of the subvarieties formed by the closures of its singular strata. Similarly, (3.20) yields in this case a comparison between the corresponding Chern homology classes of MacPherson:

COROLLARY 3.5 *Let* Y *be an irreducible complex algebraic* (*respectively, compact analytic*) *variety endowed with a complex algebraic* (*respectively, analytic*) *Whitney stratification* $\mathcal{V}$*. Then in the above notations we have*:

$$\chi(Y) = I\chi(Y) + \sum_{V<S} (1 - I\chi(c^\circ L_{V,Y})) \cdot \widehat{I\chi}(\bar{V}), \tag{3.21}$$

$$c_*(Y) = Ic_*(Y) + \sum_{V<S} (1 - I\chi(c^\circ L_{V,Y})) \cdot \widehat{Ic}_*(\bar{V}). \tag{3.22}$$

The stratified multiplicative property for the intersection homology Euler characteristic and for the corresponding homology characteristic classes is obtained from (3.17) and (3.18), respectively, in the case when $\alpha = ic_X$. Indeed, we have:

PROPOSITION 3.6 *Let $f : X \to Y$ be a proper complex algebraic (respectively, analytic) map, with X pure dimensional and Y irreducible (and compact in the analytic context). Assume Y is endowed with a complex algebraic (respectively, analytic) Whitney stratification $\mathcal{V}$ so that $f_*(ic_X) \in F_{\mathcal{V}}(Y)$. Then*:

$$(3.23)\qquad I\chi(X) = I\chi(F)\cdot I\chi(Y) + \sum_{V<S}\big(I\chi(f^{-1}(c^\circ L_{V,Y})) - I\chi(F)\cdot I\chi(c^\circ L_{V,Y})\big)\cdot \widehat{I\chi}(\bar{V}),$$

$$(3.24)\qquad f_*(Ic_*(X)) = I\chi(F)\cdot Ic_*(Y) + \sum_{V<S}\big(I\chi(f^{-1}(c^\circ L_{V,Y})) - I\chi(F)\cdot I\chi(c^\circ L_{V,Y})\big)\cdot \widehat{Ic}_*(\bar{V}).$$

PROOF: Based on the above considerations, it suffices to show that

$$(3.25)\qquad \chi(ic_X|_F) = I\chi(F)$$

and

$$(3.26)\qquad \chi(ic_X|_{F_V}) = I\chi(f^{-1}(c^\circ L_{V,Y})).$$

Since the general fiber F of f is locally normally nonsingular embedded in X, we have a quasi-isomorphism [9, §5.4.1]:

$$IC'_X|_F \simeq IC'_F,$$

hence an equality $ic_X|_F = ic_F$, thus proving (3.25).

Similarly, since $\chi(ic_X|_{F_V}) = f_*(ic_X)(v)$ for some $v \in V$, in order to prove (3.26), it suffices to show that

$$(3.27)\qquad \mathcal{H}^j(Rf_*IC'_X)_v \cong IH^j(f^{-1}(c^\circ L_{V,Y});\mathbb{Q}).$$

Let N be a normal slice to V at v in local analytic coordinates $(Y, v) \hookrightarrow (\mathbb{C}^n, v)$, that is, a germ of a complex manifold $(N, v) \hookrightarrow (\mathbb{C}^n, v)$, intersecting V transversally only at v, and with $\dim V + \dim N = n$. Recall that the link $L_{V,Y}$ of the stratum V in Y is defined as

$$L_{V,Y} := Y \cap N \cap \partial B_r(v),$$

where $B_r(v)$ is an open ball of (very small) radius r around v. Moreover, $Y \cap N \cap B_r(v)$ is isomorphic (in a stratified sense) to the open cone $c^\circ L_{V,Y}$ on the link [2, p. 44]. By factoring i_v as the composition,

$$\{v\} \overset{\phi}{\hookrightarrow} Y \cap N \overset{\psi}{\hookrightarrow} Y,$$

we can now write

$$\begin{aligned}\mathcal{H}^j(Rf_*IC'_X)_v &\cong \mathbb{H}^j(Y, i_{v*}i_v^*Rf_*IC'_X)\\ &\cong \mathbb{H}^j(v, \phi^*\psi^*Rf_*IC'_X)\\ &\cong \mathcal{H}^j(\psi^*Rf_*IC'_X)_v\end{aligned}$$

$$\cong \mathbb{H}^j(c^\circ L_{V,Y}, Rf_* IC'_X)$$

$$\cong \mathbb{H}^j(f^{-1}(c^\circ L_{V,Y}), IC'_X)$$

$$\overset{(*)}{\cong} \mathbb{H}^j(f^{-1}(c^\circ L_{V,Y}), IC'_{f^{-1}(c^\circ L_{V,Y})})$$

$$\cong IH^j(f^{-1}(c^\circ L_{V,Y}); \mathbb{Q}),$$

where in $(*)$ we used the fact that the inverse image of a normal slice to a stratum of Y in a stratification of f is (locally) normally nonsingular embedded in X (this fact is a consequence of the first isotopy lemma). □

Acknowledgment. We are grateful to the anonymous referee for his valuable comments and suggestions regarding the functorial approach used in the present paper. The first and third authors were partially supported by grants from the National Science Foundation and the Defense Advanced Research Projects Agency. The second author was partially supported by a grant from the New York University Research Challenge Fund.

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Received April 2006.
Revised October 2006.