A geometric perspective on generalized weighted Ehrhart theory

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(joint work with J. Schürmann)

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  - face $Q \preceq P \rightsquigarrow$ cone $\sigma_Q \in \Sigma_P \rightsquigarrow$ orbit O_{σ_Q} in X_P

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- consider the associated (possibly singular) projective *toric variety* X_P with ample Cartier divisor D_P.
- $$\mathrm{Ehr}_P(\ell) = \chi(X_P, \mathcal{O}(\ell D_P)) \overset{(RR)}{=} \sum_{k \geq 0} \left(\frac{1}{k!} \int_{X_P} [D_P]^k \cap td_k(X_P) \right) \ell^k = \sum_{k \geq 0} a_k \ell^k,$$

  with $td_k(X_P) \in H_{2k}(X_P; \mathbb{Q})$ the degree k component of the Baum-Fulton-MacPherson Todd class $td_*(X_P)$.

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  with $td_k(X_P) \in H_{2k}(X_P; \mathbb{Q})$ the degree k component of the Baum-Fulton-MacPherson Todd class $td_*(X_P)$.
- $a_n = \mathrm{vol}(P)$, $a_{n-1} = \frac{1}{2}\mathrm{vol}(\partial P)$, $a_0 = \chi(P) = 1$.

Reciprocity

♣ Ehrhart reciprocity:

$$\mathrm{Ehr}_P(-\ell) = (-1)^n \cdot \#\left(\mathrm{Int}(\ell P) \cap M\right) = (-1)^n \cdot \mathrm{Ehr}_{\mathrm{Int}(P)}(\ell)$$

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♣ Geometrically,

$$\#(\mathrm{Int}(\ell P) \cap M) = (-1)^n \cdot \chi(X_P, \mathcal{O}(-\ell D_P))$$

Weighted Ehrhart theory

♣ Face decomposition for P:

$$\#(\ell P \cap M) = \sum_{Q \preceq P} \#(\mathrm{Relint}(\ell Q) \cap M),$$

with $\mathrm{Relint}(\ell Q)$ the relative interior of the face ℓQ of the dilated polytope ℓP.

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♣ Assign Laurent polynomial weights $f_Q(y) \in \mathbb{Z}[y^{\pm 1}]$ to each face $Q \preceq P$ of P, and define for any $\ell \in \mathbb{Z}_{>0}$ the weighted Ehrhart "polynomial" of P and $f = \{f_Q\}_{Q \preceq P}$ by

$$\mathrm{Ehr}_{P,f}(\ell, y) := \sum_{Q \preceq P} f_Q(y) \cdot (1+y)^{\dim(Q)} \cdot \#(\mathrm{Relint}(\ell Q) \cap M)$$

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♣ If $f = \mathbf{1} := \{1\}$, get for $y = 0$: $\mathrm{Ehr}_{P,\mathbf{1}}(\ell, 0) = \mathrm{Ehr}_P(\ell)$

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- (Constant term) For $\ell = 0$,

$$\mathrm{Ehr}_{P,f}(0, y) = \sum_{Q \preceq P} f_Q(y) \cdot (-1-y)^{\dim(Q)},$$

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- (Reciprocity formula) For $\ell \in \mathbb{Z}_{>0}$,

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♣ There is an order-reversing one-to-one correspondence between the faces Q of P, and the faces Q° of the polar polytope P°, switching the roles of polytopes and emptysets seen as faces. For a proper face $\emptyset \neq Q \prec P$, one has $\dim_{\mathbb{R}}(Q) + \dim_{\mathbb{R}}(Q^\circ) = n - 1$.

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♣ If P is a *simple* polytope, the polar polytope P° is *simplicial*, so that $g_{Q^\circ}(-y) = 1$, for all faces Q of P.

Purity

Theorem (Beck-Gunnells-Materov)

For $f_Q(y) = g_{Q^\circ}(-y)$ the weight vector given by Stanley's g-polynomials for the faces of the polar polytope P° of P, the following purity property holds:

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♣ Our aim is to explain a *geometric proof* of this result, and to prove a form of reciprocity/purity for *any* weight vector f.

♣ We use Hodge theory, and recover all properties of $\mathrm{Ehr}_{P,f}(\ell)$ from the calculus of characteristic classes of mixed Hodge modules on X_P (via a generalized Hirzebruch-Riemann-Roch).

Hodge polynomial

♣ There is a Hodge polynomial ring homomorphism

$$\chi_y : K_0(\mathrm{MHS}) \longrightarrow \mathbb{Z}[y^{\pm 1}]$$

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In particular, for $\mathcal{M} = \mathbb{Q}_X$ set $\chi_y(X) = \chi_y([H^\bullet(X)])$,
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♣ M.-Saito-Schürmann: $I\chi_1(X) = \sigma(X)$ is the intersection cohomology signature of X (Goresky-MacPherson).

Lemma (M.-Schürmann)

For any weight vector $f = \{f_Q\}_{Q \preceq P}$ *on the faces of the lattice polytope* P*, there exists some* $\mathcal{M} \in D^b\mathrm{MHM}(X_P)$ *with constant cohomology sheaves along the torus orbits such that*

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Example (Fieseler, Denef-Loeser)

For a lattice polytope P with Stanley's g-polynomials $f_Q(y) = g_{Q^\circ}(-y)$, one can choose $\mathcal{M} = IC_{X_P}[-n]$.

Homology Hirzebruch classes

♣ For X projective,

$$\chi_y(X;\mathcal{M}) := \chi_y([H^\bullet(X;\mathcal{M})]) = \int_X T_{y*}([\mathcal{M}]),$$

with

$$T_{y*} : K_0(\mathrm{MHM}(X)) \to H_{2*}(X) \otimes \mathbb{Q}[y^{\pm 1}]$$

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$$T_{0*}(X) = td_*(X).$$

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Theorem A (M.-Schürmann)

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$$\chi_y(X_P;\mathcal{M}) = \sum_{Q \preceq P} \chi_y(\mathcal{H}^\bullet(\mathcal{M})_{x_Q}) \cdot (-1-y)^{\dim(Q)}.$$

For $\mathcal{M} = \mathbb{Q}_{X_P}$, get:

Corollary

(a) The Hodge polynomial $\chi_y(X_P)$ is computed by:

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Corollary

(a) *The Hodge polynomial* $\chi_y(X_P)$ *is computed by:*

$$\chi_y(X_P) = \sum_{Q \preceq P} (-1-y)^{\dim(Q)}.$$

(b) *The Euler characteristic* $e(X)$ *is computed by:*

$$e(X_P) = \text{number of vertices of } P.$$

Stanley's g-polynomials and intersection cohomology

Corollary

Assume $0 \in \mathrm{Int}(P)$, *and* $\mathcal{M} = IC_{X_P}[-n]$. *Then:*

$$I\chi_y(X_P) := \chi_y([IH^\bullet(X_P)]) = \sum_{Q \preceq P} g_{Q^\circ}(-y) \cdot (-1-y)^{\dim(Q)}.$$

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Remark

Upon substituting $y = -t^2$ *in* $I\chi_y(X_P)$, *we recover Fieseler's formula for the intersection cohomology Poincaré polynomial* $IP_{X_P}(t) = h_P(t^2)$ *of* X_P.

Weighted Ehrhart theory via generalized HRR for X_P

Theorem B (M.-Schürmann)

Let $P \subset M_{\mathbb{R}} \cong \mathbb{R}^n$ be a full-dimensional lattice polytope with associated toric variety X_P and ample Cartier divisor D_P.

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$$\begin{aligned} \mathrm{Ehr}_{P,f}(\ell, y) &= \int_{X_P} e^{\ell [D_P]} \cap T_{y*}([\mathcal{M}]) \\ &= \sum_{k=0}^{n} \left(\frac{1}{k!} \int_X [D_P]^k \cap T_{y,k}([\mathcal{M}]) \right) \cdot \ell^k \end{aligned}$$

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with $\mathcal{M} \in D^b\mathrm{MHM}(X_P)$ a mixed Hodge module complex with constant cohomology sheaves along orbits chosen so that $f_Q(y) = \chi_y(\mathcal{H}^\bullet(\mathcal{M})_{x_Q})$ for some (any) $x_Q \in O_{\sigma_Q} \subset X_P$.

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$$\mathrm{Ehr}_{P,f}(0,y) = \chi_y(X_P; \mathcal{M}) = \sum_{Q \preceq P} f_Q(y) \cdot (-1-y)^{\dim(Q)}$$

Reciprocity and Purity for arbitrary weight vectors

Theorem C (M.-Schürmann)

For any $\mathcal{M} \in D^b\mathrm{MHM}(X_P)$ *with constant cohomology sheaves along the torus orbits, we have the reciprocity property*

$$\mathrm{Ehr}_{P,\mathcal{M}}(-\ell, y) = \mathrm{Ehr}_{P,\mathcal{D}_X\mathcal{M}}(\ell, \frac{1}{y}).$$

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More generally, for any weight vector f on the faces of P, we have

$$\mathrm{Ehr}_{P,f}(-\ell, y) = \sum_{Q \preceq P} f_Q(y) \cdot (-1-y)^{\dim(Q)} \cdot \#(\ell Q \cap M).$$

Remark

The purity of Beck-Gunnells-Materov for $\mathrm{Ehr}_{P,f}(\ell, y)$*, with* f *given by Stanley's* g*-polynomials of faces of the polar polytope* P°*, follows for the special case of* IC_{X_P}*, which is self-dual pure Hodge module of weight* n *on* X_P*.*

Generalized Ehrhart theory

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Theorem (Brion-Vergne, 1997)

$\mathrm{Ehr}_P^{\varphi}(\ell)$ *is a polynomial in* ℓ *of degree* $\dim(P) + \deg(\varphi)$, *with constant term* $\varphi(0)$, *which satisfies the reciprocity law*

$$\begin{aligned}\mathrm{Ehr}_P^{\varphi}(-\ell) &= (-1)^{\dim(P)+\deg(\varphi)} \sum_{m \in \mathrm{Int}(\ell P) \cap M} \varphi(m) \\ &= (-1)^{\dim(P)+\deg(\varphi)} \cdot \mathrm{Ehr}_{\mathrm{Int}(P)}^{\varphi}(\ell)\end{aligned}$$

- Let $\varphi\colon M_{\mathbb{R}} \cong \mathbb{R}^n \to \mathbb{C}$ be a homogeneous polynomial function.

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$$\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y) := \sum_{Q \preceq P} f_Q(y) \cdot (1+y)^{\dim(Q)+\deg(\varphi)} \cdot \sum_{m \in \mathrm{Relint}(\ell Q) \cap M} \varphi(m)$$

with $\mathrm{Relint}(\ell Q)$ denoting the relative interior of the face ℓQ of the dilated polytope ℓP.

Properties of $\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y)$

The Brion-Vergne combinatorial approach to reciprocity can be linearly extended (over the faces of P) to this generalized weighted Ehrhart theory, so that $\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y)$ has the following properties:

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- (Reciprocity formula) For $\ell \in \mathbb{Z}_{>0}$,

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Theorem (Beck-Gunnells-Materov)

The following purity property holds:

$$\mathrm{Ehr}^{\varphi}_{P,f}(-\ell, y) = (-y)^{\dim(P)+\deg(\varphi)} \cdot \mathrm{Ehr}^{\varphi}_{P,f}(\ell, 1/y).$$

♣ To prove all these properties for $\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y)$ *geometrically*, we need to work equivariantly, with torus equivariant mixed Hodge modules on X_P.

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♣ To explain geometrically the polynomial behavior in ℓ of $\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y)$ we work in equivariant homology, using equivariant localization at torus fixed points (a combinatorial proof can be given using work of Brion-Vergne).

Remark

If P is a simple lattice polytope, $\mathrm{Ehr}^{\varphi}_{P,f}(\ell, y)$ can be computed by Euler-Maclaurin type formulae, like in works of Beck-Gunnells-Materov (combinatorially) or Cappell-M.-Schürmann-Shaneson (via the equivariant Hirzebruch-Riemann-Roch formalism).

THANK YOU !!!