EQUIVARIANT TORIC GEOMETRY AND EULER-MACLAURIN FORMULAE

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Abstract. In this paper, we consider $\mathbb{T}$-equivariant versions $mC_y^{\mathbb{T}}$ and $T_{y*}^{\mathbb{T}} := \mathrm{td}_*^{\mathbb{T}} \circ mC_y^{\mathbb{T}}$ of the *motivic Chern and, resp., Hirzebruch characteristic classes* of a quasi-projective toric variety X (with corresponding torus $\mathbb{T}$), and extend many known results from non-equivariant to the equivariant setting. For example, $mC_y^{\mathbb{T}}(X)$ is the sum of the equivariant K-classes of the $\mathbb{T}$-equivariant sheaves of *Zariski p-forms* $\widehat{\Omega}_X^p$ weighted by y^p. Using the motivic as well as characteristic class nature of $T_{y*}^{\mathbb{T}}(X)$, the corresponding generalized *equivariant Hirzebruch* χ_y*-genus* $\chi_y^{\mathbb{T}}(X, \mathscr{O}_X(D))$ of a $\mathbb{T}$-invariant Cartier divisor D on X is also calculated.

Further global formulae for $T_{y*}^{\mathbb{T}}(X)$ are obtained in the simplicial context based on the Cox construction and the *equivariant Lefschetz-Riemann-Roch theorem* of Edidin-Graham. Alternative proofs of all these results are given via *localization techniques* at the torus fixed points in $\mathbb{T}$-equivariant K- and homology theories of toric varieties, due to Brion-Vergne and, resp., Brylinski-Zhang. These localization results apply to any complete toric variety with a torus fixed point. In localized $\mathbb{T}$-equivariant K-theory, we extend a classical *formula of Brion* for a full-dimensional lattice polytope P to a weighted version. We also generalize *Molien formula* of Brion-Vergne for the localized class of the structure sheaf of a simplicial variety X to the context of $mC_y^{\mathbb{T}}(X)$. Similarly, we calculate the *localized Hirzebruch class* in localized $\mathbb{T}$-equivariant homology, extending the corresponding results of Brylinski-Zhang for the *localized Todd class* (fitting with the equivariant Hirzebruch class for $y = 0$).

Furthermore, we elaborate on the relation between the *equivariant toric geometry via the equivariant Hirzebruch-Riemann-Roch* (for an ample torus invariant Cartier divisor) and *Euler-Maclaurin type formulae for full-dimensional simple lattice polytopes* (corresponding to simplicial toric varieties). Our main results provide generalizations to *arbitrary coherent sheaf coefficients*, and algebraic geometric proofs of (weighted versions of) the Euler-Maclaurin formulae of Cappell-Shaneson, Brion-Vergne, Guillemin, etc., via the equivariant Hirzebruch-Riemann-Roch formalism. Our approach, based on *motivic characteristic classes*, allows us to obtain such Euler-Maclaurin formulae also for (the interior of) a face, as well as for the polytope with several facets (i.e., codimension one faces) removed, e.g., for the interior of the polytope (as well as for equivariant characteristic class formulae for locally closed $\mathbb{T}$-invariant subsets of a toric variety). Moreover, we prove such results also in the weighted context, as well as for $\mathbb{N}$-Minkowski summands of the given full-dimensional lattice polytope (corresponding to globally generated torus invariant Cartier divisors in the toric context). Similarly, some of these results are extended to local Euler-Maclaurin formulas for the tangent cones at the vertices of the given full-dimensional lattice polytope (fitting with the localization at the torus fixed points in equivariant K-theory and equivariant (co)homology).

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1. Introduction

Many problems in mathematics, computer science or engineering can be cast in terms of counting the number of elements of a finite set. For instance, in view of its connection with integer programming, counting lattice points in (full-dimensional) polytopes is an important problem in computational and discrete geometry, as well as in operation research and computer science. Furthermore, since in many applications polytopes come endowed with some continuous function f (e.g., a probability distribution, or color intensity in a picture), it is often the case that one is interested in computing the sum of the values of f at the lattice points contained in the polytope. The resulting expression is called the *Euler-Maclaurin formula* for f and P. The lattice point counting problem then corresponds to the choice of the constant function $f = 1$. The classical (one-dimensional) Euler-Maclaurin formula computes the sum of the values of a function f at the integer points in an interval on the real line in terms of the integral of f over the interval and the values of f and its derivatives at the endpoints of that interval, thus establishing a powerful connection between integrals (which are continuous quantifiers) and sums (which encode discrete information); e.g., see [34, Section 2] for an overview. A closely related problem is the computation of the *volume* of a lattice polytope P in terms of discrete data. For example, Pick's formula computes the area of a planar triangle in terms of the number of lattice points in the triangle and those on its boundary. More generally, it is well-known that the number of lattice points in the dilatated polytope ℓP is a polynomial $E_P(\ell)$ in the dilatation factor ℓ, called the Ehrhart polynomial of P, whose highest coefficient is the volume of P.

As observed by Danilov [20], the lattice point counting problem is closely related to the celebrated Riemann-Roch theorem [4] from algebraic geometry, via the correspondence between lattice polytopes and *toric varieties*. An n-dimensional toric variety $X = X_\Sigma$ is an irreducible normal variety on which the complex affine n-torus $\mathbb{T} \simeq (\mathbb{C}^*)^n$ acts with an open orbit, e.g., see [16, 20, 27]. Toric varieties arise from combinatorial objects $\Sigma \subset N \otimes \mathbb{R} \simeq \mathbb{R}^n$ called *fans*, which are collections of cones in a lattice $N \simeq \mathbb{Z}^n$. Here N corresponds to one-parameter subgroups of $\mathbb{T}$. Let $M \simeq \mathbb{Z}^n$ be the character lattice of $\mathbb{T}$. From a full-dimensional lattice polytope $P \subset M \otimes \mathbb{R} \simeq \mathbb{R}^n$ one constructs (via the associated *inner normal fan* Σ_P of P) a toric variety $X_P := X_{\Sigma_P}$, together with an ample Cartier divisor D_P, so that the number of lattice points of P, i.e., points of $M \cap P$, is computed by the holomorphic Euler characteristic $\chi(X_P, \mathscr{O}(D_P))$. The Riemann-Roch theorem expresses the latter in terms of the Chern character of $\mathscr{O}(D_P)$ and the Baum-Fulton-MacPherson homology Todd class $td_*(X_P)$ of the toric variety X_P, thus reducing the lattice point counting problem to a characteristic class computation (see formulae (44) and (46)). Similarly, a more intricate relation between lattice point counting and the theory of Goresky-MacPherson homology L-classes was noted by the first and fourth author in [17]. Hence, the problem of finding explicit formulae for characteristic classes of toric varieties is of interest not only to topologists and algebraic geometers, but also to combinatorists, programmers, etc.

In [36, 37], the second and third author computed the *motivic Chern class* mC_y and, resp., homology *Hirzebruch classes* T_{y*}, $\widehat{T}_{y*}$ [8] of (possibly singular) toric varieties. As important special cases, they obtained new (or recovered well-known) formulae for the Baum-Fulton-MacPherson Todd classes $\mathrm{td}_*(X) = T_{0*}(X) = \widehat{T}_{0*}(X)$ (or MacPherson-Chern classes $c_*(X) = \widehat{T}_{-1*}(X)$) of a toric variety X, as well as for the Thom-Milnor L-classes $L_*(X) = \widehat{T}_{1*}(X)$ of a simplicial projective toric variety X. For instance, by taking advantage of the torus-orbit decomposition and the motivic properties of the homology Hirzebruch classes, one can express the latter in terms of the (dual) Baum-Fulton-MacPherson Todd classes of closures of orbits. (The same method also applies to torus-invariant subspaces of a given toric variety.) As a consequence, one gets *generalized Pick-type formulae*, namely, generalizing Danilov's observation one shows that the weighted lattice point counting, where each point in a face E of the polytope P carries the weight $(1+y)^{\dim(E)}$, is equivalent to the computation of the Hirzebruch class $T_{y*}(X_P)$ of the associated toric variety (see Theorem 2.3 and Remark 2.4 for a precise statement). In the case of simplicial toric varieties (e.g., coming from a simple lattice polytope), these characteristic classes were computed in [36, 37] by using the Lefschetz-Riemann-Roch theorem of Edidin-Graham [24] in the context of the geometric quotient description of such varieties [19]; see Subsection 2.4 for an overview of such formulae.

A natural generalization of the classical one-dimensional Euler-Maclaurin formula is to relate the sum $\sum_{m\in P\cap M} f(m)$ of the values of a suitable function f at the lattice points in a lattice polytope $P \subset M_{\mathbb{R}} := M \otimes \mathbb{R}$ to integrals over the polytope and/or its faces. In this paper, we consider f to be either a polynomial on $M_{\mathbb{R}}$ or an exponential function $f(m) = e^{\langle m,z\rangle}$, where $\langle\cdot,\cdot\rangle : M \times N \to \mathbb{Z}$ is the canonical pairing and $z \in N_{\mathbb{C}} := N \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(M_{\mathbb{R}}, \mathbb{C})$.

Khovanskii and Pukhlikov [38] extended the classical (one-dimensional) Euler-Maclaurin identity to a formula for the sum of the values of a polynomial over the lattice points in higher-dimensional *regular* lattice polytopes (corresponding to smooth projective toric varieties). The first substantial advance for non-regular polytopes was a different type of Euler-Maclaurin formula for *simple* polytopes achieved by the first and fourth authors in [18, 43], by using their theory of topological *characteristic L-classes* of singular spaces (agreeing with $\widehat{T}_{1*}$ in this context). A few years later, the Khovanskii-Pukhlikov formula was extended to simple lattice polytopes by Brion-Vergne [12] [13], the latter using the *equivariant Hirzebruch-Riemann-Roch* theorem for the corresponding complete simplicial toric varieties, together with localization techniques in equivariant cohomology. Other Euler-Maclaurin type formulae were obtained by Guillemin [31] by using methods from symplectic geometry and geometric quantization, by Karshon et al. [34] by combinatorial means, etc. While most of above-mentioned Euler-Maclaurin formulae are obtained by integrating the function f over a dilatation of the polytope, the Cappell-Shaneson approach involves a summation over the faces of the polytope of integrals (over such faces) of linear differential operators with constant coefficients, applied to the function; see also [14] for some formulae of this type. Later on, these Euler-Maclaurin formulae have been extended (even for non-simplicial toric varieties, resp., non-simple lattice polytopes) by Berline–Vergne [6, 7] and Garoufalidis–Pommersheim [29], together with Fischer–Pommersheim [25], to local formulae satisfying a *Danilov condition* (see also the work of Pommershein–Thomas [41] for such formulae for the Todd class in the non-equivariant context). In the geometric context of toric varieties, all of these Danilov type formulae depend in a crucial way on the birational invariance of the (equivariant) Todd class of the structure sheaf. However, this birational invariance property does not apply to the more general context studied in this paper.

In the present work we consider $\mathbb{T}$-equivariant versions $mC_y^{\mathbb{T}}$ and $T_{y*}^{\mathbb{T}} := \mathrm{td}_*^{\mathbb{T}} \circ mC_y^{\mathbb{T}}$ of the motivic Chern and, resp., Hirzebruch characteristic classes of [8], and we extend the formulae from [36, 37] to the equivariant setting. Such constructions and results are first discussed in Section 3. The proofs given in Section 3 are based on the Cox construction and the equivariant Lefschetz-Riemann-Roch theorem of Edidin-Graham [24], and they resemble those of [36, 37], up to some technical modifications dictated by the equivariant context. Alternative proofs of all these results are given in Section 4.3 via localization at the torus fixed points.

Furthermore, we elaborate on the relation (cf. also [13]) between the equivariant toric geometry via the (equivariant) Hirzebruch-Riemann-Roch (abbreviated HRR for short) and Euler-Maclaurin type formulae for simple lattice polytopes (corresponding to simplicial toric varieties), fitting into the following picture:

$$\begin{array}{ccc} \text{Euler-Maclaurin formulae for } f & \longleftarrow & \text{equivariant HRR for } D_P \\ {\scriptstyle f=1}\big\downarrow & & \big\downarrow \\ \text{counting lattice points} & \longleftarrow & \text{HRR for } D_P\,, \end{array}$$

with D_P the $\mathbb{T}$-invariant ample Cartier divisor associated to the simple lattice polytope P.

Our main results provide generalizations to *arbitrary coherent sheaf coefficients*, which for natural choices related to the toric variety (or the polytope) give uniform geometric proofs of the Euler-Maclaurin formulae of Brion-Vergne and, resp., Cappell-Shaneson, via the equivariant Hirzebruch-Riemann-Roch formalism. Our approach, based on motivic characteristic classes, allows us to obtain such Euler-Maclaurin formulae also for (the interior of) a face, as well as for the polytope with several facets (i.e., codimension one faces) removed, e.g., for the interior of the polytope. Moreover, we prove such results also in the weighted context, as well as for $\mathbb{N}$-Minkowski summands of the given full-dimensional lattice polytope (corresponding to globally generated torus invariant Cartier divisors in the toric context). Similarly, some of these results are extended to local Euler-Maclaurin formulae for the tangent cones at the vertices of the given full-dimensional lattice polytope (fitting with localization at the torus fixed points in equivariant K-theory and equivariant (co)homology).

In what follows we give a brief overview of our main results. We first introduce some notations. For simplicity, in this introduction we formulate the results for a projective simplicial toric variety (e.g., associated to a simple full-dimensional lattice polytope).

Let $X = X_\Sigma$ be an n-dimensional projective simplicial toric variety with fan $\Sigma \subset N_\mathbb{R} = N \otimes \mathbb{R}$ and torus $\mathbb{T} = T_N$. Denote by $H^*_\mathbb{T}(X;\mathbb{Q})$ the (Borel-type) rational equivariant cohomology of X, and note that for a point space one has

$$H^*_\mathbb{T}(pt;\mathbb{Q}) \simeq \mathbb{Q}[t_1,\dots,t_n] =: (\Lambda_\mathbb{T})_\mathbb{Q}.$$

Let M be the dual lattice of $N \simeq \mathbb{Z}^n$. Viewing characters $m \in M$ (resp., $\chi^m \in \mathbb{Z}[M] \simeq K_0^\mathbb{T}(pt)$) of $\mathbb{T}$ as $\mathbb{T}$-equivariant line bundles $\mathbb{C}_{\chi^m}$ over a point space pt gives an isomorphism $M \simeq Pic_\mathbb{T}(pt)$. Taking the first equivariant Chern class $c^1_\mathbb{T}$ (or the dual $-c^1_\mathbb{T}$) gives an isomorphism

$$c = c^1_\mathbb{T}, \text{ resp., } s = -c^1_\mathbb{T} : M \simeq H^2_\mathbb{T}(pt;\mathbb{Z}).$$

Hence, upon choosing a basis m_i $(i = 1,\dots,n)$ of $M \simeq \mathbb{Z}^n$, one has that $H^*_\mathbb{T}(pt;\mathbb{Q}) = (\Lambda_\mathbb{T})_\mathbb{Q} \simeq \mathbb{Q}[t_1,\dots,t_n]$, with $t_i = \pm c^1_\mathbb{T}(\mathbb{C}_{\chi^{m_i}})$ for $i = 1,\dots,n$. As explained in Subsection 2.5, $H^*_\mathbb{T}(X;\mathbb{Q})$ can be described as a $H^*_\mathbb{T}(pt;\mathbb{Q}) = (\Lambda_\mathbb{T})_\mathbb{Q}$-algebra. This fact plays an important role for proving Euler-Maclaurin type formulae. In fact, we will be working with the completions $\widehat{H}^*_\mathbb{T}(X;\mathbb{Q})$ and $(\widehat{\Lambda}_\mathbb{T})_\mathbb{Q} \simeq \mathbb{Q}[[t_1,\dots,t_n]]$ of these rings. Moreover, the equivariant Chern character $\mathrm{ch}^\mathbb{T}$ and Todd homology classes $\mathrm{td}^\mathbb{T}_*$ take values in an *analytic subring* (cf. Proposition 5.17)

$$(H^*_\mathbb{T}(X;\mathbb{Q}))^{an} \subset \widehat{H}^*_\mathbb{T}(X;\mathbb{Q}),$$

with $\mathbb{Q}\{t_1,\dots,t_n\} \simeq (H^*_\mathbb{T}(pt;\mathbb{Q}))^{an} =: (\Lambda^{an}_\mathbb{T})_\mathbb{Q} \subset (\widehat{\Lambda}_\mathbb{T})_\mathbb{Q}$ the subring of *convergent power series* (around zero) with rational coefficients, i.e., after pairing with $z \in N_\mathbb{C}$ one gets a convergent power series *function* in z around zero, whose corresponding Taylor polynomials have rational coefficients.

Let $X = X_\Sigma$ be a projective simplicial toric variety, with a $\mathbb{T}$-equivariant coherent sheaf $\mathscr{F}$. The cohomology spaces $H^i(X;\mathscr{F})$ are finite dimensional $\mathbb{T}$-representations, vanishing for i large enough. Using the corresponding $\mathbb{T}$-eigenspaces as in (68), the (cohomological) *Euler* *characteristic* of $\mathscr{F}$ is defined by

$$\chi^{\mathbb{T}}(X,\mathscr{F}) = \sum_{m\in M}\sum_{i=0}^{n}(-1)^i \dim_{\mathbb{C}} H^i(X;\mathscr{F})_{\chi^m}\cdot e^{c(m)} \in (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}. \tag{1}$$

For $\mathscr{E}$, resp., $\mathscr{F}$, a $\mathbb{T}$-equivariant vector bundle, resp., coherent sheaf on X, one then has the following *equivariant Hirzebruch-Riemann-Roch formula*:

$$\chi^{\mathbb{T}}(X,\mathscr{E}\otimes\mathscr{F}) = \int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{E})\cap \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]), \tag{2}$$

where $\int_X : \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q}) \to \widehat{H}^*_{\mathbb{T}}(pt;\mathbb{Q}) = (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$ is the equivariant pushforward for the constant map $X\to pt$. See Subsection 2.7 for examples and applications of this formula.

In Subsection 3.2, formula (2) is extended to a *generalized equivariant Hirzebruch-Riemann-Roch formula*, cf. Theorem 3.14 (which is proved more generally for closed algebraic subsets of X defined by $\mathbb{T}$-invariant closed subsets):

$$\begin{aligned}\chi_y^{\mathbb{T}}(X,\mathscr{O}_X(D)) &:= \sum_{p=0}^{n}\chi^{\mathbb{T}}(X,\widehat{\Omega}^p_X\otimes\mathscr{O}_X(D))\cdot y^p \\ &= \int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_X(D))\cap T^{\mathbb{T}}_{y*}(X),\end{aligned} \tag{3}$$

with D a $\mathbb{T}$-invariant Cartier divisor on X, $\mathrm{ch}^{\mathbb{T}}$ the equivariant Chern character, and $\widehat{\Omega}^p_X$ the sheaf of Zariski p-forms on X. Here we use the following explicit description of the equivariant motivic Chern class $mC_y^{\mathbb{T}}(X)$ and, resp., equivariant Hirzebruch class $T^{\mathbb{T}}_{y*}(X)$ of X (see Proposition 3.5):

$$mC_y^{\mathbb{T}}(X) = \sum_{p=0}^{\dim(X)}[\widehat{\Omega}^p_X]_{\mathbb{T}}\cdot y^p \in K_0^{\mathbb{T}}(X)[y] \quad\text{and}\quad T^{\mathbb{T}}_{y*}(X) = \sum_{p=0}^{\dim(X)}\mathrm{td}_*^{\mathbb{T}}([\widehat{\Omega}^p_X]_{\mathbb{T}})\cdot y^p. \tag{4}$$

Let now P be a full-dimensional simple lattice polytope in $M_{\mathbb{R}}\simeq\mathbb{R}^n$ with associated toric variety $X=X_P$ with torus $\mathbb{T}$, inner normal fan $\Sigma=\Sigma_P$ and ample Cartier divisor $D=D_P$. As a consequence of (3), we obtain the following weighted formula (see Corollary 3.15):

$$\chi_y^{\mathbb{T}}(X,\mathscr{O}_X(D)) = \sum_{E\preceq P}(1+y)^{\dim(E)}\cdot \sum_{m\in \mathrm{Relint}(E)\cap M} e^{s(m)}, \tag{5}$$

where the first sum is over the faces E of P and $\mathrm{Relint}(E)$ denotes the relative interior of the face E.

Let us next consider a globally generated $\mathbb{T}$-invariant Cartier divisor D' on $X=X_\Sigma$, with associated (not necessarily full-dimensional) lattice polytope $P_{D'}\subset M_{\mathbb{R}}$. Let $X_{D'}$ be the toric variety of the lattice polytope $P_{D'}$, defined via the corresponding generalized fan Σ'. There is a proper toric morphism $f: X\to X_{D'}$, induced by the corresponding lattice projection $N\to N_{D'}$ given by dividing out by the minimal cone of the generalized fan of $P_{D'}$. In particular, $f: X\to X_{D'}$ is a toric fibration. For σ' a cone in the generalized fan Σ' of $P_{D'}$, let

$$d_\ell(X/\sigma') := |\Sigma_\ell(X/\sigma')|, \tag{6}$$

with

$$\Sigma_\ell(X/\sigma') := \{\sigma\in\Sigma \mid O_\sigma\subset X,\ f(O_\sigma)=O_{\sigma'},\ \ell=\dim(O_\sigma)-\dim(O_{\sigma'})\}, \tag{7}$$

and $|-|$ denoting the cardinality of a finite set. If E is the face of $P_{D'}$ corresponding to $\sigma' \in \Sigma'$, we denote these multiplicities by $d_\ell(X/E)$. Then we have the following generalization of formula (5) (see Corollary 3.17 for a more general statement)

$$(8) \qquad \chi_y^{\mathbb{T}}(X, \mathscr{O}_X(D')) = \sum_{E \preceq P_{D'}} \left(\sum_{\ell \geq 0} (-1)^\ell \cdot d_\ell(X/E) \cdot (1+y)^{\ell + \dim(E)} \right) \cdot \sum_{m \in \mathrm{Relint}(E) \cap M} e^{s(m)}.$$

By forgetting the $\mathbb{T}$-action (i.e., setting $s(m) = 0$ for all $m \in M$), we get the following weighted lattice point counting for lattice polytopes associated to globally generated $\mathbb{T}$-invariant Cartier divisors:

$$(9) \qquad \chi_y(X, \mathscr{O}_X(D')) = \sum_{E \preceq P_{D'}} \left(\sum_{\ell \geq 0} (-1)^\ell \cdot d_\ell(X/E) \cdot (1+y)^{\ell + \dim(E)} \right) \cdot |\mathrm{Relint}(E) \cap M|.$$

In Section 3, we extend the characteristic class formulae from [36, 37] for the motivic Chern and Hirzebruch classes to the equivariant setting. Here we use the global Cox construction (see Section 2.1.2) and the equivariant Lefschetz-Riemann-Roch theorem of Edidin-Graham [24]. Let $X := X_\Sigma$ be an n-dimensional simplicial projective toric variety with fan Σ. Then the equivariant Hirzebruch class $T_{y*}^{\mathbb{T}}(X)$ is computed by (cf. Theorem 3.22):

$$(10) \qquad T_{y*}^{\mathbb{T}}(X) = (1+y)^{n-r} \cdot \sum_{g \in G_\Sigma} \prod_{\rho \in \Sigma(1)} \frac{F_\rho \cdot \left(1 + y \cdot a_\rho(g) \cdot e^{-F_\rho}\right)}{1 - a_\rho(g) \cdot e^{-F_\rho}} \in \widehat{H}_{\mathbb{T}}^*(X; \mathbb{Q})[y],$$

with $r = |\Sigma(1)|$ the number of rays of Σ, and $F_\rho = [D_\rho]_{\mathbb{T}}$ denoting the equivariant fundamental class of the $\mathbb{T}$-invariant divisor D_ρ corresponding to the ray $\rho \in \Sigma(1)$. See Section 2.1.2 for the meaning of G_Σ and $a_\rho(g)$ in the context of the Cox construction. Note that if X is smooth, then G_Σ is just the identity element, and all $a_\rho(g) = 1$. For $y = 0$, with $T_{0*}^{\mathbb{T}}(X) = \mathrm{td}_*^{\mathbb{T}}(X)$ the equivariant Todd class of X, formula (10) specializes to the classical counterpart of Brion-Vergne [13] for the equivariant Todd class of X. A more general statement is obtained in Theorem 3.28, for the equivariant Hirzebruch classes of complements of $\mathbb{T}$-invariant divisors in X. Alternative proofs of all these characteristic class formulae are given in Section 4.3 via localization at the torus fixed points.

In Section 4, we apply localization techniques in $\mathbb{T}$-equivariant K- and (Borel-Moore) homology theories of toric varieties, due to Brion-Vergne [13] and, resp., Brylinski-Zhang [15], for the calculation of the $\mathbb{T}$-equivariant motivic Chern and Hirzebruch classes in the toric context.

Let $X = X_\Sigma$ be an n-dimensional toric variety with torus $\mathbb{T} = T_N$ such that the fixed-point set $X^{\mathbb{T}} \neq \emptyset$, e.g., X is projective. Let $x_\sigma \in X^{\mathbb{T}}$ be a fixed point corresponding to $\sigma \in \Sigma(n)$, with U_σ the corresponding $\mathbb{T}$-invariant open affine variety. Consider the multiplicative subset $S \subset \mathbb{Z}[M] = K_0^{\mathbb{T}}(pt)$ generated by the elements $1 - \chi^m$, for $0 \neq m \in M$. Then the projection map

$$pr_{x_\sigma} : K_0^{\mathbb{T}}(X)_S \simeq K_0^{\mathbb{T}}(X^{\mathbb{T}})_S \to K_0^{\mathbb{T}}(x_\sigma)_S = \mathbb{Z}[M]_S$$

can be calculated, after restriction to U_σ, as $pr_{x_\sigma} = \mathbb{S} \circ \chi_\sigma^{\mathbb{T}}$, with $\chi_\sigma^{\mathbb{T}} : K_0^{\mathbb{T}}(U_\sigma) \to \mathbb{Z}[M]_{\mathrm{sum}} \subset \mathbb{Z}[[M]]$ the local counterpart of the equivariant Euler characteristic (1) (with $\chi^m \in \mathbb{Z}[M]$ instead of $e^{c(m)}$), and $\mathbb{S}$ the corresponding summation map as introduced by Brion-Vergne [13]. We then have (see formula (155) for a more general version):

$$\chi_\sigma^{\mathbb{T}}(mC_y^{\mathbb{T}}(X)|_{U_\sigma}) = \sum_{\tau\preceq\sigma}(1+y)^{\dim(O_\tau)} \sum_{m\in \mathrm{Relint}(\sigma^\vee\cap\tau^\perp)\cap M} \chi^{-m} \in \mathbb{Z}[M]_{\mathrm{sum}}\otimes_{\mathbb{Z}}\mathbb{Z}[y], \tag{11}$$

where the first sum is over the faces of σ. For $y=0$, this specializes to $\chi_\sigma^{\mathbb{T}}(\mathscr{O}_X|_{U_\sigma})$, since X has rational singularities, so that $mC_0^{\mathbb{T}}(X) = [\mathscr{O}_X]_{\mathbb{T}} \in K_0^{\mathbb{T}}(X)$. As a consequence, we get the following weighted version of Brion's formula (see Corollary 4.8).

Corollary 1.1. *Let P be a full-dimensional lattice polytope with associated projective toric variety $X=X_P$ and ample Cartier divisor $D=D_P$. For each vertex v of P, consider the cone $C_v = \mathrm{Cone}(P\cap M - v) = \sigma_v^\vee$, with faces $E_v = \mathrm{Cone}(E\cap M-v)$ for $v\in E$. Then the following identity holds in $\mathbb{Z}[M]_S\otimes_{\mathbb{Z}}\mathbb{Z}[y]$:*

$$\chi^{\mathbb{T}}(X, mC_y^{\mathbb{T}}(X)\otimes\mathscr{O}_X(D)) = \sum_{v \text{ vertex}} \chi^{-v}\cdot\mathbb{S}\left(\sum_{v\in E\preceq P}(1+y)^{\dim(E)}\cdot\sum_{m\in\mathrm{Relint}(E_v)\cap M}\chi^{-m}\right). \tag{12}$$

Brion's formula [9] is obtained from (12) by specializing to $y=0$.

In the case of a simplicial cone, we get more explicit formulae by using a Lefschetz type variant $tr_\sigma^{\mathbb{T}'}$ of the Euler characteristic $\chi_\sigma^{\mathbb{T}}$, and a corresponding summation map. Let $\sigma\in\Sigma(n)$ be a simplicial cone with $u_1,\dots,u_n\in N=N_\sigma$ the generators of the the rays $\rho_j\in\sigma(1)$, $j=1,\dots,n$. Let $N'=N'_\sigma$ be the finite index sublattice of N generated by $u_1,\dots,u_n$, and consider $\sigma\in N'_{\mathbb{R}}=N_{\mathbb{R}}$ so that it is smooth with respect to the lattice N'. With $\mathbb{T}$, $\mathbb{T}'$ the corresponding n-dimensional tori of the lattices N, resp., N', the inclusion $N'\hookrightarrow N$ induces a toric morphism $\pi: U'_\sigma\to U_\sigma$ of the associated affine toric varieties. Let G_σ be the finite kernel of the epimorphism $\pi:\mathbb{T}'\to\mathbb{T}$, so that $U'_\sigma/G_\sigma\simeq U_\sigma$. Let $m'_{\sigma,1},\dots,m'_{\sigma,n}$ be the dual basis in the dual lattice $M'=M'_\sigma$ of N', with corresponding characters $a_{\rho_j}: G_\sigma\to\mathbb{C}^*$ of G_σ as in the global context of the Cox construction mentioned before (see (33)). With these notations, we have:

$$\mathbb{S}(\chi_\sigma^{\mathbb{T}}(mC_y^{\mathbb{T}}(X)|_{U_\sigma})) = \frac{1}{|G_\sigma|}\sum_{g\in G_\sigma}\prod_{i=1}^n \frac{1+y\cdot a_{\rho_i}(g^{-1})\cdot\chi^{-m'_{\sigma,i}}}{1-a_{\rho_i}(g^{-1})\cdot\chi^{-m'_{\sigma,i}}}, \tag{13}$$

which can be regarded as a local K-theoretical version of the global formula (10). For $y=0$, this specializes to the *Molien formula* of Brion-Vergne [13].

We next discuss some (co)homological counterparts of the above localization formulae. Let $L\subset(\Lambda_{\mathbb{T}})_{\mathbb{Q}} = H^*_{\mathbb{T}}(pt;\mathbb{Q})$ be the multiplicative subset generated by the elements $\pm c(m)$, for $0\neq m\in M$. With $x_\sigma\in X^{\mathbb{T}}$ a fixed point corresponding to $\sigma\in\Sigma(n)$, there is an associated *homological localization map* (for equivariant Borel-Moore homology) at x_σ,

$$pr_{x_\sigma}: \widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})_L \simeq \widehat{H}^{\mathbb{T}}_*(X^{\mathbb{T}};\mathbb{Q})_L \longrightarrow \widehat{H}^{\mathbb{T}}_*(x_\sigma;\mathbb{Q})_L = L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}.$$

These K-theoretic and (co)homological localization maps are compatible with the equivariant Todd class transformation of Edidin-Graham [23] (and Brylinski-Zhang [15]), in the following sense:

Proposition 1.2. *Let $\mathscr{F}$ be a $\mathbb{T}$-equivariant coherent sheaf on $X = X_\Sigma$, and let $x_\sigma \in X^\mathbb{T}$ be a given fixed point of the $\mathbb{T}$-action. Then:*

$$\mathrm{td}_*^\mathbb{T}([\mathscr{F}])_{x_\sigma} := pr_{x_\sigma}(\mathrm{td}_*^\mathbb{T}([\mathscr{F}]) = \mathrm{ch}^\mathbb{T}((\mathbb{S}\circ\chi_\sigma^\mathbb{T})(\mathscr{F})) \in L^{-1}(\Lambda_\mathbb{T}^{an})_\mathbb{Q} \subset L^{-1}(\widehat{\Lambda}_\mathbb{T})_\mathbb{Q}, \tag{14}$$

with $\mathrm{ch}^\mathbb{T} : \mathbb{Z}[M]_S \to L^{-1}(\Lambda_\mathbb{T}^{an})_\mathbb{Q}$ induced by the $\mathbb{T}$-equivariant Chern character on a point space.

As an example, we get by (11) the following localized equivariant Hirzebruch class formula for the toric variety $X = X_\Sigma$:

(15)

$$T_{y*}^\mathbb{T}(X)_{x_\sigma} := \mathrm{td}_*^\mathbb{T}([mC_y^\mathbb{T}(X)])_{x_\sigma} = \sum_{\tau\preceq\sigma}(1+y)^{\dim(O_\tau)}\cdot(\mathrm{ch}^\mathbb{T}\circ\mathbb{S})\left(\sum_{m\in\mathrm{Relint}(\sigma^\vee\cap\tau^\perp)\cap M}\chi^{-m}\right)$$
$$= \sum_{\tau\preceq\sigma}(1+y)^{\dim(O_\tau)}\cdot\mathbb{S}\left(\sum_{m\in\mathrm{Relint}(\sigma^\vee\cap\tau^\perp)\cap M}e^{s(m)}\right).$$

Specializing the above formula to $y = 0$ reduces it to a corresponding localized Todd class formula of Brylinski-Zhang [15] for $\mathrm{td}_*^\mathbb{T}(X)_{x_\sigma} = T_{0*}^\mathbb{T}(X)_{x_\sigma}$. In the case of a simplicial cone $\sigma \in \Sigma(n)$ corresponding to $x_\sigma \in X^\mathbb{T}$, formula (15) implies the following:

$$T_{y*}^\mathbb{T}(X)_{x_\sigma} = \frac{1}{|G_\sigma|}\sum_{g\in G_\sigma}\prod_{i=1}^n\frac{1+y\cdot a_{\rho_i}(g^{-1})\cdot e^{-c(m'_{\sigma,i})}}{1-a_{\rho_i}(g^{-1})\cdot e^{-c(m'_{\sigma,i})}}, \tag{16}$$

This is exactly the localization of the global formula (10), as explained in Section 4.3.

We now describe applications of the above results to Euler-Maclaurin formulae. Let P be as above a full-dimensional lattice polytope in $M_\mathbb{R} \simeq \mathbb{R}^n$, with toric variety $X = X_P$, inner normal fan $\Sigma = \Sigma_P$ and ample Cartier divisor $D = D_P$. Let $\Sigma(1)$ be the set of rays of Σ, corresponding to the facets F of P. For each ray $\rho \in \Sigma(1)$, let $u_\rho \in N$ be the corresponding ray generator. We also let $F_\rho := [D_\rho]_\mathbb{T}$ be the equivariant fundamental class of the $\mathbb{T}$-equivariant divisor D_ρ on X corresponding to the ray $\rho \in \Sigma(1)$. Let $P(h)$ be the dilatation of P with respect to the vector $h = (h_\rho)_{\rho\in\Sigma(1)}$ with real entries indexed by the rays of Σ. So, if P is defined by inequalities of the form

$$\langle m, u_\rho\rangle + c_\rho \geq 0,$$

with u_ρ the ray generators and $c_\rho \in \mathbb{Z}$, for each $\rho \in \Sigma(1)$, then $P(h)$ is defined by inequalities

$$\langle m, u_\rho\rangle + c_\rho + h_\rho \geq 0,$$

for each $\rho \in \Sigma(1)$. In these notations, we have that $D = D_P = \sum_{\rho\in\Sigma(1)} c_\rho \cdot D_\rho$.

Using localization techniques and adapting arguments of Brion-Vergne to our setup (see Theorem 5.1, Theorem 5.8 and Theorem 7.12), we get the following abstract Euler-Maclaurin formula coming from the equivariant Hirzebruch-Riemann-Roch theorem (see Theorem 5.18, Corollary 5.21 and Proposition 5.22):

Theorem 1.3. *Let $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$ be fixed, and choose a convergent power series $p(x_\rho) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}$ so that $p(F_\rho) = \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) \in \left(H_{\mathbb{T}}^*(X;\mathbb{Q})\right)^{an}$. Then, with $p(\frac{\partial}{\partial h})$ the corresponding infinite order differential operator obtained from $p(x_\rho)$ by substituting $x_\rho \mapsto \frac{\partial}{\partial h_\rho}$, for all $\rho \in \Sigma(1)$, we have for any polynomial function f on $M_{\mathbb{R}}$:*

$$(17)\quad p(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\cdot e^{\langle m,z\rangle}\, dm\right)_{|_{h=0}} = \\ = \sum_{m\in M}\left(\sum_{i=0}^{n}(-1)^i\cdot \dim_{\mathbb{C}} H^i(X;\mathscr{O}_X(D)\otimes\mathscr{F})_{\chi^{-m}}\right)\cdot f(m)\cdot e^{\langle m,z\rangle},$$

as analytic functions in $z \in N_{\mathbb{C}}$ with z small enough.

If $v \in P$ is a vertex with tangent cone $Tan(P,v)$ and corresponding cone $\sigma \in \Sigma(n)$, then for z small enough with $-z \in Int(\sigma)$ we get:

$$(18)\qquad p(\frac{\partial}{\partial h})\left(\int_{Tan(P,v)(h)} e^{\langle m,z\rangle}\, dm\right)_{|_{h=0}} = \langle \frac{e^v\cdot\left(i_\sigma^* p(F_\rho)\right)}{Eu_X^{\mathbb{T}}(x_\sigma)}, z\rangle = \langle e^v\cdot \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}])_{x_\sigma}, z\rangle\,.$$

Here, we use a more explicit description of the homological localization map pr_{x_σ} given as $pr_{x_\sigma} = \frac{i_\sigma^*}{Eu_X^{\mathbb{T}}(x_\sigma)}$, see (176) for more details.

It is important to note that the operator $p(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\cdot e^{\langle m,z\rangle}\, dm\right)_{|_{h=0}}$ (with $f=1$ in the local case) depends only on the class $\mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) \in \left(H_{\mathbb{T}}^*(X;\mathbb{Q})\right)^{an}$ and not on the chosen convergent power series representative. Such a convergent power series representative exists since $\mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) \in \left(H_{\mathbb{T}}^*(X;\mathbb{Q})\right)^{an}$. If one uses only polynomials in formula (17) (i.e., setting $z=0$), this formula even holds for $p(x_\rho) \in \mathbb{Q}[[x_\rho \mid \rho \in \Sigma(1)]]$ a formal power series with $p(F_\rho) = \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) \in \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})$.

Remark 1.4. The formulae of Theorem 1.3 hold with the same operator $p(\frac{\partial}{\partial h})$ (which is fixed by the choice of a convergent power series $p(x_\rho) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}$ so that $p(F_\rho) = \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) \in \left(H_{\mathbb{T}}^*(X;\mathbb{Q})\right)^{an}$), but applied to different full-dimensional simple lattice polytopes P corresponding to different choices of ample divisors $D = D_P$ on the toric variety X_Σ (e.g., for dilatations of a given polytope).

Note that by evaluating formula (17) at $z=0$ and for $f=1$ (i.e., forgetting the $\mathbb{T}$-action), we get a generalized *volume formula*, namely,

$$p(\frac{\partial}{\partial h})\left(vol\ P(h)\right)_{|_{h=0}} = \chi(X, \mathscr{O}_X(D)\otimes\mathscr{F}),$$

with $vol\ P(h) = \int_{P(h)} dm$ the volume of $P(h)$ and the Lebesgue measure normalized so that the unit cube in $M \subset M_{\mathbb{R}}$ has volume 1. See [12][Thm.2.15] for the case when $\mathscr{F} = \mathscr{O}_X$ (corresponding to counting points in $P \cap M$) and $\mathscr{F} = \omega_X$ (corresponding to counting points in $\mathrm{Int}(P)\cap M$).

In Section 5.1 we explain how formula (17) can be specialized to yield old and new Euler-Maclaurin type formulae. In particular, we obtain an Euler-Maclaurin formula for a simple lattice polytope with some facets removed (generalizing the classical case of (the interior of) a polytope), see formula (210). We also obtain Euler-Maclaurin formulae for a face E of P (see Theorem 5.27) and for the interior of E (see Theorem 5.26).

Another way to obtain examples of explicit weighted Euler-Maclaurin formulae is by considering the classes $[\mathscr{F}] := [mC_y^{\mathbb{T}}(X)] \in K_0^{\mathbb{T}}(X)[y]$, or twisting these by $\mathscr{O}_X(D'-D)$, for $D = D_P$ the original ample divisor associated to the full-dimensional simple lattice polytope P, and D' any $\mathbb{T}$-invariant Cartier divisor on X (see Theorem 6.2, Corollary 6.3 and Example 6.9).

Let D' be a globally generated $\mathbb{T}$-invariant Cartier divisor on X, with associated (not necessarily full-dimensional) lattice polytope $P_{D'} \subset M_{\mathbb{R}}$. Let $D'-D = \sum_{\rho\in\Sigma(1)} d_\rho D_\rho$ as a $\mathbb{T}$-invariant Cartier divisor. Let $X_{D'}$ be the toric variety of the lattice polytope $P_{D'}$, defined via the corresponding generalized fan. Consider the infinite order differential operator

$$T'_y(\frac{\partial}{\partial h}) := e^{\sum_{\rho\in\Sigma(1)} d_\rho\cdot\frac{\partial}{\partial h_\rho}} \cdot T_y(\frac{\partial}{\partial h}) \in \mathbb{Q}\{\frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)\}[y], \tag{19}$$

with $T_y(\frac{\partial}{\partial h})$ obtained by substituting $F_\rho \mapsto \frac{\partial}{\partial h_\rho}$ (for each $\rho \in \Sigma(1)$) into the right-hand side formula (10) for the equivariant Hirzebruch class $T^{\mathbb{T}}_{y*}(X)$. For any polynomial f on $M_{\mathbb{R}}$, one then has by Theorem 1.3 the following new weighted Euler-Maclaurin formula:

$$\begin{aligned} T'_y(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\cdot e^{\langle m,z\rangle}\, dm\right)_{|h=0} &= \\ &= \sum_{E\preceq P_{D'}}\left(\sum_{\ell\geq 0}(-1)^\ell\cdot d_\ell(X/E)\cdot(1+y)^{\ell+\dim(E)}\right)\cdot\sum_{m\in \mathrm{Relint}(E)\cap M} f(m)\cdot e^{\langle m,z\rangle}, \end{aligned} \tag{20}$$

with multiplicities $d_\ell(X/E) = d_\ell(X/\sigma')$ as in (6), and the face E of $P_{D'}$ corresponding to the cone $\sigma' \in \Sigma'$. Note that in this context $P_{D'}$ is an $\mathbb{N}$-Minkowski summand of the original polytope P. Forgetting the $\mathbb{T}$-action (i.e., for $f = 1$ and $z = 0$), one gets the following volume formula (fitting with (9)):

$$T'_y(\frac{\partial}{\partial h})\,(vol\ P(h))_{|h=0} = \sum_{E\preceq P_{D'}}\left(\sum_{\ell\geq 0}(-1)^\ell\cdot d_\ell(X/E)\cdot(1+y)^{\ell+\dim(E)}\right)\cdot|\mathrm{Relint}(E)\cap M|. \tag{21}$$

Specializing (20) to the case $D = D'$, with $P = P_{D'}$, one gets the following weighted Euler-Maclaurin formula:

$$T_y(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\cdot e^{\langle m,z\rangle}\, dm\right)_{|h=0} = \sum_{E\preceq P}(1+y)^{\dim(E)}\cdot\sum_{m\in\mathrm{Relint}(E)\cap M} f(m)\cdot e^{\langle m,z\rangle}. \tag{22}$$

For $y = 0$ this further reduces to the classical Euler-Maclaurin formula of Brion-Vergne [13] for simple lattice polytopes, and Khovanskii-Pukhlikov [38] for Delzant lattice polytopes (corresponding to smooth projective toric varieties):

$$Todd(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\cdot e^{\langle m,z\rangle}\, dm\right)_{|h=0} = \sum_{m\in P\cap M} f(m)\cdot e^{\langle m,z\rangle}, \tag{23}$$

with $Todd(\frac{\partial}{\partial h}) := T_0(\frac{\partial}{\partial h})$.

Following [34], in Section 7 we introduce power series of the form

$$p(t_i,x_\rho) := \sum_{\alpha=(\alpha_i)\in\mathbb{N}_0^n} p_\alpha(x_\rho)\prod_{i=1}^n t_i^{\alpha_i} \in \mathbb{Q}\{t_1,\dots,t_n,x_\rho \mid \rho\in\Sigma(1)\},$$

resp., $\mathbb{Q}[[t_1,\dots,t_n,x_\rho \mid \rho\in\Sigma(1)]]$ as convergent, resp., formal power series in the t_i,x_ρ, with corresponding differential operator

$$p(\partial_i,\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\cdot e^{\langle m,z\rangle}\,dm\right)_{|h=0} := \sum_{\alpha=(\alpha_i)\in\mathbb{N}_0^n} p_\alpha(\frac{\partial}{\partial h})\left(\int_{P(h)}\prod_{i=1}^n \partial_i^{\alpha_i} f(m)\cdot e^{\langle m,z\rangle}\,dm\right)_{|h=0}$$

for f a polynomial function on $M_\mathbb{R}$, obtained by substituting $t_i\mapsto\partial_i, x_\rho\mapsto\frac{\partial}{\partial h_\rho}$ into the power series $p(t_i,x_\rho)$. We can now state our second abstract Euler-Maclaurin formula coming from equivariant Hirzebruch-Riemann-Roch, see Theorem 7.10 and Corollary 7.11.

Theorem 1.5. *Let $X=X_P$ be the projective simplicial toric variety associated to a full-dimensional simple lattice polytope $P\subset M_\mathbb{R}$. Let $\Sigma:=\Sigma_P$ be the inner normal fan of P, and $D:=D_P$ the ample Cartier divisor associated to P. Let $[\mathscr{F}]\in K_0^\mathbb{T}(X)$ be fixed, and choose a convergent power series $p(t_i,x_\rho)\in\mathbb{Q}\{t_1,\dots,t_n,x_\rho\mid\rho\in\Sigma(1)\}$ so that $p(s(m_i),F_\rho)=$ $\mathrm{td}_*^\mathbb{T}([\mathscr{F}])\in\left(H_\mathbb{T}^*(X;\mathbb{Q})\right)^{an}$. Then, for a polynomial function f on $M_\mathbb{R}$,*

$$(24)\quad p(\partial_i,\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\cdot e^{\langle m,z\rangle}\,dm\right)_{|h=0} = \\ = \sum_{m\in M}\left(\sum_{i=0}^n(-1)^i\cdot\dim_\mathbb{C} H^i(X;\mathscr{O}_X(D)\otimes\mathscr{F})_{\chi^{-m}}\right)\cdot f(m)\cdot e^{\langle m,z\rangle},$$

as analytic functions in z with z small enough.

Of course, Theorem 1.5 reduces to the first part of Theorem 1.3 in case

$$p(t_i,x_\rho)\in\mathbb{Q}\{x_\rho\mid\rho\in\Sigma(1)\}\subset\mathbb{Q}\{t_1,\dots,t_n,x_\rho\mid\rho\in\Sigma(1)\}$$

does not depend on the variables t_i. As before, the operator $p(\partial_i,\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\cdot e^{\langle m,z\rangle}\,dm\right)_{|h=0}$ depends only on the class $\mathrm{td}_*^\mathbb{T}([\mathscr{F}])\in\left(H_\mathbb{T}^*(X;\mathbb{Q})\right)^{an}$ and not on the chosen convergent power series representative. If one uses only polynomials in formula (24) (i.e., setting $z=0$), this formula even holds for $p(t_i,x_\rho)\in\mathbb{Q}[[t_1,\dots,t_n,x_\rho\mid\rho\in\Sigma(1)]]$ a formal power series with $p(s(m_i),F_\rho)=\mathrm{td}_*^\mathbb{T}([\mathscr{F}])\in\widehat{H}_\mathbb{T}^*(X;\mathbb{Q})$.

Let us next denote by σ_E the cone in $\Sigma=\Sigma_P$ corresponding to the face E of P, and let V_{σ_E} be the closure of the $\mathbb{T}$-orbit in X corresponding to σ_E. As explained in Remark 7.15, the equivariant fundamental classes $[V_{\sigma_E}]_\mathbb{T}$ generate $\widehat{H}_\mathbb{T}^*(X;\mathbb{Q})$ as a $\widehat{H}_\mathbb{T}^*(pt;\mathbb{Q})=(\widehat{\Lambda}_\mathbb{T})_\mathbb{Q}$-algebra. Let now $[\mathscr{F}]\in K_0^\mathbb{T}(X)$ be fixed, and choose elements $p_{\sigma_E}(t_i)\in\widehat{H}_\mathbb{T}^*(pt;\mathbb{Q})=(\widehat{\Lambda}_\mathbb{T})_\mathbb{Q}\simeq\mathbb{Q}[[t_1,\dots,t_n]]$ with

$$(25)\qquad \mathrm{td}_*^\mathbb{T}([\mathscr{F}])=\sum_{E\preceq P} p_{\sigma_E}(t_i)[V_{\sigma_E}]_\mathbb{T}\in\widehat{H}_\mathbb{T}^*(X;\mathbb{Q}).$$

Then $\mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) = p(s(m_i), F_\rho) \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$ for

$$p(t_i,x_\rho) := \sum_{E\preceq P}\left(\mathrm{mult}(\sigma_E)\cdot\prod_{\rho\in\sigma_E(1)} x_\rho\right)\cdot p_{\sigma_E}(t_i) \in \mathbb{Q}[[t_1,\ldots,t_n,x_\rho \mid \rho\in\Sigma(1)]]\,. \tag{26}$$

With these notations, we can now state our final abstract Euler-Maclaurin formula coming from the equivariant Hirzebruch-Riemann-Roch theorem, which also provides a generalization of the Cappell-Shaneson Euler-Maclaurin formula, see Theorem 7.16 and Remark 7.17.

Theorem 1.6. *Let $X = X_P$ be the projective simplicial toric variety associated to a full-dimensional simple lattice polytope $P \subset M_{\mathbb{R}}$. Let $\Sigma := \Sigma_P$ be the inner normal fan of P, and $D := D_P$ the ample Cartier divisor associated to P. Let $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$ be fixed, and choose the formal power series $p(t_i,x_\rho) \in \mathbb{Q}[[t_1,\ldots,t_n,x_\rho \mid \rho\in\Sigma(1)]]$ as in (26). Then for a polynomial function f on $M_{\mathbb{R}}$, we have:*

$$\sum_{E\preceq P}\int_E p_{\sigma_E}(\partial_i) f(m)\, dm = \sum_{m\in M}\left(\sum_{i=0}^n (-1)^i\cdot \dim_{\mathbb{C}} H^i(X;\mathscr{O}_X(D)\otimes\mathscr{F})_{\chi^{-m}}\right)\cdot f(m)\,. \tag{27}$$

Remark 1.7. Also the formula of Theorem 1.6 holds with the same operators $p_{\sigma_E}(\partial_i)$ (which are fixed by formula (25)), but applied to different full-dimensional simple lattice polytopes P corresponding to different choices of ample divisors $D = D_P$ on the toric variety X_Σ (e.g., for dilatations of a given polytope).

In the classical case $\mathscr{F} := \mathscr{O}_X$, this is exactly Cappell-Shaneson's recipe for the definition of the differential operators $p_{\sigma_E}(\partial_i)$, described here geometrically in terms of the equivariant Todd class $\mathrm{td}_*^{\mathbb{T}}(X) := \mathrm{td}_*^{\mathbb{T}}([\mathscr{O}_X]) \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$ (see [18][Thm.2] or [43, Sect.6.2]). In this case, (27) reduces to (see Remark 7.17 for more details):

$$\sum_{E\preceq P}\int_E p_{\sigma_E}(\partial_i) f(m)\, dm = \sum_{m\in P\cap M} f(m).$$

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2. Preliminaries

2.1. **Background on toric varieties and lattice polytopes.** In this section we review some basic facts and terminology from the theory of toric varieties and lattice polytopes. For complete details, the interested reader is referred to [16, 20, 27, 39].

2.1.1. *Toric varieties.* Let $M \simeq \mathbb{Z}^n$ be a n-dimensional lattice in $\mathbb{R}^n$, and let $N = \mathrm{Hom}(M,\mathbb{Z})$ be the dual lattice. Denote the natural pairing by

$$\langle \cdot,\cdot \rangle : M \times N \to \mathbb{Z}.$$

A *rational polyhedral cone* $\sigma \subset N_{\mathbb{R}} := N \otimes \mathbb{R}$ is a cone $\sigma = Cone(S)$ on a finite set $S \subset N$. Such a cone σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$. The dimension of a cone σ is the dimension of the subspace of $N_{\mathbb{R}}$ spanned by σ. A *face* τ of a cone σ (we write $\tau \preceq \sigma$) is a subset

$$\tau := \{u \in \sigma \mid \langle m,u \rangle = 0\} \subset \sigma$$

for some $m \in M \cap \check{\sigma}$, where

$$\check{\sigma} = \{m \in M_{\mathbb{R}} \mid \langle m,u \rangle \geq 0, \forall u \in \sigma\}$$

is the *dual cone* of σ. Every face of σ is again a rational polyhedral cone, and a face of a face is a face. A one-dimensional face ρ of a cone σ is called a *ray*. The collection of rays of a cone σ is denoted by $\sigma(1)$. For each ray $\rho \in \sigma(1)$, let u_ρ be the unique generator of the semigroup $\rho \cap N$. The $\{u_\rho\}_{\rho \in \sigma(1)}$ are called the *generators* of σ. A cone σ is called *smooth* if it is generated by a part of the $\mathbb{Z}$-basis of N. A cone σ is *simplicial* if its generators are linearly independent over $\mathbb{R}$. The *multiplicity* $\mathrm{mult}(\sigma)$ of a simplicial cone σ with generators $u_{\rho_1},\dots,u_{\rho_k}$ is $|N_\sigma/(u_{\rho_1},\dots,u_{\rho_k})|$, where N_σ is the sublattice of N spanned by points in $\sigma \cap N$, and $|-|$ denotes the cardinality of a (finite) set.

A *fan* Σ in $N_{\mathbb{R}}$ is a finite collection of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ so that any face of $\sigma \in \Sigma$ is also in Σ, and if $\sigma_1,\sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2$ is a face of each (so $\sigma_1 \cap \sigma_2 \in \Sigma$). The *support* of a fan Σ is the set $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}}$. The fan is called *complete* if $|\Sigma| = N_{\mathbb{R}}$. A fan Σ is called *smooth* if every cone $\sigma \in \Sigma$ is smooth. Σ is called *simplicial* if every cone in $\sigma \in \Sigma$ is simplicial. We denote by $\Sigma(i)$ the set of i-dimensional cones of Σ, and similarly, by $\sigma(i)$ the collection of i-dimensional faces of a cone σ.

To any fan Σ one associates a *toric variety* X_Σ as follows. Each cone $\sigma \in \Sigma$ gives rise to an affine toric variety

$$U_\sigma = \mathrm{Spec}\big(\mathbb{C}[M \cap \check{\sigma}]\big)$$

where $\mathbb{C}[M \cap \check{\sigma}]$ is the $\mathbb{C}$-algebra with generators $\{\chi^m \mid m \in M \cap \check{\sigma}\}$. The toric variety X_Σ is obtained by gluing these affine pieces together, so that U_{σ_1} and U_{σ_2} get glued along $U_{\sigma_1 \cap \sigma_2}$. The affine toric variety corresponding to the trivial cone $\{0\}$ is the torus $T_N = N \otimes \mathbb{C}^* = \mathrm{Spec}(\mathbb{C}[M])$, so T_N is an affine open subset of X_Σ. It follows that M can be identified with the character lattice of T_N, i.e., $M \simeq \mathrm{Hom}_{\mathbb{Z}}(T_N,\mathbb{C}^*)$, while $N \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}^*,T_N)$ is the lattice of one-parameter subgroups of T_N. In particular, the natural pairing between $m \in M$ and $n \in N$ yields the integer $\langle m,n \rangle$ such that $m(n(t)) = t^{\langle m,n \rangle}$ for all $t \in \mathbb{C}^*$.

The action of the torus T_N on itself extends to an algebraic action of T_N on X_Σ with finitely many orbits O_σ, one for each cone $\sigma \in \Sigma$. In fact, by the orbit-cone correspondence, to a k-dimensional cone $\sigma \in \Sigma$ there corresponds a $(n-k)$-dimensional torus-orbit $O_\sigma \cong (\mathbb{C}^*)^{n-k}$, and the closure V_σ of O_σ (which is the same in both classical and Zariski topology) is itself a toric variety and a T_N-invariant subvariety of X_Σ. In particular, each ray $\rho \in \Sigma(1)$ corresponds to an irreducible T_N-invariant divisor $D_\rho := V_\rho$ on X_Σ. Moreover, if σ is a face of ν, i.e., $\sigma \preceq \nu$, then $O_\nu \subseteq \overline{O}_\sigma = V_\sigma$ and

$$V_\sigma = \bigsqcup_{\sigma \preceq \nu} O_\nu.$$

So the toric variety X_Σ is stratified by the orbits of the T_N-action.

Let us also describe here, for future reference, the fan of the toric variety V_σ, i.e., the closure of the orbit corresponding to the cone $\sigma \in \Sigma$. Let

$$N(\sigma) = N/N_\sigma,$$

with N_σ denoting as before the sublattice of N spanned by the points in $\sigma \cap N$. Let $T_{N(\sigma)} = N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^*$ be the torus associated to $N(\sigma)$. For each cone $\nu \in \Sigma$ containing σ, let $\overline{\nu}$ be the image cone in $N(\sigma)_{\mathbb{R}}$ under the quotient map $N_{\mathbb{R}} \to N(\sigma)_{\mathbb{R}}$. Then

$$Star(\sigma) = \{\overline{\nu} \subseteq N(\sigma)_{\mathbb{R}} \mid \sigma \preceq \nu\} \tag{28}$$

is a fan in $N(\sigma)_{\mathbb{R}}$, with associated toric variety isomorphic to V_σ (see, e.g., [16][Prop.3.2.7]). Note that T_N acts on V_σ via the morphism

$$T_N \to T_{N(\sigma)} \tag{29}$$

induced by the quotient map $N \to N(\sigma)$.

If Σ is a fan in $N_{\mathbb{R}} \simeq \mathbb{R}^n$, then the toric variety X_Σ is a complex algebraic variety of dimension n, whose geometry is subtly related to the properties of the fan Σ. For example:

(1) X_Σ is complete iff Σ is a complete fan.
(2) X_Σ contains no torus factor iff $N_{\mathbb{R}}$ is spanned by the ray generators u_ρ, $\rho \in \Sigma(1)$.
(3) X_Σ is smooth iff Σ is a smooth fan.
(4) X_Σ is an orbifold iff Σ is a simplicial fan.

The toric variety X_Σ associated to a simplicial fan is called a *simplicial toric variety*. Such a variety is an orbifold and therefore also a rational homology manifold, so it satisfies Poincaré duality over $\mathbb{Q}$. The singular locus of a (simplicial) toric variety X_Σ is $\bigcup_{\sigma \in \Sigma_{\text{sing}}} V_\sigma$, the union being taken over the collection Σ_{sing} of all singular (non-smooth) cones in the fan Σ.

We conclude this section with a quick description of the rational cohomology ring $H^*(X_\Sigma;\mathbb{Q})$ of a complete *simplicial* toric variety X_Σ. Fix a numbering $\rho_1, \dots, \rho_r$ for the rays in $\Sigma(1)$, and let u_i denote the generator of ρ_i, for $i = 1, \dots, r$. We denote by D_i the T_N-invariant divisor corresponding to ρ_i, and let $[D_i] \in H^2(X_\Sigma;\mathbb{Q})$ be the cohomology class of D_i (under rational Poincaré duality). Introduce a variable x_i of degree 2 for each ρ_i, and define the graded ring

$$R_{\mathbb{Q}}(\Sigma) := \mathbb{Q}[x_1, \dots, x_r]/(\mathscr{I} + \mathscr{J}),$$

where:

- $\mathscr{I}$ is the *Stanley-Reisner ideal* generated by monomials $x_{i_1} \dots x_{i_s}$ with all i_j's distinct and $\rho_{i_1} + \cdots + \rho_{i_s}$ not a cone of Σ;
- $\mathscr{J}$ is the ideal generated by the linear forms $\sum_{i=1}^r \langle m, u_i \rangle x_i$, where m ranges over (some basis of) M.

Then the assignment $x_i \mapsto [D_i]$ induces an isomorphism (see, e.g., [16][Thm.12.4.1]):

$$R_{\mathbb{Q}}(\Sigma) \xrightarrow{\simeq} H^*(X_\Sigma;\mathbb{Q}). \tag{30}$$

2.1.2. *Cox quotient construction of simplicial toric varieties.* In this section, we recall the Cox construction of simplicial toric varieties as geometric quotients, see [19] and [16][Sect.5.1].

Let Σ be a fan in $N_{\mathbb{R}} \simeq \mathbb{R}^n$, with associated toric variety $X := X_\Sigma$ and torus $\mathbb{T} := T_N = (\mathbb{C}^*)^n$. For each ray $\rho \in \Sigma(1)$, denote by u_ρ the corresponding ray generator. Let

$$r = |\Sigma(1)|$$

be the number of rays in the fan Σ. For simplicity, we also assume that $X = X_\Sigma$ contains no torus factor.

Using the fact that $N \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{C}^*, \mathbb{T})$ is identified with the one-parameter subgroups of $\mathbb{T}$, define the map of tori

$$\gamma : \widetilde{\mathbb{T}} := (\mathbb{C}^*)^r \longrightarrow \mathbb{T} \quad \text{by} \quad (t_\rho)_\rho \mapsto \prod_{\rho \in \Sigma(1)} u_\rho(t_\rho),$$

and let $G := \ker(\gamma)$. Then G is a product of a torus and a finite abelian group, so G is reductive. Let $Z(\Sigma) \subset \mathbb{C}^r$ be the variety defined by the monoidal ideal

$$B(\Sigma) := \langle \hat{x}_\sigma : \sigma \in \Sigma \rangle$$

where

$$\hat{x}_\sigma := \prod_{\rho \notin \sigma(1)} x_\rho$$

and $(x_\rho)_{\rho \in \Sigma(1)}$ are coordinates on $\mathbb{C}^r$. Then the variety

$$W := \mathbb{C}^r \setminus Z(\Sigma)$$

is a toric manifold, and there is a toric morphism

$$\pi : W \to X.$$

The group G acts on W by the restriction of the diagonal action of $(\mathbb{C}^*)^r$, and the toric morphism π is constant on G-orbits. Moreover, Cox [19] (see also [16][Thm.5.1.11]) proved that if $X = X_\Sigma$ is a simplicial toric variety containing no torus factor, then X is the geometric quotient W/G.

Under the quotient map $\pi : W \to X$, the coordinate hyperplane $\{x_\rho = 0\}$ maps to the invariant Weil divisor D_ρ. More generally, if σ is a k-dimensional cone generated by rays $\rho_1, \dots, \rho_k$, then the orbit closure V_σ is the image under π of the linear subspace W_σ defined by the ideal $(x_{\rho_1}, \dots, x_{\rho_k})$ of the total coordinate ring $\mathbb{C}\big[x_\rho \mid \rho \in \Sigma(1)\big]$. Let G_σ be the stabilizer of W_σ. Then, if

$$a_\rho : (\mathbb{C}^*)^r \to \mathbb{C}^*$$

is the projection onto the ρ-th factor (and similarly for any restriction of this projection), we have by definition that

$$\begin{aligned} G_\sigma &= \{ g \in G \mid a_\rho(g) = 1, \forall \rho \notin \sigma(1) \} \\ &\simeq \{ (t_\rho)_{\rho \in \sigma(1)} \mid t_\rho \in \mathbb{C}^*, \prod_{\rho \in \sigma(1)} u_\rho(t_\rho) = 1 \}, \end{aligned} \tag{31}$$

so G_σ depends only on σ (and not on the fan Σ nor the group G).

For a k-dimensional rational simplicial cone σ generated by the rays $\rho_1, \dots, \rho_k$ one can moreover show that

$$G_\sigma \simeq N_\sigma / (u_1, \dots, u_k), \tag{32}$$

so $|G_\sigma| = \mathrm{mult}(\sigma)$ is just the multiplicity of σ, with $\mathrm{mult}(\sigma) = 1$ exactly in the case of a smooth cone. Let $m_i \in M_\sigma$ for $1 \leq i \leq k$ be the unique primitive elements in the dual lattice M_σ of N_σ satisfying $\langle m_i, u_j \rangle = 0$ for $i \neq j$ and $\langle m_i, u_i \rangle > 0$, so that the dual lattice M'_σ of $(u_1, \dots, u_k)$ is generated by the elements $\frac{m_j}{\langle m_j, u_j \rangle}$. Then, for $g = n + (u_1, \dots, u_k) \in G_\sigma = N_\sigma/(u_1, \dots, u_k)$, the character $a_{\rho_j}(g)$ is also given by (see [27][page 34]):

$$a_{\rho_j}(g) = \exp\left(2\pi i \cdot \gamma_{\rho_j}(g)\right), \quad \text{with} \quad \gamma_{\rho_j}(g) := \frac{\langle m_j, n \rangle}{\langle m_j, u_j \rangle}. \tag{33}$$

Consider next the finite set

$$G_\Sigma := \bigcup_{\sigma \in \Sigma} G_\sigma,$$

and note that the smooth cones of Σ only contribute the identity element id_G of G. For $g \in G$, the set of coordinates of $\mathbb{C}^r$ (with $r = |\Sigma(1)|$) on which g acts non-trivially is identified with:

$$g(1) = \{\rho \in \Sigma(1) \mid a_\rho(g) \neq 1\}.$$

Thus the fixed locus of g in $\mathbb{C}^r$ is the linear subspace defined by the ideal $\left(x_\rho : \rho \in g(1)\right)$ of the total coordinate ring. So the fixed locus W^g of g in W is the intersection of this linear subspace with $W = \mathbb{C}^r \setminus Z(\Sigma)$. Note that $W^g \neq \emptyset$ if and only if the linear space defined by the ideal $\left(x_\rho : \rho \in g(1)\right)$ is not contained in $Z(\Sigma)$, and the latter is equivalent to the existence of a cone $\sigma \in \Sigma$ which is generated by the rays of $g(1)$. For such a cone σ, we have that $g \in G_\sigma$ and $W^g = W_\sigma$.

In what follows we write as usual $\tau \preceq \sigma$ for a face τ of σ, and we use the notation $\tau \prec \sigma$ for a proper face τ of σ. Note that

$$\tau \preceq \sigma \Longrightarrow G_\tau \subseteq G_\sigma.$$

Set

$$G^\circ_\sigma := G_\sigma \setminus \bigcup_{\tau \prec \sigma} G_\tau = \{g \in G_\sigma \mid a_\rho(g) \neq 1, \forall \rho \in \sigma(1)\}. \tag{34}$$

Since $G_\sigma = \{id_G\}$ for a smooth cone σ, it follows that $G^\circ_\sigma = \emptyset$ if σ is a smooth cone of positive dimension, while $G^\circ_{\{0\}} = G_{\{0\}} = \{id_G\}$. Moreover, as

$$G_\sigma = \bigsqcup_{\tau \preceq \sigma} G^\circ_\tau,$$

(with $\bigsqcup$ denoting a disjoint union) it follows that

$$G_\Sigma := \bigcup_{\sigma \in \Sigma} G_\sigma = \bigsqcup_{\sigma \in \Sigma} G^\circ_\sigma = \{id_G\} \sqcup \bigsqcup_{\sigma \in \Sigma_{\mathrm{sing}}} G^\circ_\sigma, \tag{35}$$

with Σ_{sing} the collection of singular cones in Σ. Moreover, in the above notations, for any $\sigma \in \Sigma_{\mathrm{sing}}$ one has the following:

$$g \in G^\circ_\sigma \iff g(1) = \sigma(1) \iff W^g = W_\sigma.$$

2.1.3. *Lattice polytopes and their associated toric varieties.* A *polytope* $P \subset M_{\mathbb{R}} = M \otimes \mathbb{R}$ is the convex hull of a finite set $S \subset M_{\mathbb{R}}$. P is called a *lattice polytope* if its vertices lie in M. The dimension $\dim(P)$ of a polytope P is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing P. A polytope P whose dimension equals $n = \dim(M_{\mathbb{R}})$ is called full-dimensional. Faces of codimension one of P are called *facets*. A polytope P is called *simple* if every vertex of P is the intersection of precisely $\dim(P)$ facets.

Let P be a full-dimensional lattice polytope. For each facet F of P there is a unique pair $(u_F, c_F) \in N \times \mathbb{Z}$ so that P is uniquely described by its *facet presentation*:

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle + c_F \geq 0, \text{ for all facets } F \text{ of } P\}. \tag{36}$$

Here u_F is the unique ray generator of the inward-pointing facet normal of F.

To a full-dimensional lattice polytope P one associates a fan Σ_P in $N_{\mathbb{R}}$, called the *inner normal fan* of P, which is defined as follows: to each face E of P associate the cone σ_E by

$$\sigma_E := Cone(u_F \mid F \text{ contains } E).$$

In particular, $\rho_F := \sigma_F = Cone(u_F)$ is the ray generated by u_F. Let

$$X_P := X_{\Sigma_P}$$

be the corresponding toric variety, which is commonly referred to as the toric variety associated to the polytope P. As Σ_P is a complete fan, it follows that X_P is proper. If P is simple, then X_P is simplicial. Moreover, if P is a *Delzant* (or regular) polytope, then X_P is smooth.

For a cone $\sigma_E \in \Sigma_P$ associated to a face E of P, the corresponding orbit closure V_{σ_E} can be identified with the toric variety X_E, with corresponding lattice polytope defined as follows (cf. [16][Prop.3.2.9]): translate P by a vertex m_0 of E so that the origin is a vertex of $E_0 := E - m_0$; while this translation by m_0 does not change Σ_P or X_P, E_0 is now a full-dimensional polytope in $Span(E_0)$. So E_0 is a full-dimensional lattice polytope relative to $Span(E_0) \cap M$, and X_E is the associated toric variety.

With the above notations, the *divisor D_P of the polytope P* is defined as:

$$D_P := \sum_F c_F D_F, \tag{37}$$

where F runs over the collection of facets of P and D_F is the torus-invariant divisor corresponding to the ray ρ_F. Then D_P is a torus-invariant Cartier divisor on X_P, which is ample and basepoint free (e.g., see [16][Prop.6.1.10]). Hence X_P is a projective variety.

Remark 2.1. The above construction also works for $P \subset M_{\mathbb{R}}$ a full-dimensional lattice *polyhedron* as in [16][Def.7.1.3]. In this case, the toric variety X_P has a fan Σ_P with full-dimensional convex support in the sense of [16][page 265] (see [16][Thm.7.1.6]). Then the toric divisor D_P of the polyhedron P is a basepoint free (hence, by [16][Thm.6.3.12], a *nef*) Cartier divisor (see [16][page 322 and Thm.7.2.2]). Moreover, the toric variety X_P is quasi-projective, and therefore semi-projective in the sense of [16][Prop.7.2.9].

If P is a full-dimensional lattice polytope (and even for full-dimensional lattice polyhedra), one has (e.g., see [16][Prop.4.3.3])

$$\Gamma(X_P; \mathscr{O}_{X_P}(D_P)) = \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m \subset \mathbb{C}[M], \tag{38}$$

where χ^m denotes the character defined by $m \in M$, and $\mathbb{C}[M]$ the coordinate ring of the torus T_N. The nefness of D_P also yields that (e.g., see [16][Prop.9.2.3])

$$H^i(X_P; \mathscr{O}_{X_P}(D_P)) = 0, \text{ for all } i > 0. \tag{39}$$

Moreover, if P is a full-dimesional lattice polytope, one has (e.g., see [16][Prop.9.2.7])

$$H^i(X_P; \mathscr{O}_{X_P}(-D_P)) = 0, \text{ for all } i \neq n, \tag{40}$$

and

$$H^n(X_P; \mathscr{O}_{X_P}(-D_P)) = \bigoplus_{m \in \mathrm{Int}(P) \cap M} \mathbb{C} \cdot \chi^{-m}, \tag{41}$$

with $\mathrm{Int}(P)$ denoting the interior of P. Also, by toric Serre duality [16][Thm.9.2.10],

$$\bigoplus_{m \in \mathrm{Int}(P) \cap M} \mathbb{C} \cdot \chi^m = (H^n(X_P; \mathscr{O}_{X_P}(-D_P)))^\vee \simeq H^0(X_P; \mathscr{O}_{X_P}(D_P) \otimes \omega_{X_P}), \tag{42}$$

with ω_{X_P} the *dualizing sheaf* of X_P, and $H^i(X_P; \mathscr{O}_{X_P}(D_P) \otimes \omega_{X_P}) = 0$ for all $i \neq 0$.

Remark 2.2. If one wishes to apply the above formulas to the toric variety X_E associated to a face E of a full-dimensional lattice polytope P as previously described, one needs to work with the corresponding divisor

$$D_{E_0} = D_{P-m_0}|_{X_E} = D_P|_{X_E} + div(\chi^{m_0}). \tag{43}$$

2.2. **Counting lattice points via Todd classes.** In [20], Danilov used the Hirzebruch-Riemann-Roch theorem to establish a direct connection between the problem of counting the number of lattice points in a lattice polytope and the Todd classes of the associated toric variety (see also [9, 12]).

Let $P \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a full-dimensional lattice polytope with associated projective toric variety $X := X_P$ and ample Cartier divisor $D := D_P$. Then one has by (38) and (39) the following:

$$\begin{aligned} |P \cap M| = \chi(X, \mathscr{O}_X(D)) &= \int_X \mathrm{ch}(\mathscr{O}_X(D)) \cap \mathrm{td}_*(X), \\ &= \sum_{k \geq 0} \frac{1}{k!} \int_X [D]^k \cap \mathrm{td}_k(X), \end{aligned} \tag{44}$$

with $\mathrm{td}_* : K_0(X) \to H_*(X) \otimes \mathbb{Q}$ the Baum-Fulton-MacPherson Todd class transformation [4] and $\mathrm{td}_*(X) := \mathrm{td}_*([\mathscr{O}_X])$. Here, $K_0(X)$ is the Grothendieck group of coherent sheaves, and for $H_*(-)$ one can use either the Chow group CH_* or the even degree Borel-Moore homology group H_{2*}^{BM}. The second equality of (44) follows from the module property for td_*, i.e., if $\beta \in K^0(X)$ and $\alpha \in K_0(X)$, then (see [26][Thm.18.3]):

$$\mathrm{td}_*(\beta \otimes \alpha) = \mathrm{ch}(\beta) \cap \mathrm{td}_*(\alpha). \tag{45}$$

Similarly, (42) and the module property (45) yield:

$$|\mathrm{Int}(P) \cap M| = \chi(X, \mathscr{O}_X(D) \otimes \omega_X) = \int_X \mathrm{ch}(\mathscr{O}_X(D)) \cap \mathrm{td}_*([\omega_X]). \tag{46}$$

It thus follows from (44) and (46) that counting lattice points in (the interior of) a full-dimensional lattice polytope P amounts to computing the Todd class $\mathrm{td}_*(X)$ (resp., the dual Todd class $\mathrm{td}_*([\omega_X])$) of the associated projective toric variety $X = X_P$.

2.3. **Weighted lattice point counting via the generalized Hirzebruch-Riemann-Roch.** In this section, we recall how the homology Hirzebruch classes of [8] can be used for counting lattice points in a full-dimensional lattice polytope P, with certain weights reflecting the face decomposition

$$P = \bigcup_{E \preceq P} \mathrm{Relint}(E), \tag{47}$$

with $\mathrm{Relint}(E)$ denoting the relative interior of a face E of P. Recall that Brasselet-Schürmann-Yokura defined in [8] un-normalized homology Hirzebruch classes $T_{y*}(X)$ of a complex algebraic variety X as the image of the distinguished element $[id_X]$ under a natural transformation:

$$T_{y*} : K_0(var/X) \longrightarrow H_*(X) \otimes \mathbb{Q}[y]$$

defined on the relative Grothendieck group $K_0(var/X)$ of complex algebraic varieties over X. A normalized version $\widehat{T}_{y*}(X)$ is obtained by multiplying each degree k-piece of $T_{y*}(X)$ by $(1+y)^{-k}$. With these notations, one has the following *generalized Pick-type formula* (see [36][Thm.1.3]):

Theorem 2.3. *Let M be a lattice of rank n and let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope with associated projective toric variety $X := X_P$ and ample Cartier divisor $D := D_P$. Then the following formula holds:*

$$\sum_{E \preceq P} (1+y)^{\dim(E)} \cdot |\mathrm{Relint}(E) \cap M| = \int_X \mathrm{ch}(\mathscr{O}_X(D)) \cap T_{y*}(X), \tag{48}$$

where the summation on the left is over the faces E of P, and the number of points inside a face E is counted with respect to the lattice $\mathrm{Span}(E) \cap M$.

Remark 2.4. In the notations of Theorem 2.3, we make the following observations:

(a) If $y = 0$, then formula (48) reduces to (44); indeed, in the context of toric varieties, one has the specialization: $T_{y*}(X)|_{y=0} = \mathrm{td}_*(X)$ (see [8]).

(b) The proof of (48) in [36] uses motivic properties of the homology Hirzebruch classes (see [36][Thm.3.3]).

(c) The right-hand side of formula (48) is computed by the *generalized Hirzebruch-Riemann-Roch theorem* (see [36][Thm.2.4]), namely:

$$\begin{aligned} \chi_y(X, \mathscr{O}_X(D)) := \sum_{p \geq 0} \chi(X, \widehat{\Omega}^p_X \otimes \mathscr{O}_X(D)) \cdot y^p &= \int_X \mathrm{ch}(\mathscr{O}_X(D)) \cap T_{y*}(X) \\ &= \int_X \mathrm{ch}_{(1+y)}(\mathscr{O}_X(D)) \cap \widehat{T}_{y*}(X), \end{aligned} \tag{49}$$

with $\widehat{\Omega}^p_X := j_* \Omega^p_{X_{\mathrm{reg}}}$ the sheaf of Zariski p-forms on X and $j : X_{\mathrm{reg}} \hookrightarrow X$ the inclusion of the non-singular locus. This follows from the following calculation of the un-normalized homology Hirzebruch class (see [36][(1.2)]):

$$T_{y*}(X) = \sum_{p \geq 0} \mathrm{td}_*([\widehat{\Omega}^p_X]) \cdot y^p. \tag{50}$$

The last equality in (49) is just a renormalization of the Chern character by powers of $(1+y)$, namely $\mathrm{ch}_{(1+y)}(\mathscr{O}_X(D)) := e^{(1+y)c_1(\mathscr{O}_X(D))}$, which makes up for the switch from $T_{y*}(X)$ to $\widehat{T}_{y*}(X)$.

(d) For future reference, let us elaborate here on the meaning of formula (50). First recall that the Hirzebruch class transformation T_{y*} for a variety X is defined by the composition

$$T_{y*} := \mathrm{td}_* \circ mC_y,$$

with $mC_y : K_0(var/X) \to K_0(X)[y]$ the *motivic Chern class transformation* from [8]. Setting $mC_y(X) := mC_y([id_X])$, it was shown in [8] that $mC_y(X)$ can be computed from the filtered *Du Bois complex* $(\underline{\Omega}^\bullet_X, F)$ of X [22]. More precisely, the following identification holds:

$$mC_y(X) = \sum_{p\geq 0} [\,\underline{\underline{\Omega}}^p_X] \cdot y^p := \sum_{i,p\geq 0} (-1)^i\, [\mathscr{H}^i(\underline{\underline{\Omega}}^p_X)] \cdot y^p \in K_0(X)[y], \tag{51}$$

where

$$\underline{\underline{\Omega}}^p_X := \mathrm{Gr}^p_F(\underline{\Omega}^\bullet_X)[p] \in D^b_{\mathrm{coh}}(X)$$

is a bounded complex of sheaves with coherent cohomology, which coincides with the sheaf of p-forms Ω^p_X on a smooth variety X. If X is a toric variety, there is a natural quasi-isomorphism

$$\underline{\underline{\Omega}}^p_X \simeq \widehat{\Omega}^p_X,$$

with $\widehat{\Omega}^p_X$ denoting as before the sheaf of Zariski p-forms on X. In particular, for a toric variety X this further yields:

$$mC_y(X) = \sum_{p\geq 0} [\,\widehat{\Omega}^p_X] \cdot y^p \tag{52}$$

and (50) is obtained by applying the Todd class transformation td_* to (52).

2.4. **Todd and Hirzebruch classes of a simplicial toric variety.** In the context of a simplicial toric variety X_Σ (e.g., X_P for P a simple full-dimensional lattice polytope), a formula for the Todd class $\mathrm{td}_*(X_\Sigma)$ was obtained in [13] via equivariant cohomology, in [24] by using the Lefschetz-Riemann-Roch theorem, and in [40] by using a resolution of singularities and the birational invariance of Todd classes.

A formula for the Todd class $\mathrm{td}_*(X_\Sigma)$ of a simplicial toric variety X_Σ can also be deduced from a more general result of [36] on the computation of the (homology) Hirzebruch classes of [8] via the Lefschetz-Riemann-Roch theorem [24]. More precisely, with the notations from Section 2.1.2, the following result holds (see [36][Thm.5.4]):

Theorem 2.5. *Let $X = X_\Sigma$ be a n-dimensional simplicial toric variety, with $r = |\Sigma(1)|$. The un-normalized and, resp., normalized Hirzebruch classes of X are computed by:*

$$T_{y*}(X) = (1+y)^{n-r} \cdot \left(\sum_{g\in G_\Sigma} \prod_{\rho\in\Sigma(1)} \frac{[D_\rho]\cdot\big(1+y\cdot a_\rho(g)\cdot e^{-[D_\rho]}\big)}{1-a_\rho(g)\cdot e^{-[D_\rho]}} \right) \cap [X]\,. \tag{53}$$

$$\widehat{T}_{y*}(X) = \left(\sum_{g\in G_\Sigma} \prod_{\rho\in\Sigma(1)} \frac{[D_\rho]\cdot\big(1+y\cdot a_\rho(g)\cdot e^{-[D_\rho](1+y)}\big)}{1-a_\rho(g)\cdot e^{-[D_\rho](1+y)}} \right) \cap [X]\,. \tag{54}$$

In particular, for $y = 0$, either of (53) or (54) yields the following formula for the homology Todd class $\mathrm{td}_*(X)$ of a simplicial toric variety $X = X_\Sigma$ (compare with [13, 24]):

$$\mathrm{td}_*(X) := \left(\sum_{g\in G_\Sigma} \prod_{\rho\in\Sigma(1)} \frac{[D_\rho]}{1-a_\rho(g)e^{-[D_\rho]}} \right) \cap [X]\,. \tag{55}$$

A different formula for the Todd class $\mathrm{td}_*(X)$ of a simplicial toric variety $X := X_\Sigma$ was obtained in [36][Cor.1.16], by also using the motivic properties of Hirzebruch classes in the context of the orbit stratification of a toric variety. More precisely, with the notations from Section 2.1.2, we have the following:

Theorem 2.6. *Let $X = X_\Sigma$ be a simplicial toric variety. Then:*

$$\mathrm{td}_*(X) = \sum_{\sigma\in\Sigma} \alpha(\sigma)\cdot\left(\sum_{\{\tau|\sigma\preceq\tau\}} \mathrm{mult}(\tau)\prod_{\rho\in\tau(1)}\frac{1}{2}[D_\rho]\prod_{\rho\notin\tau(1)}\frac{\frac{1}{2}[D_\rho]}{\tanh(\frac{1}{2}[D_\rho])}\right)\cap[X], \tag{56}$$

where, for a singular cone $\sigma\in\Sigma_{\mathrm{sing}}$,

$$\begin{aligned}\alpha(\sigma) &:= \frac{1}{\mathrm{mult}(\sigma)}\cdot\sum_{g\in G_\sigma^\circ}\prod_{\rho\in\sigma(1)}\frac{1+a_\rho(g)\cdot e^{-[D_\rho]}}{1-a_\rho(g)\cdot e^{-[D_\rho]}}\\ &= \frac{1}{\mathrm{mult}(\sigma)}\cdot\sum_{g\in G_\sigma^\circ}\prod_{\rho\in\sigma(1)}\coth\left(\pi i\cdot\gamma_\rho(g)+\frac{1}{2}[D_\rho]\right),\end{aligned} \tag{57}$$

and $\alpha(\{0\}) := 1$ and $\alpha(\sigma) := 0$ for any other smooth cone $\sigma\in\Sigma$.

Similar formulae were obtained by Cappell and Shaneson in the early 1990s in the case of complete simplicial toric varieties, see [17, 43].

2.5. **Rational equivariant cohomology of a (complete simplicial) toric variety.** Let X be a complex algebraic variety with an algebraic action of the torus $\mathbb{T}$ (e.g., X is a toric variety with torus $\mathbb{T} = T_N = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{C}^*)$). Recall that the (Borel-type) rational *equivariant cohomology* of X is defined as

$$H^*_{\mathbb{T}}(X;\mathbb{Q}) := H^*(E\mathbb{T}\times_{\mathbb{T}} X;\mathbb{Q}), \tag{58}$$

where $\mathbb{T}\hookrightarrow E\mathbb{T}\to B\mathbb{T}$ is the universal principal $\mathbb{T}$-bundle, i.e., $B\mathbb{T} = (B\mathbb{C}^*)^n = (\mathbb{C}P^\infty)^n$ and $E\mathbb{T}$ is contractible. In particular, if $X = pt$ is a point space,

$$H^*_{\mathbb{T}}(pt;\mathbb{Q}) = H^*(B\mathbb{T};\mathbb{Q}) \simeq \mathbb{Q}[t_1,\dots,t_n] =: (\Lambda_{\mathbb{T}})_{\mathbb{Q}}. \tag{59}$$

For $m\in M$, with M the character lattice of the torus $\mathbb{T}$, one can view the corresponding character $\chi^m:\mathbb{T}\to\mathbb{C}^*$ as a $\mathbb{T}$-equivariant line bundle $\mathbb{C}_{\chi^m}$ over a point space pt, where the $\mathbb{T}$-action on $\mathbb{C}$ is induced via χ^m. This gives rise to an isomorphism

$$M \simeq Pic_{\mathbb{T}}(pt)\,.$$

Note that $m\mapsto -m$ corresponds to the duality involution $(-)^\vee$. Taking the first equivariant Chern class $c^1_{\mathbb{T}}$ (or the dual $-c^1_{\mathbb{T}} = c^1_{\mathbb{T}}\circ(-)^\vee$) of $\mathbb{C}_{\chi^m}$ gives an isomorphism

$$c = c^1_{\mathbb{T}},\ \text{resp.},\ s = -c^1_{\mathbb{T}} : M \simeq H^2_{\mathbb{T}}(pt;\mathbb{Z}) \tag{60}$$

and

$$c,\ \text{resp.},\ s : Sym_{\mathbb{Q}}(M) \simeq H^*_{\mathbb{T}}(pt;\mathbb{Q}) = (\Lambda_{\mathbb{T}})_{\mathbb{Q}}\,, \tag{61}$$

with $Sym_{\mathbb{Q}}(M) = \bigoplus_{k=0}^\infty Sym^k_{\mathbb{Q}}(M)$ the (rational) symmetric algebra of M. So if m_i ($i = 1,\dots,n$) are a basis of $M\simeq\mathbb{Z}^n$, then $H^*_{\mathbb{T}}(pt;\mathbb{Q}) = (\Lambda_{\mathbb{T}})_{\mathbb{Q}} \simeq \mathbb{Q}[t_1,\dots,t_n]$, with $t_i = \pm c^1_{\mathbb{T}}(\mathbb{C}_{\chi^{m_i}})$ for $i = 1,\dots,n$.

Note that $E\mathbb{T}\times_{\mathbb{T}} X$ is a fiber bundle over $B\mathbb{T}$ with fiber X, so $H^*_{\mathbb{T}}(X;\mathbb{Q})$ is a $H^*(B\mathbb{T};\mathbb{Q}) = (\Lambda_{\mathbb{T}})_{\mathbb{Q}}$-algebra. Furthermore, restriction to fibers defines a graded ring homomorphism

$$i_X^* : H^*_{\mathbb{T}}(X;\mathbb{Q}) \to H^*(X;\mathbb{Q})$$

called "forgetting the $\mathbb{T}$-action".

If X is moreover a simplicial toric variety with torus $\mathbb{T}$, then X is a rational homology manifold so that equivariant Poincaré duality holds rationally. For any cone $\sigma\in\Sigma$, the orbit closure $V_\sigma = \overline{O}_\sigma$ then defines via Poincaré duality an equivariant cohomology class

$$[V_\sigma]_{\mathbb{T}} \in H^{2\dim(\sigma)}_{\mathbb{T}}(X;\mathbb{Q}).$$

We reserve the notation $[D_\rho]_{\mathbb{T}}\in H^2_{\mathbb{T}}(X;\mathbb{Q})$ for the equivariant cohomology class corresponding to a ray $\rho\in\Sigma(1)$. Then one has the relation:

$$[V_\sigma]_{\mathbb{T}} = \mathrm{mult}(\sigma)\cdot \prod_{\rho\in\sigma(1)} [D_\rho]_{\mathbb{T}}. \tag{62}$$

We assume now that X is a simplicial toric variety which has a fan Σ of of full-dimensional convex support (see [16][page 265]), e.g., $X = X_P$ for P a full-dimensional lattice polyhedron. Then, by [21][Thm.3.6], the cohomology ring $H^*(X;\mathbb{Q})$ is even (i.e., it vanishes in odd degrees). Then the proof of [16][Prop.12.4.7] yields an isomorphism of $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}$-modules

$$H^*_{\mathbb{T}}(X;\mathbb{Q}) \simeq (\Lambda_{\mathbb{T}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} H^*(X;\mathbb{Q}). \tag{63}$$

In particular, the equivariant cohomology ring $H^*_{\mathbb{T}}(X;\mathbb{Q})$ is a free $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}$-module. Moreover, in this case the ordinary cohomology ring $H^*(X;\mathbb{Q})$ is determined by the $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}$-algebra structure of $H^*_{\mathbb{T}}(X;\mathbb{Q})$ via the isomorphism

$$H^*(X;\mathbb{Q}) \simeq H^*_{\mathbb{T}}(X;\mathbb{Q})/I_{\mathbb{T}}H^*_{\mathbb{T}}(X;\mathbb{Q}), \tag{64}$$

where $I_{\mathbb{T}}\subset(\Lambda_{\mathbb{T}})_{\mathbb{Q}}$ is the ideal generated by elements of positive degree, as in [13][Prop.3.2] or [16][Cor.12.4.8]. (In particular, the quotient $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}/I_{\mathbb{T}} \simeq \mathbb{Q}$ gives $\mathbb{Q}$ the structure of a $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}$-module.)

For the remaining of this Subsection 2.5, assume moreover that X is a *complete simplicial* toric variety.

There are several different descriptions of the rational equivariant cohomology $H^*_{\mathbb{T}}(X;\mathbb{Q})$ of a complete simplicial toric variety $X = X_\Sigma$ associated to the fan Σ in $N\otimes\mathbb{R}$, e.g., see [11], [16][Sect.12.4] or [28][Lecture 13]. Such descriptions, which are recalled below, are related to various aspects of the Euler-Maclaurin formulae for simple lattice polytopes (see Sections 5 and 7 below):

(a) $H^*_{\mathbb{T}}(X;\mathbb{Q})$ can be realized as the *Stanley-Reisner ring* $SR_{\mathbb{Q}}(\Sigma)$ of the fan Σ (see, e.g., [16][Sect.12.4]), i.e.,

$$SR_{\mathbb{Q}}(\Sigma) := \mathbb{Q}[x_1,\dots,x_r]/\sim_{SR} \simeq H^*_{\mathbb{T}}(X;\mathbb{Q})$$

$$x_i \longmapsto [D_i]_{\mathbb{T}} = c^1_{\mathbb{T}}(\mathscr{O}(D_i)),$$

with x_i of degree two and D_i the $\mathbb{T}$-invariant divisor corresponding to the ray $\rho_i = \langle u_i \rangle \in \Sigma(1)$ with ray generator u_i ($i = 1, \dots, r$), and the Stanley-Reisner relation $\sim_{SR}$ (or, in the notations of Section 2.1.1, the ideal $\mathscr{I}$) generated by:

$$x_{i_1} \cdots x_{i_s} = 0$$

for distinct i_j with $\{\rho_{i_j} | j = 1, \dots, s\}$ not spanning a cone of Σ.

(b) $H^*_{\mathbb{T}}(X;\mathbb{Q})$ can be realized as the ring R_Σ of continuous piecewise polynomial functions (aka, polynomial splines) with rational coefficients on the fan Σ (e.g., see [13][Prop.3.2] or [16][page 606]). This can be seen by starting with the following exact sequence in equivariant cohomology (as in the proof of [16][Lem.12.4.17]):

$$0 \to H^*_{\mathbb{T}}(X;\mathbb{Q}) \to \bigoplus_{\sigma \in \Sigma(n)} H^*_{\mathbb{T}}(U_\sigma;\mathbb{Q}) \to \bigoplus_{\tau \in \Sigma(n-1)} H^*_{\mathbb{T}}(U_\tau;\mathbb{Q}), \tag{65}$$

where $U_\sigma \subset X_\Sigma$ denotes as usual the unique $\mathbb{T}$-invariant open affine subset containing the corresponding $\mathbb{T}$-orbit O_σ, and $O_\sigma = x_\sigma \in X$ are the $\mathbb{T}$-fixed points in the case $\sigma \in \Sigma(n)$ (i.e., σ is of top dimension n). Note also that x_σ corresponds to a vertex of the lattice polytope P in the case when $\Sigma = \Sigma_P$ is the corresponding inner normal fan. There are equivariant deformation retracts $U_\sigma \to O_\sigma$ for any $\sigma \in \Sigma$ (e.g., see [16][Prop.12.1.9(a)]), hence

$$H^*_{\mathbb{T}}(U_\sigma;\mathbb{Q}) \simeq H^*_{\mathbb{T}}(O_\sigma;\mathbb{Q}).$$

Moreover, for $\sigma \in \Sigma(n)$ one gets by (61):

$$H^*_{\mathbb{T}}(O_\sigma;\mathbb{Q}) \simeq H^*_{\mathbb{T}}(x_\sigma;\mathbb{Q}) \simeq Sym_{\mathbb{Q}}(M),$$

and for $\tau \in \Sigma(n-1)$ one has an isomorphism

$$H^*_{\mathbb{T}}(O_\tau;\mathbb{Q}) \simeq Sym_{\mathbb{Q}}(M_\tau),$$

for $M_\tau = M/\tau^\perp \cap M$. So the localization sequence (65) translates into the exact sequence

$$0 \to H^*_{\mathbb{T}}(X;\mathbb{Q}) \to \bigoplus_{\sigma \in \Sigma(n)} Sym_{\mathbb{Q}}(M) \to \bigoplus_{\tau \in \Sigma(n-1)} Sym_{\mathbb{Q}}(M_\tau), \tag{66}$$

with $Sym_{\mathbb{Q}}(M)$, resp., $Sym_{\mathbb{Q}}(M_\tau)$ the ring of polynomial functions with rational coefficients on $\sigma \in \Sigma(n)$, resp., $\tau \in \Sigma(n-1)$. In other words, an element of $H^*_{\mathbb{T}}(X;\mathbb{Q})$ can be thought of as a collection $\{f_\sigma\}_{\sigma \in \Sigma(n)}$, where f_σ is a polynomial function on σ, such that $f_\sigma|_\tau = f_{\sigma'}|_\tau$ whenever $\sigma \cap \sigma' = \tau \in \Sigma(n-1)$.

(c) $H^*_{\mathbb{T}}(X;\mathbb{Q})$ can also be described as the $H^*_{\mathbb{T}}(pt;\mathbb{Q}) = (\Lambda_{\mathbb{T}})_{\mathbb{Q}}$-algebra

$$(\Lambda_{\mathbb{T}})_{\mathbb{Q}}[x_1, \dots, x_r]/\sim \simeq H^*_{\mathbb{T}}(X;\mathbb{Q})$$

$$x_i \longmapsto [D_i]_{\mathbb{T}} = c^1_{\mathbb{T}}(\mathscr{O}(D_i)),$$

and $\sim$ generated by the Stanley-Reisner relation $\sim_{SR}$ as well as

$$\pm m := c(m) = -s(m) = c^1_{\mathbb{T}}(\mathbb{C}_{\chi^m}) = \sum_{i=1}^r \langle m, u_i \rangle x_i \quad \text{for all } m \in M. \tag{67}$$

Here we consider the $+$-sign in the case when we use the isomorphism

$$c : Sym_{\mathbb{Q}}(M) \simeq H^*_{\mathbb{T}}(pt;\mathbb{Q}) =: (\Lambda_{\mathbb{T}})_{\mathbb{Q}}$$

coming from $c^1_{\mathbb{T}}$ as usual in algebraic geometry (see [28][Lecture 13]). But the use of the minus-sign and the isomorphism

$$s : Sym_{\mathbb{Q}}(M) \simeq H^*_{\mathbb{T}}(pt;\mathbb{Q}) =: (\Lambda_{\mathbb{T}})_{\mathbb{Q}}$$

coming from $c^1_{\mathbb{T}} \circ (-)^\vee$ (as in [16][Sect.12.4]) fits better with the corresponding Euler-Maclaurin formulae and the equivariant Riemann-Roch theorem (for instance, see [16][Sect.13.3]).

Remark 2.7. The inclusion $\mathbb{Q}[x_1,\dots,x_r] \subset (\Lambda_{\mathbb{T}})_{\mathbb{Q}}[x_1,\dots,x_r]$ induces the natural map

$$\mathbb{Q}[x_1,\dots,x_r]/\sim_{SR} \longrightarrow (\Lambda_{\mathbb{T}})_{\mathbb{Q}}[x_1,\dots,x_r]/\sim ,$$

which is an isomorphism relating parts (a) and (c) above, see Lemma 7.1 and compare also with [28][Lecture13, Lem.2.2 and Rem.2.7]. Note that in (a) and (b) the $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}$-algebra structure of $H^*_{\mathbb{T}}(X;\mathbb{Q})$ is not explicitly mentioned, but follows then from relation (67) (compare also with [16][Sect.12.4]).

In the context of a complete simplicial toric variety $X = X_\Sigma$ there is also no difference between the rational equivariant (co)homology ring and the corresponding rational equivariant Chow ring of X (see [11][Sect.5]).

Finally note that by (64) one has that $c^1_{\mathbb{T}}(\mathbb{C}_{\chi^m}) \mapsto 0$ under the natural map $H^*_{\mathbb{T}}(X;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ obtained by forgetting the $\mathbb{T}$-action, so the relation (67) maps to the corresponding additional relation in the description (30) of $H^*(X;\mathbb{Q})$ as a quotient of the Stanley-Reisner ring.

As it shall be indicated below, the equivariant Chern character $\mathrm{ch}^{\mathbb{T}}(\mathscr{E})$ of a $\mathbb{T}$-equivariant line or vector bundle $\mathscr{E}$ on X, as well as the image of the equivariant Todd homology class $\mathrm{td}^{\mathbb{T}}_*(X) \in \widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q}) \simeq \widehat{CH}^{\mathbb{T}}_*(X) \otimes \mathbb{Q}$ under equivariant Poincaré duality, live in the completion

$$\widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q}) := \prod_{i \geq 0} H^i_{\mathbb{T}}(X;\mathbb{Q}),$$

with

$$(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}} := \widehat{H}^*_{\mathbb{T}}(pt;\mathbb{Q}) \simeq \mathbb{Q}[[t_1,\dots,t_n]] .$$

Under the isomorphisms described in this section, this completion corresponds to:

(a) $\mathbb{Q}[[x_1,\dots,x_r]]/\sim_{SR} \simeq \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$.
(b) the corresponding completion $\widehat{R}_\Sigma \simeq \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$ of the ring of continuous piecewise polynomial functions with rational coefficients on the fan Σ.
(c) $(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}[[x_1,\dots,x_r]]/\sim \simeq \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$.

Abstractly, these are completions of connected integer graded commutative rings R^* with respect to the maximal ideal $R^{>0}$ given by positive degree elements.

In fact, as we will see and use later on, the equivariant Chern character and Todd homology classes live in an *analytic subring*

$$(H^*_{\mathbb{T}}(X;\mathbb{Q}))^{an} \subset \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$$

(see (194)), with $\mathbb{Q}\{t_1,\dots,t_n\} \simeq (H^*_{\mathbb{T}}(pt;\mathbb{Q}))^{an} =: (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$ the subring of *convergent power series* (around zero) with rational coefficients, i.e., after pairing with $z \in N_{\mathbb{K}} = N \otimes_{\mathbb{Z}} \mathbb{K}$ (for $\mathbb{K} = \mathbb{R}, \mathbb{C}$) one gets a convergent power series *function* in z around zero, whose corresponding Taylor polynomials have rational coefficients.

Remark 2.8. The injectivity part of the sequence (65) still holds (as in [16][Cor.12.4.9]) even if X is only a simplicial toric variety with a fan of full-dimensional convex support, since the equivariant cohomology of X is a free $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}$-module.

2.6. **Equivariant Euler characteristic.** Assume that a torus $\mathbb{T}$ with character lattice M acts linearly on a finite dimensional complex vector space W, with eigenspace

$$W_{\chi^m} := \{w \in W \mid t \cdot w = \chi^m(t)w \text{ for all } t \in \mathbb{T}\}, \tag{68}$$

for χ^m a character of T. Then, as in [16][Prop.1.1.2], one has an *eigenspace* decomposition

$$W = \bigoplus_{m \in M} W_{\chi^m}.$$

This induces an isomorphism

$$K^0_{\mathbb{T}}(pt) \xrightarrow{\simeq} \mathbb{Z}[M], \;\; [W] \mapsto \sum_{m \in M} \dim_{\mathbb{C}} W_{\chi^m} \cdot \chi^m. \tag{69}$$

Let $X = X_\Sigma$ be a complete toric variety of dimension n, with a $\mathbb{T}$-equivariant coherent sheaf $\mathscr{F}$. The cohomology spaces $H^i(X;\mathscr{F})$ are finite dimensional $\mathbb{T}$-representations, vanishing for i large enough. We define the *K-theoretic Euler characteristic* of $\mathscr{F}$ as:

$$\begin{aligned} \chi^{\mathbb{T}}(X,\mathscr{F}) &:= [H^*(X;\mathscr{F})] := \sum_i (-1)^i [H^i(X;\mathscr{F})] \in K^0_{\mathbb{T}}(pt) \\ &= \sum_{m \in M} \sum_{i=0}^n (-1)^i \dim_{\mathbb{C}} H^i(X;\mathscr{F})_{\chi^m} \cdot \chi^m \in \mathbb{Z}[M]. \end{aligned} \tag{70}$$

By applying the equivariant Chern character $\mathrm{ch}^{\mathbb{T}} : \mathbb{Z}[M] \simeq K^0_{\mathbb{T}}(pt) \hookrightarrow (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$, we further get the *cohomological Euler characteristic* of $\mathscr{F}$:

$$\chi^{\mathbb{T}}(X,\mathscr{F}) = \sum_{m \in M} \sum_{i=0}^n (-1)^i \dim_{\mathbb{C}} H^i(X;\mathscr{F})_{\chi^m} \cdot e^{c(m)} \in (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}. \tag{71}$$

Let D be a torus-invariant Cartier divisor on X. For each $i \geq 0$, consider the *weight* decomposition of $H^i(X;\mathscr{O}_X(D))$ given by (e.g., see [16][Sect.9.1]):

$$H^i(X;\mathscr{O}_X(D)) = \bigoplus_{m \in M} H^i(X;\mathscr{O}_X(D))_m, \tag{72}$$

induced via a Čech resolution from

$$H^0(U_\sigma;\mathscr{O}_X(D)) = \bigoplus_m H^0(U_\sigma;\mathscr{O}_X(D))_m \subset \mathbb{C}[M], \tag{73}$$

where each $H^0(U_\sigma;\mathscr{O}_X(D))_m$ is either 0 or $\mathbb{C} \cdot \chi^m$.

For comparing this weight decomposition with the eigenspace decomposition of (68), we recall here that in this paper the torus $\mathbb{T}$ acts on $\mathbb{C}[M]$ as follows: if $t \in \mathbb{T}$ and $f \in \mathbb{C}[M]$, then $t \cdot f \in \mathbb{C}[M]$ is given by $p \mapsto f(t^{-1} \cdot p)$, for $p \in \mathbb{T}$ (see [16][pag.18]). In particular, $t \cdot \chi^m = \chi^m(t^{-1})\chi^m$, so that

$$H^i(X;\mathscr{O}_X(D))_m = H^i(X;\mathscr{O}_X(D))_{\chi^{-m}}.$$

So, via $s(m) = c(-m) = -c_1^{\mathbb{T}}(\mathbb{C}_{\chi^m})$, formula (71) translates into the *equivariant Euler characteristic* of D, as defined in [16][Def.13.3.2]:

$$\chi^{\mathbb{T}}(X,\mathscr{O}_X(D)) = \sum_{m\in M}\sum_{i=0}^{n}(-1)^i \dim H^i(X;\mathscr{O}_X(D))_m \cdot e^{s(m)} \in (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}. \tag{74}$$

Note that $i^*_{pt}\chi^{\mathbb{T}}(X,\mathscr{O}_X(D)) = \chi(X,\mathscr{O}_X(D))$, where $i^*_{pt} : (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}} \to \mathbb{Q}$ sends elements of positive degree to zero (i.e., one forgets the $\mathbb{T}$-action).

Example 2.9. In the special case when X_P is the projective toric variety associated to a full-dimensional lattice polytope $P \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$, and D_P is the corresponding ample Cartier divisor, then the formulae (38)–(41) yield

$$\chi^{\mathbb{T}}(X_P,\mathscr{O}_{X_P}(D_P)) = \sum_{m\in P\cap M} e^{s(m)} \tag{75}$$

and

$$\chi^{\mathbb{T}}(X_P,\mathscr{O}_{X_P}(-D_P)) = (-1)^n \sum_{m\in \mathrm{Int}(P)\cap M} e^{s(m)}. \tag{76}$$

□

Example 2.10. For later applications, let us also consider the following more general context. Let X be a toric variety and D a torus-invariant Cartier divisor. Then

$$\Gamma(X;\mathscr{O}_X(D)) = \bigoplus_{m\in P_D\cap M} \mathbb{C}\cdot\chi^m \subset \mathbb{C}[M], \tag{77}$$

where χ^m denotes the character defined by $m \in M$, and P_D is the polyhedron associated to the divisor D (see [16][Prop.4.3.3]). If, moreover, the fan of X has full-dimensional convex support and $\mathscr{O}_X(D)$ is globally generated, then P_D is a *lattice* polyhedron (see [16][Thm.6.1.7]), and

$$H^i(X;\mathscr{O}_X(D)) = 0, \text{ for all } i > 0 \tag{78}$$

(e.g., see [16][Prop.9.2.3]). Hence, for X complete and D as above (with P_D the corresponding lattice polytope), we get

$$\chi^{\mathbb{T}}(X,\mathscr{O}_X(D)) = \sum_{m\in P_D\cap M} e^{s(m)} \in (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}. \tag{79}$$

□

2.7. **Equivariant Chern character and equivariant Riemann-Roch map.** Let X be a complex algebraic variety with an algebraic action of the torus $\mathbb{T}$ with character lattice M (e.g., X is a toric variety). For any $\mathbb{T}$-equivariant vector bundle $\mathscr{E} \to X$, one can define equivariant Chern classes $c_i^{\mathbb{T}}(\mathscr{E})$, equivariant Todd classes $\mathrm{Td}^{\mathbb{T}}(\mathscr{E})$ and an equivariant Chern character $\mathrm{ch}^{\mathbb{T}}(\mathscr{E})$ in $\widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$, by means of the corresponding Chern roots, see [23][Sect.3.1] for complete details. Forgetting the $\mathbb{T}$-action, these notions reduce under $i_X^* : \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ to the classical non-equivariant counterparts. For example, if D is a $\mathbb{T}$-invariant Cartier divisor on X, then the equivariant Chern character of the line bundle $\mathscr{O}_X(D)$ is given by:

$$\mathrm{ch}^{\mathbb{T}}(\mathscr{O}_X(D)) = e^{[D]_{\mathbb{T}}} = 1 + [D]_{\mathbb{T}} + \frac{1}{2!}[D]_{\mathbb{T}}^2 + \cdots \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q}), \tag{80}$$

where $[D]_{\mathbb{T}} = c_1^{\mathbb{T}}(\mathscr{O}_X(D)) \in H^2_{\mathbb{T}}(X;\mathbb{Q})$ is the equivariant cohomology class of the torus-invariant Cartier divisor D. Forgetting the $\mathbb{T}$-action yields: $i_X^*\mathrm{ch}^{\mathbb{T}}(\mathscr{O}_X(D)) = \mathrm{ch}(\mathscr{O}_X(D))$.

We denote as usual by $K^0_{\mathbb{T}}(X)$ the Grothendieck group of $\mathbb{T}$-equivariant vector bundles on X, and we let $K_0^{\mathbb{T}}(X)$ denote the Grothendieck group of $\mathbb{T}$-equivariant coherent sheaves on X. If X is smooth and quasi-projective, then $K^0_{\mathbb{T}}(X) \simeq K_0^{\mathbb{T}}(X)$, since in this case any $\mathbb{T}$-equivariant coherent sheaf has a finite resolution by $\mathbb{T}$-equivariant locally free sheaves. In general, the tensor product gives a ring structure on $K^0_{\mathbb{T}}(X)$, and $K_0^{\mathbb{T}}(X)$ is a module for this ring. Note that the Chern character $\mathrm{ch}^{\mathbb{T}}$ induces a contravariant, functorial, ring homomorphism

$$\mathrm{ch}^{\mathbb{T}} : K^0_{\mathbb{T}}(X) \longrightarrow \widehat{H}^{2*}_{\mathbb{T}}(X;\mathbb{Q}).$$

Edidin-Graham defined in [24][Thm.3.1] an *equivariant Riemann-Roch map*, which in our setup can be given as

$$\mathrm{td}_*^{\mathbb{T}} : K_0^{\mathbb{T}}(X) \longrightarrow \widehat{CH}_*^{\mathbb{T}}(X) \otimes \mathbb{Q} \longrightarrow \widehat{H}^{\mathbb{T}}_{2*}(X;\mathbb{Q}),$$

with the same functoriality as in the nonequivariant case of Baum-Fulton-MacPherson [4]. Here, we use the completions

$$\widehat{CH}_*^{\mathbb{T}}(X) \otimes \mathbb{Q} := \prod_{i \leq \dim(X)} CH_i^{\mathbb{T}}(X) \otimes \mathbb{Q}, \ \text{ and } \ \widehat{H}_*^{\mathbb{T}}(X;\mathbb{Q}) := \prod_{i \leq \dim(X)} \widehat{H}_i^{\mathbb{T}}(X;\mathbb{Q}).$$

(These equivariant Chow and Borel-Moore homology groups can be non-zero also in negative degrees.) Compare also with [15] for the homological version of this transformation.

The transformation $\mathrm{td}_*^{\mathbb{T}}$ has the following properties:

(a) (functoriality) $\mathrm{td}_*^{\mathbb{T}}$ is covariant for $\mathbb{T}$-equivariant proper morphisms;
(b) (module property) if $\beta \in K^0_{\mathbb{T}}(X)$ and $\alpha \in K_0^{\mathbb{T}}(X)$, then:

$$\mathrm{td}_*^{\mathbb{T}}(\beta \otimes \alpha) = \mathrm{ch}^{\mathbb{T}}(\beta) \cap \mathrm{td}_*^{\mathbb{T}}(\alpha). \tag{81}$$

Forgetting the $\mathbb{T}$-action, one recovers under the forgetful map $\widehat{CH}_*^{\mathbb{T}}(X) \to CH_*(X)$ the classical nonequivariant Todd class transformation of Baum-Fulton-MacPherson [4].

The *equivariant Todd class* of X is defined as:

$$\mathrm{td}_*^{\mathbb{T}}(X) := \mathrm{td}_*^{\mathbb{T}}([\mathscr{O}_X]_{\mathbb{T}}).$$

If X is smooth, one also has the *normalization property* (see [2][Sect.6.1]):

$$\mathrm{td}_*^{\mathbb{T}}(X) = \mathrm{Td}^{\mathbb{T}}(T_X) \cap [X]_{\mathbb{T}}, \tag{82}$$

with $\mathrm{Td}^{\mathbb{T}}(T_X)$ the equivariant cohomological Todd class of the tangent bundle T_X of X.

Example 2.11. Assume $X = pt$ is a point space. By the normalization property, $\mathrm{td}_*^{\mathbb{T}}(pt) = [pt]$. So by the module property, one gets via the equivariant Poincaré duality that $\mathrm{td}_*^{\mathbb{T}} : K_0^{\mathbb{T}}(pt) \to \widehat{H}_*^{\mathbb{T}}(pt;\mathbb{Q})$ reduces to the equivariant Chern character, resp., Euler characteristic map

$$\mathrm{ch}^{\mathbb{T}} = \chi^{\mathbb{T}}(pt,-) : \mathbb{Z}[M] \simeq K^0_{\mathbb{T}}(pt) \hookrightarrow (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}.$$

□

For a compact variety X, using the functoriality and module properties of $\mathrm{td}_*^{\mathbb{T}}$ for the constant map $f : X \to pt$, one gets the following equivariant Hirzebruch-Riemann-Roch formula (see [24][Cor.3.1]):

Theorem 2.12 (Equivariant Hirzebruch-Riemann-Roch). *Let X be a compact complex algebraic variety with an algebraic action of the torus $\mathbb{T}$, and let $\mathscr{E}$, resp., $\mathscr{F}$, be a $\mathbb{T}$-equivariant vector bundle, resp., coherent sheaf on X. Then*

$$\chi^{\mathbb{T}}(X, \mathscr{E} \otimes \mathscr{F}) = \int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{E}) \cap \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]), \tag{83}$$

where $\int_X : \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q}) \to \widehat{H}^*_{\mathbb{T}}(pt;\mathbb{Q}) = (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$ is the equivariant Gysin map (or, equivalently, the equivariant pushforward) for the constant map $X \to pt$.

Example 2.9 and Theorem 2.12 yield the following:

Corollary 2.13. *Let $P \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a full-dimensional lattice polytope, with corresponding projective toric variety $X = X_P$ and ample Cartier divisor $D = D_P$. Then:*

$$\sum_{m \in P \cap M} e^{s(m)} = \int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_X(D)) \cap \mathrm{td}_*^{\mathbb{T}}(X), \tag{84}$$

$$\sum_{m \in \mathrm{Int}(P) \cap M} e^{s(m)} = \int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_X(D)) \cap \mathrm{td}_*^{\mathbb{T}}([\omega_X]_{\mathbb{T}}). \tag{85}$$

Proof. Formula (84) follows directly from (75) and (83). For (85) we proceed as in (42) and (46), using the fact that toric Serre duality holds equivariantly. □

Similarly, Example 2.10 and Theorem 2.12 yield:

Corollary 2.14. *Let X be a complete toric variety, and D a globally generated torus-invariant Cartier divisor with associated lattice polytope $P_D \subset M_{\mathbb{R}}$. Then*

$$\sum_{m \in P_D \cap M} e^{s(m)} = \int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_X(D)) \cap \mathrm{td}_*^{\mathbb{T}}(X). \tag{86}$$

3. Equivariant characteristic classes of toric varieties.

3.1. **Definition. Properties.** In this section we recall the definitions of equivariant motivic Chern and Hirzebruch classes of complex quasi-projective varieties with an algebraic $\mathbb{T}$-action.

3.1.1. *Equivariant motivic Chern class transformation.* Recall that if X is a complex quasi-projective variety with an algebraic $\mathbb{T}$-action, the (relative) equivariant motivic Grothendieck group $K_0^{\mathbb{T}}(var/X)$ of varieties over X is the free abelian group generated by symbols $[Z \to X]$, where Z is a quasi-projective $\mathbb{T}$-variety and $Z \to X$ is a $\mathbb{T}$-equivariant morphism, modulo the additivity relation:

$$[Z \to X] = [U \to X] + [Z \setminus U \to X],$$

for $U \subset Z$ an open $\mathbb{T}$-invariant subvariety. If $X = pt$ is a point space, then $K_0^{\mathbb{T}}(var/pt)$ is a ring with product given by the external product of morphisms, and the group $K_0^{\mathbb{T}}(var/X)$ is a module over $K_0^{\mathbb{T}}(var/pt)$ with respect to the external product.

For any equivariant morphism $f : X \to Y$ of quasi-projective $\mathbb{T}$-varieties, there is a well-defined push-forward $f_! : K_0^{\mathbb{T}}(var/X) \to K_0^{\mathbb{T}}(var/Y)$ defined by composition. One can also define an exterior product: $K_0^{\mathbb{T}}(var/X) \boxtimes K_0^{\mathbb{T}}(var/X') \to K_0^{\mathbb{T}}(var/X \times X')$ via the cross-product: $[Z \to X] \times [Z' \to X'] = [Z \times Z' \to X \times X']$.

The following result provides an equivariant analogue of the motivic Chern class transformation of [8]; see [1][Thm.4.2]:

Theorem 3.1. *Let X be a complex quasi-projective variety with an action of the torus $\mathbb{T}$. There exists a unique natural transformation*

$$mC_y^{\mathbb{T}} : K_0^{\mathbb{T}}(var/X) \longrightarrow K_0^{\mathbb{T}}(X)[y]$$

satisfying the following properties:

(a) *(functoriality) $mC_y^{\mathbb{T}}$ is covariant for $\mathbb{T}$-equivariant proper morphisms;*
(b) *(normalization) if X is smooth, then*

$$mC_y^{\mathbb{T}}([id_X]) = \Lambda_y(T_X^*) := \sum_{p=0}^{\dim(X)} [\Lambda^p T_X^*]_{\mathbb{T}} \cdot y^p \in K^0_{\mathbb{T}}(X)[y] \simeq K_0^{\mathbb{T}}(X)[y]. \tag{87}$$

Moreover, $mC_y^{\mathbb{T}}$ commutes with exterior products.

As a consequence of these properties, the transformation $mC_y^{\mathbb{T}}$ is determined by its image on classes $[f : Z \to X] = f_*[id_Z]$, where Z is a *smooth* quasi-projective variety and $f : Z \to X$ is a $\mathbb{T}$-equivariant *proper* morphism. For such $[f]$, one has:

$$mC_y^{\mathbb{T}}([f : Z \to X]) = f_* mC_y^{\mathbb{T}}([id_Z]) = f_*(\Lambda_y(T_Z^*) \otimes [\mathscr{O}_Z]).$$

So the uniqueness of $mC_y^{\mathbb{T}}$ follows from equivariant resolution of singularities.

Definition 3.2. The *equivariant motivic Chern class* of a complex quasi-projective variety X with an algebraic $\mathbb{T}$-action is defined as:

$$mC_y^{\mathbb{T}}(X) := mC_y^{\mathbb{T}}([id_X]).$$

We next discuss a calculation of motivic Chern classes in terms of a cubical hyper-resolution, which will be used for proving a generalized equivariant Hirzebruch-Riemann-Roch theorem in the toric context. Choose an equivariant simplicial resolution $f_\bullet : X_\bullet \to X$ of X derived from a cubical hyper-resolution in the sense of [32][Théorème 2.15], with each X_i a smooth $\mathbb{T}$-variety and $f_i : X_i \to X$ a proper $\mathbb{T}$-equivariant morphism (with $\dim(X_i) \leq \dim(X) - i$). Then

$$[id_X] = \sum_i (-1)^i [f_i : X_i \to X] \in K_0^{\mathbb{T}}(var/X). \tag{88}$$

Indeed, (88) follows from the inductive construction of an equivariant cubical hyper-resolution, by the following abstract equivariant blowup relation. Let $f:\widetilde{X}\to X$ be a proper $\mathbb{T}$-equivariant map of complex quasi-projective varieties. Let $D\subset X$ be a $\mathbb{T}$-invariant subvariety, with $E:=f^{-1}(D)\subset\widetilde{X}$. Assume $f:\widetilde{X}\setminus E\to X\setminus D$ is an isomorphism. Then

$$[\widetilde{X}\to X]-[E\to X]=[id_X]-[D\to X]\in K_0^{\mathbb{T}}(var/X). \tag{89}$$

Applying the transformation $mC_y^{\mathbb{T}}$ to (88) yields:

$$\begin{aligned} mC_y^{\mathbb{T}}(X) &= \sum_i (-1)^i mC_y^{\mathbb{T}}([f_i:X_i\to X]) \\ &= \sum_i (-1)^i f_{i*}(\Lambda_y(T^*_{X_i})\otimes[\mathscr{O}_{X_i}]) \\ &= \sum_{p\geq 0}\left(\sum_i (-1)^i f_{i*}([\Omega^p_{X_i}]_{\mathbb{T}})\right)\cdot y^p. \end{aligned} \tag{90}$$

In particular, for any $p\geq 0$, the Grothendieck class

$$\sum_i (-1)^i f_{i*}([\Omega^p_{X_i}]_{\mathbb{T}}) =: [Rf_{\bullet *}\Omega^p_{X_\bullet}]_{\mathbb{T}}\in K_0^{\mathbb{T}}(X)$$

is independent of the choice of an equivariant cubical hyper-resolution, and it provides an equivariant analogue of the Grothendieck class $[\underline{\underline{\Omega}}^p_X]\in K_0(X)$ appearing in (51), corresponding to the graded pieces of the filtered Du Bois complex of X. So, while an equivariant version of the Du Bois complex is not available in the literature, the Grothendieck classes of its graded pieces are well defined equivariantly by the equivariant blow-up relation. This is sufficient for the purpose of this paper.

As an application of the above formula (90), Weber [44][Thm.5.1, Rem.5.2] deduces the following result:

Theorem 3.3. *Let X be a smooth quasi-projective $\mathbb{T}$-variety, and $U\subset X$ an open subvariety so that $X\setminus U=:D=\bigcup_{i=1}^m D_i$ is a $\mathbb{T}$-invariant simple normal crossing divisor. Then:*

$$mC_y^{\mathbb{T}}([U\hookrightarrow X])=[\mathscr{O}_X(-D)\otimes\Lambda_y\Omega^1_X(\log D)]_{\mathbb{T}}\in K^0_{\mathbb{T}}(X)[y]\simeq K_0^{\mathbb{T}}(X)[y]\,. \tag{91}$$

In fact Weber [44][Thm.5.1, Rem.5.2] formulated his results in the non-equivariant context, but his proof also works $\mathbb{T}$-equivariantly as follows. By the inclusion-exclusion formula for the $\mathbb{T}$-equivariant motivic Chern class one has

$$mC_y^{\mathbb{T}}([U\hookrightarrow X])=\sum_{I\subset\{1,\dots,m\}}(-1)^{|I|}mC_y^{\mathbb{T}}([D_I\hookrightarrow X])=\sum_{I\subset\{1,\dots,m\}}(-1)^{|I|}[\Lambda_y\Omega^1_{D_I}]_{\mathbb{T}}\,, \tag{92}$$

with $D_I:=\bigcap_{i\in I}D_i$ smooth $\mathbb{T}$-invariant closed subvarieties (and $D_\emptyset:=X$). Similarly

$$[\mathscr{O}_X(-D)]_{\mathbb{T}}=\sum_{I\subset\{1,\dots,m\}}(-1)^{|I|}[\mathscr{O}_{D_I}]_{\mathbb{T}}\in K^0_{\mathbb{T}}(X)\simeq K_0^{\mathbb{T}}(X)\,.$$

Then Deligne's weight filtration of the sheaf of logarithmic p-forms $\Omega^p_X(\log D)=\Lambda^p\Omega^1_X(\log D)$ is also $\mathbb{T}$-equivariant (and similarly for the $\mathbb{T}$-invariant normal crossing divisor $D_{>I}:=\bigcup_{I\subset J}D_J\subset D_I$), so that the calculation of Weber [44][Thm.5.1, Rem.5.2] holds $\mathbb{T}$-equivariantly:

$$\sum_{I\subset\{1,\dots,m\}}(-1)^{|I|}[\Lambda_y\Omega^1_{D_I}]_{\mathbb{T}}=[\mathscr{O}_X(-D)\otimes\Lambda_y\Omega^1_X(\log D)]_{\mathbb{T}}\in K^0_{\mathbb{T}}(X)[y]\simeq K_0^{\mathbb{T}}(X)[y]\,. \tag{93}$$

3.1.2. *Equivariant Hirzebruch class transformation.* Let X be a complex quasi-projective variety X with an algebraic $\mathbb{T}$-action. An equivariant version of the Hirzebruch class transformation of [8] can be defined as follows (see also [2, 44]).

Definition 3.4. The un-normalized *equivariant Hirzebruch class* transformation $T^{\mathbb{T}}_{y*}$ is the composition:

$$T^{\mathbb{T}}_{y*} := \mathrm{td}^{\mathbb{T}}_{*} \circ mC^{\mathbb{T}}_{y} : K^{\mathbb{T}}_{0}(var/X) \longrightarrow \widehat{CH}^{\mathbb{T}}_{*}(X) \otimes \mathbb{Q}[y] \longrightarrow \widehat{H}^{\mathbb{T}}_{2*}(X) \otimes \mathbb{Q}[y],$$

while a normalized version $\widehat{T}^{\mathbb{T}}_{y*}$ is obtained by precomposing $T^{\mathbb{T}}_{y*}$ with the normalization functor

$$\Psi_{(1+y)} : \widehat{CH}^{\mathbb{T}}_{*}(X) \otimes \mathbb{Q}[y] \to \widehat{CH}^{\mathbb{T}}_{*}(X;\mathbb{Q}[y,(1+y)^{-1}]) := \prod_{i \leq \dim(X)} CH^{\mathbb{T}}_{i}(X) \otimes \mathbb{Q}[y,(1+y)^{-1}]$$

given in degree k by multiplication by $(1+y)^{-k}$.

By construction, both transformations $T^{\mathbb{T}}_{y*}$ and $\widehat{T}^{\mathbb{T}}_{y*}$ are covariant for $\mathbb{T}$-equivariant proper morphisms and they commute with exterior products. The corresponding *equivariant Hirzebruch classes of X* are defined as:

$$T^{\mathbb{T}}_{y*}(X) := T^{\mathbb{T}}_{y*}([id_X]) \ , \ \ \widehat{T}^{\mathbb{T}}_{y*}(X) := \widehat{T}^{\mathbb{T}}_{y*}([id_X]).$$

If X is smooth, then by (87) we get

$$T^{\mathbb{T}}_{y*}(X) = \sum_{p=0}^{\dim(X)} \mathrm{td}^{\mathbb{T}}_{*}([\Omega^p_X]_{\mathbb{T}}) \cdot y^p \tag{94}$$

The following result is an equivariant version of [36][(1.1) and (2.13)], generalizing (94) to the singular toric context:

Proposition 3.5. *Let X be a quasi-projective toric variety with torus $\mathbb{T}$. Then*

$$mC^{\mathbb{T}}_{y}(X) = \sum_{p=0}^{\dim(X)} [\widehat{\Omega}^p_X]_{\mathbb{T}} \cdot y^p \in K^{\mathbb{T}}_0(X)[y], \tag{95}$$

where $\widehat{\Omega}^p_X$ is the sheaf of Zariski p-forms on X. Moreover,

$$T^{\mathbb{T}}_{y*}(X) = \sum_{p=0}^{\dim(X)} \mathrm{td}^{\mathbb{T}}_{*}([\widehat{\Omega}^p_X]_{\mathbb{T}}) \cdot y^p. \tag{96}$$

In particular, the top degree in y of $mC^{\mathbb{T}}_{y}(X)$, resp., $T^{\mathbb{T}}_{y*}(X)$, is given by the equivariant class $[\omega_X]_{\mathbb{T}}$ of the dualizing sheaf, resp., $\mathrm{td}^{\mathbb{T}}_{*}([\omega_X]_{\mathbb{T}})$. Similarly, for $y=0$, we get $mC^{\mathbb{T}}_{0}(X) = [\mathscr{O}_X]_{\mathbb{T}}$ and $T^{\mathbb{T}}_{0*}(X) = \mathrm{td}^{\mathbb{T}}_{*}(X)$.

Proof. First note that (96) is obtained by applying the transformation $\mathrm{td}^{\mathbb{T}}_{*}$ to (95).

Choose an equivariant cubic hyper-resolution $f_{\bullet} : X_{\bullet} \to X$ of X (obtained by adapting to the equivariant context the results from [32][Exposé V, Sect. 4]). The natural map $\widehat{\Omega}^p_X \longrightarrow Rf_{\bullet *}\Omega^p_{X_{\bullet}}$ of $\mathbb{T}$-equivariant sheaf complexes is, by [3][Thm.4.3], a quasi-isomorphism after forgetting the $\mathbb{T}$-action. In particular, $Rf_{\bullet *}\Omega^p_{X_{\bullet}}$ is concentrated in degree zero, and also given by a reflexive sheaf (since this is the case for $\widehat{\Omega}^p_X$, see [16][Prop.8.0.1, eqn.(8.0.5)]). This map then has to be a $\mathbb{T}$-equivariant isomorphism since the restrictions of these two sheaves to the $\mathbb{T}$-invariant smooth locus X_{sm} of X are canonically isomorphic to $\Omega^p_{X_{sm}}$ as $\mathbb{T}$-equivariant sheaves. □

Remark 3.6. The statement of Proposition 3.5 holds more generally, for X a $\mathbb{T}$-invariant closed subvariety (i.e., a closed union of torus orbits, corresponding to a star-closed subset $\Sigma' \subset \Sigma$) in a quasi-projective toric variety with torus $\mathbb{T}$ and fan Σ. Instead of Zariski differential forms, one has to use the corresponding sheaves of Ishida differentials $\widetilde{\Omega}^p_X$ from [33], with canonical sheaf $\omega_X := \widetilde{\Omega}^{\dim(X)}_X$. To endow each $\widetilde{\Omega}^p_X$ with a canonical $\mathbb{T}$-action, one proceeds as indicated in [3][Introduction], by realizing them as subsheaves of $f_*\Omega^p_M$ with $f : M \to X$ any $\mathbb{T}$-equivariant resolution of singularities. Note that these pushforward sheaves are independent of the choice of resolution by the birational invariance of Ω^p_M. In the notations used in the proof of Proposition 3.5, the natural map $\widetilde{\Omega}^p_X \longrightarrow R f_{\bullet *}\Omega^p_{X_\bullet}$ of sheaf complexes is a quasi-isomorphism. If the $\mathbb{T}$-equivariant cubic hyper-resolution starts with a resolution of singularities $\varepsilon_0 : X_0 \to X$, then this morphism is even $\mathbb{T}$-equivariant. This implies that

$$mC^{\mathbb{T}}_y(X) = \sum_{p=0}^{\dim(X)} [\widetilde{\Omega}^p_X]_{\mathbb{T}} \cdot y^p \in K^{\mathbb{T}}_0(X)[y]. \tag{97}$$

Note also that $\widetilde{\Omega}^0_X \simeq \mathscr{O}_X$, as $\mathbb{T}$-equivariant sheaves. If X is already a toric variety, these Ishida sheaves coincide with the Zariski sheaves, with the canonical $\mathbb{T}$-action, as used above.

Let X_Σ be a quasi-projective toric variety defined by a fan Σ, with $X := X_{\Sigma'} \subset X_\Sigma$ a $\mathbb{T}$-invariant closed algebraic subset of X_Σ defined by a star-closed subset $\Sigma' \subset \Sigma$. Let O_σ be the orbit of the cone $\sigma \in \Sigma'$, with $i_\sigma : V_\sigma \hookrightarrow X$ the orbit closure inclusion. Then, by additivity, the following equivariant version of [36][Prop.3.1] holds:

Proposition 3.7.

$$mC^{\mathbb{T}}_y(X) = \sum_{\sigma\in\Sigma'} mC^{\mathbb{T}}_y([O_\sigma \hookrightarrow X]) = \sum_{\sigma\in\Sigma'} (i_\sigma)_* mC^{\mathbb{T}}_y([O_\sigma \hookrightarrow V_\sigma]). \tag{98}$$

A similar formula holds for the equivariant Hirzebruch classes $T^{\mathbb{T}}_{y*}(X)$ *and* $\widehat{T}^{\mathbb{T}}_{y*}(X)$.

Moreover, toric geometry can be used as in [36][Prop.3.2] to get the following (see also [44][Cor.11.2]):

Proposition 3.8. *For any cone* $\sigma \in \Sigma$, *one has:*

$$mC^{\mathbb{T}}_y([O_\sigma \hookrightarrow V_\sigma]) = (1+y)^{\dim(\sigma)} \cdot [\omega_{V_\sigma}]_{\mathbb{T}}, \tag{99}$$

with ω_{V_σ} *the canonical sheaf of* V_σ*, viewed as a* $\mathbb{T} = T_N$*-equivariant sheaf via the quotient map* $T_N \to T_{N(\sigma)}$ *given by* (29). *Therefore,*

$$T^{\mathbb{T}}_{y*}([O_\sigma \hookrightarrow V_\sigma]) = (1+y)^{\dim(\sigma)} \cdot \mathrm{td}^{\mathbb{T}}_*([\omega_{V_\sigma}]_{\mathbb{T}}). \tag{100}$$

In particular, if V_σ *is projective, then*

$$\chi^{\mathbb{T}}_y([O_\sigma \hookrightarrow V_\sigma]) := T^{\mathbb{T}}_{y*}([O_\sigma \hookrightarrow V_\sigma \to pt]) = (-1-y)^{\dim(O_\sigma)} \in \mathbb{Q}[y] \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}[y]. \tag{101}$$

Proof. Let $\widetilde{V}_\sigma \xrightarrow{f_\sigma} V_\sigma$ be a toric resolution of singularities of V_σ. Then $\widetilde{V}_\sigma$ is a toric variety obtained by refining the fan of V_σ. Let $O_\sigma \hookrightarrow \widetilde{V}_\sigma$ be the natural open inclusion, with complement the simple normal crossing divisor D_σ whose irreducible components correspond to the rays in the fan of $\widetilde{V}_\sigma$. Note that D_σ is a $T_{N(\sigma)}$-invariant (hence a $\mathbb{T}$-invariant) divisor. Then formula (91) applied to the open inclusion $O_\sigma \hookrightarrow \widetilde{V}_\sigma$ yields by the functoriality of $mC_y^{\mathbb{T}}$ with respect to the proper morphism f_σ that:

$$mC_y^{\mathbb{T}}\big([O_\sigma \hookrightarrow V_\sigma]\big) = (f_\sigma)_*\big([\mathscr{O}_{\widetilde{V}_\sigma}(-D_\sigma)\otimes \Lambda_y\Omega^1_{\widetilde{V}_\sigma}(\log D_\sigma)]_{\mathbb{T}}\big). \tag{102}$$

However, since $\widetilde{V}_\sigma$ is a smooth toric variety, the locally free sheaf $\Omega^1_{\widetilde{V}_\sigma}(\log D_\sigma)$ is in fact $T_{N(\sigma)}$-equivariantly, hence also $\mathbb{T}$-equivariantly, a trivial sheaf of rank equal to $\dim(\widetilde{V}_\sigma) = \dim(O_\sigma)$, e.g., see [16][(8.1.5)]. So we get that:

$$[\mathscr{O}_{\widetilde{V}_\sigma}(-D_\sigma)\otimes \Lambda_y\Omega^1_{\widetilde{V}_\sigma}(\log D_\sigma)]_{\mathbb{T}} = (1+y)^{\dim(O_\sigma)}\cdot[\mathscr{O}_{\widetilde{V}_\sigma}(-D_\sigma)]_{\mathbb{T}} \in K_0^{\mathbb{T}}(\widetilde{V}_\sigma)[y]. \tag{103}$$

Furthermore, note that the canonical dualizing sheaf $\omega_{\widetilde{V}_\sigma}$ on the toric variety $\widetilde{V}_\sigma$ is precisely given by $\mathscr{O}_{\widetilde{V}_\sigma}(-D_\sigma)$, e.g., see [16][Thm.8.2.3]. And since the toric morphism $f_\sigma : \widetilde{V}_\sigma \to V_\sigma$ is induced by a refinement of the fan of V_σ, it follows from [16][Thm.8.2.15] that there is a $T_{N(\sigma)}$-equivariant, hence also $\mathbb{T}$-equivariant, isomorphism:

$$f_{\sigma*}\omega_{\widetilde{V}_\sigma} \simeq \omega_{V_\sigma}. \tag{104}$$

Altogether, we obtain the following sequence of equalities:

$$\begin{aligned} mC_y^{\mathbb{T}}\big([O_\sigma \hookrightarrow V_\sigma]\big) &= (1+y)^{\dim(O_\sigma)}\cdot (f_\sigma)_*\big([\mathscr{O}_{\widetilde{V}_\sigma}(-D_\sigma)]_{\mathbb{T}}\big)\\ &= (1+y)^{\dim(O_\sigma)}\cdot (f_\sigma)_*\big([\omega_{\widetilde{V}_\sigma}]_{\mathbb{T}}\big)\\ &= (1+y)^{\dim(O_\sigma)}\cdot [\omega_{V_\sigma}]_{\mathbb{T}}, \end{aligned} \tag{105}$$

where the last equality is a consequence of (104) and the vanishing of the higher derived image sheaves of $\omega_{\widetilde{V}_\sigma}$, i.e., $R^i(f_\sigma)_*\omega_{\widetilde{V}_\sigma} = 0$, for all $i > 0$, see [16][Thm.9.3.12].

By applying the Todd class transformation $\mathrm{td}_*^{\mathbb{T}}$ to equation (99), we obtain formula (100).

Finally, formula (101) follows from

$$\chi^{\mathbb{T}}(V_\sigma, \omega_{V_\sigma}) = (-1)^{\dim(O_\sigma)},$$

which is obtained by equivariant Serre duality and $\chi^{\mathbb{T}}(V_\sigma, \mathscr{O}_{V_\sigma}) = 1$. The latter equality is just a special case of (79) for D the zero divisor, so that $\mathscr{O}(D) = \mathscr{O}$. □

Using the functoriality of the equivariant motivic Chern and Hirzebruch classes, we get a relative version of Proposition 3.7 as follows. Let $f : X \to X'$ be a proper toric morphism of quasi-projective toric varieties, with the corresponding lattice homomorphism $f_N : N \to N'$ surjective (i.e., f is a *toric fibration* in the sense of [21][Prop.2.1]). Let $f_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}'$ be the corresponding map of the associated tori, so that $\mathbb{T}$ acts on X' via f. Let Σ, Σ' be the fans of X, resp., X'. Since f is a toric fibration, a $\mathbb{T}$-orbit O_σ ($\sigma \in \Sigma$) is mapped by f to a $\mathbb{T}'$-orbit $f(O_\sigma) = O_{\sigma'}$ ($\sigma' \in \Sigma'$), such that the restriction map $f_\sigma = f|_{O_\sigma} : O_\sigma \to O_{\sigma'}$ is isomorphic to a projection $O_\sigma \simeq O_{\sigma'} \times O_{\sigma/\sigma'} \to O_{\sigma'}$, with $O_{\sigma/\sigma'} \simeq (\mathbb{C}^*)^\ell$ and $\ell = \dim(O_\sigma) - \dim(O_{\sigma'})$ the relative dimension of f_σ (see [21][Lem.2.6 and Prop.2.7]). Let $U \subset X$ be a locally closed $\mathbb{T}$-invariant subset (i.e., a locally closed union of $\mathbb{T}$-orbits of X), with

$$d_\ell(U/\sigma') := |\Sigma_\ell(U/\sigma')| \tag{106}$$

and

$$\Sigma_\ell(U/\sigma') := \{\sigma \in \Sigma \mid O_\sigma \subset U,\ f(O_\sigma) = O_{\sigma'},\ \ell = \dim(O_\sigma) - \dim(O_{\sigma'})\}. \tag{107}$$

Proposition 3.9. *Under the above notations and assumptions, we have*

$$\begin{aligned} f_* mC_y^{\mathbb{T}}([U \hookrightarrow X]) &= \sum_{\sigma' \in \Sigma'} \sum_{\ell \geq 0} d_\ell(U/\sigma') \cdot (-y-1)^\ell \cdot mC_y^{\mathbb{T}}([O_{\sigma'} \hookrightarrow X']) \\ &= \sum_{\sigma' \in \Sigma'} \sum_{\ell \geq 0} (-1)^\ell \cdot d_\ell(U/\sigma') \cdot (1+y)^{\ell + \dim(O'_\sigma)} \cdot [\omega_{V_{\sigma'}}]_{\mathbb{T}}. \end{aligned} \tag{108}$$

A similar formula holds for $f_* T_{y*}^{\mathbb{T}}([U \hookrightarrow X])$.

Proof. By functoriality and additivity of $mC_y^{\mathbb{T}}$, we have

$$f_* mC_y^{\mathbb{T}}([U \hookrightarrow X]) = mC_y^{\mathbb{T}}([U \hookrightarrow X \xrightarrow{f} X']) = \sum_{\sigma' \in \Sigma'} \sum_{\ell \geq 0} \sum_{\sigma \in \Sigma_\ell(U/\sigma')} mC_y^{\mathbb{T}}([O_\sigma \hookrightarrow X \xrightarrow{f} X']).$$

For $\ell \geq 0$, $\sigma' \in \Sigma'$ and $\sigma \in \Sigma_\ell(U/\sigma')$ fixed, we have

$$[O_\sigma \hookrightarrow X \xrightarrow{f} X'] = [O_\sigma \xrightarrow{f_\sigma} O_{\sigma'} \hookrightarrow X'].$$

Let us choose a splitting of the surjection $f_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}'$ so that $\mathbb{T} = \mathbb{T}' \oplus \mathbb{T}''$. By [21][Lem.2.6] and the proof of [21][Prop.2.7], there is a $\mathbb{T}$-equivariant isomorphism $O_\sigma \simeq O_{\sigma'} \times O_{\sigma/\sigma'} \to O_{\sigma'}$, with $\mathbb{T}$ acting on $O_{\sigma'}$ by $f_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}'$ and on $O_{\sigma/\sigma'}$ via the projection $\mathbb{T} \to \mathbb{T}'' \to O_{\sigma/\sigma'}$. Here, the surjective group homomorphism $\mathbb{T}'' \to O_{\sigma/\sigma'}$ is given in the proof of [21][Prop.2.7]. We then have

$$[O_\sigma \xrightarrow{f_\sigma} O_{\sigma'} \hookrightarrow X'] = [O_{\sigma'} \hookrightarrow X'] \times [O_{\sigma/\sigma'} \to pt].$$

Using the multiplicativity of $mC_y^{\mathbb{T}}$, we get

$$mC_y^{\mathbb{T}}([O_\sigma \xrightarrow{f_\sigma} O_{\sigma'} \hookrightarrow X']) = mC_y^{\mathbb{T}}([O_{\sigma'} \hookrightarrow X']) \boxtimes mC_y^{\mathbb{T}}([O_{\sigma/\sigma'} \to pt]).$$

Here, $mC_y^{\mathbb{T}}([O_{\sigma'} \hookrightarrow X'])$ is calculated by (99) using the factorization $O_{\sigma'} \hookrightarrow V_{\sigma'} \hookrightarrow X'$. Finally, $mC_y^{\mathbb{T}}([O_{\sigma/\sigma'} \to pt]) = (-1-y)^\ell$ by (101) and using a projective toric compactification of $O_{\sigma/\sigma'}$.

The corresponding formula for the equivariant Hirzebruch class follows by applying the equivariant Todd transformation to (108). □

Remark 3.10. By forgetting the action, the corresponding non-equivariant version of (108) holds with the same proof, and even without the quasi-projectivity assumption.

Back to the general context, a choice of a splitting of a surjection of tori $\mathbb{T} \to \mathbb{T}'$ (as in the above results), yields an identification of equivariant characteristic classes with respect to $\mathbb{T}$ and $\mathbb{T}'$, as we explain next.

Proposition 3.11. *Let the complex torus* $\mathbb{T}$ *act on a quasi-projective variety* X *with a subtorus* $\mathbb{T}''$ *acting trivially. Let* $\mathbb{T}' := \mathbb{T}/\mathbb{T}''$ *and choose a compatible splitting of these tori and of their corresponding character lattices*

$$\mathbb{T} = \mathbb{T}' \oplus \mathbb{T}'' \text{ and } M = M' \oplus M''.$$

Regard $X = X \times pt$, *with the* $\mathbb{T}$*-action on the left corresponding to the product action of* $\mathbb{T}'$ *on* X *and* $\mathbb{T}''$ *on* pt *on the right.*

(i) *These splittings then induce the following factorizations:*

(a) *(equivariant K-theory)*

$$K^0_{\mathbb{T}}(X) \simeq K^0_{\mathbb{T}'}(X) \otimes_{\mathbb{Z}} K^0_{\mathbb{T}''}(pt) \tag{109}$$

$$\mathbb{Z}[M] = K^0_{\mathbb{T}}(pt) \simeq K^0_{\mathbb{T}'}(pt) \otimes_{\mathbb{Z}} K^0_{\mathbb{T}''}(pt) = \mathbb{Z}[M'] \otimes_{\mathbb{Z}} \mathbb{Z}[M''] \tag{110}$$

$$K_0^{\mathbb{T}}(X) \simeq K_0^{\mathbb{T}'}(X) \otimes_{\mathbb{Z}} K_0^{\mathbb{T}''}(pt), \tag{111}$$

with (109) *and* (110) *compatible under pullback, and* (109) *and* (111) *compatible under the module structure induced from the tensor product.*

(b) *(equivariant (co)homology)*

$$H^*_{\mathbb{T}}(X,\mathbb{Q}) \simeq H^*_{\mathbb{T}'}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} (\Lambda_{\mathbb{T}''})_{\mathbb{Q}} \tag{112}$$

$$Sym_{\mathbb{Q}}(M) = (\Lambda_{\mathbb{T}})_{\mathbb{Q}} \simeq (\Lambda_{\mathbb{T}'})_{\mathbb{Q}} \otimes_{\mathbb{Q}} (\Lambda_{\mathbb{T}''})_{\mathbb{Q}} = Sym_{\mathbb{Q}}(M') \otimes_{\mathbb{Q}} Sym_{\mathbb{Q}}(M'') \tag{113}$$

$$H_*^{\mathbb{T}}(X,\mathbb{Q}) \simeq H_*^{\mathbb{T}'}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} (\Lambda_{\mathbb{T}''})_{\mathbb{Q}}, \tag{114}$$

with (112) *and* (113) *compatible under pullback, and* (112) *and* (114) *compatible under the module structure induced from the cap product. Similar statements hold for the corresponding completions, as well as for equivariant Chow groups and their completions.*

(ii) *The above factorizations are compatible with the corresponding equivariant Chern character and Todd class transformations.*

(iii) *Tensoring with the distinguished elements $\mathbb{C}_{\chi^0} \in K^0_{\mathbb{T}''}(pt)$ and $1 \in (\Lambda_{\mathbb{T}''})_{\mathbb{Q}}$ induces inclusions of the $\mathbb{T}'$-equivariant groups on X (or pt) into the corresponding $\mathbb{T}$-equivariant groups on X (or pt). Moreover, these inclusions are independent of the choice of the compatible splittings of $\mathbb{T}$ and M, respectively.*

Proof. The factorizations of (a) and (b) of (i) follow from suitable Künneth formulae.

Statement (ii) follows by construction from the multiplicativity under cross-products of the equivariant Chern character and equivariant Todd class transformation, see [23].

It remains to justify the independence statement of part (iii). Two choices of splittings of $\mathbb{T}$ differ by a group homomorphism $h : \mathbb{T}' \to \mathbb{T}''$ acting on product classes $[-]_{\mathbb{T}'} \otimes \mathbb{C}_{\chi^{m''}}$ of type (a) with $m'' \in M''$ via (compare [35][Prop.2.14]):

$$[-]_{\mathbb{T}'} \otimes \mathbb{C}_{\chi^{m''}} \mapsto \left([-]_{\mathbb{T}'} \cdot \mathbb{C}_{\chi^{m''\circ h}}\right) \otimes \mathbb{C}_{\chi^{m''}}.$$

So for $m'' = 0$, such an h acts trivially. Similarly, by choosing a basis $m''_1, \dots, m''_r$ of M'', one gets a basis $\prod_{i=1}^r c(m''_i)^{k_i}$, $k_i \geq 0$, of the $\mathbb{Q}$-vector space $(\Lambda_{\mathbb{T}''})_{\mathbb{Q}}$. Then h acts on product classes $[-]_{\mathbb{T}'} \otimes \prod_{i=1}^r c(m''_i)^{k_i}$ of type (b) via

$$[-]_{\mathbb{T}'} \otimes \prod_{i=1}^r c(m''_i)^{k_i} \mapsto \left([-]_{\mathbb{T}'} \cdot \prod_{i=1}^r c(m''_i \circ h)^{k_i}\right) \otimes \prod_{i=1}^r c(m''_i)^{k_i}.$$

So on product classes of type $[-]_{\mathbb{T}'} \otimes 1$ (i.e., with all k_i's being 0) it acts trivially. □

Remark 3.12. Results of Proposition 3.11 dealing only with rational equivariant (co)homology hold with the same proof under the weaker assumption of a splitting of rational vector spaces $M_{\mathbb{Q}} = M'_{\mathbb{Q}} \oplus M''_{\mathbb{Q}}$. Here $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$, and similarly for $M'_{\mathbb{Q}}$, $M''_{\mathbb{Q}}$.

As a consequence, we have the following.

Corollary 3.13. *In the setup of Proposition 3.11, let $\mathscr{E}, \mathscr{F}$ be a $\mathbb{T}'$-equivariant vector bundle, resp., coherent sheaf on X, with induced $\mathbb{T}$-action via the projection $\mathbb{T} \to \mathbb{T}'$. Then*

$$\mathrm{ch}^{\mathbb{T}}([\mathscr{E}]_{\mathbb{T}}) = \mathrm{ch}^{\mathbb{T}'}([\mathscr{E}]_{\mathbb{T}'}) \otimes 1 \quad \text{and} \quad \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]_{\mathbb{T}}) = \mathrm{td}_*^{\mathbb{T}'}([\mathscr{F}]_{\mathbb{T}'}) \otimes 1. \tag{115}$$

Proof. Choose a splitting $\mathbb{T} = \mathbb{T}' \oplus \mathbb{T}''$. Then, by part (ii) of Proposition 3.11, we get:

$$\begin{aligned} \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]_{\mathbb{T}}) &= \mathrm{td}_*^{\mathbb{T}'}([\mathscr{F}]_{\mathbb{T}'}) \otimes \mathrm{td}_*^{\mathbb{T}''}(\mathbb{C}_{\chi^0}) \\ &= \mathrm{td}_*^{\mathbb{T}'}([\mathscr{F}]_{\mathbb{T}'}) \otimes 1, \end{aligned}$$

and similarly for the equivariant Chern character. □

In particular, in the notations of Proposition 3.8, a choice of splitting of the surjection of tori $\mathbb{T} \to \mathbb{T}' := T_{N(\sigma)}$ yields an identification

$$\mathrm{td}_*^{\mathbb{T}}([\omega_{V_\sigma}]_{\mathbb{T}}) = \mathrm{td}_*^{\mathbb{T}'}([\omega_{V_\sigma}]_{\mathbb{T}'}), \tag{116}$$

which is independent of the splitting.

3.2. **Generalized equivariant Hirzebruch-Riemann-Roch.** Let X_Σ be a projective toric variety defined by a fan Σ, with $X := X_{\Sigma'} \subset X_\Sigma$ a $\mathbb{T}$-invariant closed algebraic subset of X_Σ defined by a star-closed subset $\Sigma' \subset \Sigma$. Let D be a $\mathbb{T}$-invariant Cartier divisor on X_Σ. The *equivariant Hirzebruch polynomial of $D|_X$* is defined by the formula:

$$\chi_y^{\mathbb{T}}(X, \mathscr{O}_{X_\Sigma}(D)|_X) := \sum_{p=0}^{\dim(X)} \chi^{\mathbb{T}}(X, \widetilde{\Omega}_X^p \otimes \mathscr{O}_{X_\Sigma}(D)|_X) \cdot y^p \in (\Lambda_{\mathbb{T}}^{an})_{\mathbb{Q}}[y] \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}[y], \tag{117}$$

with $\widetilde{\Omega}_X^p$ denoting as in Remark 3.6 the sheaf of Ishida p-forms on X.

Then we have the following result:

Theorem 3.14. *(generalized equivariant Hirzebruch-Riemann-Roch)*
In the above setup, the equivariant Hirzebruch polynomial of $D|_X$ is computed by the formula:

$$\chi_y^{\mathbb{T}}(X, \mathscr{O}_{X_\Sigma}(D)|_X) = \int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_{X_\Sigma}(D)|_X) \cap T_{y*}^{\mathbb{T}}(X). \tag{118}$$

Proof. This follows as in [36][Theorem 2.4], by making use of Proposition 3.5, Remark 3.6 and the module property (81) of $\mathrm{td}_*^{\mathbb{T}}$. □

As a consequence, we obtain the following weighted version of formula (75):

Corollary 3.15. *Let P be a full-dimensional lattice polytope with associated projective toric variety X_P and ample Cartier divisor $D = D_P$. Let $X := X_{P'}$ be the $\mathbb{T}$-invariant closed algebraic subset of X_P corresponding to a polytopal subcomplex $P' \subset P$ (i.e., a closed union of faces of P). Then:*

$$\chi_y^{\mathbb{T}}(X, \mathscr{O}_{X_P}(D)|_X) = \sum_{E \preceq P'} (1+y)^{\dim(E)} \cdot \sum_{m \in \mathrm{Relint}(E) \cap M} e^{s(m)}. \tag{119}$$

Proof. For a face E of P', denote by $i_E : V_{\sigma_E} := X_E \hookrightarrow X$ the inclusion of the orbit closure associated to the (cone of the) face E. Note that we have $\dim(E) = \dim(O_{\sigma_E})$. Let $\mathbb{T}' := T_{N(\sigma_E)}$ be the quotient torus of $\mathbb{T}$ corresponding to X_E.

Then, by Theorem 3.14, Proposition 3.7 and Proposition 3.8, the following equality holds:

$$\begin{aligned}\chi_y^{\mathbb{T}}(X, \mathscr{O}_{X_P}(D)|_X) &= \int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_{X_P}(D)|_X) \cap T_{y*}^{\mathbb{T}}(X) \\ &= \sum_{E \preceq P'} (1+y)^{\dim(E)} \int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_{X_P}(D)|_X) \cap (i_E)_* \mathrm{td}_*^{\mathbb{T}}([\omega_{X_E}]_{\mathbb{T}}).\end{aligned}$$

It remains to prove that for any face E of P', we have that:

$$\int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_{X_P}(D)|_X) \cap (i_E)_* \mathrm{td}_*^{\mathbb{T}}([\omega_{X_E}]_{\mathbb{T}}) = \sum_{m \in \mathrm{Relint}(E) \cap M} e^{s(m)}. \tag{120}$$

This follows from the functorial properties of the cap product and formula (43). Indeed,

$$\begin{aligned}\int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_{X_P}(D)|_X) \cap (i_E)_* \mathrm{td}_*^{\mathbb{T}}([\omega_{X_E}]_{\mathbb{T}}) &= \int_{X_E} (i_E)^* \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_{X_P}(D)|_X) \cap \mathrm{td}_*^{\mathbb{T}}([\omega_{X_E}]_{\mathbb{T}}) \\ &= \int_{X_E} \mathrm{ch}^{\mathbb{T}}((i_E)^*(\mathscr{O}_{X_P}(D)|_X)) \cap \mathrm{td}_*^{\mathbb{T}}([\omega_{X_E}]_{\mathbb{T}}) \\ &\overset{(43)}{=} \int_{X_E} \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_{X_E}(D_{E_0} - div(\chi^{m_0}))) \cap \mathrm{td}_*^{\mathbb{T}}([\omega_{X_E}]) \\ &= e^{-c(m_0)} \int_{X_E} \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_{X_E}(D_{E_0})) \cap \mathrm{td}_*^{\mathbb{T}}([\omega_{X_E}]_{\mathbb{T}}) \\ &\overset{(*)}{=} e^{-c(m_0)} \int_{X_E} \mathrm{ch}^{\mathbb{T}'}(\mathscr{O}_{X_E}(D_{E_0})) \cap \mathrm{td}_*^{\mathbb{T}'}([\omega_{X_E}]_{\mathbb{T}'}) \\ &\overset{(85)}{=} e^{s(m_0)} \cdot \left(\sum_{m \in \mathrm{Relint}(E-m_0) \cap M} e^{s(m)} \right) \\ &= \sum_{m \in \mathrm{Relint}(E) \cap M} e^{s(m)},\end{aligned}$$

where (*) uses Proposition 3.11 and Corollary 3.13. Here m_0 is a vertex of E as in (43), so that $E_0 := E - m_0$ is a full-dimensional lattice polytope in $Span(E_0)$ relative to the lattice $Span(E_0) \cap M$, with X_E the associated toric variety. □

Remark 3.16. For future use, we include here the following formula, which one gets as in the proof of (120), but using $\mathscr{O}_{X_E}$ instead of ω_{X_E}:

$$\int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_{X_P}(D)|_X) \cap \mathrm{td}_*^{\mathbb{T}}((i_E)_*[\mathscr{O}_{X_E}]_{\mathbb{T}}) = \sum_{m \in E \cap M} e^{s(m)}. \tag{121}$$

This is also a special case of Corollary 3.15 for $y = 0$ and $P' = E$.

As another application, we employ Theorem 3.14 in the context of a *globally generated* $\mathbb{T}$-invariant Cartier divisor D on a projective toric variety X with associated torus $\mathbb{T}$. Let $P_D \subset M_{\mathbb{R}}$ be the lattice polytope corresponding to D, and let X_D be the toric variety of the lattice polytope P_D, defined via the corresponding *generalized fan* as in [16][Prop.6.2.3]. By [16][Thm.6.2.8], there is a proper toric morphism $f : X \to X_D$, induced by the corresponding lattice projection $N \to N_D$ given by dividing out by the minimal cone of the generalized fan of P_D. In particular, $f : X \to X_D$ is a toric fibration. Let $M_D \hookrightarrow M$ be the associated inclusion of dual lattices. Choosing a vertex m_0 of P_D, we get that $P' := P_D - m_0 \subset M_D$ is a full-dimensional lattice polytope. Let D' be the ample divisor on X_D associated to P', with Σ' the inner normal fan of P' (defining X_D), so that there is a one-to-one correspondence between cones $\sigma' \in \Sigma'$ and faces E' of the lattice polytope P', and by translation to the faces E of P_D. Then, by the proof of [16][Thm.6.2.8], one gets that $\mathscr{O}(D - div(\chi^{m_0})) \simeq f^*\mathscr{O}(D')$, as $\mathbb{T}$-equivariant sheaves.

Following the notations of Proposition 3.9, let $U \subset X$ be a closed $\mathbb{T}$-invariant subset with associated multiplicities $d_\ell(U/\sigma')$, denoted here via the above correspondence as $d_\ell(U/E)$. We can now prove the following.

Corollary 3.17. *With the above notations,*

$$\chi_y^{\mathbb{T}}(U, \mathscr{O}_X(D)|_U) = \sum_{E \preceq P_D} \left(\sum_{\ell \geq 0} (-1)^\ell \cdot d_\ell(U/E) \cdot (1+y)^{\ell + \dim(E)} \right) \cdot \sum_{m \in \mathrm{Relint}(E) \cap M} e^{s(m)}. \tag{122}$$

Proof. By the generalized Hirzebruch-Riemann-Roch Theorem 3.14, the projection formula, and functoriality of Hirzebruch classes, we have

$$\begin{aligned}
\chi_y^{\mathbb{T}}(U, \mathscr{O}_X(D)|_U) &= \int_U \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_X(D)|_U) \cap T_{y*}^{\mathbb{T}}(U) \\
&= \int_X \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_X(D)) \cap T_{y*}^{\mathbb{T}}([U \hookrightarrow X]) \\
&= e^{s(m_0)} \int_X \mathrm{ch}^{\mathbb{T}}(f^*\mathscr{O}_X(D')) \cap T_{y*}^{\mathbb{T}}([U \hookrightarrow X]) \\
&= e^{s(m_0)} \int_{X_D} \mathrm{ch}^{\mathbb{T}}(\mathscr{O}_X(D')) \cap f_* T_{y*}^{\mathbb{T}}([U \hookrightarrow X]) \\
&\overset{(*)}{=} e^{s(m_0)} \left(\sum_{E' \preceq P'} \sum_{\ell \geq 0} (-1)^\ell \cdot d_\ell(U/E') \cdot (1+y)^{\ell + \dim(E')} \cdot \sum_{m \in \mathrm{Relint}(E') \cap M} e^{s(m)} \right) \\
&= \sum_{E \preceq P_D} \sum_{\ell \geq 0} (-1)^\ell \cdot d_\ell(U/E) \cdot (1+y)^{\ell + \dim(E)} \cdot \sum_{m \in \mathrm{Relint}(E) \cap M} e^{s(m)},
\end{aligned}$$

where the equality $(*)$ follows by using (108) and (120). □

Example 3.18 (Rigidity of the equivariant χ_y-genus). If, in Corollary 3.17, we take $D = 0$, then $P_D = \{0\} \subset M \subset M_{\mathbb{R}}$, so $d_\ell(U/\{0\})$ is just the number of ℓ-dimensional torus orbits in U. Hence, formula (122) becomes in this case

$$\chi_y^{\mathbb{T}}(U, \mathscr{O}_U) = \left(\sum_{\ell \geq 0} (-1)^\ell \cdot d_\ell(U/\{0\}) \cdot (1+y)^\ell \right) \cdot e^{s(0)}. \tag{123}$$

Note that this formula does not take into account any equivariant information. So, forgetting the $\mathbb{T}$-action,

$$\left(\sum_{\ell \geq 0} (-1)^\ell \cdot d_\ell(U/\{0\}) \cdot (1+y)^\ell \right) = \chi_y(U, \mathscr{O}_U).$$

In particular, for $U = X$ and $y = 0$, this becomes $\chi(X, \mathscr{O}_X) = 1$. □

Remark 3.19. Note that if D is an ample $\mathbb{T}$-invariant Cartier divisor, then the above morphism f is the identity of X, so formula (122) reduces to (119) since the multiplicities $d_\ell(U/E)$ are given as follows: $d_\ell(U/E) = 0$ for $\ell > 0$, and $d_0(U/E)$ is either 1 or 0, depending wether the orbit associated to E is included or not in U.

Remark 3.20. By forgetting the $\mathbb{T}$-action in formula (122), we get the following weighted lattice point counting for lattice polytopes associated to globally generated $\mathbb{T}$-invariant Cartier divisors. More precisely, in the above notations, we get the following generalization of Theorem 2.3 and of [36][Cor.1.8].

$$\chi_y(U, \mathscr{O}_X(D)|_U) = \sum_{E \preceq P_D} \left(\sum_{\ell \geq 0} (-1)^\ell \cdot d_\ell(U/E) \cdot (1+y)^{\ell + \dim(E)} \right) \cdot |\mathrm{Relint}(E) \cap M|. \tag{124}$$

Here, it is enough to assume that X is complete, since the non-equivariant Hirzebruch class transformation is defined in this generality.

3.3. **Equivariant Hirzebruch and Todd characteristic classes of simplicial toric varieties.** The previously mentioned formulae of Section 2.4 for the Todd class and Hirzebruch classes of quasi-projective simplicial toric varieties hold with the same proofs (up to some small modifications, as explained below) for the equivariant versions of these characteristic classes, compare also with [24][Rem.4.3]. A different proof of all these results, independent of the equivariant version of the Lefschetz-Riemann-Roch from [24][Thm.3.1 and Rem.4.3], will be given in Section 4.3 via localization at the torus fixed points. Nevertheless, the approach of this section is useful to derive global expressions for the equivariant characteristic classes of interest, which will then be localized.

Let $X := X_\Sigma$ be a simplicial quasi-projective toric variety associated to a fan Σ of full-dimensional convex support, e.g., $X = X_P$ is the semi-projective simplicial toric variety corresponding to a simple full-dimensional lattice polyhedron $P \subset M_\mathbb{R}$. This includes the cases of full-dimensional simple lattice polytopes, as well as full-dimensional rational pointed polyhedral cones. The assumption that Σ is of full-dimensional convex support implies that X contains no torus factors, so we can use the Cox construction as described in Section 2.1.2. Moreover, this allows us to make use of Remark 2.8.

In addition, we also prove formulae for the equivariant Hirzebruch classes $T^{\mathbb{T}}_{y*}([U \hookrightarrow X])$, with U the open complement of a $\mathbb{T}$-invariant divisor $D_K := \bigcup_{\rho \in K} D_\rho$, for $K \subset \Sigma(1)$. Similarly, we indicate here the argument even for the Hirzebruch classes of an orbit closure V_τ, for $\tau \in \Sigma$, pointing out the needed modifications in the proof.

The arguments in this section are based on the Cox construction. With the notations from Subsection 2.1.2, let $\pi : W := \mathbb{C}^r \setminus Z(\Sigma) \to X$ be the toric morphism, with G the kernel of the corresponding map of tori $\gamma : \widetilde{\mathbb{T}} := (\mathbb{C}^*)^r \to \mathbb{T}$ (here $r = |\Sigma(1)|$). Since X is a simplicial toric variety containing no torus factor, it follows that X is the geometric quotient W/G, with G acting with finite stabilizers. If τ is a cone of Σ, then the orbit closure V_τ is the image under π of a linear subspace $W_\tau \subset W$. We have the following formula (in which we omit the symbols for pushforwards under closed embeddings).

Lemma 3.21.

$$\mathrm{td}^{\mathbb{T}}_*((\pi_* \Lambda_y T^*_{W_\tau})^G) = (1+y)^{r-n} \cdot \mathrm{td}^{\mathbb{T}}_*(mC^{\mathbb{T}}_y(V_\tau)) \in \widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})[y] \simeq \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})[y]. \tag{125}$$

Proof. By (65), it suffices to check formula (125) after restriction to each U_σ, with U_σ the $\mathbb{T}$-invariant open affine subset of X containing the corresponding $\mathbb{T}$-fixed point x_σ, for $\sigma \in \Sigma(n)$. We have $\pi^{-1}(U_\sigma) \simeq \mathbb{C}^n \times (\mathbb{C}^*)^{r-n}$, with $\widetilde{\mathbb{T}} \simeq \mathbb{T} \times (\mathbb{C}^*)^{r-n} =: \mathbb{T} \times \mathbb{T}'$ acting on the respective factors, and the factor $\mathbb{C}^n$ corresponding to the rays of σ and $\mathbb{T}' = (\mathbb{C}^*)^{r-n}$ acting freely by multiplication on itself. Similarly, $G \simeq G_\sigma \times \mathbb{T}'$, with $G_\sigma \subset \mathbb{T}$ a finite subgroup. So, above U_σ, π can be factorized as a composition of the free quotient $\widetilde{\pi} : \mathbb{C}^n \times (\mathbb{C}^*)^{r-n} \to \mathbb{C}^n$ by the $\mathbb{T}'$-action, followed by a finite quotient map $\pi = \pi_\sigma : \mathbb{C}^n \to \mathbb{C}^n/G_\sigma = U_\sigma$. Similarly, $\pi^{-1}(U_\sigma \cap V_\tau) \simeq L_\tau \times (\mathbb{C}^*)^{r-n}$, with $L_\tau \subset \mathbb{C}^n$ a linear subspace. Then

$$\left(\widetilde{\pi}_* \Lambda_y T^*_{L_\tau \times (\mathbb{C}^*)^{r-n}}\right)^{\mathbb{T}'} = \Lambda_y T^*_{L_\tau} \cdot (1+y)^{r-n},$$

since $T^*_{(\mathbb{C}^*)^{r-n}} \simeq (\mathbb{C}^*)^{r-n} \times T^*_{id}$, with $\mathbb{T}'$ acting by the co-adjoint action on the cotangent space T^*_{id} of $\mathbb{T}'$ at the identity element $id \in \mathbb{T}'$. This is the trivial $\mathbb{T}'$-bundle of rank $r-n$, since $\mathbb{T}'$ is an abelian group, so that $\Lambda_y T^*_{(\mathbb{C}^*)^{r-n}} = (1+y)^{r-n}$. Finally, in $K_0^{\mathbb{T}}(U_\sigma)[y]$, we have by Proposition 3.5 and Corollary 3.13 that

$$((\pi_\sigma)_* \Lambda_y T^*_{L_\tau})^{G_\sigma} = mC_y^{\mathbb{T}}(V_\tau \cap U_\sigma).$$

The desired formula (125) follows by applying the equivariant Todd transformation. □

By the equivariant version of the Lefschetz-Riemann-Roch theorem (see [24][Thm.3.1 and Rem.4.3]), applied to the left-hand side of formula (125), the proof of the formulae of Section 2.4 for the Todd class and Hirzebruch classes of projective simplicial toric varieties hold with the same proofs in the equivariant setting. For later applications, we work with the cohomological images $\mathrm{td}_*^{\mathbb{T}}(X) \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$ and $T^{\mathbb{T}}_{y*}(X) \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})[y]$, resp., $\widehat{T}^{\mathbb{T}}_{y*}(X) \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q}[y]) := \prod_{i \geq 0} H^i_{\mathbb{T}}(X;\mathbb{Q}[y])$ of these classes under equivariant Poincaré duality. In the notations of Section 2.1.2, one has the following:

Theorem 3.22. *Let $X := X_\Sigma$ be a simplicial quasi-projective toric variety associated to a fan Σ of full-dimensional convex support. Then the equivariant Hirzebruch classes $T^{\mathbb{T}}_{y*}(X)$ and $\widehat{T}^{\mathbb{T}}_{y*}(X)$ are computed by:*

$$T^{\mathbb{T}}_{y*}(X) = (1+y)^{n-r} \cdot \sum_{g \in G_\Sigma} \prod_{\rho \in \Sigma(1)} \frac{F_\rho \cdot \left(1 + y \cdot a_\rho(g) \cdot e^{-F_\rho}\right)}{1 - a_\rho(g) \cdot e^{-F_\rho}} \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})[y], \tag{126}$$

$$\widehat{T}^{\mathbb{T}}_{y*}(X) = \sum_{g \in G_\Sigma} \prod_{\rho \in \Sigma(1)} \frac{F_\rho \cdot \left(1 + y \cdot a_\rho(g) \cdot e^{-F_\rho(1+y)}\right)}{1 - a_\rho(g) \cdot e^{-F_\rho(1+y)}} \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q}[y]), \tag{127}$$

with $r = |\Sigma(1)|$, and $F_\rho = [D_\rho]_{\mathbb{T}}$ denoting the equivariant fundamental class of the $\mathbb{T}$-invariant divisor D_ρ corresponding to the ray $\rho \in \Sigma(1)$.

Corollary 3.23. *If $X := X_\Sigma$ is a smooth quasi-projective toric variety associated to a fan Σ of full-dimensional convex support, then G_Σ is the trivial group. Hence, the equivariant Hirzebruch classes of X are given by:*

$$T^{\mathbb{T}}_{y*}(X) = (1+y)^{n-r} \cdot \prod_{\rho \in \Sigma(1)} \frac{F_\rho \cdot \left(1 + y \cdot e^{-F_\rho}\right)}{1 - e^{-F_\rho}} \tag{128}$$

$$\widehat{T}^{\mathbb{T}}_{y*}(X) = \prod_{\rho\in\Sigma(1)} \frac{F_\rho\cdot\left(1+y\cdot e^{-F_\rho(1+y)}\right)}{1-e^{-F_\rho(1+y)}}\,. \tag{129}$$

Remark 3.24. In particular, by setting $y=0$ in Theorem 3.22, we recover the equivariant Todd class formula of [13] given by the following expression in $\widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$:

$$\mathrm{td}^{\mathbb{T}}_*(X) = \sum_{g\in G_\Sigma}\prod_{\rho\in\Sigma(1)} \frac{F_\rho}{1-a_\rho(g)\cdot e^{-F_\rho}}. \tag{130}$$

Similarly, setting $y=0$ in Corollary 3.23, one gets the equivariant Todd class for such a smooth quasi-projective toric variety. There is also an equivariant version of formula (56). Finally, if X is projective, the specialization at $y=1$ fits with suitable L-classes, i.e., $\widehat{T}_{1*}(X)=L(X)$ is the Thom-Milnor L-class of a projective toric variety X.

Moreover, taking the top degree of y in (126) and using (96), yields the following.

Corollary 3.25. *Under the assumptions and notations of Theorem 3.22, we have:*

$$\mathrm{td}^{\mathbb{T}}_*([\omega_X]_{\mathbb{T}}) = \sum_{g\in G_\Sigma}\prod_{\rho\in\Sigma(1)} \frac{a_\rho(g)\cdot F_\rho\cdot e^{-F_\rho}}{1-a_\rho(g)e^{-F_\rho}} \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q}). \tag{131}$$

Remark 3.26. Note that formula (131) coincides with (130) upon substituting $-F_\rho$ for F_ρ, for each $\rho\in\Sigma(1)$, i.e., these classes are exchanged by the *cohomological duality involution* on $\widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$ given by multiplication with $(-1)^i$ in degree $2i$. (Recall that under our assumptions $\widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$ is even, by (63).)

In Section 5 below, formula (131) will be used for proving Euler-Maclaurin type formulae for lattice points in the interior of a full-dimensional polytope, generalizing formula (46) for lattice point counting. The next results are motivated by the fact that, instead of deleting all facets of the polytope P, one can just delete some of the facets F_i, $i\in K\subset\Sigma_P(1)$.

Lemma 3.27. *Let $X:=X_\Sigma$ be a simplicial quasi-projective toric variety associated to a fan Σ of full-dimensional convex support. Let $U\subset X$ be the open complement of the divisor $D_K:=\bigcup_{\rho\in K}D_\rho$, for $K\subset\Sigma(1)$. In the notations of the Cox construction, let $W_\rho=\{x_\rho=0\}\subset W$ be the inverse image of D_ρ under the quotient map $\pi: W\to X$. Then the preimage $\widetilde{U}$ of U under π is the complement of the $\widetilde{\mathbb{T}}$-invariant normal crossing divisor $W_K=\bigcup_{\rho\in K}W_\rho$ in W, and*

$$\mathrm{td}^{\mathbb{T}}_*\left((\pi_* mC^{\widetilde{\mathbb{T}}}_y([\widetilde{U}\hookrightarrow W]))^G\right) = (1+y)^{r-n}\cdot \mathrm{td}^{\mathbb{T}}_*(mC^{\mathbb{T}}_y([U\hookrightarrow X]))\in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})[y]. \tag{132}$$

Proof. By the inclusion-exclusion formula for the equivariant motivic Chern classes, one has as in (92),

$$mC^{\mathbb{T}}_y([U\hookrightarrow X]) = \sum_{I\subset K}(-1)^{|I|} mC^{\mathbb{T}}_y([D_I\hookrightarrow X])\,, \tag{133}$$

with $D_I=\bigcap_{\rho\in I}D_\rho$. Similarly,

$$mC^{\widetilde{\mathbb{T}}}_y([\widetilde{U}\hookrightarrow W]) = \sum_{I\subset K}(-1)^{|I|} mC^{\widetilde{\mathbb{T}}}_y([W_I\hookrightarrow W]), \tag{134}$$

with $W_I=\bigcap_{\rho\in I}W_\rho$. The assertion follows now by applying formula (125) to the summands on the right-hand side of the two identities above. □

By formula (91), in the above notations we have:

$$mC_y^{\widetilde{\mathbb{T}}}([\widetilde{U} \hookrightarrow W]) = [\mathscr{O}_W(-W_K) \otimes \Lambda_y \Omega_W^1(\log W_K)]_{\widetilde{\mathbb{T}}} \in K_0^{\widetilde{\mathbb{T}}}(W)[y]\,. \tag{135}$$

Then by the equivariant version of the Lefschetz-Riemann-Roch theorem (see [24][Thm.3.1 and Rem.4.3]), applied to the right-hand side of formula (135), we get the following generalization of Theorem 3.22:

Theorem 3.28. *Let $X := X_\Sigma$ be a simplicial quasi-projective toric variety associated to a fan Σ of full-dimensional convex support. Let $U \subset X$ be the open complement of the divisor $D_K := \bigcup_{\rho \in K} D_\rho$, for $K \subset \Sigma(1)$. The equivariant Hirzebruch classes $T_{y*}^{\mathbb{T}}([U \hookrightarrow X])$ and $\widehat{T}_{y*}^{\mathbb{T}}([U \hookrightarrow X])$ are computed by:*

(136)

$$T_{y*}^{\mathbb{T}}([U \hookrightarrow X]) = (1+y)^{n-r} \cdot \sum_{g \in G_\Sigma} \prod_{\rho \in K} \frac{F_\rho \cdot (1+y) \cdot a_\rho(g) \cdot e^{-F_\rho}}{1 - a_\rho(g) \cdot e^{-F_\rho}} \prod_{\rho \notin K} \frac{F_\rho \cdot \left(1 + y \cdot a_\rho(g) \cdot e^{-F_\rho}\right)}{1 - a_\rho(g) \cdot e^{-F_\rho}}\,,$$

(137)

$$\widehat{T}_{y*}^{\mathbb{T}}([U \hookrightarrow X]) = \sum_{g \in G_\Sigma} \prod_{\rho \in K} \frac{F_\rho \cdot (1+y) \cdot a_\rho(g) \cdot e^{-F_\rho(1+y)}}{1 - a_\rho(g) \cdot e^{-F_\rho(1+y)}} \prod_{\rho \notin K} \frac{F_\rho \cdot \left(1 + y \cdot a_\rho(g) \cdot e^{-F_\rho(1+y)}\right)}{1 - a_\rho(g) \cdot e^{-F_\rho(1+y)}}\,,$$

with $r = |\Sigma(1)|$, and $F_\rho = [D_\rho]_{\mathbb{T}}$ denoting the equivariant fundamental class of the $\mathbb{T}$-invariant divisor D_ρ corresponding to the ray $\rho \in \Sigma(1)$.

Proof. The proof is similar to that of Theorem 3.22 and is based on the equivariant Lefschetz-Riemann-Roch theorem. We only indicate here the changes. Instead of calculating the equivariant twisted Chern character $ch^{\widetilde{\mathbb{T}}}(i_g^*-)(g)$, $g \in G$, of $\Lambda_y T_W^*$ for $i_g : W^g \hookrightarrow W$ the fixed point set inclusion, one now has to calculate this for $\mathscr{O}_W(-W_K) \otimes \Lambda_y \Omega_W^1(\log W_K)$. Here, for $\mathscr{E} \in K^0_{\widetilde{\mathbb{T}}}(W)$, the twisted equivariant Chern character is defined as

$$ch^{\widetilde{\mathbb{T}}}(i_g^*\mathscr{E})(g) = \sum_\chi \chi(g) \cdot ch^{\widetilde{\mathbb{T}}}(\mathscr{E}_\chi),$$

where $\mathscr{E} \simeq \oplus_\chi \mathscr{E}_\chi$ is the finite decomposition of $\mathscr{E}$ into sheaves $\mathscr{E}_\chi$ on which g acts by a (complex-valued) character χ. These sheaves $\mathscr{E}_\chi$ are also $\widetilde{\mathbb{T}}$-equivariant since $\widetilde{\mathbb{T}}$ is an abelian group.

We now have to evaluate $ch^{\widetilde{\mathbb{T}}}(i_g^*-)(g)$ on

$$\mathscr{O}_W(-W_K) \otimes \Lambda_y \Omega_W^1(\log W_K) \simeq \prod_{\rho \in K} \left(\mathscr{O}_W(-W_\rho) \otimes pr_\rho^* \Lambda_y \Omega_{\mathbb{C}}^1(\log\{0\})\right) \prod_{\rho \notin K} \Lambda_y \mathscr{O}_W(-W_\rho).$$

Here $pr_\rho : W \to \mathbb{C}$ is the projection to the ρ-th factor, with $\widetilde{\mathbb{T}} \simeq (\mathbb{C}^*)^r$ acting factorwise via the projection $a_\rho : \widetilde{\mathbb{T}} \to \mathbb{C}^*$ to the corresponding ρ-th factor. So the calculation can be done factorwise, with

$$ch^{\widetilde{\mathbb{T}}}(i_g^* \Lambda_y \mathscr{O}_W(-W_\rho))(g) = i_g^*\left(1 + y \cdot a_\rho(g^{-1}) \cdot e^{-z_\rho}\right),$$

for $\rho \notin K$, and $z_\rho := [W_\rho]_{\widetilde{\mathbb{T}}} \in H^*_{\widetilde{\mathbb{T}}}(W;\mathbb{Q})$ the corresponding equivariant fundamental class as in [36][eqn.(5.13)]. By the inclusion-exclusion formula, for $\{0\} \subset \mathbb{C}$ and $\rho \in K$, we get:

$$[\mathscr{O}_W(-W_\rho) \otimes pr_\rho^* \Lambda_y \Omega_{\mathbb{C}}^1(\log\{0\})] = [\Lambda_y \mathscr{O}_W(-W_\rho)] - [\mathscr{O}_{D_\rho}].$$

So

$$\begin{aligned}
& ch^{\widetilde{\mathbb{T}}}\left(i_g^*\big(\mathscr{O}_W(-W_\rho)\otimes pr_\rho^*\Lambda_y\Omega^1_{\mathbb{C}}(\log\{0\})\big)\right)(g) \\
&= i_g^*\big(1+y\cdot a_\rho(g^{-1})\cdot e^{-z_\rho}\big) - ch^{\widetilde{\mathbb{T}}}\big(i_g^*\left([\mathscr{O}_W]-[\mathscr{O}_W(-D_\rho)]\right)\big)(g) \\
&= i_g^*\big(1+y\cdot a_\rho(g^{-1})\cdot e^{-z_\rho}\big) - i_g^*\big(1-a_\rho(g^{-1})\cdot e^{-z_\rho}\big) \\
&= i_g^*\left((1+y)\cdot a_\rho(g^{-1})\cdot e^{-z_\rho}\right).
\end{aligned}$$

Then these formulae are pushed forward via $(i_g)_*$ using the projection formula, and one applies the ring isomorphism $\phi : H^*_{\widetilde{\mathbb{T}}}(W;\mathbb{Q}) \simeq H^*_{\mathbb{T}}(X;\mathbb{Q})$. Note that z_ρ maps to F_ρ under the identification $\phi : H^2_{\widetilde{\mathbb{T}}}(W;\mathbb{Q}) \simeq H^2_{\mathbb{T}}(X;\mathbb{Q})$. Finally, note that $g \in G_\Sigma$ if and only if $g^{-1} \in G_\Sigma$. This completes the proof for the un-normalized equivariant Hirzebruch classes. To get the formula for the normalized equivariant Hirzebruch classes, we just have to substitute $(1+y)^{-n}\cdot[X]_{\mathbb{T}}$ for $[X]_{\mathbb{T}} \in H^{\mathbb{T}}_{2n}(X;\mathbb{Q})$ (implicitly used in the equivariant Poincaré duality), and $(1+y)\cdot F_\rho$ for $F_\rho \in H^2(X;\mathbb{Q})$. □

For later use, and since the needed notation was already introduced in the previous result, let us sketch a proof of the following extension of the Todd class formula (130) to the $\mathbb{T}$-equivariant coherent sheaf $\pi_*(\mathscr{O}_W\otimes\mathbb{C}_{\chi^{\widetilde{m}}})^G$, for $\pi : W \to X$ the quotient map of the Cox construction and $\chi^{\widetilde{m}}$ a character of $\widetilde{\mathbb{T}}$.

Lemma 3.29. *With the above notations, we have*

$$\operatorname{td}^{\mathbb{T}}_*(\pi_*(\mathscr{O}_W\otimes\mathbb{C}_{\chi^{\widetilde{m}}})^G) = \sum_{g\in G_\Sigma}\chi^{\widetilde{m}}(g^{-1})\prod_{\rho\in\Sigma(1)}\frac{F_\rho\cdot e^{\langle\widetilde{m},e_\rho\rangle\cdot F_\rho}}{1-a_\rho(g)e^{-F_\rho}}, \tag{138}$$

with $\{e_\rho\}_{\rho\in\Sigma(1)}$ the standard basis of the lattice $\widetilde{N} = \mathbb{Z}^{|\Sigma(1)|}$.

Proof. Comparing to the proof of the classical Todd class formula (130), we need to calculate $ch^{\widetilde{\mathbb{T}}}(i_g^*(\mathscr{O}_W\otimes\mathbb{C}_{\chi^{\widetilde{m}}}))(g)$ instead of $ch^{\widetilde{\mathbb{T}}}(i_g^*(\mathscr{O}_W))(g) = 1$. This is given by

$$ch^{\widetilde{\mathbb{T}}}(i_g^*(\mathscr{O}_W\otimes\mathbb{C}_{\chi^{\widetilde{m}}}))(g) = i_g^*\left(\chi^{\widetilde{m}}(g)e^{c(\widetilde{m})}\right),$$

with $c(\widetilde{m}) = \sum_\rho\langle\widetilde{m},e_\rho\rangle z_\rho \in H^2_{\widetilde{\mathbb{T}}}(W;\mathbb{Q})$, e.g., see [16][Prop.12.4.13(b)]. □

Remark 3.30. Note that if $K=\emptyset$, Theorem 3.28 reduces to Theorem 3.22. At the other extreme, if $K=\Sigma(1)$, one gets for the un-normalized class first by specializing to $y=0$ just Corollary 3.25, and using this one then recovers for arbitrary y formula (100) in the case of the zero cone.

Remark 3.31. For $y=0$, both formulae of Theorem 3.28 specialize to the following Todd class type formula, which will fit later on with the Euler-Maclaurin formulas for polytopes with some facets removed:

$$T^{\mathbb{T}}_{0*}([U\hookrightarrow X]) = \sum_{g\in G_\Sigma}\prod_{\rho\in K}\frac{F_\rho\cdot a_\rho(g)\cdot e^{-F_\rho}}{1-a_\rho(g)\cdot e^{-F_\rho}}\prod_{\rho\notin K}\frac{F_\rho}{1-a_\rho(g)\cdot e^{-F_\rho}}. \tag{139}$$

Note that in view of Remark 3.6, one also has

$$T^{\mathbb{T}}_{0*}([U\hookrightarrow X]) = td^{\mathbb{T}}_*(X) - td^{\mathbb{T}}_*(D_K). \tag{140}$$

4. LOCALIZATION IN EQUIVARIANT K-THEORY AND APPLICATIONS

In this section we apply localization techniques in $\mathbb{T}$-equivariant K- and homology theories of toric varieties, due to Brion-Vergne [13] and, resp., Brylinski-Zhang [15], for the calculation of the $\mathbb{T}$-equivariant motivic Chern and Hirzebruch classes in this toric context.

4.1. **Localization in equivariant K-theory.** Let $X = X_\Sigma$ be an n-dimensional toric variety with torus $\mathbb{T} = T_N$ such that the fixed-point set $X^\mathbb{T} \neq \emptyset$, e.g., X is projective.

By [13][Prop.1.5], the inclusion $i : X^\mathbb{T} \hookrightarrow X$ induces an injective morphism of $K_0^\mathbb{T}(pt)$-modules

$$i_* : K_0^\mathbb{T}(X^\mathbb{T}) \hookrightarrow K_0^\mathbb{T}(X)$$

which becomes an isomorphism

$$i_* : K_0^\mathbb{T}(X^\mathbb{T})_S \simeq K_0^\mathbb{T}(X)_S$$

upon localization at the multiplicative subset $S \subset \mathbb{Z}[M] = K_0^\mathbb{T}(pt)$ generated by the elements $1-\chi^m$, for $0 \neq m \in M$. Note that

$$K_0^\mathbb{T}(X^\mathbb{T}) = \bigoplus_{x \in X^\mathbb{T}} K_0^\mathbb{T}(x),$$

from which one gets via the isomorphism i_* a projection map of $K_0^\mathbb{T}(pt)$-modules, called here the *K-theoretic localization map at x*,

$$pr_x : K_0^\mathbb{T}(X)_S \simeq \bigoplus_{x \in X^\mathbb{T}} K_0^\mathbb{T}(x)_S \longrightarrow K_0^\mathbb{T}(x)_S = \mathbb{Z}[M]_S.$$

Let $x = x_\sigma \subset U_\sigma$ be the $\mathbb{T}$-fixed point corresponding to a cone $\sigma \in \Sigma(n)$ of maximal dimension, as in (65), where $U_\sigma \subset X$ is the unique $\mathbb{T}$-invariant open affine subset containing $x_\sigma = O_\sigma$. Then the localization map at x factorizes via restriction over U_σ as:

$$pr_x : K_0^\mathbb{T}(X)_S \longrightarrow K_0^\mathbb{T}(U_\sigma)_S \simeq K_0^\mathbb{T}(x_\sigma)_S = \mathbb{Z}[M]_S.$$

We now explain a different description of the isomorphism (denoted also by pr_x)

$$pr_x : K_0^\mathbb{T}(U_\sigma)_S \simeq K_0^\mathbb{T}(x_\sigma)_S = \mathbb{Z}[M]_S$$

by using the eigenspace decomposition (68). As in [13][Sect.1.3] (see also [16][Def.13.2.2]), a formal power series $f \in \mathbb{Z}[[M]]$ is called *summable* if there is $g \in \mathbb{Z}[M]$ and a finite subset $I \subset M \setminus \{0\}$ such that in $\mathbb{Z}[[M]]$ one has: $f \cdot \prod_{m \in I}(1-\chi^m) = g$. Let

$$\mathbb{S}(f) := g \cdot \prod_{m \in I}(1-\chi^m)^{-1} \in \mathbb{Z}[M]_S$$

be the *sum* of f, which is easily seen to be independent of the factorization. Let $\mathbb{Z}[[M]]_{\mathrm{Sum}} \subset \mathbb{Z}[[M]]$ be the subset of summable elements in $\mathbb{Z}[[M]]$. This is a $\mathbb{Z}[M]$-submodule of $\mathbb{Z}[[M]]$, and the summation map $\mathbb{S}$ induces a homomorphism of $\mathbb{Z}[M]$-modules

$$\mathbb{S} : \mathbb{Z}[[M]]_{\mathrm{Sum}} \to \mathbb{Z}[M]_S.$$

Let $\mathscr{F}$ be a $\mathbb{T}$-equivariant coherent sheaf on U_σ. Then $W := H^0(U_\sigma; \mathscr{F})$ has an eigenspace decomposition $W = \bigoplus_{m \in M} W_{\chi^m}$ (see [13][Sect.1.3]) with eigenspaces W_{χ^m} of finite dimension as in (68) (although $W = H^0(U_\sigma;\mathscr{F})$ could be infinite dimensional). Then

$$\chi_\sigma^{\mathbb{T}}(\mathscr{F}) := \sum_{m\in M} \dim_{\mathbb{C}} W_{\chi^m}\cdot \chi^m \in \mathbb{Z}[[M]] \tag{141}$$

is summable (cf. [13][Prop.1.3]). This induces a homomorphism of $\mathbb{Z}[M]$-modules (see [13][Rem.1.3])

$$\chi_\sigma^{\mathbb{T}} : K_0^{\mathbb{T}}(U_\sigma) \longrightarrow \mathbb{Z}[M]_{\mathrm{sum}} \tag{142}$$

such that the composition $\mathbb{S}\circ\chi_\sigma^{\mathbb{T}}$ induces after localization the map pr_x, since

$$\chi_\sigma^{\mathbb{T}}(i_*([\mathbb{C}_{\chi^m}]) = \chi^m \in \mathbb{Z}[M] \subset \mathbb{Z}[M]_{\mathrm{sum}}. \tag{143}$$

For applications to the case of simplicial cones, we next introduce a Lefschetz type variant of the Euler characteristic $\chi_\sigma^{\mathbb{T}}$, and a corresponding summation map $\mathbb{S}$. Let $\sigma\in\Sigma(n)$ be a simplicial cone with $u_1,\dots,u_n \in N = N_\sigma$ the generators of the the rays $\rho_j\in\sigma(1)$, $j=1,\dots,n$. Let $N' = N'_\sigma$ be the finite index sublattice of N generated by $u_1,\dots,u_n$, and consider $\sigma\in N'_{\mathbb{R}} = N_{\mathbb{R}}$ so that it is smooth with respect to the lattice N'. With $\mathbb{T}$, $\mathbb{T}'$ the corresponding n-dimensional tori of the lattices N, resp., N', the inclusion $N'\hookrightarrow N$ induces a toric morphism $\pi: U'_\sigma \to U_\sigma$ of the associated affine toric varieties. Let G_σ be the finite kernel of the epimorphism $\pi:\mathbb{T}'\to\mathbb{T}$, so that $U'_\sigma/G_\sigma \simeq U_\sigma$ (e.g., see [16][Prop.1.3.18]).

Let $m_i \in M = M_\sigma$, $1\le i\le n$, be the unique primitive elements in the dual lattice M of N satisfying $\langle m_i,u_j\rangle = 0$ for $i\neq j$ and $q_i := \langle m_i,u_i\rangle > 0$ so that the dual lattice $M' = M'_\sigma$ of N' is generated by the elements $m'_j := \frac{m_j}{q_j}$. Let $a_{\rho_j} : G_\sigma \to \mathbb{C}^*$ be the characters of G_σ as introduced in (33).

For $\mathscr{F}'$ a $\mathbb{T}'$-equivariant coherent sheaf on U'_σ, the vector space $W' := H^0(U'_\sigma;\mathscr{F}')$ has an eigenspace decomposition $W' = \bigoplus_{m'\in M'} W'_{\chi^{m'}}$ as before. Since $\mathbb{T}'$ is abelian, its finite subgroup G_σ acts on W' respecting this eigenspace decomposition. We can then introduce the Lefschetz type Euler characteristic

$$tr_\sigma^{\mathbb{T}'}(\mathscr{F}') := \frac{1}{|G_\sigma|}\bigoplus_{m'\in M'}\sum_{g\in G_\sigma} tr(g : W'_{\chi^{m'}} \to W'_{\chi^{m'}})\cdot\chi^{m'} \in \mathbb{C}[[M']]_{\mathrm{sum}}. \tag{144}$$

In this context, the notion of *summable* is defined almost as above, but using the multiplicative subset $S' \subset \mathbb{C}[M']$ generated by elements $1 - a\cdot\chi^{m'}$, for $0\neq m'\in M'$ and $a\in\mathbb{C}^*$. This induces a homomorphism of $\mathbb{Z}[M']$-modules

$$tr_\sigma^{\mathbb{T}'} : K_0^{\mathbb{T}}(U'_\sigma) \longrightarrow \mathbb{C}[M']_{\mathrm{sum}}.$$

The fact that $tr_\sigma^{\mathbb{T}'}(\mathscr{F}')$ is summable will be explained below for sheaves $\mathscr{F}' = \mathscr{O}_{U'_\sigma}\otimes\mathbb{C}_{\chi^{m'}}$, with $m'\in M'$, whose classes generate $K_0^{\mathbb{T}}(U'_\sigma)$ (as in the proof of [13][Cor.1.2]). There is also a corresponding summation map

$$\mathbb{S}' : \mathbb{C}[[M']]_{\mathrm{sum}} \to \mathbb{C}[M']_{S'},$$

so that the following diagram of $\mathbb{Z}[M]$-linear maps commutes:

$$\begin{array}{ccccc} K_0^{\mathbb{T}'}(U'_\sigma) & \xrightarrow{tr_\sigma^{\mathbb{T}'}} & \mathbb{C}[[M']]_{\text{sum}} & \xrightarrow{\mathbb{S}'} & \mathbb{C}[M']_{S'} \\ \pi_*^{G_\sigma}\downarrow & & \uparrow & & \uparrow \\ K_0^{\mathbb{T}}(U_\sigma) & \xrightarrow{\chi_\sigma^{\mathbb{T}}} & \mathbb{Z}[[M]]_{\text{sum}} & \xrightarrow{\mathbb{S}} & \mathbb{Z}[M]_S\ . \end{array} \tag{145}$$

Here, $\pi_*^{G_\sigma}$ is induced by the corresponding exact functor given by taking the G_σ-invariant part of the pushforward for the finite map π, which is $\mathbb{T} = \mathbb{T}'/G_\sigma$-equivariant. The monomorphism $\mathbb{Z}[[M]]_{\text{sum}} \to \mathbb{C}[[M']]_{\text{sum}}$ and the algebra map $\mathbb{Z}[M]_S \to \mathbb{C}[M']_{S'}$ are induced by the lattice injection $M \hookrightarrow M'$.

Example 4.1. If $\mathscr{F} = \mathscr{O}_X|_{U_\sigma}$, then

$$\chi_\sigma^{\mathbb{T}}(\mathscr{O}_X|_{U_\sigma}) = \sum_{m\in\sigma^\vee\cap M} \chi^{-m} \in \mathbb{Z}[M]_{\text{sum}}.$$

If, moreover, σ is a smooth cone with $m_{\sigma,i}$, $i = 1,\dots,n$, the minimal generators of $\sigma^\vee$, then

$$\chi_\sigma^{\mathbb{T}}(\mathscr{O}_X|_{U_\sigma}) = \prod_{i=1}^n (\sum_{k\geq 0} (\chi^{-m_{\sigma,i}})^k),$$

hence (as in [16][Lem.13.2.4])

$$\mathbb{S}(\chi_\sigma^{\mathbb{T}}(\mathscr{O}_X|_{U_\sigma})) = \prod_{i=1}^n \frac{1}{1-\chi^{-m_{\sigma,i}}}. \tag{146}$$

If σ is only a simplicial cone, with the same notation for the minimal generators of $\sigma^\vee$, we get for $\mathscr{F}' = \mathscr{O}_{U'_\sigma}$ that

$$tr_\sigma^{\mathbb{T}'}(\mathscr{O}_{U'_\sigma}) = \frac{1}{|G_\sigma|}\sum_{g\in G_\sigma}\prod_{i=1}^n(\sum_{k\geq 0}(a_{\rho_i}(g^{-1})\cdot\chi^{-m'_{\sigma,i}})^k),$$

hence this series is summable, and by applying diagram (145) to $\mathscr{F}' = \mathscr{O}_{U'_\sigma}$, with $\mathscr{O}_{U_\sigma} \simeq \pi_*^{G_\sigma}(\mathscr{O}_{U'_\sigma})$, we recover *Molien's formula* (see [13][page 24]):

$$\mathbb{S}(\chi_\sigma^{\mathbb{T}}(\mathscr{O}_X|_{U_\sigma})) = \frac{1}{|G_\sigma|}\sum_{g\in G_\sigma}\prod_{i=1}^n \frac{1}{1-a_{\rho_i}(g^{-1})\cdot\chi^{-m'_{\sigma,i}}} \in \mathbb{C}[M']_{S'}. \tag{147}$$

Recall here that, by our convention (following [16]), $\mathbb{T}$ acts on χ^m by $t\cdot\chi^m = \chi^m(t^{-1})\chi^m$. Note that [13] uses Oda's convention [39][pag.6]: $t\cdot\chi^m = \chi^m(t)\chi^m$, which explains the sign difference. □

Example 4.2. Consider the case of a simplicial cone $\sigma \in \Sigma(n)$ with minimal generators of $\sigma^\vee$ as before. We get for $\mathscr{F}' = \mathscr{O}_{U'_\sigma} \otimes \mathbb{C}_{\chi^{m'}}$ (with $m' \in M'$) that

$$tr_\sigma^{\mathbb{T}'}(\mathscr{O}_{U'_\sigma}\otimes\mathbb{C}_{\chi^{m'}}) = \frac{1}{|G_\sigma|}\sum_{g\in G_\sigma}\chi^{m'}(g)\cdot\chi^{m'}\prod_{i=1}^n(\sum_{k\geq 0}(a_{\rho_i}(g^{-1})\cdot\chi^{-m'_{\sigma,i}})^k),$$

hence this series is summable, and by applying diagram (145) to $\mathscr{F}' = \mathscr{O}_{U'_\sigma} \otimes \mathbb{C}_{\chi^{m'}}$, with $\mathscr{F} := \pi_*^{G_\sigma}(\mathscr{O}_{U'_\sigma} \otimes \mathbb{C}_{\chi^{m'}})$, we get

$$\mathbb{S}(\chi_\sigma^{\mathbb{T}}(\mathscr{F})) = \frac{1}{|G_\sigma|} \sum_{g \in G_\sigma} \chi^{m'}(g) \cdot \chi^{m'} \prod_{i=1}^{n} \frac{1}{1 - a_{\rho_i}(g^{-1}) \cdot \chi^{-m'_{\sigma,i}}} \in \mathbb{C}[M']_{S'}. \tag{148}$$

□

Example 4.3. Back to the general case, for $\sigma \in \Sigma(n)$, consider $\mathscr{F} = \mathscr{O}_X(D)|_{U_\sigma}$, with D a $\mathbb{T}$-invariant Cartier divisor on X. Then one gets

$$\chi_\sigma^{\mathbb{T}}(\mathscr{O}_X(D)|_{U_\sigma}) = \chi^{-m_\sigma} \sum_{m \in \sigma^\vee \cap M} \chi^{-m} \in \mathbb{Z}[M]_{\mathrm{sum}},$$

where $m_\sigma \in M$ is uniquely defined so that $D|_{U_\sigma} = div(\chi^{-m_\sigma})|_{U_\sigma}$. The sequence $\{m_\sigma \in M \mid \sigma \in \Sigma(n)\}$ is the Cartier data of D, in the sense of [16][Thm.4.2.8]. In other words, $\mathscr{O}_X(D)|_{U_\sigma} \simeq U_\sigma \times \mathbb{C}_{\chi^{-m_\sigma}}$ as $\mathbb{T}$-equivariant line bundles (e.g., see [16][page 609]), so that for any $\mathbb{T}$-equivariant coherent sheaf on X, we have

$$\chi_\sigma^{\mathbb{T}}((\mathscr{F} \otimes \mathscr{O}_X(D))|_{U_\sigma}) = \chi^{-m_\sigma} \cdot \chi_\sigma^{\mathbb{T}}(\mathscr{F}|_{U_\sigma}). \tag{149}$$

□

Example 4.4. Generalizing the case of $\mathscr{O}_X$, by [16][Prop.8.2.18], one has for any $0 \le p \le n = \dim(X)$,

$$\chi_\sigma^{\mathbb{T}}(\widehat{\Omega}_X^p|_{U_\sigma}) = \sum_{\tau \preceq \sigma} \binom{\dim(O_\tau)}{p} \sum_{m \in \mathrm{Relint}(\sigma^\vee \cap \tau^\perp) \cap M} \chi^{-m} \in \mathbb{Z}[M]_{\mathrm{sum}}.$$

Hence,

$$\chi_\sigma^{\mathbb{T}}(\omega_X|_{U_\sigma}) = \sum_{m \in \mathrm{Relint}(\sigma^\vee) \cap M} \chi^{-m} \in \mathbb{Z}[M]_{\mathrm{sum}}. \tag{150}$$

In particular, if X is quasi-projective, then

$$\chi_\sigma^{\mathbb{T}}(mC_y^{\mathbb{T}}(X)|_{U_\sigma}) = \sum_{\tau \preceq \sigma} (1+y)^{\dim(O_\tau)} \sum_{m \in \mathrm{Relint}(\sigma^\vee \cap \tau^\perp) \cap M} \chi^{-m} \in \mathbb{Z}[M]_{\mathrm{sum}} \otimes_{\mathbb{Z}} \mathbb{Z}[y].$$

If, moreover, σ is a smooth cone with $m_{\sigma,i}$, $i = 1, \ldots, n$, the minimal generators of $\sigma^\vee$, then

$$\chi_\sigma^{\mathbb{T}}(mC_y^{\mathbb{T}}(X)|_{U_\sigma}) = \prod_{i=1}^{n} \left(1 + (1+y) \cdot \sum_{k \ge 1} (\chi^{-m_{\sigma,i}})^k \right),$$

hence

$$\mathbb{S}(\chi_\sigma^{\mathbb{T}}(mC_y^{\mathbb{T}}(X)|_{U_\sigma})) = \prod_{i=1}^{n} \frac{1 + y \cdot \chi^{-m_{\sigma,i}}}{1 - \chi^{-m_{\sigma,i}}} \in \mathbb{Z}[M]_S. \tag{151}$$

Consider now the case of a simplicial cone σ with minimal generators of $\sigma^\vee$ as before, and let $\mathscr{F}' = \Omega^p_{U'_\sigma}$, $p \ge 0$, resp., $[\mathscr{F}'] = mC_y^{\mathbb{T}'}(U'_\sigma) = \sum_{p=0}^n [\Omega^p_{U'_\sigma}] \cdot y^p \in K_0^{\mathbb{T}'}(U'_\sigma)[y]$. Then

$$tr_\sigma^{\mathbb{T}'}(mC_y^{\mathbb{T}'}(U'_\sigma)) = \frac{1}{|G_\sigma|} \sum_{g \in G_\sigma} \prod_{i=1}^{n} \left(1 + (1+y) \cdot \sum_{k \ge 1} (a_{\rho_i}(g^{-1}) \cdot \chi^{-m'_{\sigma,i}})^k \right).$$

By applying diagram (145) to $\mathscr{F}' = \Omega^p_{U'_\sigma}$, with $\mathscr{F} = \pi_*^{G_\sigma}(\Omega^p_{U'_\sigma}) \simeq \widehat{\Omega}^p_X|_{U_\sigma}$, resp., $[\mathscr{F}'] = mC_y^{\mathbb{T}'}(U'_\sigma)$, with $[\mathscr{F}] = mC_y^{\mathbb{T}}(X)|_{U_\sigma}$, we get

$$\mathbb{S}(\chi_\sigma^{\mathbb{T}}(mC_y^{\mathbb{T}}(X)|_{U_\sigma})) = \frac{1}{|G_\sigma|}\sum_{g\in G_\sigma}\prod_{i=1}^n \frac{1+y\cdot a_{\rho_i}(g^{-1})\cdot \chi^{-m'_{\sigma,i}}}{1-a_{\rho_i}(g^{-1})\cdot \chi^{-m'_{\sigma,i}}} \in \mathbb{C}[M']_{S'}. \tag{152}$$

Finally, one can twist the sheaves $\widehat{\Omega}^p_X$ by $\mathscr{O}_X(D)$, for D a $\mathbb{T}$-invariant Cartier divisor, and use (149) to get the corresponding identities. □

Remark 4.5. If, more generally, we work with $X = X_{\Sigma'}$ a closed $\mathbb{T}$-invariant algebraic subset of X_Σ corresponding to a star-closed subset $\Sigma' \subset \Sigma$, similar localization formulas hold for the sheaves $\widetilde{\Omega}^p_X$ of Ishida p-forms of X (extended by zero to X_Σ). More precisely, using now [3][Sect.3], one gets for $0 \le p \le \dim(X)$

$$\chi_\sigma^{\mathbb{T}}(\widetilde{\Omega}^p_X|_{U_\sigma}) = \sum_{\tau\preceq\sigma,\tau\in\Sigma'}\binom{\dim(O_\tau)}{p}\sum_{m\in \mathrm{Relint}(\sigma^\vee\cap\tau^\perp)\cap M}\chi^{-m} \in \mathbb{Z}[M]_{\mathrm{sum}}. \tag{153}$$

In particular, if $X = V_\tau \subset X_\Sigma$ is the orbit closure for $\tau \preceq \sigma$ of dimension d, then $\widetilde{\Omega}^d_X = \omega_{V_\tau}$, so we get by (99) that

$$\chi_\sigma^{\mathbb{T}}(mC_0([O_\tau \hookrightarrow X_\Sigma])|_{U_\sigma}) = \chi_\sigma^{\mathbb{T}}(\omega_{V_\tau}|_{U_\sigma}) = \sum_{m\in \mathrm{Relint}(\sigma^\vee\cap\tau^\perp)\cap M}\chi^{-m} \in \mathbb{Z}[M]_{\mathrm{sum}}. \tag{154}$$

Example 4.6. Let $Z = X_{\Sigma'}$ be a closed $\mathbb{T}$-invariant algebraic subset of $X = X_\Sigma$ corresponding to a star-closed subset $\Sigma' \subset \Sigma$, with open complement $V := X \setminus Z$. Then

(155)
$$\chi_\sigma^{\mathbb{T}}(mC_y^{\mathbb{T}}([V \hookrightarrow X])|_{U_\sigma}) = \sum_{\tau\preceq\sigma;\tau\notin\Sigma'}(1+y)^{\dim(O_\tau)}\sum_{m\in \mathrm{Relint}(\sigma^\vee\cap\tau^\perp)\cap M}\chi^{-m} \in \mathbb{Z}[M]_{\mathrm{sum}}\otimes_{\mathbb{Z}}\mathbb{Z}[y].$$

We now consider the case when $Z = D_K := \bigcup_{\rho\in K} D_\rho$, for $K \subset \Sigma(1)$, and $\sigma \in \Sigma(n)$ is a smooth, resp., simplicial cone.

Let us first assume that σ is smooth, with $m_{\sigma,i}$, $i = 1,\dots,n$, the minimal generators of $\sigma^\vee$. Then

$$\begin{aligned}\chi_\sigma^{\mathbb{T}}&(mC_y^{\mathbb{T}}([V \hookrightarrow X])|_{U_\sigma})\\ &= \prod_{\rho_i\in\sigma(1)\cap K}\left((1+y)\cdot\sum_{k\ge1}(\chi^{-m_{\sigma,i}})^k\right)\cdot\prod_{\rho_i\in\sigma(1)\setminus K}\left(1+(1+y)\cdot\sum_{k\ge1}(\chi^{-m_{\sigma,i}})^k\right),\end{aligned}$$

hence

(156)
$$\mathbb{S}(\chi_\sigma^{\mathbb{T}}(mC_y^{\mathbb{T}}([V \hookrightarrow X])|_{U_\sigma})) = \prod_{\rho_i\in\sigma(1)\cap K}\frac{(1+y)\cdot\chi^{-m_{\sigma,i}}}{1-\chi^{-m_{\sigma,i}}}\prod_{\rho_i\in\sigma(1)\setminus K}\frac{1+y\cdot\chi^{-m_{\sigma,i}}}{1-\chi^{-m_{\sigma,i}}} \in \mathbb{Z}[M]_S.$$

Consider now the case of a simplicial cone σ with minimal generators of $\sigma^\vee$ as before, and let $V' = U'_\sigma \setminus D_{K'}$, with $D_{K'} = \bigcup_{\rho\in K'} D'_\rho$, for $K' := \sigma(1)\cap K$. Let $[\mathscr{F}'] = mC_y^{\mathbb{T}'}([V' \hookrightarrow U'_\sigma])$.

Then

$$tr_{\sigma}^{\mathbb{T}'}(mC_y^{\mathbb{T}'}([V \hookrightarrow U'_{\sigma}])) =$$
$$\frac{1}{|G_{\sigma}|}\sum_{g\in G_{\sigma}}\prod_{\rho_i\in K'}\left((1+y)\sum_{k\geq 1}(a_{\rho_i}(g^{-1})\chi^{-m'_{\sigma,i}})^k\right)\cdot\prod_{\rho_i\in\sigma(1)\setminus K'}\left(1+(1+y)\sum_{k\geq 1}(a_{\rho_i}(g^{-1})\chi^{-m'_{\sigma,i}})^k\right).$$

By applying (145) to $[\mathscr{F}'] = mC_y^{\mathbb{T}'}([V' \hookrightarrow U'_{\sigma}])$, with $[\mathscr{F}] = \pi_*^{G_{\sigma}}([\mathscr{F}']) = mC_y^{\mathbb{T}}([V \hookrightarrow X])|_{U_{\sigma}}$, we get

$$\mathbb{S}(\chi_{\sigma}^{\mathbb{T}}(mC_y^{\mathbb{T}}([V \hookrightarrow X])|_{U_{\sigma}}) = \frac{1}{|G_{\sigma}|}\sum_{g\in G_{\sigma}}\prod_{\rho_i\in\sigma(1)\cap K}\frac{(1+y)\cdot a_{\rho_i}(g^{-1})\cdot\chi^{-m'_{\sigma,i}}}{1-a_{\rho_i}(g^{-1})\cdot\chi^{-m'_{\sigma,i}}}\cdot\prod_{\rho_i\in\sigma(1)\setminus K}\frac{1+y\cdot a_{\rho_i}(g^{-1})\cdot\chi^{-m'_{\sigma,i}}}{1-a_{\rho_i}(g^{-1})\cdot\chi^{-m'_{\sigma,i}}}. \tag{157}$$

□

We now explain the following equality, originally due to Brion [9], where we follow the approach of [13][Cor.1.3] (see also [16][Thm.13.2.8] for a special case):

Theorem 4.7. *Let $X = X_{\Sigma}$ be a complete toric variety of complex dimension n, with $\mathscr{F}$ a $\mathbb{T}$-equivariant coherent sheaf on X. Then the K-theoretic Euler characteristic of $\mathscr{F}$ can be calculated via localization at the $\mathbb{T}$-fixed points as:*

$$\chi^{\mathbb{T}}(X,\mathscr{F}) = \sum_{\sigma\in\Sigma(n)}\mathbb{S}(\chi_{\sigma}^{\mathbb{T}}(\mathscr{F}|_{U_{\sigma}})) \in \mathbb{Z}[M]_S. \tag{158}$$

Proof. Let $x_{\sigma} = O_{\sigma} \subset U_{\sigma}$ be the torus fixed point corresponding to $\sigma \in \Sigma(n)$. The assertion follows from the commutativity of the lower right square of the following diagram:

$$\begin{array}{ccccc}
\bigoplus_{\sigma\in\Sigma(n)} K_0^{\mathbb{T}}(x_{\sigma}) & \xrightarrow{i_*} & K_0^{\mathbb{T}}(X) & \xrightarrow{\chi^{\mathbb{T}}} & \mathbb{Z}[M] \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{\sigma\in\Sigma(n)} K_0^{\mathbb{T}}(x_{\sigma})_S & \xrightarrow{\sim} & K_0^{\mathbb{T}}(X)_S & \xrightarrow{\chi^{\mathbb{T}}} & \mathbb{Z}[M]_S \\
\| & & \wr\downarrow & & \| \\
\bigoplus_{\sigma\in\Sigma(n)} K_0^{\mathbb{T}}(x_{\sigma})_S & \xrightarrow{\sim} & \bigoplus_{\sigma\in\Sigma(n)} K_0^{\mathbb{T}}(U_{\sigma})_S & \xrightarrow{\sum_{\sigma}(\mathbb{S}\circ\chi_{\sigma}^{\mathbb{T}})} & \mathbb{Z}[M]_S\,.
\end{array}$$

The vertical maps are the natural localization maps (upon restriction to the U_{σ}'s for the middle map). The commutativity of the outer square follows from (143). The lower left square commutes by the functoriality of restriction and localization. This yields the desired commutativity of the lower right square. □

As a consequence, we get the following weighted version of Brion's equality [9][page 655], see also [16][Cor.13.2.10(a)].

Corollary 4.8. *Let P be a full-dimensional lattice polytope with associated projective toric variety $X = X_P$ and ample Cartier divisor $D = D_P$. For each vertex v of P, consider the cone*

$C_v = \mathrm{Cone}(P \cap M - v) = \sigma_v^\vee$, *with faces* $E_v = \mathrm{Cone}(E \cap M - v)$ *for* $v \in E$. *Then the following identity holds in* $\mathbb{Z}[M]_S \otimes_{\mathbb{Z}} \mathbb{Z}[y]$*:*

$$\chi^{\mathbb{T}}(X, mC_y^{\mathbb{T}}(X) \otimes \mathscr{O}_X(D)) = \sum_{v \text{ vertex}} \chi^{-v} \cdot \mathbb{S}\left(\sum_{v \in E \preceq P} (1+y)^{\dim(E)} \cdot \sum_{m \in \mathrm{Relint}(E_v) \cap M} \chi^{-m} \right). \tag{159}$$

Proof. By the definition of the inner normal fan Σ_P of P, one has for any vertex v of P that $\sigma_v = C_v^\vee$, with C_v defined as above (see [16][page 76]). By dualizing, it follows that $C_v = \sigma_v^\vee$ (cf. [16][Prop.1.2.4]). By [16][Prop.1.2.10], the faces E_v of C_v are exactly of the form $\sigma_v^\vee \cap \tau^\perp$, with $\tau \preceq \sigma_v$. Note also that the Cartier data for $D = D_P$ over U_{σ_v} is exactly given by $m_{\sigma_v} = v \in M$, see [16][(4.2.8)].

The desired identity (159) follows now from Theorem 4.7 together with Example 4.4 and formula (149). □

Remark 4.9. A direct calculation of $\chi^{\mathbb{T}}(X, mC_y^{\mathbb{T}}(X) \otimes \mathscr{O}_X(D))$, without using localization techniques, can be obtained from [16][Lem.9.4.8 and Thm.9.3.1], as follows:

$$\chi^{\mathbb{T}}(X, mC_y^{\mathbb{T}}(X) \otimes \mathscr{O}_X(D)) = \sum_{E \preceq P} (1+y)^{\dim(E)} \cdot \sum_{m \in \mathrm{Relint}(E) \cap M} \chi^{-m} \in \mathbb{Z}[M] \otimes_{\mathbb{Z}} \mathbb{Z}[y]. \tag{160}$$

This is a K-theoretic version of Corollary 3.15. For $y = 0$, formula (160) reduces to

$$\chi^{\mathbb{T}}(X, \mathscr{O}_X(D)) = \sum_{m \in P \cap M} \chi^{-m} \in \mathbb{Z}[M],$$

which can already be deduced from (38) and (39).

For $y = 0$, formula (159) therefore reduces to Brion's equality:

$$\sum_{m \in P \cap M} \chi^{-m} = \chi^{\mathbb{T}}(X, \mathscr{O}_X(D)) = \sum_{v \text{ vertex}} \chi^{-v} \cdot \mathbb{S}\left(\sum_{m \in C_v \cap M} \chi^{-m} \right). \tag{161}$$

4.2. **Localization in equivariant homology.** Let X_Σ be an n-dimensional toric variety with torus $\mathbb{T} = T_N$. Let $X = X_{\Sigma'}$ be a $\mathbb{T}$-invariant closed algebraic subset of X_Σ, defined by a star-closed subset $\Sigma' \subset \Sigma$, such that the fixed-point set $X^{\mathbb{T}} \neq \emptyset$.

By [15][Lem.8.4, Lem.8.5], the inclusion $i : X^{\mathbb{T}} \hookrightarrow X$ induces an injective morphism of $H^*_{\mathbb{T}}(pt;\mathbb{Q})$-modules

$$i_* : \widehat{H}^{\mathbb{T}}_*(X^{\mathbb{T}};\mathbb{Q}) \hookrightarrow \widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})$$

which becomes an isomorphism

$$i_* : \widehat{H}^{\mathbb{T}}_*(X^{\mathbb{T}};\mathbb{Q})_L \simeq \widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})_L$$

upon localization at the multiplicative subset $L \subset (\Lambda_{\mathbb{T}})_{\mathbb{Q}} = H^*_{\mathbb{T}}(pt;\mathbb{Q})$ generated by the elements $\pm c(m)$, for $0 \neq m \in M$ (cf. [15][Cor.8.9]).

Remark 4.10. In particular, by applying the localization isomorphism to both X and X_Σ, it follows that the homomorphism $\widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})_L \to \widehat{H}^{\mathbb{T}}_*(X_\Sigma;\mathbb{Q})_L$ induced by inclusion is injective. So in the following it suffices to work in the localized homology $\widehat{H}^{\mathbb{T}}_*(X_\Sigma;\mathbb{Q})_L$ of the ambient toric variety.

Let us now assume that $X = X_\Sigma$. Since

$$\widehat{H}_*^{\mathbb{T}}(X^{\mathbb{T}};\mathbb{Q}) = \bigoplus_{x\in X^{\mathbb{T}}} \widehat{H}_*^{\mathbb{T}}(x;\mathbb{Q}),$$

one gets via the isomorphism i_* a projection map of $H^*_{\mathbb{T}}(pt;\mathbb{Q})$-modules, called here the *homological localization map at x*,

$$pr_x : \widehat{H}_*^{\mathbb{T}}(X;\mathbb{Q})_L \simeq \bigoplus_{x\in X^{\mathbb{T}}} \widehat{H}_*^{\mathbb{T}}(x;\mathbb{Q})_L \longrightarrow \widehat{H}_*^{\mathbb{T}}(x;\mathbb{Q})_L = L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}.$$

Remark 4.11. If X is a toric variety with $H_*^{\mathbb{T}}(X;\mathbb{Q})$ a finitely generated free $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}$-module (e.g., X is a complete simplicial toric variety, see [16][Prop.12.4.7]), then the localization map $\widehat{H}_*^{\mathbb{T}}(X;\mathbb{Q}) \to \widehat{H}_*^{\mathbb{T}}(X;\mathbb{Q})_L$ is injective.

The equivariant Todd class transformation $\mathrm{td}_*^{\mathbb{T}}$ commutes with pushforward for closed inclusions i_* and pullback j^* under open inclusions. So $\mathrm{td}_*^{\mathbb{T}}$ is compatible with the two diagrams below:

$$\begin{array}{ccccccc} K_0^{\mathbb{T}}(X^{\mathbb{T}}) & \xrightarrow{i_*} & K_0^{\mathbb{T}}(X) & & \widehat{H}_{2*}^{\mathbb{T}}(X^{\mathbb{T}};\mathbb{Q}) & \xrightarrow{i_*} & \widehat{H}_{2*}^{\mathbb{T}}(X;\mathbb{Q}) \\ \downarrow & & \downarrow j^* & \xrightarrow{\mathrm{td}_*^{\mathbb{T}}} & \downarrow & & \downarrow j^* \\ K_0^{\mathbb{T}}(x_\sigma) & \xrightarrow{i_*} & K_0^{\mathbb{T}}(U_\sigma) & & \widehat{H}_{2*}^{\mathbb{T}}(x_\sigma;\mathbb{Q}) & \xrightarrow{i_*} & \widehat{H}_{2*}^{\mathbb{T}}(U_\sigma;\mathbb{Q}) \end{array} \tag{162}$$

To extend this property to the localized versions, we need to show that the ring homomorphism $\mathrm{ch}^{\mathbb{T}} : \mathbb{Z}[M] \simeq K^0_{\mathbb{T}}(pt) \to (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$ given by the equivariant Chern character is compatible with the multiplicative subsets used for localization. This fact follows as in [16][pag.644]; indeed, if $0 \neq m \in M$, then

$$\mathrm{ch}^{\mathbb{T}}(1-\chi^m) = 1 - e^{c(m)} = c(m)\frac{1-e^{c(m)}}{c(m)} \in (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}\,,$$

with $c(m) \in L$ and the *convergent* power series $\frac{1-e^{c(m)}}{c(m)} \in (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$ being invertible (since its constant coefficient is a unit). So $\mathrm{ch}^{\mathbb{T}}$ induces a ring homomorphism of localized rings

$$\mathrm{ch}^{\mathbb{T}} : \mathbb{Z}[M]_S \simeq K^0_{\mathbb{T}}(pt)_S \longrightarrow L^{-1}(\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}\,.$$

In particular, this localized version of the equivariant Chern character $\mathrm{ch}^{\mathbb{T}}$ can be applied to the image of the summation map $\mathbb{S} : \mathbb{Z}[[M]]_{\mathrm{Sum}} \to \mathbb{Z}[M]_S$, where by abuse of notation we denote

$$(\mathrm{ch}^{\mathbb{T}} \circ \mathbb{S})\left(\sum_{m\in M} a_m \cdot \chi^m\right) =: \mathbb{S}\left(\sum_{m\in M} a_m \cdot e^{c(m)}\right),$$

extending the corresponding formula $\mathrm{ch}^{\mathbb{T}}(\sum_{m\in M} a_m \cdot \chi^m) = \sum_{m\in M} a_m \cdot e^{c(m)}$ for finite sums in $\mathbb{Z}[M] \subset \mathbb{Z}[[M]]_{\mathrm{Sum}}$.

Remark 4.12. In the context of diagram (145), we have to also consider the extension of the Chern character to the complexified K-theory

$$\mathrm{ch}^{\mathbb{T}'} : \mathbb{C}[M'] \simeq K^0_{\mathbb{T}'}(pt) \otimes_{\mathbb{Z}} \mathbb{C} \to (\Lambda^{an}_{\mathbb{T}'})_{\mathbb{C}} \subset (\widehat{\Lambda}_{\mathbb{T}'})_{\mathbb{C}},$$

which is compatible with localization

$$\mathrm{ch}^{\mathbb{T}'}: \mathbb{C}[M']_{S'} \simeq K^0_{\mathbb{T}'}(pt)_{S'} \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow L'^{-1}(\Lambda^{an}_{\mathbb{T}'})_{\mathbb{C}} \subset L'^{-1}(\widehat{\Lambda}_{\mathbb{T}'})_{\mathbb{C}}\,.$$

Here, $S' \subset \mathbb{C}[M']$ is as before the multiplicative subset generated by elements $1 - a \cdot \chi^{m'}$, for $0 \neq m' \in M'$ and $a \in \mathbb{C}^*$, and $L' \subset (\Lambda_{\mathbb{T}'})_{\mathbb{C}} = H^*_{\mathbb{T}'}(pt;\mathbb{C})$ is the multiplicative set generated by the elements $\pm a \cdot c(m')$, for $0 \neq m' \in M'$ and $a \in \mathbb{C}^*$. The compatibility with localization follows as above, using in addition that

$$\mathrm{ch}^{\mathbb{T}'}(1 - a \cdot \chi^{m'}) = 1 - a \cdot e^{c(m')} \in (\Lambda^{an}_{\mathbb{T}'})_{\mathbb{C}} \subset (\widehat{\Lambda}_{\mathbb{T}'})_{\mathbb{C}}\,,$$

for $a \neq 1$ and $0 \neq m' \in M'$, is an invertible convergent power series since its constant coefficient $1-a$ is a unit. These two Chern characters $\mathrm{ch}^{\mathbb{T}'}$ and $\mathrm{ch}^{\mathbb{T}}$ fit into a commutative diagram

$$\begin{array}{ccccc} \mathbb{C}[M']_{S'} & \xrightarrow{\mathrm{ch}^{\mathbb{T}'}} & L'^{-1}(\Lambda^{an}_{\mathbb{T}'})_{\mathbb{C}} & \longrightarrow & L'^{-1}(\widehat{\Lambda}_{\mathbb{T}'})_{\mathbb{C}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{Z}[M]_S & \xrightarrow{\mathrm{ch}^{\mathbb{T}}} & L^{-1}(\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} & \longrightarrow & L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}\,. \end{array} \tag{163}$$

The last two vertical arrows are injections (between convergent, resp., formal Laurent series with rational or complex coefficients).

By the module property (81) of the equivariant Todd class transformation and the functoriality of $\mathrm{ch}^{\mathbb{T}}$ under pullbacks, it follows that $\mathrm{td}^{\mathbb{T}}_*$ is compatible with the corresponding localized versions of the diagrams appearing in (162), with the corresponding maps labeled i_* being now isomorphisms, as used in the definitions of the localization maps pr_x in K-theory and homology, respectively.

Proposition 4.13. *For a $\mathbb{T}$-equivariant coherent sheaf $\mathscr{F}$ on $X = X_\Sigma$, and $x_\sigma \in X^{\mathbb{T}}$ a given fixed point of the $\mathbb{T}$-action, we have:*

$$\mathrm{td}^{\mathbb{T}}_*([\mathscr{F}])_{x_\sigma} := pr_{x_\sigma}(\mathrm{td}^{\mathbb{T}}_*([\mathscr{F}]) = \mathrm{ch}^{\mathbb{T}}((\mathbb{S} \circ \chi^{\mathbb{T}}_\sigma)(\mathscr{F})) \in L^{-1}(\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}} \tag{164}$$

and

$$\begin{aligned} L^{-1}\mathrm{td}^{\mathbb{T}}_*([\mathscr{F}]) &= \bigoplus_{\sigma \in \Sigma(n)} \mathrm{td}^{\mathbb{T}}_*([\mathscr{F}])_{x_\sigma} \cdot [x_\sigma]_{\mathbb{T}} \\ &= \bigoplus_{\sigma \in \Sigma(n)} \mathrm{ch}^{\mathbb{T}}((\mathbb{S} \circ \chi^{\mathbb{T}}_\sigma)(\mathscr{F})) \cdot [x_\sigma]_{\mathbb{T}} \in \widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})_L. \end{aligned} \tag{165}$$

Proof. Indeed, using the explicit calculation of the localization map in equivariant K-theory (as in the previous section), we have:

$$pr_{x_\sigma}(\mathrm{td}^{\mathbb{T}}_*([\mathscr{F}]) = \mathrm{td}^{\mathbb{T}}_*(pr_{x_\sigma}([\mathscr{F}])) = \mathrm{ch}^{\mathbb{T}}((\mathbb{S} \circ \chi^{\mathbb{T}}_\sigma)(\mathscr{F})).$$

The second formula just follows from the isomorphism $i_*: \widehat{H}^{\mathbb{T}}_*(X^{\mathbb{T}};\mathbb{Q})_L \simeq \widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})_L$. □

Example 4.14 (Localized equivariant Hirzebruch and Todd classes). Assume now that X_Σ is a quasi-projective toric variety. Let, moreover, $X := X_{\Sigma'}$ be the $\mathbb{T}$-invariant closed algebraic subset defined by a star-closed subset $\Sigma' \subset \Sigma$, with $x_\sigma \in X^{\mathbb{T}}$ a given fixed point of the $\mathbb{T}$-action.

By using (164) and (153), the following identity holds in $L^{-1}(\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathbb{Z}[y] \subset L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathbb{Z}[y]$:

(166)

$$\begin{aligned} T^{\mathbb{T}}_{y*}(X)_{x_\sigma} &:= \mathrm{td}^{\mathbb{T}}_*([mC^{\mathbb{T}}_y(X)])_{x_\sigma} = \sum_{\tau\preceq\sigma,\tau\in\Sigma'} (1+y)^{\dim(O_\tau)} \cdot (\mathrm{ch}^{\mathbb{T}} \circ \mathbb{S}) \left(\sum_{m\in \mathrm{Relint}(\sigma^\vee\cap\tau^\perp)\cap M} \chi^{-m} \right) \\ &= \sum_{\tau\preceq\sigma,\tau\in\Sigma'} (1+y)^{\dim(O_\tau)} \cdot \mathbb{S} \left(\sum_{m\in \mathrm{Relint}(\sigma^\vee\cap\tau^\perp)\cap M} e^{s(m)} \right). \end{aligned}$$

We thus have in $\widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})_L \otimes_{\mathbb{Z}} \mathbb{Z}[y]$ (compare also with [42][Thm.5.3], [44][Thm.11.3]):

$$L^{-1}T^{\mathbb{T}}_{y*}(X) = \bigoplus_{\sigma\in\Sigma'(n)} \left(\sum_{\tau\preceq\sigma,\tau\in\Sigma'} (1+y)^{\dim(O_\tau)} \cdot \mathbb{S} \left(\sum_{m\in \mathrm{Relint}(\sigma^\vee\cap\tau^\perp)\cap M} e^{s(m)} \right) \right) \cdot [x_\sigma]_{\mathbb{T}}. \tag{167}$$

Specializing the above formulae to $y = 0$ for X quasi-projective, or applying (164) to the structure sheaf $\mathscr{F} = \mathscr{O}_X$ (extended by zero to the ambient toric variety, which now can be arbitrary) we get the following result:

$$\mathrm{td}^{\mathbb{T}}_*(X)_{x_\sigma} = \sum_{\tau\preceq\sigma,\tau\in\Sigma'} \mathbb{S} \left(\sum_{m\in \mathrm{Relint}(\sigma^\vee\cap\tau^\perp)\cap M} e^{s(m)} \right) \in L^{-1}(\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}, \tag{168}$$

and

$$L^{-1}\mathrm{td}^{\mathbb{T}}_*(X) = \bigoplus_{\sigma\in\Sigma'(n)} \left(\sum_{\tau\preceq\sigma,\tau\in\Sigma'} \mathbb{S} \left(\sum_{m\in \mathrm{Relint}(\sigma^\vee\cap\tau^\perp)\cap M} e^{s(m)} \right) \right) \cdot [x_\sigma]_{\mathbb{T}} \in \widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})_L. \tag{169}$$

In the case $\Sigma' = \Sigma$, formulas (168) and (169) reduce to a result of Brylinski-Zhang [15][Thm.9.4]:

$$\mathrm{td}^{\mathbb{T}}_*(X_\Sigma)_{x_\sigma} = \mathbb{S} \left(\sum_{m\in\sigma^\vee\cap M} e^{s(m)} \right) \in L^{-1}(\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}, \tag{170}$$

and

$$L^{-1}\mathrm{td}^{\mathbb{T}}_*(X_\Sigma) = \bigoplus_{\sigma\in\Sigma(n)} \mathbb{S} \left(\sum_{m\in\sigma^\vee\cap M} e^{s(m)} \right) \cdot [x_\sigma]_{\mathbb{T}} \in \widehat{H}^{\mathbb{T}}_*(X_\Sigma;\mathbb{Q})_L. \tag{171}$$

□

Example 4.15 (Localized equivariant Hirzebruch classes for complements of divisors). Let $X = X_\Sigma$ be a toric variety with $V := X \setminus D_K$, and $D_K = \bigcup_{\rho\in K} D_\rho$ for $K \subset \Sigma(1)$. Let $\sigma \in \Sigma(n)$ be a simplicial cone with $x_\sigma \in X^{\mathbb{T}}$ the corresponding $\mathbb{T}$-fixed point. Then we get by (157) and

Proposition 4.13 the following identity in $L'^{-1}(\Lambda^{an}_{\mathbb{T}'})_{\mathbb{C}}[y] \subset L'^{-1}(\widehat{\Lambda}_{\mathbb{T}'})_{\mathbb{C}}[y]$:

$$\begin{aligned} T^{\mathbb{T}}_{y*}([V \hookrightarrow X])_{x_\sigma} &:= \mathrm{td}^{\mathbb{T}}_*([mC^{\mathbb{T}}_y([V \hookrightarrow X])])_{x_\sigma} \\ &= \frac{1}{|G_\sigma|} \sum_{g \in G_\sigma} \prod_{\rho_i \in \sigma(1) \cap K} \frac{(1+y) \cdot a_{\rho_i}(g^{-1}) \cdot e^{-c(m'_{\sigma,i})}}{1 - a_{\rho_i}(g^{-1}) \cdot e^{-c(m'_{\sigma,i})}} \cdot \prod_{\rho_i \in \sigma(1) \setminus K} \frac{1 + y \cdot a_{\rho_i}(g^{-1}) \cdot e^{-c(m'_{\sigma,i})}}{1 - a_{\rho_i}(g^{-1}) \cdot e^{-c(m'_{\sigma,i})}}. \end{aligned} \tag{172}$$

In particular, if σ is a smooth cone, then

$$T^{\mathbb{T}}_{y*}([V \hookrightarrow X])_{x_\sigma} = \prod_{\rho_i \in \sigma(1) \cap K} \frac{(1+y) \cdot e^{-c(m_{\sigma,i})}}{1 - e^{-c(m_{\sigma,i})}} \cdot \prod_{\rho_i \in \sigma(1) \setminus K} \frac{1 + y \cdot e^{-c(m_{\sigma,i})}}{1 - e^{-c(m_{\sigma,i})}},$$

with $m'_{\sigma,i} = m_{\sigma,i}$ for all i (since σ is a smooth cone).

By specializing (172) to $y = 0$, we get a local version of the Todd type formula (139). In particular, for $y = 0$ and σ smooth, we get:

$$T^{\mathbb{T}}_{0*}([V \hookrightarrow X])_{x_\sigma} = \prod_{\rho_i \in \sigma(1) \cap K} \frac{e^{-c(m_{\sigma,i})}}{1 - e^{-c(m_{\sigma,i})}} \cdot \prod_{\rho_i \in \sigma(1) \setminus K} \frac{1}{1 - e^{-c(m_{\sigma,i})}}. \tag{173}$$

In the case when $K = \emptyset$, we thus obtain a more explicit expression for the equivariant Hirzebruch and Todd classes of X localized at x_σ. □

As a consequence of Proposition 4.13 and Theorem 4.7, we also have the following.

Corollary 4.16. *Let $X = X_\Sigma$ be a complete toric variety of complex dimension n, with $\mathscr{F}$ a $\mathbb{T}$-coherent sheaf on X. Then the cohomological Euler characteristic of $\mathscr{F}$ can be calculated via localization at the $\mathbb{T}$-fixed points as:*

$$\begin{aligned} \chi^{\mathbb{T}}(X, \mathscr{F}) &= \sum_{\sigma \in \Sigma(n)} (\mathrm{ch}^{\mathbb{T}} \circ \mathbb{S})(\chi^{\mathbb{T}}_\sigma(\mathscr{F}|_{U_\sigma})) \\ &= \sum_{\sigma \in \Sigma(n)} \mathrm{td}^{\mathbb{T}}_*([\mathscr{F}])_{x_\sigma} \in L^{-1}(\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} \subset L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}. \end{aligned} \tag{174}$$

Before giving a more classical description of the homological localization map pr_x in the context of simplicial toric varieties, let us formulate the following compatibility of $pr_x : \widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})_L \to \widehat{H}^{\mathbb{T}}_*(x;\mathbb{Q})_L$ with cap products. As before, $X = X_\Sigma$ is a toric variety, with $x \in X^{\mathbb{T}}$, and let $i_x : \{x\} \hookrightarrow X$ be the inclusion map. Then, for any $a \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})_L$ and $b \in \widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})_L$, we have:

$$pr_x(a \cap b) = i^*_x(a) \cap pr_x(b). \tag{175}$$

This follows from the definition of pr_x, since the cap product $\cap$ commutes with pullback for open inclusions, together with the projection formula for a closed inclusion.

For the rest of this section we assume that $X = X_\Sigma$ is a *simplicial* toric variety, so that one has equivariant Poincaré duality with $\mathbb{Q}$-coefficients (recall that in the non-compact case, $H^{\mathbb{T}}_*$ denotes equivariant Borel-Moore homology):

$$\cap[X]_{\mathbb{T}} : \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})_L \xrightarrow{\sim} \widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q})_L.$$

For this reason, in the following we use interchangeably homology and cohomology, with a cap product in homology corresponding to multiplication in cohomology.

Proposition 4.17. *Let $X = X_\Sigma$ be a simplicial toric variety of dimension n, with $x_\sigma \in X^{\mathbb{T}}$ a torus fixed point and inclusion map $i_\sigma : \{x_\sigma\} \hookrightarrow X$. Then*

$$pr_{x_\sigma} = \frac{i_\sigma^*}{Eu_X^{\mathbb{T}}(x_\sigma)} : \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})_L \to \widehat{H}_{\mathbb{T}}^*(x_\sigma;\mathbb{Q})_L, \tag{176}$$

with the generalized Euler class of the fixed point x_σ in X defined by

$$0 \neq Eu_X^{\mathbb{T}}(x_\sigma) := i_\sigma^* \left(\mathrm{mult}(\sigma) \cdot \prod_{\rho \in \sigma(1)} [D_\rho]_{\mathbb{T}} \right) \in \mathbb{Q} \cdot L.$$

If X is, moreover, a complete simplicial toric variety, let $\int_X : \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})_L \to \widehat{H}_{\mathbb{T}}^*(pt;\mathbb{Q})_L = L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$ be the equivariant Gysin map (or, equivalently, the equivariant pushforward) for the constant map $X \to pt$. Then

$$\int_X = \sum_{\sigma \in \Sigma(n)} \frac{i_\sigma^*}{Eu_X^{\mathbb{T}}(x_\sigma)} : \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})_L \to \widehat{H}_{\mathbb{T}}^*(pt;\mathbb{Q})_L. \tag{177}$$

Proof. For $\sigma \in \Sigma(n)$ a cone of maximal dimension, formula (62) written in homological terms becomes:

$$\left(\mathrm{mult}(\sigma) \cdot \prod_{\rho \in \sigma(1)} [D_\rho]_{\mathbb{T}} \right) \cap [X]_{\mathbb{T}} = (i_\sigma)_* [x_\sigma]_{\mathbb{T}}.$$

Using (175), we get

$$i_\sigma^* \left(\mathrm{mult}(\sigma) \cdot \prod_{\rho \in \sigma(1)} [D_\rho]_{\mathbb{T}} \right) \cap pr_{x_\sigma}([X]_{\mathbb{T}}) = pr_{x_\sigma}((i_\sigma)_*[x_\sigma]_{\mathbb{T}}) = [x_\sigma]_{\mathbb{T}},$$

where the last equality follows from the fact that the composition $pr_{x_\sigma} \circ (i_\sigma)_*$ is by definition just the identity. The factor $i_\sigma^* \left(\mathrm{mult}(\sigma) \cdot \prod_{\rho \in \sigma(1)} [D_\rho]_{\mathbb{T}}\right)$ is by definition the generalized Euler class $0 \neq Eu_X^{\mathbb{T}}(x_\sigma)$ (this is non-zero by the formula above, since $[x_\sigma]_{\mathbb{T}} \neq 0$), and it is an element in $\mathbb{Q} \cdot L$ (see [16][Lem..13.3.5]) since any divisor on a simplicial toric variety is $\mathbb{Q}$-Cartier ([16][Prop.4.2.7]). So formula (176) follows now from the projection formula (175), i.e., for $a \in \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})_L$, we have:

$$pr_{x_\sigma}(a \cap [X]_{\mathbb{T}}) = i_\sigma^*(a) \cap \frac{[x_\sigma]_{\mathbb{T}}}{Eu_X^{\mathbb{T}}(x_\sigma)}.$$

For the second assertion, note that the map $\int_X : \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})_L \to \widehat{H}_{\mathbb{T}}^*(pt;\mathbb{Q})_L = L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$ corresponds (by the functoriality of pushforward) under the localization isomorphism

$$\widehat{H}_*^{\mathbb{T}}(X;\mathbb{Q})_L \simeq \widehat{H}_*^{\mathbb{T}}(X^{\mathbb{T}};\mathbb{Q})_L = \bigoplus_{x \in X^{\mathbb{T}}} \widehat{H}_*^{\mathbb{T}}(x;\mathbb{Q})_L$$

to the sum of localization maps pr_x, $x \in X^{\mathbb{T}}$, i.e.,

$$\int_X = \sum_{\sigma \in \Sigma(n)} pr_{x_\sigma} : \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})_L \to \widehat{H}_{\mathbb{T}}^*(pt;\mathbb{Q})_L.$$

□

Remark 4.18. If the fixed point x_σ corresponds to a smooth cone $\sigma \in \Sigma$, then $\mathrm{mult}(\sigma) = 1$ and the divisors D_ρ ($\rho \in \sigma(1)$) are Cartier divisors. Moreover, the normal bundle $N_{x_\sigma}U_\sigma$ of x_σ in the smooth affine variety U_σ is given by

$$N_{x_\sigma}U_\sigma = \bigoplus_{\rho\in\sigma(1)} \mathscr{O}(D_\rho)|_{x_\sigma}.$$

Hence $Eu_X^{\mathbb{T}}(x_\sigma) = c_n^{\mathbb{T}}(N_{x_\sigma}U_\sigma)$ is the classical equivariant Euler class of the fixed point x_σ, given by the top equivariant Chern class of the normal bundle $N_{x_\sigma}U_\sigma$.

Translating Theorem 4.7 into the (co)homological context, using Proposition 4.13 and, in the simplicial context, Proposition 4.17, we get the following consequence.

Corollary 4.19. *Let $X = X_\Sigma$ be a complete toric variety, D a $\mathbb{T}$-invariant Cartier divisor, with $\mathscr{F}$ a $\mathbb{T}$-equivariant coherent sheaf on X. Then the cohomological equivariant Euler characteristic of $\mathscr{F}$ is computed by*

$$\chi^{\mathbb{T}}(X, \mathscr{O}_X(D)\otimes\mathscr{F}) = \sum_{\sigma\in\Sigma(n)} pr_{x_\sigma}(\mathrm{td}_*^{\mathbb{T}}([\mathscr{O}_X(D)\otimes\mathscr{F}]) \in L^{-1}(\Lambda_{\mathbb{T}}^{an})_{\mathbb{Q}}. \tag{178}$$

If, moreover, X is a simplicial toric variety, then

$$\chi^{\mathbb{T}}(X, \mathscr{O}_X(D)\otimes\mathscr{F}) = \sum_{\sigma\in\Sigma(n)} \frac{i_\sigma^*(e^{[D]_{\mathbb{T}}}\mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]))}{Eu_X^{\mathbb{T}}(x_\sigma)} \in L^{-1}(\Lambda_{\mathbb{T}}^{an})_{\mathbb{Q}}. \tag{179}$$

4.3. **Equivariant Hirzebruch classes of simplicial toric varieties via localization.** In this section, we explain how to reprove Theorems 3.22, 3.28 and Lemma 3.29 by localization techniques, instead of using the equivariant Lefschetz-Riemann-Roch theorem of [24][Thm.3.1 and Rem.4.3] (as done in Section 3.3).

Let $X := X_\Sigma$ be a simplicial quasi-projective toric variety associated to a fan Σ of full-dimensional convex support. Then the equivariant cohomology of X is a free $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}$-module (by (63)), so that the injectivity part of the sequence (65), resp., (66), still holds for the equivariant cohomology (as pointed out in Remark 2.8). Hence the corresponding map for the completed equivariant cohomology rings is injective, i.e.,

$$\bigoplus_{\sigma\in\Sigma(n)} i_\sigma^* : \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q}) \hookrightarrow \bigoplus_{\sigma\in\Sigma(n)} \widehat{H}_{\mathbb{T}}^*(x_\sigma;\mathbb{Q}).$$

By localizing at the multiplicative set L, and using the exactness of localization, we get an injective map

$$\bigoplus_{\sigma\in\Sigma(n)} i_\sigma^* : \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})_L \hookrightarrow \bigoplus_{\sigma\in\Sigma(n)} \widehat{H}_{\mathbb{T}}^*(x_\sigma;\mathbb{Q})_L.$$

So it is enough to check these characteristic class formulae by using for each fixed point x_σ, $\sigma \in \Sigma(n)$, the induced restriction map pr_{x_σ}

$$\widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q}) \to \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})_L \stackrel{pr_{x_\sigma}}{\to} \widehat{H}_{\mathbb{T}}^*(x_\sigma;\mathbb{Q})_L \to \widehat{H}_{\mathbb{T}'}^*(x_\sigma;\mathbb{C})_{L'}, \tag{180}$$

with the middle arrow pr_{x_σ} as in (176). Even the direct sum $\bigoplus_{\sigma\in\Sigma(n)} pr_{x_\sigma}$ of these induced restriction maps is still injective, since the localization map on the integral domain

$\widehat{H}^*_{\mathbb{T}}(x_\sigma;\mathbb{Q}) \to \widehat{H}^*_{\mathbb{T}}(x_\sigma;\mathbb{Q})_L$ is injective, and $pr_{x_\sigma} = \frac{i^*_\sigma}{Eu^{\mathbb{T}}_X(x_\sigma)}$ differs from i^*_σ by the unit

$$Eu^{\mathbb{T}}_X(x_\sigma) = i^*_\sigma \left(\mathrm{mult}(\sigma) \cdot \prod_{\rho\in\sigma(1)} [D_\rho]_{\mathbb{T}} \right) = |G_\sigma| \prod_{\rho\in\sigma(1)} i^*_\sigma F_\rho \in \widehat{H}^*_{\mathbb{T}}(x_\sigma;\mathbb{Q})_L.$$

Moreover, as mentioned after diagram (163), no information is lost if we consider complex instead of rational coefficients.

4.3.1. *Proof of Lemma 3.29 by localization.* In order to reprove formula (138) by localization, we start with the local version discussed in the following example.

Example 4.20. Let $\sigma \in \Sigma(n)$ be a simplicial cone with minimal generators $\{m_{\sigma,i}\}_i$ of $\sigma^\vee$ as in the context of diagram (145). For $\mathscr{F} := \pi^{G_\sigma}_*(\mathscr{O}_{U'_\sigma} \otimes \mathbb{C}_{\chi^{m'}})$ with $m' \in M'$, we get by Proposition 4.13 and formula (148) the identity

$$\mathrm{td}^{\mathbb{T}}_*([\mathscr{F}])_{x_\sigma} = \frac{1}{|G_\sigma|} \sum_{g\in G_\sigma} \chi^{m'}(g) e^{c(m')} \prod_{i=1}^{n} \frac{1}{1-a_{\rho_i}(g^{-1})\cdot e^{-c(m'_{\sigma,i})}} \in \widehat{H}^*_{\mathbb{T}'}(x_\sigma;\mathbb{C})_{L'}. \tag{181}$$

□

Using Example 4.20, we can now reprove formula (138) of Lemma 3.29 by localization techniques. Recall that the global formula

$$\mathrm{td}^{\mathbb{T}}_*(\pi_*(\mathscr{O}_W \otimes \mathbb{C}_{\chi^{\widetilde{m}}})^G) = \sum_{g\in G_\Sigma} \chi^{\widetilde{m}}(g^{-1}) \prod_{\rho\in\Sigma(1)} \frac{F_\rho \cdot e^{\langle \widetilde{m}, e_\rho\rangle \cdot F_\rho}}{1-a_\rho(g)e^{-F_\rho}} \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q}), \tag{182}$$

with $\{e_\rho\}_{\rho\in\Sigma(1)}$ the standard basis of the lattice $\widetilde{N} = \mathbb{Z}^{|\Sigma(1)|}$ and $\widetilde{m}$ in the dual character lattice $\widetilde{M}$, is formulated in terms of the Cox construction $\pi : W \to X$. So let us now compare, for a fixed $\sigma \in \Sigma(n)$ with U_σ the $\mathbb{T}$-invariant open affine subset of X containing the corresponding fixed point x_σ, the local and global formulae (181) and, resp., (182).

To simplify the notation, assume $\sigma(1) = \{\rho_1,\dots,\rho_n\}$. We next adapt the arguments of [13][page 24] to our context. If $g \in G_\Sigma \setminus G_\sigma$, by (31) there exists a $\rho \in \Sigma(1)$ with $a_\rho(g) \neq 1$, so that the restriction of $\frac{F_\rho \cdot e^{\langle \widetilde{m}, e_\rho\rangle \cdot F_\rho}}{1-a_\rho(g)e^{-F_\rho}}$ to U_σ becomes 0. Thus, the summation on the right-hand side of (182) reduces after restriction to U_σ to a summation over $g \in G_\sigma$. If $g \in G_\sigma$ and $\rho \notin \sigma(1)$, then $a_\rho(g) = 1$, so that the restriction of $\frac{F_\rho \cdot e^{\langle \widetilde{m}, e_\rho\rangle \cdot F_\rho}}{1-a_\rho(g)e^{-F_\rho}}$ to U_σ becomes 1. So the product on the right hand side of (182) reduces to a product over $\rho \in \sigma(1)$. As in the proof of formula (125), we have $\pi^{-1}(U_\sigma) = U'_\sigma \times (\mathbb{C}^*)^{r-n}$, with $\widetilde{\mathbb{T}} \simeq \mathbb{T}' \times (\mathbb{C}^*)^{r-n}$, $G \simeq G_\sigma \times (\mathbb{C}^*)^{r-n}$, and $U'_\sigma/G_\sigma = U_\sigma$. Let $\widetilde{m} = m' + m'' \in M' \oplus M''$ be the corresponding character decomposition. Note that $\mathbb{C}_{\chi^{m''}}$ is a trivial $(\mathbb{C}^*)^{r-n}$-equivariant line bundle on $(\mathbb{C}^*)^{r-n}$, so we can assume that $m'' = 0$ is the trivial character. By the projection formula, the restriction of $\pi_*(\mathscr{O}_W \otimes \mathbb{C}_{\chi^{\widetilde{m}}})^G$ to U_σ is isomorphic to the $\mathscr{F}$ appearing in (181). Finally,

$$
\begin{aligned}
&pr_{x_\sigma}\left(\sum_{g\in G_\Sigma}\chi^{\widetilde{m}}(g^{-1})\prod_{\rho\in\Sigma(1)}\frac{F_\rho\cdot e^{\langle\widetilde{m},e_\rho\rangle\cdot F_\rho}}{1-a_\rho(g)e^{-F_\rho}}\right)\\
&=\frac{1}{Eu_X^{\mathbb{T}}(x_\sigma)}\cdot i_\sigma^*\left(\sum_{g\in G_\Sigma}\chi^{\widetilde{m}}(g^{-1})\prod_{\rho\in\Sigma(1)}\frac{F_\rho\cdot e^{\langle\widetilde{m},e_\rho\rangle\cdot F_\rho}}{1-a_\rho(g)e^{-F_\rho}}\right)\\
&=\frac{1}{Eu_X^{\mathbb{T}}(x_\sigma)}\cdot i_\sigma^*\left(\sum_{g\in G_\sigma}\chi^{m'}(g^{-1})\prod_{i=1}^{n}\frac{F_{\rho_i}\cdot e^{\langle m',e_{\rho_i}\rangle\cdot F_{\rho_i}}}{1-a_{\rho_i}(g)e^{-F_{\rho_i}}}\right)\\
&=\frac{1}{|G_\sigma|}\sum_{g\in G_\sigma}\chi^{m'}(g^{-1})\cdot i_\sigma^*\left(\prod_{i=1}^{n}\frac{e^{\langle m',e_{\rho_i}\rangle\cdot F_{\rho_i}}}{1-a_{\rho_i}(g)e^{-F_{\rho_i}}}\right)\\
&=\frac{1}{|G_\sigma|}\sum_{g\in G_\sigma}\chi^{m'}(g)e^{c(m')}\prod_{i=1}^{n}\frac{1}{1-a_{\rho_i}(g^{-1})\cdot e^{-c(m'_{\sigma,i})}},
\end{aligned}
$$

where the last equality uses (67) for the torus $\mathbb{T}'$ to show that $i_\sigma^* F_{\rho_i} = c(m'_{\sigma,i})$ and $c(m') = \sum_{i=1}^n \langle m', e_{\rho_i}\rangle i_\sigma^* F_{\rho_i}$, as well as changing g by g^{-1} in G_σ. Also note that $a_\rho(g)$ for $g \in G_\Sigma$ and $\rho \in \Sigma(1)$ fits with the corresponding $a_\rho(g)$ for $g \in G_\sigma$ and $\rho \in \sigma(1)$, as in (33).

Altogether, the local formula (181) is obtained from the global formula (182) upon applying the homological localization map pr_{x_σ} of (180). □

4.3.2. *Proof of Theorem 3.28 by localization.* We can now explain a proof of formula (136) of Theorem 3.28 via localization. With $X := X_\Sigma$ as before, let $V \subset X$ be the open complement of the divisor $D_K := \bigcup_{\rho\in K} D_\rho$, for $K \subset \Sigma(1)$. Recall that the equivariant Hirzebruch class $T_{y*}^{\mathbb{T}}([V \hookrightarrow X])$ is computed by:

(183)

$$
T_{y*}^{\mathbb{T}}([V\hookrightarrow X]) = (1+y)^{n-r}\cdot\sum_{g\in G_\Sigma}\prod_{\rho\in K}\frac{F_\rho\cdot(1+y)\cdot a_\rho(g)\cdot e^{-F_\rho}}{1-a_\rho(g)\cdot e^{-F_\rho}}\prod_{\rho\notin K}\frac{F_\rho\cdot\left(1+y\cdot a_\rho(g)\cdot e^{-F_\rho}\right)}{1-a_\rho(g)\cdot e^{-F_\rho}},
$$

with $r = |\Sigma(1)|$, and $F_\rho = [D_\rho]_{\mathbb{T}}$ the equivariant fundamental class of the $\mathbb{T}$-invariant divisor D_ρ corresponding to the ray $\rho \in \Sigma(1)$. As above, it suffices to prove the equality of both sides of formula (183) after applying the homological localization map pr_{x_σ} of (180), with $pr_{x_\sigma}(T_{y*}^{\mathbb{T}}([V \hookrightarrow X])) = T_{y*}^{\mathbb{T}}([V \hookrightarrow X])_{x_\sigma}$ computed by formula (172).

If $g \in G_\Sigma \setminus G_\sigma$, by (31) there exists a $\rho \in \Sigma(1)$ with $a_\rho(g) \neq 1$, so that one factor on the restriction of the right-hand side of (183) to U_σ becomes 0. Hence the summation on the right-hand side of (183) reduces after restriction to U_σ to a summation over $g \in G_\sigma$. If $g \in G_\sigma$ and $\rho \notin \sigma(1)$, then $a_\rho(g) = 1$, so that the restriction of the factor $\frac{F_\rho\cdot(1+y)\cdot a_\rho(g)\cdot e^{-F_\rho}}{1-a_\rho(g)\cdot e^{-F_\rho}}$ (for $\rho \in K$) or $\frac{F_\rho\cdot\left(1+y\cdot a_\rho(g)\cdot e^{-F_\rho}\right)}{1-a_\rho(g)\cdot e^{-F_\rho}}$ (for $\rho \notin K$) to U_σ becomes $1+y$. With $r-n = |\Sigma(1) \setminus \sigma(1)|$, this will in turn cancel the factor $(1+y)^{n-r}$ of formula (183). Therefore, we get

$$
\begin{aligned}
& pr_{x_\sigma}\left((1+y)^{n-r} \cdot \sum_{g\in G_\Sigma} \prod_{\rho\in K} \frac{F_\rho\cdot(1+y)\cdot a_\rho(g)\cdot e^{-F_\rho}}{1-a_\rho(g)\cdot e^{-F_\rho}} \prod_{\rho\notin K} \frac{F_\rho\cdot\left(1+y\cdot a_\rho(g)\cdot e^{-F_\rho}\right)}{1-a_\rho(g)\cdot e^{-F_\rho}} \right) \\
&= \frac{1}{Eu_X^{\mathbb{T}}(x_\sigma)} \cdot i_\sigma^* \left(\sum_{g\in G_\sigma} \prod_{\rho\in\sigma(1)\cap K} \frac{F_\rho\cdot(1+y)\cdot a_\rho(g)\cdot e^{-F_\rho}}{1-a_\rho(g)\cdot e^{-F_\rho}} \prod_{\rho\in\sigma(1)\setminus K} \frac{F_\rho\cdot\left(1+y\cdot a_\rho(g)\cdot e^{-F_\rho}\right)}{1-a_\rho(g)\cdot e^{-F_\rho}} \right) \\
&= \frac{1}{|G_\sigma|} \sum_{g\in G_\sigma} i_\sigma^* \left(\prod_{\rho\in\sigma(1)\cap K} \frac{(1+y)\cdot a_\rho(g)\cdot e^{-F_\rho}}{1-a_\rho(g)\cdot e^{-F_\rho}} \prod_{\rho\in\sigma(1)\setminus K} \frac{1+y\cdot a_\rho(g)\cdot e^{-F_\rho}}{1-a_\rho(g)\cdot e^{-F_\rho}} \right).
\end{aligned}
$$

This expression reduces to formula (172), as desired, after using (67) for the torus $\mathbb{T}'$ to show that for $\rho=\rho_i$ the rays of $\sigma(1)$, one gets as before $i_\sigma^* F_{\rho_i} = c(m'_{\sigma,i})$, as well as by changing g by g^{-1} in G_σ. □

Remark 4.21. Specializing to $K=\emptyset$, the above arguments also reprove formula (126) of Theorem 3.22.

5. Euler-Maclaurin formulae via equivariant Hirzebruch-Riemann-Roch

An Euler-Maclaurin formula relates the sum $\sum_{m\in P\cap M} f(m)$ of the values of a suitable function f at the lattice points in a polyhedron $P\subset M_{\mathbb{R}} := M\otimes\mathbb{R}$ to integrals over polyhedra. Here, we are interested in the case where P is a polytope or a cone.

We begin this section with a short overview of some of the literature describing relations between the equivariant Hirzebruch-Riemann-Roch theorem and Euler-Maclaurin type formulae. We aim to relate the equivariant geometric context for projective simplicial toric varieties $X=X_\Sigma$ to a purely combinatorial geometric context for polytopes $P\subset M_{\mathbb{R}}$. The corresponding results for pointed cones will fit with the localization results developed in the previous section.

We further derive an abstract Euler-Maclaurin type formula, inspired by a corresponding proof of Brion-Vergne in the classical case. Our abstract formula is based on the equivariant Hirzebruch-Riemann-Roch theorem and the motivic formalism of equivariant characteristic classes developed in the previous sections. It can be specialized to various situations, recovering many known Euler-Maclaurin type formulae as well as obtaining several new ones in a uniform way dictated by toric geometry.

The theory of valuations for rational polyhedra $P\subset M_{\mathbb{R}}$ was applied by Khovanskii and Pukhlikov in [38] to obtain Euler-Maclaurin type formulae for Delzant polytopes (corresponding to smooth projective toric varieties) in terms of infinitesimal movements $P(h)$ of the polytope. More precisely, for a polynomial f one has the identity:

$$
Todd\left(\frac{\partial}{\partial h}\right)\left(\int_{P(h)} f(m)\,dm\right)_{|h=0} = \sum_{m\in P\cap M} f(m)\,. \tag{184}
$$

Here, if the polytope P is the lattice polytope defined by inequalities of the form

$$\langle m, u_F\rangle + c_F \geq 0,$$

with u_F the facet normals, the polytope $P(h)$ with shifted faces is defined by inequalities

$$\langle m, u_F\rangle + c_F + h_F \geq 0,$$

with $h = (h_F)_F$ a vector with real entries indexed by the facets F of P. Moreover, $Todd(\frac{\partial}{\partial h})$ is the differential operator defined by:

$$Todd(\frac{\partial}{\partial h}) := \prod_{F \text{ facet}} \frac{\frac{\partial}{\partial h_F}}{1 - e^{-\frac{\partial}{\partial h_F}}}.$$

The relation between formula (184) and the equivariant Hirzebruch-Riemann-Roch theorem is only indicated in [38], but not used (but see, e.g., the proof of [16][Thm.13.5.6]). Moreover, it is clearly pointed out that exponential functions $f(x) = e^{\langle x,z \rangle}$ are needed to work with rational polyhedral cones. From this, one then gets the result for polynomials f (compare also with [16][Sect.13.5]).

For a direct extension of (184) to simple lattice polytopes, with a corresponding more complicated Todd operator $Todd(\frac{\partial}{\partial h})$, see [12] as well as the nice survey [34]. The approach to the Euler-Maclaurin formula (184) through the equivariant Hirzebruch-Riemann-Roch theorem for projective simplicial toric varieties corresponding to simple lattice polytopes is due to Brion-Vergne, see [13][Thm.4.5].

In the following, we explain a uniform approach to various Euler-Maclaurin formulae for simple lattice polytopes or pointed cones, formulated in terms of dilatations of the polytope or cone. To have a unified language, assume P is a full-dimensional simple lattice polyhedron in $M_{\mathbb{R}}$ with associated (semi-projective) toric variety $X = X_P$ and inner normal fan $\Sigma = \Sigma_P$. Let $P(h)$ be the dilatation of P with respect to the vector $h = (h_\rho)_{\rho \in \Sigma(1)}$ with real entries indexed by the rays of Σ (as we use the related toric geometry), or equivalently, by the facets of P (if one would like to formulate everything directly in terms of P). So, if P is defined by inequalities of the form

$$\langle m, u_\rho \rangle + c_\rho \geq 0,$$

with u_ρ the ray generators and $c_\rho \in \mathbb{Z}$, for each $\rho \in \Sigma(1)$, then $P(h)$ is defined by inequalities

$$\langle m, u_\rho \rangle + c_\rho + h_\rho \geq 0,$$

for each $\rho \in \Sigma(1)$. For later use in the context of weighted Euler-Maclaurin formulae, we also define a parametrized dilatation $P_y(h)$ of P by:

$$P_y(h) := \{ m \in M \mid \langle m, u_\rho \rangle + (1+y) \cdot c_\rho + h_\rho \geq 0, \text{ for all } \rho \in \Sigma(1) \},$$

with $h = (h_\rho)_\rho$ as before a vector with real entries indexed by the rays ρ of Σ. Note that $P_0(h) = P(h)$, and $P_y(0) =: P_y$ is the dilatation of P by the factor $1+y$.

Let us first consider the case when P is a full-dimensional lattice polytope in $M_{\mathbb{R}}$. In what follows, we adapt the arguments of Brion-Vergne [13][Thm.4.5] to our context. If h is in a small enough open neighborhood U of the origin in $\mathbb{R}^r$ (with r the number of facets of P), then $P(h)$ is again a simple polytope of the same combinatorial type, and similarly for $P_y(h)$. Let us fix $h \in U \cap \mathbb{Q}^r$ and $y \in \mathbb{Q}$, and choose $k \in \mathbb{N}$ so that $k \cdot P_y(h)$ is a lattice polytope in $M_{\mathbb{R}}$ with ample Cartier divisor

$$D_{k \cdot P_y(h)} = k \cdot D_{P_y(h)} = k \left(\sum_{\rho \in \Sigma(1)} ((1+y) \cdot c_\rho + h_\rho) \cdot D_\rho \right),$$

where $D_P = \sum_{\rho\in\Sigma(1)} c_\rho \cdot D_\rho$ is the ample Cartier divisor of P. On a simplicial toric variety any divisor is $\mathbb{Q}$-Cartier ([16][Prop.4.2.7]), so that $D_{P_y(h)}$ has an equivariant fundamental class with rational coefficients satisfying $k\cdot[D_{P_y(h)}]_{\mathbb{T}} := [k\cdot D_{P_y(h)}]_{\mathbb{T}} = c_1^{\mathbb{T}}(\mathscr{O}_X(k\cdot D_{P_y(h)}))$, e.g., see [16][Prop.12.4.13]. Applying formula (84) for $k\cdot P(h)$, and using the fact that the associated toric variety does not change under polytope dilatation, we get

$$\sum_{m\in k\cdot P_y(h)\cap M} e^{s(m)} = \int_X e^{k\cdot[D_{P_y(h)}]_{\mathbb{T}}} \cap \mathrm{td}_*^{\mathbb{T}}(X) \in (\Lambda_{\mathbb{T}}^{an})_{\mathbb{Q}} \subset (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}\,. \tag{185}$$

A localized version of formula (185) can be derived from (177) as:

$$\sum_{m\in k\cdot P_y(h)\cap M} e^{s(m)} = \sum_{\sigma\in\Sigma(n)} \frac{i_\sigma^*\left(e^{k\cdot[D_{P_y(h)}]_{\mathbb{T}}}\right)}{Eu_X^{\mathbb{T}}(x_\sigma)} \cdot i_\sigma^*(\mathrm{td}_*^{\mathbb{T}}(X)) \in L^{-1}(\Lambda_{\mathbb{T}}^{an})_{\mathbb{Q}} \subset L^{-1}(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}\,. \tag{186}$$

Recall the identification $s : Sym_{\mathbb{Q}}(M) \simeq H_{\mathbb{T}}^*(pt;\mathbb{Q}) =: (\Lambda_{\mathbb{T}})_{\mathbb{Q}}$. Let $z\in N_{\mathbb{C}} := N\otimes_{\mathbb{Z}}\mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(M_{\mathbb{R}},\mathbb{C})$ be given. By the universal property of Sym, z induces $\mathbb{R}$-algebra homomorphisms

$$\langle -, z\rangle : Sym_{\mathbb{R}}(M) \to \mathbb{C}\,,$$

by which we can view $\langle p, z\rangle$ for z now variable and $p\in Sym_{\mathbb{R}}(M)$, resp., $p\in (Sym_{\mathbb{R}}(M))^{an}$ fixed, as a $\mathbb{C}$-valued polynomial on $N_{\mathbb{R}}$, resp., as a *convergent power series function* around zero in $N_{\mathbb{R}}$. Assume now that z is chosen so that $\langle Eu_X^{\mathbb{T}}(x_\sigma), z\rangle \neq 0$ for each $\sigma\in\Sigma(n)$, i.e., $\langle i_\sigma^* F_\rho, z\rangle \neq 0$, for each ray $\rho\in\sigma(1)$ of $\sigma\in\Sigma(n)$. Applying the function $\langle -, \frac{1}{k}\cdot z\rangle$ to (186), we get

$$\sum_{m\in k\cdot P_y(h)\cap M} e^{\langle m, \frac{1}{k}\cdot z\rangle} = \sum_{\sigma\in\Sigma(n)} \frac{e^{\langle i_\sigma^*[D_{P_y(h)}]_{\mathbb{T}}, z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma), \frac{1}{k}\cdot z\rangle} \cdot \langle i_\sigma^*(\mathrm{td}_*^{\mathbb{T}}(X)), \frac{1}{k}\cdot z\rangle. \tag{187}$$

Note that by the Riemann sum approximation of an integral, we have:

$$\lim_{k\to\infty} \frac{1}{k^n} \sum_{m\in k\cdot P_y(h)\cap M} e^{\langle m, \frac{1}{k}\cdot z\rangle} = \lim_{k\to\infty} \frac{1}{k^n} \sum_{m\in P_y(h)\cap \frac{1}{k}M} e^{\langle m, z\rangle} = \int_{P_y(h)} e^{\langle m, z\rangle}\, dm,$$

with the Lebesgue measure dm normalized so that the unit cube in $M\subset M_{\mathbb{R}}$ has volume 1 (which explains the use of the factor $\frac{1}{k^n}$). We next study the limits on the right-hand side of (187). Note that

$$k^n \cdot \langle Eu_X^{\mathbb{T}}(x_\sigma), \frac{1}{k}\cdot z\rangle = \langle Eu_X^{\mathbb{T}}(x_\sigma), z\rangle$$

since $\langle Eu_X^{\mathbb{T}}(x_\sigma), z\rangle$ is a (non-zero) homogeneous polynomial of degree n in z. Finally, we have that

$$\lim_{k\to\infty} \langle i_\sigma^*(\mathrm{td}_*^{\mathbb{T}}(X)), \frac{1}{k}\cdot z\rangle = 1. \tag{188}$$

Indeed, by formula (130), we have

$$\langle i_\sigma^*(\mathrm{td}_*^{\mathbb{T}}(X)), \frac{1}{k}\cdot z\rangle = \sum_{g\in G_\Sigma} \prod_{\rho\in\Sigma(1)} \frac{\langle i_\sigma^* F_\rho, \frac{1}{k}\cdot z\rangle}{1 - a_\rho(g)\cdot e^{-\langle i_\sigma^* F_\rho, \frac{1}{k}\cdot z\rangle}},$$

with $F_\rho = [D_\rho]_{\mathbb{T}}$ denoting the equivariant fundamental class of the $\mathbb{T}$-invariant divisor D_ρ corresponding to the ray $\rho \in \Sigma(1)$, and note that for k fixed (or $k = 1$) this is a *convergent power series function* around zero in z. Finally,

$$\lim_{k\to\infty} \frac{\langle i_\sigma^* F_\rho, \frac{1}{k}\cdot z\rangle}{1 - a_\rho(g)\cdot e^{-\langle i_\sigma^* F_\rho, \frac{1}{k}\cdot z\rangle}} = \begin{cases} 0, & \text{if } a_\rho(g) \neq 1, \\ 1, & \text{if } a_\rho(g) = 1. \end{cases}$$

So only $g = id_G$ contributes a non-zero limit to (188), and this contribution is 1. Altogether, we get the following result.

Theorem 5.1. *In the above notations, we have*

$$\begin{aligned} \int_{P_y(h)} e^{\langle m,z\rangle}\, dm &= \sum_{\sigma\in\Sigma(n)} \frac{e^{\langle i_\sigma^*[D_{P_y(h)}],z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma), z\rangle} \\ &= \sum_{\sigma\in\Sigma(n)} \frac{e^{\langle (1+y)\cdot i_\sigma^* c_1^{\mathbb{T}}(\mathscr{O}_X(D_P)),z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma), z\rangle} \cdot e^{\sum_\rho h_\rho \langle i_\sigma^* F_\rho, z\rangle}. \end{aligned} \tag{189}$$

Remark 5.2. The left hand side of (189) is a *continuous* function in h near zero, and for all $z \in N_{\mathbb{C}}$, resp., in y, whereas the right hand side is an *analytic* function in h near zero, and for $z \in N_{\mathbb{C}}$ away from the linear hyperplanes $\langle i_\sigma^* F_\rho, z\rangle = 0$ for each ray $\rho \in \sigma(1)$ of $\sigma \in \Sigma(n)$, resp., in y. But then both sides of this equality have to be *analytic functions in h near zero and all $z \in N_{\mathbb{C}}$*, resp., in $y \in \mathbb{R}$, with the corresponding Taylor series around zero converging uniformly on small compact neighborhoods of zero in the variables h and z (cf. also [34][page 27]).

For later use, we state the following consequence of Theorem 5.1 (see also [12][Lem.3.11]).

Corollary 5.3. *Let f be a polynomial function of degree d on $M_{\mathbb{R}}$. Then $\int_{P_y(h)} f(m)\, dm$ is a polynomial function in h of degree bounded above by $d+n$.*

Proof. We follow the idea of proof from [12][Lem.3.11], adapted to our language. By replacing in (189) z by tz, with $0 \neq t \in \mathbb{R}$ small (so that also $\langle i_\sigma^* F_\rho, tz\rangle \neq 0$ for each ray $\rho \in \sigma(1)$ of $\sigma \in \Sigma(n)$), and multiplying both sides by t^n , we get an equality of analytic functions in t, with each term on the right hand side also analytic even in zero, since $\langle Eu_X^{\mathbb{T}}(x_\sigma), z\rangle \neq 0$ is homogeneous in z of order n. Now taking the Taylor expansion at $t = 0$ of these two analytic functions, we get for z small and away from the linear hyperplanes $\langle i_\sigma^* F_\rho, z\rangle = 0$ for each ray $\rho \in \sigma(1)$ of $\sigma \in \Sigma(n)$, that the assertion holds for $f(m) = \langle m, z\rangle^k$, for any given non-negative integer k. This then implies the statement for any polynomial $f(m)$. □

Corollary 5.4. *Let $X = X_P$ be the toric variety associated to a simple full-dimensional lattice polytope $P \subset M_{\mathbb{R}}$, with $D = D_P$ the corresponding ample divisor. Let $\mathscr{F}$ be a $\mathbb{T}$-equivariant coherent sheaf on X. Then, for a polynomial function f on $M_{\mathbb{R}}$, the expression*

$$\sum_{m\in M} \left(\sum_{i=0}^{n} (-1)^i \cdot \dim_{\mathbb{C}} H^i(X; \mathscr{O}_X((1+y)D)\otimes \mathscr{F})_{\chi^{-m}} \right) \cdot f(m) \tag{190}$$

is a polynomial in $1+y$*. Moreover, the value of this polynomial at* 0 *(i.e., for* $y=-1$*) is given by*

$$\sum_{m\in M}\left(\sum_{i=0}^{n}(-1)^i\cdot\dim_{\mathbb{C}}H^i(X;\mathscr{F})_{\chi^{-m}}\right)\cdot f(m).$$

Proof. Using the ideas of the above corollary (as in [12][Prop.4.1]), the assertion follows from the formula

$$\begin{aligned}&\sum_{m\in M}\left(\sum_{i=0}^{n}(-1)^i\cdot\dim_{\mathbb{C}}H^i(X;\mathscr{O}_X((1+y)D)\otimes\mathscr{F})_{\chi^{-m}}\right)\cdot e^{\langle m,z\rangle}\\ &=\sum_{\sigma\in\Sigma(n)}\frac{e^{\langle i_\sigma^*[D_{P_y}],z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma),z\rangle}\cdot\langle i_\sigma^*(td_*^{\mathbb{T}}([\mathscr{F}])),z\rangle\\ &=\sum_{\sigma\in\Sigma(n)}\frac{e^{\langle(1+y)\cdot i_\sigma^*c_1^{\mathbb{T}}(\mathscr{O}_X(D_P)),z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma),z\rangle}\cdot\langle i_\sigma^*(td_*^{\mathbb{T}}([\mathscr{F}])),z\rangle\,,\end{aligned}$$

which can be deduced from (179), upon pairing with $z\in N_{\mathbb{C}}$. $\square$

Remark 5.5. For $\mathscr{F}=\mathscr{O}_X$, the expression (190) becomes

$$\sum_{m\in M\cap P_y}f(m)$$

as in [12]. For the function $f=1$, this further specializes to the classical *Ehrhart polynomial*. As another special case, for $[\mathscr{F}]=mC_0^{\mathbb{T}}([O_\tau\hookrightarrow X])\in K_0^{\mathbb{T}}(X)$, (190) calculates

$$\sum_{m\in M\cap\mathrm{Relint}(E_y)}f(m)$$

with E the face of P corresponding to a cone $\tau\in\Sigma$.

We next explain (in Theorem 5.8 below) a result analogous to Theorem 5.1 for the dilatation

$$Tan(P,v)(h):=(v+C_v)(h)=v(h)+C_v$$

of the *tangent cone* $Tan(P,v)=v+C_v$ of a vertex v of the polytope P, with $C_v=Cone(P\cap M-v)\subset M_{\mathbb{R}}$ a full-dimensional simple lattice cone with vertex 0. The arguments from the case of a full-dimensional lattice polytope apply similarly to the tangent cone, once the approximation of the integral $\int_{C_v(h)}e^{\langle m,z\rangle}\,dm$ by the corresponding Riemann sum is explained. The corresponding equivariant Hirzebruch-Riemann-Roch formula in this setting is (170), with $\sigma^\vee=C_v$ and v corresponding to the torus fixed point x_σ in the $\mathbb{T}$-invariant open affine neighborhood U_σ associated to $\sigma\in\Sigma(n)$.

Proposition 5.6. *With the above notations, one has that* $\sum_{m\in C_v\cap M}e^{\langle m,z\rangle}$ *and* $\int_{C_v}e^{\langle m,z\rangle}\,dm$ *are convergent (locally uniformly) to meromorphic functions in* $L^{-1}(\Lambda_{\mathbb{T}}^{an})_{\mathbb{C}}$*, for* $z\in N_{\mathbb{C}}$ *satisfying* $-z\in Int(\sigma)$*. Moreover,*

$$\lim_{k\to\infty}\frac{1}{k^n}\sum_{m\in k\cdot C_v(h)\cap M}e^{\langle m,\frac{1}{k}\cdot z\rangle}=\lim_{k\to\infty}\frac{1}{k^n}\sum_{m\in C_v(h)\cap\frac{1}{k}M}e^{\langle m,z\rangle}=\int_{C_v(h)}e^{\langle m,z\rangle}\,dm,$$

with $h\in\mathbb{Q}^r$ *a small enough dilatation vector, and* $k\in\mathbb{N}$ *so that* $k\cdot C_v(h)$ *is a simple pointed lattice cone in* $M_{\mathbb{R}}$*.*

Proof. Let $\sigma \in \Sigma(n)$ be a smooth cone with generators $u_{\sigma,1},\dots,u_{\sigma,n}$ of the rays $\rho_1,\dots,\rho_n \in \sigma(1)$, and let $m_{\sigma,1},\dots,m_{\sigma,n}$ be the dual basis of $M = \bigoplus_{i=1}^n \mathbb{N}\cdot m_{\sigma,i}$. Then

$$\sum_{m\in C_v\cap M} e^{\langle m,z\rangle} = \prod_{i=1}^n \left(\sum_{j_i=0}^{\infty} (e^{\langle m_{\sigma,i},z\rangle})^{j_i}\right) = \prod_{i=1}^n \frac{1}{1-e^{\langle m_{\sigma,i},z\rangle}} = \frac{\langle i_\sigma^*(\mathrm{td}_*^{\mathbb{T}}(X)),z\rangle}{\langle Eu_X^{\mathbb{T}}(x_\sigma),z\rangle},$$

with all $\langle m_{\sigma,i},z\rangle < 0$ for $-z \in Int(\sigma)$, and the corresponding geometric series locally uniformly convergent in these z. The last equality follows from (173) (for $K=\emptyset$), using the identification $s: Sym_{\mathbb{Q}}(M) \simeq (\Lambda_{\mathbb{T}})_{\mathbb{Q}}$. Similarly,

$$\int_{C_v} e^{\langle m,z\rangle}\, dm = \prod_{i=1}^n \frac{-1}{\langle m_{\sigma,i},z\rangle},$$

for $-z \in Int(\sigma)$ and convergence as before, with the Lebesgue measure dm normalized so that the unit cube in $M \subset M_{\mathbb{R}}$ has volume 1 (see also [29][eq.(8)] and [38][Prop.1]). By the multiplicativity of the equivariant Todd class transformation, we get as in the case of a polytope that

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{k^n} \sum_{m\in k\cdot C_v(h)\cap M} e^{\langle m,\frac{1}{k}\cdot z\rangle} &= \lim_{k\to\infty} \frac{1}{k^n} \prod_{i=1}^n \left(\frac{e^{h_i\langle m_{\sigma,i},z\rangle}}{1-e^{\langle m_{\sigma,i},\frac{1}{k}\cdot z\rangle}}\right) \\
&= \lim_{k\to\infty} \prod_{i=1}^n \left(\frac{e^{h_i\langle m_{\sigma,i},z\rangle}\cdot \langle m_{\sigma,i},\frac{1}{k}\cdot z\rangle}{(1-e^{\langle m_{\sigma,i},\frac{1}{k}\cdot z\rangle})\cdot\langle m_{\sigma,i},z\rangle}\right) \\
&= \prod_{i=1}^n \frac{-e^{h_i\langle m_{\sigma,i},z\rangle}}{\langle m_{\sigma,i},z\rangle} \\
&= e^{\sum_{i=1}^n h_i\langle m_{\sigma,i},z\rangle}\cdot \prod_{i=1}^n \frac{-1}{\langle m_{\sigma,i},z\rangle} \\
&= e^{\sum_{i=1}^n h_i\langle m_{\sigma,i},z\rangle}\cdot \int_{C_v} e^{\langle m,z\rangle}\, dm \\
&= \int_{C_v(h)} e^{\langle m,z\rangle}\, dm.
\end{aligned}$$

For the proof of the case of a simplicial cone by smooth subdivisions, let us remark that the above approximation by Riemann sums for the full cone C_v holds similarly for the cone $C_v^K := C_v \setminus \bigcup_{i\in K} F_i$ with some facets F_i ($i\in K\subset \Sigma(1)$) removed. More precisely, one gets

$$\sum_{m\in C_v^K\cap M} e^{\langle m,z\rangle} = \prod_{i\in\sigma(1)\cap K} \frac{e^{\langle m_{\sigma,i},z\rangle}}{1-e^{\langle m_{\sigma,i},z\rangle}} \cdot \prod_{i\in\sigma(1)\setminus K} \frac{1}{1-e^{\langle m_{\sigma,i},z\rangle}} = \frac{\langle i_\sigma^*(T_{0*}^{\mathbb{T}}([V\hookrightarrow X])),z\rangle}{\langle Eu_X^{\mathbb{T}}(x_\sigma),z\rangle},$$

with the last equality following from (173), for $V = X\setminus \bigcup_{i\in K} D_{\rho_i}$ (with the divisors D_{ρ_i} corresponding to the divisors of the original polytope P). The series on the left hand side are locally uniformly convergent to a meromorphic function in $L^{-1}(\Lambda_{\mathbb{T}}^{an})_{\mathbb{C}}$, for $z\in N_{\mathbb{C}}$ satisfying $-z\in Int(\sigma)$. Moreover, a similar proof yields in this case that

$$\lim_{k\to\infty}\frac{1}{k^n}\sum_{m\in k\cdot C_v^K(h)\cap M} e^{\langle m,\frac{1}{k}\cdot z\rangle} = \int_{C_v(h)} e^{\langle m,z\rangle}\, dm.$$

In the general case, let C_v be as before a full-dimensional (simple) lattice cone with vertex 0. The assumption that C_v is simple will not be needed in the following arguments. Consider a refinement $C_v = \bigcup_{i=1}^s C_i$ of C_v by smooth full-dimensional lattice cones with vertex 0, see, e.g., [16][Thm.11.1.9] in the dual context of toric resolutions by fan refinements. Then, considering each newly introduced facet F for only one of the cones it belongs to, we get a disjoint union $C_v = \bigsqcup_{i=1}^s C_i^{K_i}$ into smooth full-dimensional lattice cones with vertex 0 and with some facets removed. Let $z \in N_{\mathbb{C}}$ be so that $-z \in Int(\sigma)$, with $\sigma = C_v^\vee$. Then we also have that $-z \in Int(\sigma_i)$ with $\sigma_i = C_i^\vee$. So we can apply the case of smooth cones to all of the C_i. Then

$$\sum_{m \in C_v \cap M} e^{\langle m,z\rangle} = \sum_{i=1}^s \left(\sum_{m \in C_i^{K_i} \cap M} e^{\langle m,z\rangle} \right)$$

is convergent (locally uniformly) to a meromorphic function in $L^{-1}(\Lambda^{an}_{\mathbb{T}})_{\mathbb{C}}$, for $z \in N_{\mathbb{C}}$ satisfying $-z \in Int(\sigma)$. Moreover,

$$\lim_{k\to\infty} \frac{1}{k^n} \sum_{m \in k\cdot C_v(h) \cap M} e^{\langle m, \frac{1}{k}\cdot z\rangle} = \sum_{i=1}^s \int_{C_i(h)} e^{\langle m,z\rangle}\, dm = \int_{C_v(h)} e^{\langle m,z\rangle}\, dm,$$

with $h \in M_{\mathbb{Q}} \simeq \mathbb{Q}^r$ a small enough dilatation vector, and $k \in \mathbb{N}$ so that $k \cdot C_v(h)$ is a full-dimensional pointed lattice cone in $M_{\mathbb{R}}$. □

Remark 5.7. By the above smooth decomposition method, one can get more explicit expressions for the summation appearing in formula (170). In the context of the tangent cone $Tan(P,v)$ of a simple full-dimensional lattice polytope P at a vertex v, we also get the identification

$$\sum_{m \in C_v \cap M} e^{\langle m,z\rangle} = \langle \mathrm{td}_*^{\mathbb{T}}(X)_{x_\sigma}, z\rangle = \langle \mathbb{S}\left(\sum_{m \in \sigma^\vee \cap M} e^{s(m)} \right), z\rangle \in L^{-1}(\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}}, \tag{191}$$

for $-z \in Int(\sigma)$. The left hand side is a convergent series for such z, whereas the right hand side is a corresponding meromorphic function on $N_{\mathbb{C}}$. Furthermore, still assuming $-z \in Int(\sigma)$, we get more generally, for $\tau \preceq \sigma$ corresponding to a face E of P containing the vertex v, a similar interpretation for the following equality:

(192)

$$\sum_{m \in \mathrm{Relint}(C_v \cap E) \cap M} e^{\langle m,z\rangle} = \langle T^{\mathbb{T}}_{0*}([O_\tau \hookrightarrow X])_{x_\sigma}, z\rangle = \langle \mathbb{S}\left(\sum_{\mathrm{Relint}(\sigma^\vee \cap \tau^\perp) \cap M} e^{s(m)} \right), z\rangle \in L^{-1}(\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}}.$$

As a consequence of the above discussion, we can now give a geometric meaning to the localized summands of formula (189).

Theorem 5.8. *Let $v \in P$ be a vertex of the full-dimensional simple lattice polytope $P \subset M_{\mathbb{R}}$, with tangent cone $Tan(P,v)$, for $C_v = Cone(P \cap M - v) \subset M_{\mathbb{R}}$ a full-dimensional simple lattice cone with vertex* 0*. Consider its dilatation*

$$Tan(P,v)(h) := (v + C_v)(h).$$

Let (X,D) be the projective toric variety with ample Cartier divisor associated to P. Denote by $x_\sigma \in X$ the torus fixed point corresponding to $v \in P$, with associated cone $\sigma = C_v^\vee \in \Sigma(n)$.

Then for $z \in N_{\mathbb{C}}$ satisfying $-z \in Int(\sigma)$, we have

$$\int_{Tan(P,v)(h)} e^{\langle m,z\rangle}\, dm = \frac{e^{\langle v,z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma), z\rangle} \cdot e^{\sum_{\rho\in\sigma(1)} h_\rho \langle i_\sigma^* F_\rho, z\rangle}. \tag{193}$$

Proof. In light of the above proposition, one only has to note that $i_\sigma^* c_1^{\mathbb{T}}(\mathscr{O}(D)) = -c(v) = s(v) \in H^2_{\mathbb{T}}(pt;\mathbb{Q}) \simeq M_{\mathbb{Q}}$. □

Remark 5.9. The left hand side of the equality (193) is an analytic function in h near zero and $-z \in Int(\sigma)$, resp., in $y \in \mathbb{R}$. Moreover, this integral is locally uniformly convergent in z. The right hand side of (193) is an analytic function in h near zero and $y \in \mathbb{R}$, as well as a meromorphic function in z which is holomorphic outside the hyperplanes $\langle i_\sigma^* F_\rho, z\rangle = 0$ for each ray $\rho \in \sigma(1)$.

Back to the context of a projective simplicial toric variety associated to a full-dimensional simple lattice polytope $P \subset M_{\mathbb{R}}$, in order to relate the right-hand side of formula (189) to the equivariant Hirzebruch-Riemann-Roch formulae we have to introduce the *analytic subring*

$$(H^*_{\mathbb{T}}(X;\mathbb{Q}))^{an} \subset \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q}) \tag{194}$$

defined as the image of the *analytic Stanley-Reisner subring* (depending only on the fan Σ)

$$SR^{an}_{\mathbb{Q}}(\Sigma) := \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}/\sim_{SR} \;\subset\; \mathbb{Q}[[x_\rho \mid \rho \in \Sigma(1)]]/\sim_{SR} =: \widehat{SR_{\mathbb{Q}}(\Sigma)}$$

under the isomorphism $\widehat{SR_{\mathbb{Q}}(\Sigma)} \simeq \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$ given on generators by $x_\rho \mapsto F_\rho$, $\rho \in \Sigma(1)$. Given an element $p(F_\rho) \in \left(H^*_{\mathbb{T}}(X;\mathbb{Q})\right)^{an}$, with $p(x_\rho) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}$ a convergent power series, one gets for the restriction i_σ^* to a fixed point x_σ that

$$p(i_\sigma^* F_\rho) = i_\sigma^*(p(F_\rho)). \tag{195}$$

Here, the convergent power series on the left side corresponds to the image of $p(x_\rho)$ under the evaluation homomorphism

$$\mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\} \mapsto (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}} : x_\rho \mapsto i_\sigma^* F_\rho \in H^2_{\mathbb{T}}(pt;\mathbb{Q}) \simeq M_{\mathbb{Q}}.$$

Remark 5.10. Regarding $\sigma \in \Sigma(n)$ as a fan, we have a corresponding (analytic) Stanley-Reisner ring

$$SR^{an}_{\mathbb{Q}}(\sigma) := \mathbb{Q}\{x_\rho \mid \rho \in \sigma(1)\} \;\subset\; \mathbb{Q}[[x_\rho \mid \rho \in \sigma(1)]] =: \widehat{SR_{\mathbb{Q}}(\sigma)},$$

with an isomorphism $\widehat{SR_{\mathbb{Q}}(\sigma)} \simeq \widehat{H}^*_{\mathbb{T}}(U_\sigma;\mathbb{Q}) \overset{i_\sigma^*}{\simeq} \widehat{H}^*_{\mathbb{T}}(x_\sigma;\mathbb{Q})$ given on generators by $x_\rho \mapsto i_\sigma^* F_\rho$, $\rho \in \sigma(1)$. Then the restriction map $\left(H^*_{\mathbb{T}}(X;\mathbb{Q})\right)^{an} \to \left(H^*_{\mathbb{T}}(U_\sigma;\mathbb{Q})\right)^{an}$ is induced from a corresponding restriction map $SR^{an}_{\mathbb{Q}}(\Sigma) \to SR^{an}_{\mathbb{Q}}(\sigma)$, sending $x_\rho \mapsto 0$ if $\rho \notin \sigma(1)$. We get the following factorization of the right hand side of (195):

$$i_\sigma^* : (H^*_{\mathbb{T}}(X;\mathbb{Q}))^{an} \to (H^*_{\mathbb{T}}(U_\sigma;\mathbb{Q}))^{an} \to (H^*_{\mathbb{T}}(x_\sigma;\mathbb{Q}))^{an}.$$

As a consequence of Theorem 5.1 and Remark 5.2, we get by differentiation and convergence the following.

Corollary 5.11. *Let $p(\frac{\partial}{\partial h}) \in \mathbb{Q}\{\frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)\} \subset \mathbb{Q}[[\frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)]]$ be an infinite order differential operator with constant rational coefficients, i.e., obtained by substituting $x_\rho \mapsto \frac{\partial}{\partial h_\rho}$ into a convergent (near zero) power series with rational coefficients $p(x) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\} \subset \mathbb{Q}[[x_\rho \mid \rho \in \Sigma(1)]]$. Then, in the above notations, we get for z small enough and away from the hyperplanes $\langle i_\sigma^* F_\rho, z\rangle = 0$ for each ray $\rho \in \sigma(1)$ of $\sigma \in \Sigma(n)$ the following formula:*

$$
\begin{aligned}
p(\frac{\partial}{\partial h})\left(\int_{P_y(h)} e^{\langle m,z\rangle}\, dm\right)_{|h=0} &= \sum_{\sigma\in\Sigma(n)} \frac{e^{\langle (1+y)\cdot i_\sigma^* c_1^{\mathbb{T}}(\mathscr{O}_X(D_P)),z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma),z\rangle} \cdot \langle p(i_\sigma^* F_\rho),z\rangle \\
&= \langle \sum_{\sigma\in\Sigma(n)} \frac{i_\sigma^*\left(e^{(1+y)\cdot c_1^{\mathbb{T}}(\mathscr{O}_X(D))} p(F_\rho)\right)}{Eu_X^{\mathbb{T}}(x_\sigma)}, z\rangle \\
&= \langle \int_X e^{(1+y)\cdot c_1^{\mathbb{T}}(\mathscr{O}_X(D))} p(F_\rho), z\rangle .
\end{aligned} \tag{196}
$$

If $p_n \in \mathbb{Q}[\frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)]$ is the corresponding truncation of p up to order n, then both sides of (196) applied to p_n converge for $n \to \infty$ locally uniformly in these z to (196) applied to p.

Remark 5.12. By (196), the operator $p(\frac{\partial}{\partial h})\left(\int_{P_y(h)} e^{\langle m,z\rangle}\, dm\right)_{|h=0}$ depends only on the equivalence class of $[p(x)] \in SR^{an}_{\mathbb{Q}}(\Sigma)$ and not on the chosen convergent power series representative.

Remark 5.13. Assume in addition in Corollary 5.11 that $\langle \int_X e^{(1+y)\cdot c_1^{\mathbb{T}}(\mathscr{O}_X(D))} p(F_\rho), z\rangle$ is a convergent power series in z near zero (for $y \in \mathbb{R}$ a fixed parameter, e.g., $y = 0$). Then one gets as an application of Cauchy's integral formula (see also [34][page 27]), that both sides of (196) applied to p_n converge for $n \to \infty$ and z small locally uniformly to (196) applied to p. In particular, this limit commutes with finite order differentiations with respect to z (and z small enough).

Corollary 5.14. *Let $p(\frac{\partial}{\partial h}) \in \mathbb{Q}[[\frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)]]$ be an infinite order differential operator with constant rational coefficients, i.e., obtained by substituting $x_\rho \mapsto \frac{\partial}{\partial h_\rho}$ into a formal power series with rational coefficients $p(x) \in \mathbb{Q}[[x_\rho \mid \rho \in \Sigma(1)]]$. Then for a polynomial function f on $M_{\mathbb{R}}$, we have the following formula:*

$$
p(\frac{\partial}{\partial h})\left(\int_{P_y(h)} f(m)\, dm\right)_{|h=0} = f(\frac{\partial}{\partial z})\left(\langle \int_X e^{(1+y)\cdot c_1^{\mathbb{T}}(\mathscr{O}_X(D))} p(F_\rho), z\rangle\right)_{|z=0}, \tag{197}
$$

where on the right hand side the operator $\left(f(\frac{\partial}{\partial z})\right)_{|z=0}$ acts on a formal power series in z.

Proof. Let first $p(\frac{\partial}{\partial h}) \in \mathbb{Q}[\frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)]$ be a finite order differential operator with constant rational coefficients, i.e., obtained by substituting $x_\rho \mapsto \frac{\partial}{\partial h_\rho}$ into a polynomial with rational coefficients $p(x) \in \mathbb{Q}[x_\rho \mid \rho \in \Sigma(1)]$. Then the result follows from applying the operator $\left(f(\frac{\partial}{\partial z})\right)_{|z=0}$ to (196). Here we first need to assume that z is small enough and away from the hyperplanes $\langle i_\sigma^* F_\rho, z\rangle = 0$ for each ray $\rho \in \sigma(1)$ of $\sigma \in \Sigma(n)$, since the localization formula is used; however, formula (196) then holds for all z small enough (and $y \in \mathbb{R}$ fixed) by continuity, since the left hand side of (196) is analytic in z near zero by Remark 5.2. Moreover, by Corollary 5.3 the left hand side and therefore also the right hand side of (197) only depend on a truncation of p up to order $n+deg(f)$. Especially the left hand side is well defined and the stated equality holds for a formal power series with rational coefficients $p(x) \in \mathbb{Q}[[x_\rho \mid \rho \in \Sigma(1)]]$. □

Remark 5.15. By (197), the operator $p(\frac{\partial}{\partial h})\left(\int_{P_y(h)} f(z)\, dm\right)_{|h=0}$ depends only on the equivalence class of $[p(x)] \in \widehat{SR_{\mathbb{Q}}(\Sigma)}$ and not on the chosen formal power series representative.

As a consequence of Theorem 5.8 and Remark 5.10, we get the following local version of Corollary 5.11:

Corollary 5.16. *Let $p(\frac{\partial}{\partial h}) \in \mathbb{Q}\{\frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)\} \subset \mathbb{Q}[[\frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)]]$ be an infinite order differential operator with constant rational coefficients, i.e., obtained by substituting $x_\rho \mapsto \frac{\partial}{\partial h_\rho}$ into a convergent (near zero) power series with rational coefficients $p(x) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\} \subset \mathbb{Q}[[x_\rho \mid \rho \in \Sigma(1)]]$. Fix a vertex $v \in P$ with tangent cone $Tan(P,v)$, and let $\sigma \in \Sigma(n)$ be the corresponding cone. Then, for z small enough and $-z \in Int(\sigma)$, we have the following formula:*

$$p(\frac{\partial}{\partial h})\left(\int_{Tan(P,v)(h)} e^{\langle m,z\rangle}\, dm\right)_{|h=0} = \langle \frac{e^v \cdot \left(i_\sigma^* p(F_\rho)\right)}{Eu_X^{\mathbb{T}}(x_\sigma)}, z\rangle . \tag{198}$$

If $p_n \in \mathbb{Q}[\frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)]$ is the corresponding truncation of p up to order n, then both sides of (198) applied to p_n converge for $n \to \infty$ locally uniformly in these z to (198) applied to p.

We also have the following.

Proposition 5.17. *Let $X = X_\Sigma$ be a projective simplicial toric variety. For any class $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$, its equivariant Todd class is an element in the analytic cohomology ring of X, i.e.,*

$$\mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) \in (H_{\mathbb{T}}^*(X;\mathbb{Q}))^{an} \subset \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q}).$$

Proof. For $\mathscr{F} = \mathscr{O}_X$, this follows from the explicit formula given in (130). More generally, this holds for sheaves $\mathscr{F} = \pi_*(\mathscr{O}_W \otimes \mathbb{C}_{\chi^{\tilde{m}}})^G$ as in the explicit formula (138). As in the proof of [13][Cor.1.2], the equivariant Grothendieck group $K_0^{\mathbb{T}}(X)$ is generated by classes of sheaves of the form $\pi_*(\mathscr{O}_W \otimes \mathbb{C}_{\chi^{\tilde{m}}})^G$. □

Altogether, we get the following abstract Euler-Maclaurin formula coming from the equivariant Hirzebruch-Riemann-Roch theorem.

Theorem 5.18. *Let $X = X_P$ be the projective simplicial toric variety associated to a full-dimensional simple lattice polytope $P \subset M_{\mathbb{R}}$. Let $\Sigma := \Sigma_P$ be the inner normal fan of P, and $D := D_P$ the ample Cartier divisor associated to P. Let $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$ be fixed, and choose a convergent power series $p(x_\rho) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}$ so that $p(F_\rho) = \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) \in \left(H_{\mathbb{T}}^*(X;\mathbb{Q})\right)^{an}$. Then*

(199)
$$p(\frac{\partial}{\partial h})\left(\int_{P(h)} e^{\langle m,z\rangle}\, dm\right)_{|_{h=0}} = \langle \chi^{\mathbb{T}}(X, \mathscr{O}_X(D)\otimes \mathscr{F}), z\rangle$$
$$= \sum_{m\in M}\left(\sum_{i=0}^{n}(-1)^i \cdot \dim_{\mathbb{C}} H^i(X;\mathscr{O}_X(D)\otimes\mathscr{F})_{\chi^{-m}}\right)\cdot e^{\langle m,z\rangle},$$

as analytic functions in z with z small enough, and with $\chi^{\mathbb{T}}(X,\mathscr{O}_X(D)\otimes\mathscr{F}) \in (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}}$ the cohomological equivariant Euler characteristic of $\mathscr{O}_X(D)\otimes\mathscr{F}$.

Proof. Equation (196) for $y = 0$ can now be calculated as
$$p(\frac{\partial}{\partial h})\left(\int_{P(h)} e^{\langle m,z\rangle}\, dm\right)_{|_{h=0}} = \langle \int_X e^{c_1^{\mathbb{T}}(\mathscr{O}_X(D))}\mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]), z\rangle$$
$$= \langle \chi^{\mathbb{T}}(X,\mathscr{O}_X(D)\otimes\mathscr{F}), z\rangle,$$
where the last equality follows from the equivariant Hirzebruch-Riemann-Roch formula (83). The second equality of (199) follows from (71) which uses the eigenspace decomposition, with the minus sign of (199) due to the appearance of $c(m) = -s(m)$ in (71). Recall that we work with the identification $s : Sym_{\mathbb{Q}}(M) \simeq H^*_{\mathbb{T}}(pt;\mathbb{Q}) =: (\Lambda_{\mathbb{T}})_{\mathbb{Q}}$. Finally, in the proof we first need to assume that z is small enough and away from the hyperplanes $\langle i^*_\sigma F_\rho, z\rangle = 0$ for each ray $\rho \in \sigma(1)$ of $\sigma \in \Sigma(n)$, since the localization formula is used; however, formula (199) then holds for all z small enough, by Remark 5.13. □

Evaluating (199) at $z = 0$, we get the following generalized *volume formula*.

Corollary 5.19. *In the notations of the previous theorem, we have*

(200)
$$p(\frac{\partial}{\partial h})(vol\ P(h))_{|_{h=0}} = \chi(X,\mathscr{O}_X(D)\otimes\mathscr{F}),$$

with vol $P(h) = \int_{P(h)} dm$ the volume of $P(h)$, and the Lebesgue measure normalized so that the unit cube in $M \subset M_{\mathbb{R}}$ has volume 1.

Example 5.20. The classical volume formula [12][Thm.2.15] corresponds to $\mathscr{F} = \mathscr{O}_X$ for p given by the Todd operator $Todd(\frac{\partial}{\partial h})$ of (203) below, with $\chi(X,\mathscr{O}_X(D)) = |P\cap M|$. □

Corollary 5.21. *Let $X = X_P$ be the projective simplicial toric variety associated to a full-dimensional simple lattice polytope $P \subset M_{\mathbb{R}}$. Let $\Sigma := \Sigma_P$ be the inner normal fan of P, and $D := D_P$ the ample Cartier divisor associated to P. Let $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$ be fixed, and choose a formal power series $p(x_\rho) \in \mathbb{Q}[[x_\rho \mid \rho \in \Sigma(1)]]$ so that $p(F_\rho) = \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$. Then for a polynomial function f on $M_{\mathbb{R}}$, we have:*

(201)
$$p(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\, dm\right)_{|_{h=0}} = \sum_{m\in M}\left(\sum_{i=0}^{n}(-1)^i \cdot \dim_{\mathbb{C}} H^i(X;\mathscr{O}_X(D)\otimes\mathscr{F})_{\chi^{-m}}\right)\cdot f(m).$$

Proof. This follows from Corollary 5.14 by applying the operator $\left(f(\frac{\partial}{\partial z})\right)_{|_{z=0}}$ to the last term of formula (199), seen as a formal power series in z. □

Let us finish this section with a local counterpart of Theorem 5.18. Using Remark 5.10, Corollary 5.16 and Proposition 4.13, we get:

Proposition 5.22. *Let $X = X_P$ be the projective simplicial toric variety associated to a full-dimensional simple lattice polytope $P \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$. Let $\Sigma := \Sigma_P$ be the inner normal fan of P, and $D := D_P$ the ample Cartier divisor associated to P. Let $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$ be fixed, and choose a convergent power series $p(x_\rho) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}$ so that $p(F_\rho) = \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) \in \left(H_{\mathbb{T}}^*(X;\mathbb{Q})\right)^{an}$. Fix a vertex $v \in P$ with tangent cone $Tan(P,v)$, and let $\sigma \in \Sigma(n)$ be the corresponding cone. Then, for z small enough with $-z \in Int(\sigma)$, we have the following formula:*

$$p(\frac{\partial}{\partial h})\left(\int_{Tan(P,v)(h)} e^{\langle m,z\rangle}\, dm\right)_{|h=0} = \langle e^v \cdot \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}])_{x_\sigma}, z\rangle = \langle e^v \cdot \mathrm{ch}^{\mathbb{T}}((\mathbb{S}\circ \chi_\sigma^{\mathbb{T}})(\mathscr{F})), z\rangle \,. \tag{202}$$

5.1. **Examples of Euler-Maclaurin formulae.** In this section, we explain how various special cases of Corollary 5.21 and Proposition 5.22 yield old and new Euler-Maclaurin type formulae.

In the global context of Corollary 5.21, let $X = X_P$ be the projective simplicial toric variety associated to a full-dimensional simple lattice polytope $P \subset M_{\mathbb{R}}$. Let $\Sigma := \Sigma_P$ be the inner normal fan of P, and $D := D_P$ the ample Cartier divisor associated to P. We fix a polynomial g on $M_{\mathbb{R}}$, and let $f(m) = g(m)\cdot e^{\langle m,z\rangle}$ be a quasi-polynomial, with $z \in N_{\mathbb{C}}$ small enough. In the context of a tangent cone at a vertex v of P as in Proposition 5.22, we only use an exponential function $f(m) = e^{\langle m,z\rangle}$ with $z \in N_{\mathbb{C}}$ small enough and $-z \in Int(\sigma)$. In this local case, $p(x)$ is restricted to the variables F_ρ, $\rho \in \sigma(1)$ (as in Remark 5.10), with $\sigma \in \Sigma(n)$ the cone corresponding to v. In the concrete formulas below, this amounts to using the cone σ as a fan instead of Σ, and the finite group G_σ instead of G_Σ.

Example 5.23. The first case we consider is the classical one of the *Euler-Maclaurin formula for the polytope P*. Here we choose $\mathscr{F} = \mathscr{O}_X$ and the infinite order differential operator

$$Todd(\frac{\partial}{\partial h}) := \sum_{g\in G_\Sigma}\prod_{\rho\in\Sigma(1)} \frac{\frac{\partial}{\partial h_\rho}}{1 - a_\rho(g)\cdot e^{-\frac{\partial}{\partial h_\rho}}} \in \mathbb{Q}\{\frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)\}\,, \tag{203}$$

as dictated by formula (130) for $\mathrm{td}_*^{\mathbb{T}}(X)$. Then the equivariant Euler characteristic formula (75) yields the Euler-Maclaurin formula of Brion-Vergne [13][Thm.4.5], extended here to a quasi-polynomial f:

$$Todd(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\, dm\right)_{|h=0} = \sum_{m\in P\cap M} f(m)\,. \tag{204}$$

As explain before, the left-hand side of (204) only depends of the class of $Todd(\frac{\partial}{\partial h})$ in the analytic Stanley-Reisner ring of X. For example, another such representative corresponds to the power series fitting with (the equivariant version of) Theorem 2.6, expressing the Todd operator in terms of suitable L-class versions. If P is a Delzant polytope, one recovers formula (184) of Khovanskii-Pukhlikov, fitting also with the equivariant Todd class of smooth projective toric varieties (e.g., obtained by setting $y = 0$ in Corollary 3.23).

In the local case of the tangent cone $Tan(P,v)$ of P at a vertex v, using (191) we get (see also [38][Thm.1], [34][eqn.(B.1)]):

$$Todd(\frac{\partial}{\partial h})\left(\int_{Tan(P,v)(h)} e^{\langle m,z\rangle}\, dm\right)_{|_{h=0}} = e^{\langle v,z\rangle}\cdot \sum_{m\in Tan(P,v)\cap M} e^{\langle m,z\rangle}\,. \tag{205}$$

□

Example 5.24. An Euler-Maclaurin formula for the *interior* of a simple lattice polytope P can be obtained similarly by using the *dual Todd operator*

$$Todd^{\vee}(\frac{\partial}{\partial h}) := \sum_{g\in G_\Sigma}\prod_{\rho\in\Sigma(1)} \frac{a_\rho(g)\cdot\frac{\partial}{\partial h_\rho}\cdot e^{-\frac{\partial}{\partial h_\rho}}}{1-a_\rho(g)\cdot e^{-\frac{\partial}{\partial h_\rho}}} \in \mathbb{Q}\{\frac{\partial}{\partial h_\rho}\mid \rho\in\Sigma(1)\}\,, \tag{206}$$

corresponding to the sheaf $\mathscr{F}=\omega_X$ and its dual Todd class $\mathrm{td}_*^{\mathbb{T}}([\omega_X]_{\mathbb{T}})$ of formula (131). In particular, by the equivariant Euler characteristic formula (76) one gets:

$$Todd^{\vee}(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\, dm\right)_{|_{h=0}} = \sum_{m\in \mathrm{Int}(P)\cap M} f(m)\,. \tag{207}$$

In the local case of the tangent cone $Tan(P,v)$ of P at a vertex v, using (192) we get:

$$Todd^{\vee}(\frac{\partial}{\partial h})\left(\int_{Tan(P,v)(h)} e^{\langle m,z\rangle}\, dm\right)_{|_{h=0}} = e^{\langle v,z\rangle}\cdot \sum_{m\in Int(Tan(P,v))\cap M} e^{\langle m,z\rangle}\,. \tag{208}$$

□

More generally, one can prove an *Euler-Maclaurin formula for a polytope with some facets removed*, see also [34][Prop.7.2] for the case of Delzant polytopes. Let P be a full-dimensional polytope with r facets $F_1,\dots,F_r$. For a subset $K\subseteq\{1,\dots,r\}$, let P^K be the set obtained from P by removing the facets F_i, $i\in K$. For example, $P^{\emptyset}=P$ and $P^{\{1,\dots,r\}}=\mathrm{Int}(P)$. Let $\Sigma(1)=\{\rho_1,\dots,\rho_r\}$ be the rays of the inner normal fan of P, and denote by $h=(h_1,\dots,h_r)$ a vector of real numbers indexed by the facets of P (i.e., $h_i := h_{\rho_i}$). Consider the following operator in $\mathbb{Q}\{\frac{\partial}{\partial h_i}\mid i=1,\dots,r\}$:

$$Todd^{K}(\frac{\partial}{\partial h}) := \sum_{g\in G_\Sigma}\prod_{i\in K} \frac{a_{\rho_i}(g)\cdot\frac{\partial}{\partial h_i}\cdot e^{-\frac{\partial}{\partial h_i}}}{1-a_{\rho_i}(g)\cdot e^{-\frac{\partial}{\partial h_i}}}\cdot\prod_{i\notin K}\frac{\frac{\partial}{\partial h_i}}{1-a_{\rho_i}(g)\cdot e^{-\frac{\partial}{\partial h_i}}}, \tag{209}$$

corresponding to formula (139) for the equivariant Todd class of $[\mathscr{F}]=mC_0^{\mathbb{T}}([U\hookrightarrow X])$, with $U=X\setminus D_K$ the open complement of the divisor $D_K=\bigcup_{i\in K}D_i$ and $D_i:=D_{\rho_i}$. Moreover, by (140) and Corollary 3.15 for $y=0$, applied to X and D_K, one gets a corresponding equivariant Hirzebruch-Riemann-Roch formula for $\mathrm{td}_*^{\mathbb{T}}(\mathscr{O}_X(D)\otimes mC_0^{\mathbb{T}}([U\hookrightarrow X]))$, namely

$$\int_X \mathrm{td}_*^{\mathbb{T}}(\mathscr{O}_X(D)\otimes mC_0^{\mathbb{T}}([U\hookrightarrow X])) = \sum_{m\in P^K\cap M} e^{s(m)}.$$

For its local counterpart, we use (172) for $y=0$. With the above notations, together with (192), this then gives the following.

Theorem 5.25. *Let P be a full-dimensional simple lattice polytope in $M_{\mathbb{R}}$, and let f be a polynomial function on $M_{\mathbb{R}}$. Then:*

$$Todd^K(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\, dm\right)_{|h=0} = \sum_{m\in P^K\cap M} f(m). \tag{210}$$

In the local case of the tangent cone $Tan(P,v)$ of P at a vertex v, we get:

$$Todd^K(\frac{\partial}{\partial h})\left(\int_{Tan(P,v)(h)} e^{\langle m,z\rangle}\, dm\right)_{|h=0} = e^{\langle v,z\rangle}\cdot \sum_{m\in Tan(P,v)^K\cap M} e^{\langle m,z\rangle}. \tag{211}$$

We next explain an *Euler-Maclaurin formula for the interior of a face of a simple lattice polytope*. Let P be a full-dimensional simple lattice polytope in $M_{\mathbb{R}}$, and fix a face E of P. Let $\sigma := \sigma_E$ be the corresponding cone in the inner normal fan $\Sigma = \Sigma_P$ of P, with $V_\sigma = V_{\sigma_E} = X_E$ the closure of the orbit of σ in $X = X_P$. Denote by $i_E = i_\sigma : V_\sigma \hookrightarrow X$ the closed inclusion map. Then V_σ is a simplicial toric variety whose fan is $Star(\sigma)$, as defined in (28), which is built from cones $\tau\in\Sigma$ that have σ as a face. Recall that $\mathbb{T} = T_N$ acts on V_σ via the morphism $T_N \to T_{N(\sigma)}$. Then we get, as in Corollary 3.25 and with the above notations, the following formula:

$$\mathrm{td}_*^{\mathbb{T}}([\omega_{V_\sigma}]_{\mathbb{T}}) = i_\sigma^*\left(\sum_{g\in G_{Star(\sigma)}} \prod_{\rho\in Star(\sigma)(1)} \frac{a_\rho(g)\cdot F_\rho\cdot e^{-F_\rho}}{1-a_\rho(g)\cdot e^{-F_\rho}}\right). \tag{212}$$

Together with (62) and the projection formula, this gives the equivariant Todd class of $[\mathscr{F}] = [(i_\sigma)_*\omega_{V_\sigma}]$:

$$\mathrm{td}_*^{\mathbb{T}}([(i_\sigma)_*\omega_{V_\sigma}]) = \sum_{g\in G_{Star(\sigma)}} \mathrm{mult}(\sigma)\cdot \prod_{\rho\in\sigma(1)} F_\rho \cdot \prod_{\rho\in Star(\sigma)(1)} \frac{a_\rho(g)\cdot F_\rho\cdot e^{-F_\rho}}{1-a_\rho(g)\cdot e^{-F_\rho}}. \tag{213}$$

Using (120), this yields the following result (where we use the above notations):

Theorem 5.26. *Let P be a full-dimensional simple lattice polytope in $M_{\mathbb{R}}$, and let f be a polynomial function on $M_{\mathbb{R}}$. Then, for a fixed face E of P, with corresponding cone σ in the inner normal fan Σ of P, one has:*

$$Todd_E^\vee(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\, dm\right)_{|h=0} = \sum_{m\in \mathrm{Relint}(E)\cap M} f(m), \tag{214}$$

where

$$Todd_E^\vee(\frac{\partial}{\partial h}) := \sum_{g\in G_{Star(\sigma)}} \mathrm{mult}(\sigma)\cdot \prod_{\rho\in\sigma(1)} \frac{\partial}{\partial h_\rho}\cdot \prod_{\rho\in Star(\sigma)(1)} \frac{a_\rho(g)\cdot \frac{\partial}{\partial h_\rho}\cdot e^{-\frac{\partial}{\partial h_\rho}}}{1-a_\rho(g)\cdot e^{-\frac{\partial}{\partial h_\rho}}}. \tag{215}$$

We next discuss a similar *Euler-Maclaurin formula for a face of a simple lattice polytope*. With the same notations, we have by (130) the following formula:

$$\mathrm{td}_*^{\mathbb{T}}([\mathscr{O}_{V_\sigma}]_{\mathbb{T}}) = i_\sigma^*\left(\sum_{g\in G_{Star(\sigma)}} \prod_{\rho\in Star(\sigma)(1)} \frac{F_\rho}{1-a_\rho(g)\cdot e^{-F_\rho}}\right). \tag{216}$$

As before, the projection formula yields the equivariant Todd class of $[\mathscr{F}] = [(i_\sigma)_* \mathscr{O}_{V_\sigma}]$:

$$\mathrm{td}_*^{\mathbb{T}}([(i_\sigma)_* \mathscr{O}_{V_\sigma}]) = \sum_{g \in G_{Star(\sigma)}} \mathrm{mult}(\sigma) \cdot \prod_{\rho \in \sigma(1)} F_\rho \cdot \prod_{\rho \in Star(\sigma)(1)} \frac{F_\rho}{1 - a_\rho(g) \cdot e^{-F_\rho}}. \tag{217}$$

Using (121), this yields the following result:

Theorem 5.27. *Let P be a full-dimensional simple lattice polytope in $M_\mathbb{R}$, and let f be a polynomial function on $M_\mathbb{R}$. Then, for a fixed face E of P, with corresponding cone σ in the inner normal fan Σ of P, one has:*

$$Todd_E(\frac{\partial}{\partial h}) \left(\int_{P(h)} f(m)\, dm \right)_{|_{h=0}} = \sum_{m \in E \cap M} f(m), \tag{218}$$

where

$$Todd_E(\frac{\partial}{\partial h}) := \sum_{g \in G_{Star(\sigma)}} \mathrm{mult}(\sigma) \cdot \prod_{\rho \in \sigma(1)} \frac{\partial}{\partial h_\rho} \cdot \prod_{\rho \in Star(\sigma)(1)} \frac{\frac{\partial}{\partial h_\rho}}{1 - a_\rho(g) \cdot e^{-\frac{\partial}{\partial h_\rho}}}. \tag{219}$$

Other interesting coherent sheaves to consider are the Zariski sheaves $\widehat{\Omega}_X^p$ of p-forms on the toric variety X. In the next section, these will be considered all at once via the formal sum

$$\bigoplus_p [\widehat{\Omega}_X^p]_\mathbb{T} \cdot y^p \in K_0^\mathbb{T}(X)[y],$$

and similarly for suitable motivic Chern classes $mC_y^\mathbb{T}$ of $\mathbb{T}$-invariant constructible subsets of X.

We leave it to the reader to specialize the generalized volume formula (200) to all situations discussed in this section.

Another way to obtain examples of explicit Euler-Maclaurin formulae is by twisting the coherent sheaf $\mathscr{F}$ by $\mathscr{O}_X(D'-D)$, for $D = D_P$ the original ample divisor associated to the full-dimensional lattice polytope, and D' any $\mathbb{T}$-invariant Cartier divisor on X. By the multiplicativity of the equivariant Todd class transformation for the coherent sheaf $\mathscr{F}' = \mathscr{O}_X(D'-D) \otimes \mathscr{F}$, we have

$$\mathrm{td}_*^\mathbb{T}([\mathscr{F}']) = e^{[D'-D]_\mathbb{T}} \cdot \mathrm{td}_*^\mathbb{T}([\mathscr{F}]).$$

So, if $p(x_\rho) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}$ is a convergent power series with $p(F_\rho) = td_*^\mathbb{T}([\mathscr{F}])$, then $p'(x_\rho) := e^{\sum_{\rho \in \Sigma(1)} d_\rho x_\rho} \cdot p(x_\rho) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}$ is a convergent power series with $p'(F_\rho) = td_*^\mathbb{T}([\mathscr{F}'])$, where $D' - D = \sum_{\rho \in \Sigma(1)} d_\rho D_\rho$ as a $\mathbb{T}$-invariant Cartier divisor.

Example 5.28. As a last concrete example of this section, assume D' is a *globally generated* $\mathbb{T}$-invariant Cartier divisor on X, with associated (not necessarily full-dimensional) lattice polytope $P_{D'} \subset M_\mathbb{R}$. Consider the infinite order differential operator

$$Todd'(\frac{\partial}{\partial h}) := e^{\sum_{\rho \in \Sigma(1)} d_\rho \cdot \frac{\partial}{\partial h_\rho}} \cdot Todd(\frac{\partial}{\partial h}) \tag{220}$$

with $Todd(\frac{\partial}{\partial h})$ as in (203) and d_ρ's as above. Then the equivariant Euler characteristic formula (79) yields the following new Euler-Maclaurin formula for a quasi-polynomial f.

$$Todd'(\frac{\partial}{\partial h}) \left(\int_{P(h)} f(m)\, dm \right)_{|_{h=0}} = \sum_{m \in P_{D'} \cap M} f(m)\,. \tag{221}$$

Note that $P_{D'}$ is an $\mathbb{N}$-Minkowski summand (in the sense of [16][Def.6.2.11]) of the original polytope P, and any such $\mathbb{N}$-Minkowski summand comes from a globally generated Cartier divisor D', see [16][Cor.6.2.15]. □

6. Weighted Euler-Maclaurin formulae

Let $X = X_P$ be the projective simplicial toric variety associated to a full-dimensional simple lattice polytope $P \subset M_{\mathbb{R}} \simeq \mathbb{R}^n$. Let $\Sigma := \Sigma_P$ be the inner normal fan of P, and $D := D_P$ the ample Cartier divisor associated to P.

Let a convergent power series $p(x_\rho) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}$ be given. Additionally, we may start with a polynomial in y with coefficients consisting of such power series, as we will need in the applications of this section. Define a corresponding *renormalized series*

$$p_y(x_\rho) := \frac{p\left((1+y)x_\rho\right)}{(1+y)^n} \in \mathbb{Q}[y,(1+y)^{-1}][[x_\rho \mid \rho \in \Sigma(1)]].$$

To treat it as a convergent power series, one needs to assume that $y \in \mathbb{R} \setminus \{1\}$ is fixed or it belongs to a compact subset.

Let $p_y(\frac{\partial}{\partial h})$ be the corresponding parametrized infinite order differential operator obtained from $p_y(x_\rho)$ by substituting $x_\rho \mapsto \frac{\partial}{\partial h_\rho}$, for all $\rho \in \Sigma(1)$. Then formula (196) translates into the following:

$$(222)\quad p_y(\frac{\partial}{\partial h})\left(\int_{P_y(h)} e^{\langle m,z\rangle}\, dm\right)_{|h=0} = \sum_{\sigma\in\Sigma(n)} \frac{e^{\langle (1+y)\cdot i_\sigma^* c_1^{\mathbb{T}}(\mathscr{O}_X(D_P)),z\rangle}}{(1+y)^n \cdot \langle Eu_X^{\mathbb{T}}(x_\sigma),z\rangle} \cdot \langle p((1+y)\cdot i_\sigma^* F_\rho),z\rangle\,.$$

Let $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$ be fixed, and choose a convergent power series $p(x_\rho) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}$ so that $p(F_\rho) = \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) \in \left(H_{\mathbb{T}}^*(X;\mathbb{Q})\right)^{an}$. Applying the proof of Theorem 5.18 to formula (222), one then gets

$$(223)\quad p_y(\frac{\partial}{\partial h})\left(\int_{P_y(h)} e^{\langle m,z\rangle}\, dm\right)_{|h=0} = \sum_{m\in M}\left(\sum_{i=0}^n (-1)^i \cdot \dim_{\mathbb{C}} H^i(X;\mathscr{O}_X(D)\otimes\mathscr{F})_{\chi^{-m}}\right)\cdot e^{\langle (1+y)m,z\rangle}\,.$$

Here the use of the renormalized power series p_y and polytope $P_y(h)$ correspond to multiplying each degree $2k$ equivariant cohomology class by $(1+y)^k$. (This is just the cohomological Adams operation $\Psi^{(1+y)}$.) In particular, via the identification $s : M \simeq H_{\mathbb{T}}^2(pt;\mathbb{Z})$, $m \in M$ gets multiplied by $1+y$.

For any polynomial f defined on $M_{\mathbb{R}}$, by applying the operator $\left(f(\frac{\partial}{\partial z})\right)_{|z=0}$ to formula (223), we get the following *parametrized Euler-Maclaurin formula*:

$$(224)\quad p_y(\frac{\partial}{\partial h})\left(\int_{P_y(h)} f(m)\, dm\right)_{|h=0} = \sum_{m\in M}\left(\sum_{i=0}^n (-1)^i \cdot \dim_{\mathbb{C}} H^i(X;\mathscr{O}_X(D)\otimes\mathscr{F})_{\chi^{-m}}\right)\cdot f\left((1+y)m\right)\,.$$

Moreover, if f is homogeneous of degree $\deg(f)$, then $f\left((1+y)m\right) = (1+y)^{\deg(f)} \cdot f(m)$.

Remark 6.1. In the local case of Euler-Maclaurin formulae for a tangent cone $Tan(P,v)$, we only work with the unnormalized equivariant Hirzebruch classes $T_{y*}^{\mathbb{T}}$ and the dilatation $Tan(P,v)(h)$ of the tangent cone. In this context, a renormalization is not needed.

6.1. **Weighted Euler-Maclaurin formulae.** In this section, we apply the above renormalization not just to classes of coherent sheaves, but directly to examples of the type

$$\bigoplus_p [\widehat{\Omega}^p_X]_{\mathbb{T}} \cdot y^p \in K_0^{\mathbb{T}}(X)[y],$$

as well as for suitable motivic Chern classes $mC_y^{\mathbb{T}}$ of $\mathbb{T}$-invariant constructible subsets of X.

Recall from Section 3.3 that the equivariant Hirzebruch classes $T_{y*}^{\mathbb{T}}(X) \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})[y]$, resp., $\widehat{T}_{y*}^{\mathbb{T}}(X) \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q}[y])$ of X are computed as:

$$T_{y*}^{\mathbb{T}}(X) = (1+y)^{n-r} \cdot \sum_{g\in G_\Sigma} \prod_{\rho\in\Sigma(1)} \frac{F_\rho \cdot \left(1+y\cdot a_\rho(g)\cdot e^{-F_\rho}\right)}{1-a_\rho(g)\cdot e^{-F_\rho}}, \tag{225}$$

$$\widehat{T}_{y*}^{\mathbb{T}}(X) = \sum_{g\in G_\Sigma} \prod_{\rho\in\Sigma(1)} \frac{F_\rho \cdot \left(1+y\cdot a_\rho(g)\cdot e^{-F_\rho(1+y)}\right)}{1-a_\rho(g)\cdot e^{-F_\rho(1+y)}}, \tag{226}$$

with $r = |\Sigma(1)|$, and $F_\rho = [D_\rho]_{\mathbb{T}}$ denoting the equivariant fundamental class of the $\mathbb{T}$-invariant divisor D_ρ corresponding to the ray $\rho \in \Sigma(1)$.

The weighted lattice point counting of Theorem 2.3 (and Remark 2.4), coupled with the above expressions for the equivariant Hirzebruch classes, suggest that a weighted Euler-Maclaurin-type formula can be computed by using the Hirzebruch (or, parametrized Todd) differential operators defined by:

$$T_y(\frac{\partial}{\partial h}) := (1+y)^{n-r} \cdot \sum_{g\in G_\Sigma} \prod_{\rho\in\Sigma(1)} \frac{\frac{\partial}{\partial h_\rho} \cdot \left(1+y\cdot a_\rho(g)\cdot e^{-\frac{\partial}{\partial h_\rho}}\right)}{1-a_\rho(g)\cdot e^{-\frac{\partial}{\partial h_\rho}}}$$

and

$$\widehat{T}_y(\frac{\partial}{\partial h}) := \sum_{g\in G_\Sigma} \prod_{\rho\in\Sigma(1)} \frac{\frac{\partial}{\partial h_\rho} \cdot \left(1+y\cdot a_\rho(g)\cdot e^{-\frac{\partial}{\partial h_\rho}(1+y)}\right)}{1-a_\rho(g)\cdot e^{-\frac{\partial}{\partial h_\rho}(1+y)}}.$$

The passage from $T_y(\frac{\partial}{\partial h})$ to $\widehat{T}_y(\frac{\partial}{\partial h})$ is just a special case of our general procedure of moving from $p(\frac{\partial}{\partial h})$ to $p_y(\frac{\partial}{\partial h})$. This therefore fits with the renormalization of Hirzebruch classes, from $T_{y*}^{\mathbb{T}}(X)$ to $\widehat{T}_{y*}^{\mathbb{T}}(X)$.

Weighted Euler-Maclaurin formulae were considered from a combinatorial point of view in [30][Thm.1.1 and Rem.1.2] for $T_y(\frac{\partial}{\partial h})$, and more recently in [5][Thm.1.5] for $\widehat{T}_{y*}^{\mathbb{T}}(X)$. We reprove here the following result (see [30][Thm.1.1 and Rem.1.2] for $T_y(\frac{\partial}{\partial h})$, and [5][Thm.4.6] for $\widehat{T}_y(\frac{\partial}{\partial h})$), from the point of view of generalized Hirzebruch-Riemann-Roch formula:

Theorem 6.2. *For $z \in N_{\mathbb{C}}$ small enough, one has in the above notations:*

$$T_y(\frac{\partial}{\partial h}) \left(\int_{P(h)} e^{\langle m,z\rangle} dm \right)_{|h=0} = \sum_{E\preceq P} (1+y)^{\dim(E)} \sum_{m\in \mathrm{Relint}(E)\cap M} e^{\langle m,z\rangle}. \tag{227}$$

$$\widehat{T}_y(\frac{\partial}{\partial h}) \left(\int_{P_y(h)} e^{\langle m,z\rangle} dm \right)_{|h=0} = \sum_{E\preceq P} (1+y)^{\dim(E)} \sum_{m\in \mathrm{Relint}(E)\cap M} e^{\langle (1+y)m,z\rangle}. \tag{228}$$

In the corresponding local case of the tangent cone $Tan(P,v)$ of P at a vertex v, for $z \in N_{\mathbb{C}}$ small enough with $-z \in Int(\sigma)$, we get

(229)
$$T_y(\frac{\partial}{\partial h})\left(\int_{Tan(P,v)(h)} e^{\langle m,z\rangle}\, dm\right)_{|h=0} = \sum_{E\preceq Tan(P,v)} (1+y)^{\dim(E)}\cdot e^{\langle v,z\rangle}\cdot \sum_{m\in \mathrm{Relint}(E)\cap M} e^{\langle m,z\rangle}.$$

Proof. To deduce formula (227), instead of $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$ one considers the motivic Chern class $mC_y^{\mathbb{T}}(X) \in K_0^{\mathbb{T}}(X)[y]$, with $T_{y*}(X) := \mathrm{td}_*^{\mathbb{T}}(mC_y^{\mathbb{T}}(X))$. We work with the power series $p(x_\rho) := T_y(x_\rho) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}[y]$ fitting with the differential operator $T_y(\frac{\partial}{\partial h})$. Formula (227) follows now from (199) (linearly extended in y), using the equivariant Hirzebruch-Riemann-Roch formula of Corollary 3.15, applied to the case $P' = P$. Formula (228) is then obtained via renormalization using (223).

For the local result, we use (166) together with (192). □

Using (224), one then obtains the following result (see also [30][Thm.1.1 and Rem.1.2] and [5][Thm.1.5]):

Corollary 6.3. *For any polynomial function f on $M_{\mathbb{R}}$, one has:*

$$T_y(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\, dm\right)_{|h=0} = \sum_{E\preceq P}(1+y)^{\dim(E)} \sum_{m\in \mathrm{Relint}(E)\cap M} f(m). \tag{230}$$

$$\widehat{T}_y(\frac{\partial}{\partial h})\left(\int_{P_y(h)} f(m)\, dm\right)_{|h=0} = \sum_{E\preceq P}(1+y)^{\dim(E)} \sum_{m\in \mathrm{Relint}(E)\cap M} f((1+y)m). \tag{231}$$

Note that the Brion-Verne Euler-Maclaurin formula for the simple lattice polytope P is obtained from either (230) or (231) by specializing to $y = 0$. Moreover, for $y = 1$, one gets Euler-Maclaurin formulas corresponding to operators related to suitable L-classes, i.e., $\widehat{T}_{1*}(X) = L_*(X)$ the Thom-Milnor L-class of X (see [36]). Taking the top degree in y on both sides of (230) recovers formula (207).

Next we prove a *weighted Euler-Maclaurin formula for a polytope with some facets removed*. Here and below, for simplicity, we only indicate such a formula for a polynomial f, while we leave it to the reader to formulate the corresponding exponential formula. Let P be a full-dimensional polytope with r facets $F_1,\dots,F_r$. For a subset $K \subseteq \{1,\dots,r\}$, let P^K be as before the set obtained from P by removing the facets F_i, $i \in K$. So $P^{\emptyset} = P$ and $P^{\{1,\dots,r\}} = \mathrm{Int}(P)$. Let $\Sigma(1) = \{\rho_1,\dots,\rho_r\}$ be the rays of the inner normal fan of P, and denote by $h = (h_1,\dots,h_r)$ a vector of real numbers indexed by the facets of P (i.e., $h_i := h_{\rho_i}$). Consider the following operator in $\mathbb{Q}\{\frac{\partial}{\partial h_i} \mid i = 1,\dots,r\}[y]$:

(232)
$$T_y^K(\frac{\partial}{\partial h}) := (1+y)^{n-r}\cdot \sum_{g\in G_\Sigma}\prod_{i\in K}\frac{(1+y)\cdot a_{\rho_i}(g)\cdot \frac{\partial}{\partial h_i}\cdot e^{-\frac{\partial}{\partial h_i}}}{1-a_{\rho_i}(g)\cdot e^{-\frac{\partial}{\partial h_i}}}\cdot \prod_{i\notin K}\frac{\frac{\partial}{\partial h_i}(1+y\cdot a_{\rho_i}(g)e^{-\frac{\partial}{\partial h_i}})}{1-a_{\rho_i}(g)\cdot e^{-\frac{\partial}{\partial h_i}}},$$

corresponding to formula (136) for the equivariant Todd class of $mC^{\mathbb{T}}_y([U \hookrightarrow X])$, with $U = X \setminus D_K$ the open complement of the divisor $D_K = \bigcup_{i\in K} D_i$ and $D_i := D_{\rho_i}$. By additivity,

$$\mathrm{td}^{\mathbb{T}}_*(mC^{\mathbb{T}}_y([U \hookrightarrow X])) = T_{y*}([U \hookrightarrow X]) = T_{y*}(X) - T_{y*}(D_K)$$

By Corollary 3.15, applied to X and D_K, one gets a corresponding equivariant Hirzebruch-Riemann-Roch formula:

$$\int_X \mathrm{td}^{\mathbb{T}}_*(\mathscr{O}_X(D) \otimes mC^{\mathbb{T}}_y([U \hookrightarrow X])) = \sum_{E \preceq P^K} (1+y)^{\dim(E)} \cdot \sum_{m \in \mathrm{Relint}(E)\cap M} e^{s(m)}. \tag{233}$$

With the above notations this then gives the following.

Theorem 6.4. *Let P be a full-dimensional simple lattice polytope in $M_{\mathbb{R}}$, and let f be a polynomial function on $M_{\mathbb{R}}$. Then:*

$$T^K_y(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\ dm\right)_{|h=0} = \sum_{E \preceq P^K} (1+y)^{\dim(E)} \cdot \sum_{m \in \mathrm{Relint}(E)\cap M} f(m). \tag{234}$$

By renormalization, $T^K_y(\frac{\partial}{\partial h})$ changes to

$$\widehat{T}^K_y(\frac{\partial}{\partial h}) := \sum_{g\in G_\Sigma} \prod_{i\in K} \frac{(1+y)\cdot a_{\rho_i}(g)\cdot \frac{\partial}{\partial h_i} \cdot e^{-(1+y)\frac{\partial}{\partial h_i}}}{1 - a_{\rho_i}(g)\cdot e^{-(1+y)\frac{\partial}{\partial h_i}}} \cdot \prod_{i\notin K} \frac{\frac{\partial}{\partial h_i}(1 + y\cdot a_{\rho_i}(g) e^{-(1+y)\frac{\partial}{\partial h_i}})}{1 - a_{\rho_i}(g)\cdot e^{-(1+y)\frac{\partial}{\partial h_i}}}$$

fitting with formula (137). We thus get by (224) the following

Corollary 6.5. *Let P be a full-dimensional simple lattice polytope in $M_{\mathbb{R}}$, and let f be a polynomial function on $M_{\mathbb{R}}$. Then:*

$$\widehat{T}^K_y(\frac{\partial}{\partial h})\left(\int_{P_y(h)} f(m)\ dm\right)_{|h=0} = \sum_{E \preceq P^K} (1+y)^{\dim(E)} \cdot \sum_{m \in \mathrm{Relint}(E)\cap M} f((1+y)m). \tag{235}$$

Remark 6.6. In the corresponding local case of the tangent cone $Tan(P,v)$ of P at a vertex v, for $z \in N_{\mathbb{C}}$ small enough with $-z \in Int(\sigma)$, we get by (155), (157) (172), together with (192), the following:

(236)

$$T^K_y(\frac{\partial}{\partial h})\left(\int_{Tan(P,v)(h)} e^{\langle m,z\rangle}\ dm\right)_{|h=0} = \sum_{E \preceq Tan(P,v)^K} (1+y)^{\dim(E)} \cdot e^{\langle v,z\rangle} \cdot \sum_{m \in \mathrm{Relint}(E)\cap M} e^{\langle m,z\rangle}.$$

We next discuss a *weighted Euler-Maclaurin formula for a face of a simple lattice polytope*. Let P be a full-dimensional simple lattice polytope in $M_{\mathbb{R}}$, and fix a face E of P. Let $\sigma := \sigma_E$ be the corresponding cone in the inner normal fan $\Sigma = \Sigma_P$ of P, with $V_\sigma = V_{\sigma_E} = X_E$ the closure of the orbit of σ in $X = X_P$. Denote by $i_E = i_\sigma : V_\sigma \hookrightarrow X$ the closed inclusion map. Then V_σ is a simplicial toric variety whose fan is $Star(\sigma)$, as defined in (28), which is built from cones $\tau \in \Sigma$ that have σ as a face. Recall that $\mathbb{T} = T_N$ acts on V_σ via the morphism $T_N \to T_{N(\sigma)}$. We have by (225) the following formula:

$$T^{\mathbb{T}}_{y*}([id_{V_\sigma}]) = (1+y)^{n-r} \cdot i^*_\sigma \left(\sum_{g\in G_{Star(\sigma)}} \prod_{\rho\in Star(\sigma)(1)} \frac{F_\rho \cdot \left(1 + y\cdot a_\rho(g)\cdot e^{-F_\rho}\right)}{1 - a_\rho(g)\cdot e^{-F_\rho}} \right). \tag{237}$$

The projection formula then yields the computation of the following equivariant Hirzebruch class:
(238)
$$T^{\mathbb{T}}_{y*}([V_\sigma \hookrightarrow X]) = (1+y)^{n-r} \cdot \sum_{g\in G_{Star(\sigma)}} \mathrm{mult}(\sigma) \cdot \prod_{\rho\in\sigma(1)} F_\rho \cdot \prod_{\rho\in Star(\sigma)(1)} \frac{F_\rho \cdot \left(1+y\cdot a_\rho(g)\cdot e^{-F_\rho}\right)}{1-a_\rho(g)\cdot e^{-F_\rho}}.$$

Using Corollary 3.15, this yields the following result:

Theorem 6.7. *Let P be a full-dimensional simple lattice polytope in $M_\mathbb{R}$, and let f be a polynomial function on $M_\mathbb{R}$. Then, for a fixed face E of P, with corresponding cone σ in the inner normal fan Σ of P, one has:*

$$T^E_y(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\ dm\right)_{|h=0} = \sum_{E'\preceq E} (1+y)^{\dim(E')} \cdot \sum_{m\in \mathrm{Relint}(E')\cap M} f(m), \tag{239}$$

where
(240)
$$T^E_y(\frac{\partial}{\partial h}) := (1+y)^{n-r} \cdot \sum_{g\in G_{Star(\sigma)}} \mathrm{mult}(\sigma) \cdot \prod_{\rho\in\sigma(1)} \frac{\partial}{\partial h_\rho} \cdot \prod_{\rho\in Star(\sigma)(1)} \frac{\frac{\partial}{\partial h_\rho}\left(1+y\cdot a_\rho(g)\cdot e^{-\frac{\partial}{\partial h_\rho}}\right)}{1-a_\rho(g)\cdot e^{-\frac{\partial}{\partial h_\rho}}}.$$

For $y=0$, formula (239) specializes to (218). Taking the top degree in y on both sides of (239) gives back formula (214).

By renormalization, $T^E_y(\frac{\partial}{\partial h})$ changes to

$$\widehat{T}^E_y(\frac{\partial}{\partial h}) := \sum_{g\in G_{Star(\sigma)}} \mathrm{mult}(\sigma) \cdot \prod_{\rho\in\sigma(1)} \frac{\partial}{\partial h_\rho} \cdot \prod_{\rho\in Star(\sigma)(1)} \frac{\frac{\partial}{\partial h_\rho}\left(1+y\cdot a_\rho(g)\cdot e^{-(1+y)\frac{\partial}{\partial h_\rho}}\right)}{1-a_\rho(g)\cdot e^{-(1+y)\frac{\partial}{\partial h_\rho}}}.$$

We thus get by (224) the following

Corollary 6.8. *In the above notations, we get*

$$\widehat{T}^E_y(\frac{\partial}{\partial h})\left(\int_{P_y(h)} f(m)\ dm\right)_{|h=0} = \sum_{E'\preceq E} (1+y)^{\dim(E')} \cdot \sum_{m\in \mathrm{Relint}(E')\cap M} f((1+y)m). \tag{241}$$

For $y=1$, one gets Euler-Maclaurin formulas corresponding to operators related to suitable L-classes, i.e., $\widehat{T}_{1*}(X_E) = L_*(X_E)$ the Thom-Milnor L-class of X_E.

Another way to obtain examples of explicit weighted Euler-Maclaurin formulae is by twisting the equivariant motivic Chern class $mC^{\mathbb{T}}_y$ by sheaves of the form $\mathcal{O}_X(D'-D)$, for $D=D_P$ the original ample divisor associated to the full-dimensional lattice polytope, and D' any $\mathbb{T}$-invariant Cartier divisor on X. In the next example, we illustrate this principle in the case of $mC^{\mathbb{T}}_y(X)$, corresponding to Theorem 6.2 and Corollary 6.3.

Example 6.9. Let D' be a globally generated $\mathbb{T}$-invariant Cartier divisor on X, with associated (not necessarily full-dimensional) lattice polytope $P_{D'} \subset M_\mathbb{R}$. Let $D'-D = \sum_{\rho\in\Sigma(1)} d_\rho D_\rho$ as a $\mathbb{T}$-invariant Cartier divisor. Let $X_{D'}$ be the toric variety of the lattice polytope $P_{D'}$, defined via the corresponding generalized fan as in [16][Prop.6.2.3]. By [16][Thm.6.2.8], there is a proper toric morphism $f : X \to X_{D'}$, induced by the corresponding lattice projection $N \to N_{D'}$ given by dividing out by the minimal cone of the generalized fan of $P_{D'}$. In particular, $f : X \to X_{D'}$ is a toric fibration. For σ' a cone in the generalized fan of $P_{D'}$, define as in (106)

$$d_\ell(X/\sigma') := |\Sigma_\ell(X/\sigma')|$$

with

$$\Sigma_\ell(X/\sigma') := \{\sigma \in \Sigma \mid O_\sigma \subset X,\ f(O_\sigma) = O_{\sigma'},\ \ell = \dim(O_\sigma) - \dim(O_{\sigma'})\}.$$

Consider the infinite order differential operators

$$T'_y(\frac{\partial}{\partial h}) := e^{\sum_{\rho\in\Sigma(1)} d_\rho \cdot \frac{\partial}{\partial h_\rho}} \cdot T_y(\frac{\partial}{\partial h}) \tag{242}$$

and

$$\widehat{T}'_y(\frac{\partial}{\partial h}) := e^{(1+y)\cdot\sum_{\rho\in\Sigma(1)} d_\rho \cdot \frac{\partial}{\partial h_\rho}} \cdot \widehat{T}_y(\frac{\partial}{\partial h}), \tag{243}$$

with $T_y(\frac{\partial}{\partial h}), \widehat{T}_y(\frac{\partial}{\partial h})$ as in Theorem 6.2 and Corollary 6.3. For any quasi-polynomial f on $M_\mathbb{R}$, one then has by (122) the following new weighted Euler-Maclaurin formulae:

$$\begin{aligned} T'_y(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\, dm\right)_{|h=0} = \\ = \sum_{E\preceq P_{D'}} \left(\sum_{\ell\geq 0} (-1)^\ell \cdot d_\ell(X/E)\cdot(1+y)^{\ell+\dim(E)}\right) \cdot \sum_{m\in \mathrm{Relint}(E)\cap M} f(m). \end{aligned} \tag{244}$$

$$\begin{aligned} \widehat{T}'_y(\frac{\partial}{\partial h})\left(\int_{P_y(h)} f(m)\, dm\right)_{|h=0} = \\ = \sum_{E\preceq P_{D'}} \left(\sum_{\ell\geq 0} (-1)^\ell \cdot d_\ell(X/E)\cdot(1+y)^{\ell+\dim(E)}\right) \cdot \sum_{m\in \mathrm{Relint}(E)\cap M} f((1+y)m). \end{aligned} \tag{245}$$

As noted in Example 5.28, $P_{D'}$ is an $\mathbb{N}$-Minkowski summand (in the sense of [16][Def.6.2.11]) of the original polytope P, see [16][Cor.6.2.15]. □

7. EULER-MACLAURIN FORMULAE VIA THE CAPPELL-SHANESON ALGEBRA

7.1. **Cappell-Shaneson algebra vs. completed equivariant cohomology ring.** In this section we recall and provide generalizations of the Euler-Maclaurin formula of Cappell-Shaneson [18, 43] and explain its connection to equivariant toric geometry. The Euler-Maclaurin formula of Cappell-Shaneson [18, 43] does not use a dilation $P(h)$ of the full-dimensional lattice polytope $P \subset M_\mathbb{R}$. Instead, a summation of integrals over the faces E of P is used (compare also with [14][Sect.3.7]). More precisely, for any polynomial function f on $M_\mathbb{R}$ one has:

$$\sum_{m\in P\cap M} f(m) = \sum_{E\preceq P} \int_E p_{\sigma_E}(\partial_i) f(m) dm\,, \tag{246}$$

with $p_{\sigma_E}(\partial_i) \in \mathbb{Q}[[\partial_1,\dots,\partial_n]]$ suitable infinite order differential operators with constant rational coefficients in the partial derivatives with respect to the coordinates of the vector space

$M_{\mathbb{R}}$. Here the Lebesgue measure dm on E is normalized so that the unit cube in the lattice $Span(E_0) \cap M$ has volume 1, with $E_0 := E - m_0$ a translation of E by a vertex $m_0 \in E$.

The infinite order differential operators $p_{\sigma_E}(\partial_i)$ are defined through some relations in what is called in [34][Sect.6] the *Cappell-Shaneson algebra* of P:

$$\mathscr{A}(P) := \mathbb{Q}[[\partial_1, \dots, \partial_n]][U_F \mid F \text{ a facet of } P]/\sim$$

with relations

$$U_{F_1} \cdots U_{F_k} = 0 \quad \text{for distinct } F_i \text{ with } F_1 \cap \dots \cap F_k = \emptyset, \tag{247}$$

and

$$\frac{\partial}{\partial m} + \sum_F \langle m, n_F \rangle U_F = 0 \quad \text{for all } m \in M \text{ (or a basis of } M). \tag{248}$$

Here $\frac{\partial}{\partial m}$ is the differentiation in the direction of m, with $n_F \in M$ the minimal lattice vector orthogonal to F and pointing into P. Let us rewrite this presentation of the algebra $\mathscr{A}(P)$ (given above in terms of the polytope P) into the language of this paper using another description of the completed Stanley-Reisner ring

$$\mathbb{Q}[[x_\rho \mid \rho \in \Sigma(1)]]/\sim_{SR} =: \widehat{SR_{\mathbb{Q}}(\Sigma)}$$

of the associated toric variety X_P and the inner normal fan $\Sigma = \Sigma_P$. Note that the inner normal vector n_F corresponds to the generator u_ρ of the ray $\rho \in \Sigma(1)$ corresponding to F. For simplicity, we fix a basis $m_1, \dots, m_n$ of $M \simeq \mathbb{Z}^n$, with t_i the corresponding coordinates on $M_{\mathbb{K}}$ (for $\mathbb{K} = \mathbb{Q}, \mathbb{R}$) with respect to this basis, so that $\frac{\partial}{\partial t_i} = \partial_i$ for $i = 1, \dots, n$. Then

$$\mathbb{Q}[t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)] \simeq \mathbb{Q}[\partial_1, \dots, \partial_n, U_F \mid F \text{ a facet of } P]$$

via $t_i \mapsto \partial_i$ and $x_\rho \mapsto U_F$, with F the facet of P corresponding to $\rho \in \Sigma(1)$. The relation (247) corresponds to the Stanly-Reisner relation $\sim_{SR}$, and the relation (248) translates into

$$t_i + \sum_{\rho \in \Sigma(1)} \langle m_i, n_\rho \rangle x_\rho = 0 \quad \text{for } i = 1, \dots, n. \tag{249}$$

Denote by $\sim$ both relations together on $\mathbb{Q}[t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)]$. These are also homogeneous for the usual grading doubled (i.e., with x_ρ and t_i of degree two). Then one gets (see also [28][Lecture13, Lem.2.2]):

Lemma 7.1. *The inclusion $\mathbb{Q}[x_\rho \mid \rho \in \Sigma(1)] \hookrightarrow \mathbb{Q}[t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)]$ induces an isomorphism of graded rings*

$$SR_{\mathbb{Q}}(\Sigma) = \mathbb{Q}[x_\rho \mid \rho \in \Sigma(1)]/\sim_{SR} \xrightarrow{\sim} \mathbb{Q}[t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)]/\sim$$

and their formal power series completions

$$\widehat{SR_{\mathbb{Q}}(\Sigma)} = \mathbb{Q}[[x_\rho \mid \rho \in \Sigma(1)]]/\sim_{SR} \xrightarrow{\sim} \mathbb{Q}[[t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)]]/\sim,$$

with the analytic Stanley-Reisner subring $SR^{an}_{\mathbb{Q}}(\Sigma) = \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\}/\sim_{SR}$ identified with the image of the projection from $\mathbb{Q}\{t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)\}$.

Proof. Using the invertible variable transformation $x_\rho \mapsto x_\rho, t_i \mapsto t'_i := t_i + \sum_{\rho\in\Sigma(1)} \langle m_i, n_\rho\rangle x_\rho$ (for $i = 1,\dots,n$ and $\rho \in \Sigma(1)$), the claim is reduced to the corresponding inclusion

$$\mathbb{Q}[x_\rho \mid \rho \in \Sigma(1)] \hookrightarrow \mathbb{Q}[t'_1,\dots,t'_n, x_\rho \mid \rho \in \Sigma(1)]\,,$$

with $\sim$ given by $\sim_{SR}$ and $t'_i = 0$ for $i = 1,\dots,n$. In this case the results are obvious. □

Corollary 7.2. *Using the isomorphism*

$$s : Sym_{\mathbb{Q}}(M) \simeq (\Lambda_{\mathbb{T}})_{\mathbb{Q}} = H^*_{\mathbb{T}}(pt;\mathbb{Q});\;\; m \mapsto -c^1_{\mathbb{T}}(\mathbb{C}_{\chi^m})$$

together with $\mathbb{Q}[t_1,\dots,t_n] = Sym_{\mathbb{Q}}(M)$*, one gets isomorphisms of* $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}$*-algebras*

$$SR_{\mathbb{Q}}(\Sigma) \simeq \mathbb{Q}[t_1,\dots,t_n, x_\rho \mid \rho \in \Sigma(1)]/\sim\, \simeq (\Lambda_{\mathbb{T}})_{\mathbb{Q}}[x_\rho \mid \rho \in \Sigma(1)]/\sim\, \simeq H^*_{\mathbb{T}}(X;\mathbb{Q})$$

and

$$\widehat{SR_{\mathbb{Q}}(\Sigma)} \simeq \mathbb{Q}[[t_1,\dots,t_n, x_\rho \mid \rho \in \Sigma(1)]]/\sim\, \simeq (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}[[x_\rho \mid \rho \in \Sigma(1)]]/\sim\, \simeq \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})\,,$$

with $t_i \mapsto s(m_i) = -c^1_{\mathbb{T}}(\mathbb{C}_{\chi^{m_i}})$ *for* $i = i,\dots,n$*, and* $x_\rho \mapsto F_\rho = [D_\rho]_{\mathbb{T}} = c^1_{\mathbb{T}}(\mathscr{O}(D_\rho))$ *for* $\rho \in \Sigma(1)$*.*

Proof. It suffices to note that the relation (249) corresponds to (67). □

Remark 7.3. Let $P \subset M_{\mathbb{R}}$ be a simple full-dimensional lattice polytope with associated projective toric variety $X = X_P$ and the inner normal fan $\Sigma = \Sigma_P$. Fix a basis $m_1,\dots,m_n$ of $M \simeq \mathbb{Z}^n$. There is an algebra homomorphism from the Cappell-Shaneson algebra $\mathscr{A}(P)$ to the completed equivariant cohomology ring

$$\text{(250)}\qquad \begin{aligned} \mathscr{A}(P) = \mathbb{Q}[[\partial_1,\dots,\partial_n]][U_F]/\sim\, \to \widehat{SR_{\mathbb{Q}}(\Sigma)} \simeq\; \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q});\\ U_F \mapsto x_\rho \simeq F_\rho = [D_\rho]_{\mathbb{T}} = c^1_{\mathbb{T}}(\mathscr{O}(D_\rho)) \end{aligned}$$

as a $\mathbb{Q}[[\partial_1,\dots,\partial_n]] \simeq \widehat{H}^*_{\mathbb{T}}(pt;\mathbb{Q}) \simeq (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$-algebra by $\partial_i \simeq t_i \simeq s(m_i) = -c^1_{\mathbb{T}}(\chi^{m_i})$ for $i = 1,\dots,n$, and with the facet F of P corresponding to the ray $\rho \in \Sigma(1)$. As we shall now explain, the algebra map (250) is in fact an isomorphism. In abstract terms, the completions $\widehat{SR_{\mathbb{Q}}(\Sigma)} \simeq \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$ are completions of connected integer graded commutative rings R^* with respect to the maximal ideal $R^{>0}$ given by positive degree elements, whereas $\mathscr{A}(P)$ uses the completion with respect to the maximal ideal $I_{\mathbb{T}}$ of $\mathbb{Q}[\partial_1,\dots,\partial_n] \simeq (\Lambda_{\mathbb{T}})_{\mathbb{Q}}$. As $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}[U_F]$ is not a finitely generated $(\Lambda_{\mathbb{T}})_{\mathbb{Q}}$-module, the completion functor with respect to $I_{\mathbb{T}}$ is only left exact. In particular, there is an injective algebra homomorphism $\mathscr{A}(P) \hookrightarrow \left(\widehat{H^*_{\mathbb{T}}(X;\mathbb{Q})}\right)_{I_{\mathbb{T}}}$, which factorizes (250). Using the relations defining $\mathscr{A}(P)$, one gets as in [18][Proposition on pag. 888] or [43][6.1, pag. 621] that the image of this monomorphism is the $(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$-submodule generated by the equivalence classes $[V_E] \in \mathscr{A}(P)$ associated to all faces E of P. This also shows that one gets a natural algebra homomorphism $\mathbb{Q}[[U_F]] \to \mathscr{A}(P)$ inducing algebra homomorphisms

$$\widehat{H}^{\mathbb{T}}_*(X;\mathbb{Q}) \to \mathscr{A}(P) \to (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}[[x_\rho]]/\sim\,.$$

Since this composition is an isomorphism, the first arrow is injective and the second is surjective. Using the fact that $H^*_{\mathbb{T}}(X;\mathbb{Q}) \simeq (\Lambda_{\mathbb{T}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} H^*(X;\mathbb{Q})$, one also has that $\left(\widehat{H^*_{\mathbb{T}}(X;\mathbb{Q})}\right)_{I_{\mathbb{T}}} \simeq$

$(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}} \otimes_{\mathbb{Q}} H^*(X;\mathbb{Q})$, which is also generated as a $(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$-module by the equivariant fundamental classes of the orbit closures $[V_{\sigma_E}]_{\mathbb{T}}$ associated to all faces E of P (since $H^*(X;\mathbb{Q})$ is generated as a $\mathbb{Q}$-vector space by the fundamental classes $[V_{\sigma_E}]$). This shows that $\mathscr{A}(P) \simeq \left(\widehat{H^*_{\mathbb{T}}(X;\mathbb{Q})}\right)_{I_{\mathbb{T}}}$. Finally, the natural algebra homomorphism $\left(\widehat{H^*_{\mathbb{T}}(X;\mathbb{Q})}\right)_{I_{\mathbb{T}}} \to \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$ is an isomorphism by [10][Prop.1.4]. Therefore, the homomorphism (250) is an isomorphism.

7.2. **Euler-Maclaurin formulae via the Cappell-Shaneson algebra.** Let us now introduce the key functionals following [34][Section 6].

Definition 7.4. Let

$$p(t_i, x_\rho) := \sum_{\alpha=(\alpha_i)\in\mathbb{N}_0^n} p_\alpha(x_\rho) \prod_{i=1}^n t_i^{\alpha_i} \in \mathbb{Q}\{t_1,\ldots,t_n,x_\rho \mid \rho\in\Sigma(1)\},$$

resp., $\mathbb{Q}[[t_1,\ldots,t_n,x_\rho \mid \rho\in\Sigma(1)]]$ be a convergent, resp., formal power series in the t_i, x_ρ. Then

$$p(\partial_i, \frac{\partial}{\partial h}) \left(\int_{P(h)} e^{\langle m,z\rangle}\, dm\right)_{|h=0} := \sum_{\alpha=(\alpha_i)\in\mathbb{N}_0^n} p_\alpha(\frac{\partial}{\partial h}) \left(\int_{P(h)} \prod_{i=1}^n \partial_i^{\alpha_i} e^{\langle m,z\rangle}\, dm\right)_{|h=0} \tag{251}$$

resp.,

$$p(\partial_i, \frac{\partial}{\partial h}) \left(\int_{P(h)} f(m)\, dm\right)_{|h=0} := \sum_{\alpha=(\alpha_i)\in\mathbb{N}_0^n} p_\alpha(\frac{\partial}{\partial h}) \left(\int_{P(h)} \prod_{i=1}^n \partial_i^{\alpha_i} f(m)\, dm\right)_{|h=0} \tag{252}$$

for f a polynomial function on $M_{\mathbb{R}}$. Note that in the second case in the last term only finitely many summands are non-zero due to Corollary 5.3.

Of course in (251) one has to explain in which sense the series is convergent, as will be discussed in the following. As a consequence of Theorem 5.1 (with $y=0$), Remark 5.2 and Corollary 5.11, we get by differentiation and convergence the following.

Corollary 7.5. *Let $p(\partial_i, \frac{\partial}{\partial h}) \in \mathbb{Q}\{\partial_1,\ldots,\partial_n, \frac{\partial}{\partial h_\rho} \mid \rho\in\Sigma(1)\}$ be an infinite order differential operator with constant rational coefficients, i.e., obtained by substituting $t_i \mapsto \partial_i, x_\rho \mapsto \frac{\partial}{\partial h_\rho}$ into a convergent power series with rational coefficients $p(t_i,x_\rho) \in \mathbb{Q}\{t_1,\ldots,t_n,x_\rho \mid \rho\in\Sigma(1)\}$. Then, in the above notations, we get for z small enough and away from the hyperplanes $\langle i_\sigma^* F_\rho, z\rangle = 0$ for each ray $\rho\in\sigma(1)$ of $\sigma\in\Sigma(n)$ the following formula:*

$$\begin{aligned}
&p(\partial_i, \frac{\partial}{\partial h}) \left(\int_{P(h)} e^{\langle m,z\rangle}\, dm\right)_{|h=0} = \sum_{\alpha=(\alpha_i)\in\mathbb{N}_0^n} p_\alpha(\frac{\partial}{\partial h}) \left(\int_{P(h)} \prod_{i=1}^n \partial_i^{\alpha_i} e^{\langle m,z\rangle}\, dm\right)_{|h=0} \\
&= \sum_{\alpha=(\alpha_i)\in\mathbb{N}_0^n} \prod_{i=1}^n \langle m_i, z\rangle^{\alpha_i} \left(\sum_{\sigma\in\Sigma(n)} \frac{e^{\langle i_\sigma^* c_1^{\mathbb{T}}(\mathscr{O}_X(D_P)),z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma), z\rangle} \cdot \langle p_\alpha(i_\sigma^* F_\rho), z\rangle\right) \\
&= \langle \sum_{\sigma\in\Sigma(n)} \frac{i_\sigma^*\left(e^{(c_1^{\mathbb{T}}(\mathscr{O}_X(D))}\left(\sum_{\alpha=(\alpha_i)\in\mathbb{N}_0^n} \prod_{i=1}^n s(m_i)^{\alpha_i} p_\alpha(F_\rho)\right)\right)}{Eu_X^{\mathbb{T}}(x_\sigma)}, z\rangle \\
&= \langle \int_X e^{(c_1^{\mathbb{T}}(\mathscr{O}_X(D))} p(s(m_i), F_\rho), z\rangle\,.
\end{aligned} \tag{253}$$

If $p_n \in \mathbb{Q}[\partial_1, \dots, \partial_n, \frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)]$ is the corresponding truncation of p up to order n, then both sides of (253) applied to p_n converge for $n \to \infty$ locally uniformly in these z to (253) applied to p.

Proof. The second equality follows from $\prod_{i=1}^n \partial_i^{\alpha_i} e^{\langle m,z\rangle} = \prod_{i=1}^n \langle m_i, z\rangle^{\alpha_i} e^{\langle m,z\rangle}$ and Corollary 5.11. The third equality uses $m_i = i_\sigma^* s(m_i) \in M \simeq H^2_{\mathbb{T}}(pt;\mathbb{Q})$ for $s(m_i) \in H^2_{\mathbb{T}}(X;\mathbb{Q})$. The last equality follows from equation (177), where the element $p(s(m_i), F_\rho)$ corresponds to the image of $p(t_i, x_\rho)$ under the evaluation homomorphism

$$\mathbb{Q}\{t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)\} \mapsto (H^*_{\mathbb{T}}(X;\mathbb{Q}))^{an} : t_i \mapsto s(m_i), x_\rho \mapsto F_\rho \,.$$

□

Remark 7.6. By (253), the operator $p(\partial_i, \frac{\partial}{\partial h})\left(\int_{P(h)} e^{\langle m,z\rangle}\, dm\right)_{|_{h=0}}$ depends only on the equivalence class of $[p(t_i,x_\rho)] \in SR^{an}_{\mathbb{Q}}(\Sigma) \simeq \mathbb{Q}\{t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)\}/\sim$ and not on the chosen convergent power series representative.

Remark 7.7. Assume in addition in Corollary 7.5, that $\langle \int_X e^{(c_1^{\mathbb{T}}(\mathscr{O}_X(D))} p(s(m_i), F_\rho), z\rangle$ is a convergent power series in z near zero. Then one gets as an application of Cauchy's integral formula (see also [34][p.27]), that both sides of (253) applied to p_n converge for $n \to \infty$ and z small locally uniformly to (253) applied to p. In particular, this limit commutes with finite order differentiations with respect to z (and z small enough).

Corollary 7.8. *Let $p(\partial_i, \frac{\partial}{\partial h}) \in \mathbb{Q}[[\partial_1, \dots, \partial_n, \frac{\partial}{\partial h_\rho} \mid \rho \in \Sigma(1)]]$ be an infinite order differential operator with constant rational coefficients, i.e., obtained by substituting $t_i \mapsto \partial_i, x_\rho \mapsto \frac{\partial}{\partial h_\rho}$ into a formal power series with rational coefficients $p(t_i, x_\rho) \in \mathbb{Q}[[t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)]]$. Then for a polynomial function f on $M_{\mathbb{R}}$, we have the following formula:*

$$p(\partial_i, \frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\, dm\right)_{|_{h=0}} = f(\frac{\partial}{\partial z})\left(\langle \int_X e^{c_1^{\mathbb{T}}(\mathscr{O}_X(D))} p(s(m_i), F_\rho), z\rangle\right)_{|_{z=0}}, \tag{254}$$

where on the right hand side the operator $\left(f(\frac{\partial}{\partial z})\right)_{|_{z=0}}$ acts on a formal power series in z.

Proof. The proof is similar to that of Corollary 5.14. □

Remark 7.9. By (254), the operator $p(\partial_i, \frac{\partial}{\partial h})\left(\int_{P(h)} f(z)\, dm\right)_{|_{h=0}}$ depends only on the equivalence class of $[p(t_i,x_\rho)] \in \widehat{SR_{\mathbb{Q}}(\Sigma)}$ and not on the chosen formal power series representative.

Altogether, we get the following second abstract Euler-Maclaurin formula coming from the equivariant Hirzebruch-Riemann-Roch theorem.

Theorem 7.10. *Let $X = X_P$ be the projective simplicial toric variety associated to a full-dimensional simple lattice polytope $P \subset M_{\mathbb{R}}$. Let $\Sigma := \Sigma_P$ be the inner normal fan of P, and $D := D_P$ the ample Cartier divisor associated to P. Let $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$ be fixed, and choose a convergent power series $p(t_i, x_\rho) \in \mathbb{Q}\{t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)\}$ so that $p(s(m_i), F_\rho) = \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) \in \left(H^*_{\mathbb{T}}(X;\mathbb{Q})\right)^{an}$. Then*

(255)
$$p(\partial_i, \frac{\partial}{\partial h}) \left(\int_{P(h)} e^{\langle m,z\rangle} \, dm \right)_{|h=0} = \langle \chi^{\mathbb{T}}(X, \mathscr{O}_X(D) \otimes \mathscr{F}), z\rangle$$
$$= \sum_{m\in M} \left(\sum_{i=0}^{n} (-1)^i \cdot \dim_{\mathbb{C}} H^i(X; \mathscr{O}_X(D) \otimes \mathscr{F})_{\chi^{-m}} \right) \cdot e^{\langle m,z\rangle} ,$$

as analytic functions in z with z small enough, and with $\chi^{\mathbb{T}}(X, \mathscr{O}_X(D) \otimes \mathscr{F}) \in (\Lambda^{an}_{\mathbb{T}})_{\mathbb{Q}}$ the cohomological equivariant Euler characteristic of $\mathscr{O}_X(D) \otimes \mathscr{F}$.

Proof. Equation (253) can now be calculated as
$$p(\partial_i, \frac{\partial}{\partial h}) \left(\int_{P(h)} e^{\langle m,z\rangle} \, dm \right)_{|h=0} = \langle \int_X e^{c_1^{\mathbb{T}}(\mathscr{O}_X(D))} \mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]), z\rangle = \langle \chi^{\mathbb{T}}(X, \mathscr{O}_X(D) \otimes \mathscr{F}), z\rangle,$$
where the last equality follows from the equivariant Hirzebruch-Riemann-Roch formula (83) as in the proof of Theorem 5.18. Finally, in the proof we first need to assume that z is small enough and away from the hyperplanes $\langle i^*_\sigma F_\rho, z\rangle = 0$ for each ray $\rho \in \sigma(1)$ of $\sigma \in \Sigma(n)$, since the localization formula is used; however, formula (255) then holds for all z small enough, by Remark 7.7. ☐

Note that Theorem 7.10 reduces to Theorem 5.18 in the case when
$$p(t_i, x_\rho) \in \mathbb{Q}\{x_\rho \mid \rho \in \Sigma(1)\} \subset \mathbb{Q}\{t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)\}$$
does not depend on the variables t_i. Similarly, the next Corollary reduces in this case to Corollary 5.21.

Corollary 7.11. *Let $X = X_P$ be the projective simplicial toric variety associated to a full-dimensional simple lattice polytope $P \subset M_{\mathbb{R}}$. Let $\Sigma := \Sigma_P$ be the inner normal fan of P, and $D := D_P$ the ample Cartier divisor associated to P. Let $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$ be fixed, and choose a formal power series $p(t_i, x_\rho) \in \mathbb{Q}[[t_1, \dots, t_n, x_\rho \mid \rho \in \Sigma(1)]]$ so that $p(s(m_i), F_\rho) = td_*^{\mathbb{T}}([\mathscr{F}]) \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$. Then for a polynomial function f on $M_{\mathbb{R}}$, we have:*

(256)
$$p(\partial_i, \frac{\partial}{\partial h}) \left(\int_{P(h)} f(m) \, dm \right)_{|h=0} = \sum_{m\in M} \left(\sum_{i=0}^{n} (-1)^i \cdot \dim_{\mathbb{C}} H^i(X; \mathscr{O}_X(D) \otimes \mathscr{F})_{\chi^{-m}} \right) \cdot f(m) \, .$$

Proof. This follows from Corollary 7.8 by applying the operator $\left(f(\frac{\partial}{\partial z}) \right)_{|z=0}$ to the last term of formula (255), viewed as a formal power series in z. ☐

For the use of integration over the faces E of P instead of dilatation of the facets of P by a parameter h_ρ ($\rho \in \Sigma(1)$), we next prove an analogue of Theorem 5.1 for faces of a polytope.

Assume P is a full-dimensional simple lattice polytope in $M_{\mathbb{R}}$ with associated toric variety $X = X_P$ and inner normal fan $\Sigma = \Sigma_P$. Let $P(h)$ be the dilatation of P with respect to the vector

$h=(h_\rho)_{\rho\in\Sigma(1)}$ with real entries indexed by the rays of Σ. So, if P is defined by inequalities of the form

$$\langle m,u_\rho\rangle + c_\rho \geq 0,$$

with u_ρ the ray generators and $c_\rho \in \mathbb{Z}$, for each $\rho\in\Sigma(1)$, then $P(h)$ is defined by inequalities

$$\langle m,u_\rho\rangle + c_\rho + h_\rho \geq 0,$$

for each $\rho\in\Sigma(1)$. Similarly, we consider the dilatation $E(h)$ of a fixed face E of P, and let $\sigma_E\in\Sigma$ be the cone corresponding to E.

Theorem 7.12. *In the above notations, we have*

$$(257)\quad \begin{aligned}\int_{E(h)} e^{\langle m,z\rangle}\,dm &= \mathrm{mult}(\sigma_E)\cdot \sum_{\sigma\in\Sigma(n)} \frac{e^{\langle i_\sigma^* c_1^{\mathbb{T}}(\mathscr{O}_X(D_{P(h)})),z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma),z\rangle}\cdot \prod_{\rho\in\sigma_E(1)} \langle i_\sigma^* F_\rho, z\rangle \\ &= \mathrm{mult}(\sigma_E)\cdot \sum_{\sigma\in\Sigma(n)} \frac{e^{\langle (i_\sigma^* c_1^{\mathbb{T}}(\mathscr{O}_X(D_{P})),z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma),z\rangle}\cdot e^{\sum_\rho h_\rho \langle i_\sigma^* F_\rho,z\rangle}\cdot \prod_{\rho\in\sigma_E(1)} \langle i_\sigma^* F_\rho, z\rangle\,.\end{aligned}$$

Proof. The proof is similar to that of Theorem 5.1, with the following modifications. Instead of $\mathrm{td}_*^{\mathbb{T}}(X)$, we use $\mathrm{td}_*^{\mathbb{T}}([(i_E)_*\mathscr{O}_{V_{\sigma_E}}])$, which is given by formula (217), together with the corresponding equivariant Riemann-Roch formula of (121) which is used for the calculation of Riemann sums approximating the integral. The Lebesgue measure dm on $E(h)$ is now normalized so that the unit cube in the lattice $Span(E_0)\cap M$ has volume 1, with $E_0 := E - m_0$ a translation of E by a vertex $m_0\in E$. As before, $\langle Eu_X^{\mathbb{T}}(x_\sigma),z\rangle$ is a homogeneous polynomial of degree n in z, but the additional factor $\prod_{\rho\in\sigma_E(1)}\langle i_\sigma^* F_\rho,z\rangle$ is homogeneous of degree $\mathrm{codim}\,(E)$, fitting with the use of the multiplication factor $\frac{1}{k^{\dim(E)}}$ needed for the Riemann sums considered here. Finally, we have

$$(258)\qquad \lim_{k\to\infty}\left(k^{\mathrm{codim}\,(E)}\cdot \langle i_\sigma^*(\mathrm{td}_*^{\mathbb{T}}([(i_E)_*\mathscr{O}_{V_{\sigma_E}}])),\frac{1}{k}\cdot z\rangle\right) = \mathrm{mult}(\sigma_E)\cdot \prod_{\rho\in\sigma_E(1)} \langle i_\sigma^* F_\rho, z\rangle.$$

This follows from formula (217) by the same calculation as in the proof of Theorem 5.1. □

By combining the results of Theorem 5.1 and Theorem 7.12, we obtain as an application a formula which relates integration over faces with a differentiation with respect to the corresponding h's.

Corollary 7.13. *Let f be a polynomial on $M_{\mathbb{R}}$. Then*

$$(259)\qquad \int_{E(h)} f(m)e^{\langle m,z\rangle}\,dm = \mathrm{mult}(\sigma_E)\cdot \prod_{\rho\in\sigma_E(1)} \frac{\partial}{\partial h_\rho}\int_{P(h)} f(m)e^{\langle m,z\rangle}\,dm\,,$$

as analytic functions in z and h near zero. In particular,

(a)

(260)

$$p(\frac{\partial}{\partial h})\left(\int_{E(h)} f(m)e^{\langle m,z\rangle}\,dm\right)_{|h=0} = \mathrm{mult}(\sigma_E)\cdot \prod_{\rho\in\sigma_E(1)} \frac{\partial}{\partial h_\rho}\cdot p(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)e^{\langle m,z\rangle}\,dm\right)_{|h=0},$$

for $p(x_\rho)$ a polynomial in the variables x_ρ, $\rho\in\Sigma(1)$ (or a convergent power series as in the context of Remark 5.13).

(b)

$$p(\frac{\partial}{\partial h})\left(\int_{E(h)} f(m)\,dm\right)_{|h=0} = \mathrm{mult}(\sigma_E)\cdot \prod_{\rho\in\sigma_E(1)} \frac{\partial}{\partial h_\rho}\cdot p(\frac{\partial}{\partial h})\left(\int_{P(h)} f(m)\,dm\right)_{|h=0}, \tag{261}$$

for a formal power series $p(x_\rho)$ in the variables x_ρ, $\rho\in\Sigma(1)$.

Proof. For $f=1$, formula (259) follows by comparing the results of Theorem 5.1 and Theorem 7.12. In the general case, we apply the operator $f(\frac{\partial}{\partial z})$ to both sides of the formula obtained for $f=1$. In (b) we finally evaluate the result at $z=0$. By Corollary 5.3, a formal power series (as opposed to a convergent one) can be used in (261). □

In particular,

$$\frac{\partial}{\partial h_{\rho_1}}\cdots\frac{\partial}{\partial h_{\rho_k}}\int_{P(h)} f(m)e^{\langle m,z\rangle}\,dm = 0 \tag{262}$$

if the corresponding facets for different ρ_i $(i=1,\dots,k)$ do not intersect.

As another application of Corollary 7.13, we can mention the following reformulation of the classical Euler-Maclaurin formula (204) of Brion-Vergne (using (35)), for f a polynomial function on $M_{\mathbb{R}}$ (see also [34][page 22]):

(263)

$$\sum_{m\in P\cap M} f(m) = \sum_{E\preceq P}\sum_{g\in G^\circ_{\sigma_E}}\prod_{\rho\in\Sigma(1)}\frac{\frac{\partial}{\partial h_\rho}}{1-a_\rho(g)e^{-\frac{\partial}{\partial h_\rho}}}\left(\int_{P(h)} f(m)\,dm\right)_{|h=0}$$

$$=\sum_{E\preceq P}\frac{1}{\mathrm{mult}(\sigma_E)}\sum_{g\in G^\circ_{\sigma_E}}\prod_{\rho\notin\sigma_E(1)}\frac{\frac{\partial}{\partial h_\rho}}{1-e^{-\frac{\partial}{\partial h_\rho}}}\prod_{\rho\in\sigma_E(1)}\frac{1}{1-a_\rho(g)\cdot e^{-\frac{\partial}{\partial h_\rho}}}\left(\int_{E(h)} f(m)\,dm\right)_{|h=0}.$$

The Euler-Maclaurin formula using the last operator is due to Guillemin [31][Thm.1.3 and Eqn.3.28]

As a further application of Theorem 7.12, we give an algebro-geometric proof of the *Stokes' formula for polytopes* (see also [34][Prop.6.1]).

Theorem 7.14. *Let $P\subset M_{\mathbb{R}}$ be a full-dimensional simple lattice polytope, with corresponding inner fan Σ. Let E_ρ denote the facet of P corresponding to the ray $\rho\in\Sigma(1)$. For a fixed $m_0\in M$, let $\frac{\partial}{\partial m_0}$ be the differentiation in direction of m_0. Then,*

$$\int_{P(h)}\frac{\partial}{\partial m_0}e^{\langle m,z\rangle}\,dm = -\sum_{\rho\in\Sigma(1)}\langle m_0,u_\rho\rangle\cdot\int_{E_\rho(h)} e^{\langle m,z\rangle}\,dm. \tag{264}$$

Proof. Recall that

$$s(m_0) = -c^1_{\mathbb{T}}(\mathbb{C}_{\chi^{m_0}}) = -\sum_{\rho\in\Sigma(1)}\langle m_0,u_\rho\rangle\cdot F_\rho\in H^*_{\mathbb{T}}(X;\mathbb{Q}).$$

Then

$$\begin{aligned}\int_{P(h)} \frac{\partial}{\partial m_0} e^{\langle m,z\rangle}\, dm &= \int_{P(h)} \langle m_0, z\rangle \cdot e^{\langle m,z\rangle}\, dm \\ &\overset{(189)}{=} \sum_{\sigma\in\Sigma(n)} \frac{e^{\langle i_\sigma^* c_1^{\mathbb{T}}(\mathscr{O}_X(D_{P(h)})),z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma), z\rangle} \cdot \langle m_0, z\rangle \\ &= \sum_{\sigma\in\Sigma(n)} \frac{e^{\langle i_\sigma^* c_1^{\mathbb{T}}(\mathscr{O}_X(D_{P(h)})),z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma), z\rangle} \cdot \langle i_\sigma^* s(m_0), z\rangle \\ &= -\sum_{\rho\in\Sigma(1)} \langle m_0, u_\rho\rangle \cdot \sum_{\sigma\in\Sigma(n)} \frac{e^{\langle i_\sigma^* c_1^{\mathbb{T}}(\mathscr{O}_X(D_{P(h)})),z\rangle}}{\langle Eu_X^{\mathbb{T}}(x_\sigma), z\rangle} \cdot \langle i_\sigma^* F_\rho, z\rangle \\ &\overset{(257)}{=} -\sum_{\rho\in\Sigma(1)} \langle m_0, u_\rho\rangle \cdot \int_{E_\rho(h)} e^{\langle m,z\rangle}\, dm,\end{aligned}$$

as desired. □

Applying the operator $f(\frac{\partial}{\partial z})$ to formula (264) for f a polynomial function on $M_{\mathbb{R}}$, we get the following:

$$\int_{P(h)} \frac{\partial}{\partial m_0} f(m) e^{\langle m,z\rangle}\, dm = -\sum_{\rho\in\Sigma(1)} \langle m_0, u_\rho\rangle \cdot \int_{E_\rho(h)} f(m) e^{\langle m,z\rangle}\, dm \tag{265}$$

as analytic functions in h and z near zero. Further evaluation at $z = 0$ yields the identity

$$\int_{P(h)} \frac{\partial}{\partial m_0} f(m)\, dm = -\sum_{\rho\in\Sigma(1)} \langle m_0, u_\rho\rangle \cdot \int_{E_\rho(h)} f(m)\, dm, \tag{266}$$

as polynomial functions in h near zero.

Remark 7.15. By Remark 7.3, there is a surjection $\mathscr{A}(P) \to \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})$, as $(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$-modules, with $\mathscr{A}(P)$ generated by the equivalence classes of $[V_E] \in \mathscr{A}(P)$, which are mapped to the equivariant fundamental classes $[V_{\sigma_E}]_{\mathbb{T}}$ of orbit closures corresponding to all faces E of P. Hence, $\widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})$ is generated as a $(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$-module by the equivariant fundamental classes $[V_{\sigma_E}]_{\mathbb{T}}$.

Let $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$ be fixed, and choose elements $p_{\sigma_E}(t_i) \in \widehat{H}_{\mathbb{T}}^*(pt;\mathbb{Q}) = (\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}} \simeq \mathbb{Q}[[t_1,\dots,t_n]]$ with

$$\mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) = \sum_{E\preceq P} p_{\sigma_E}(t_i)[V_{\sigma_E}]_{\mathbb{T}} \in \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})\,. \tag{267}$$

Then $\mathrm{td}_*^{\mathbb{T}}([\mathscr{F}]) = p(s(m_i), F_\rho) \in \widehat{H}_{\mathbb{T}}^*(X;\mathbb{Q})$ for

$$p(t_i, x_\rho) := \sum_{E\preceq P} \left(\mathrm{mult}(\sigma_E) \cdot \prod_{\rho\in\sigma_E(1)} x_\rho \right) \cdot p_{\sigma_E}(t_i) \in \mathbb{Q}[[t_1,\dots,t_n, x_\rho \mid \rho \in \Sigma(1)]]\,. \tag{268}$$

Altogether, with these notations, we get our third and final abstract Euler-Maclaurin formula coming from the equivariant Hirzebruch-Riemann-Roch theorem, which provides a generalization of the Cappell-Shaneson Euler-Maclaurin formula (see Remark 7.17 below).

Theorem 7.16. *Let $X = X_P$ be the projective simplicial toric variety associated to a full-dimensional simple lattice polytope $P \subset M_{\mathbb{R}}$. Let $\Sigma := \Sigma_P$ be the inner normal fan of P, and $D := D_P$ the ample Cartier divisor associated to P. Let $[\mathscr{F}] \in K_0^{\mathbb{T}}(X)$ be fixed, and choose the formal power series $p(t_i, x_\rho) \in \mathbb{Q}[[t_1, \ldots, t_n, x_\rho \mid \rho \in \Sigma(1)]]$ as in (268). Then for a polynomial function f on $M_{\mathbb{R}}$, we have:*

$$\sum_{E \preceq P} \int_E p_{\sigma_E}(\partial_i) f(m)\, dm = \sum_{m \in M} \left(\sum_{i=0}^{n} (-1)^i \cdot \dim_{\mathbb{C}} H^i(X; \mathscr{O}_X(D) \otimes \mathscr{F})_{\chi^{-m}} \right) \cdot f(m)\,. \tag{269}$$

Proof. The assertion follows from Corollary 7.13 by the following calculation:

$$\begin{aligned}
& p(\partial_i, \frac{\partial}{\partial h}) \left(\int_{P(h)} f(m)\, dm \right)_{|h=0} \\
&= \sum_{E \preceq P} \left(\mathrm{mult}(\sigma_E) \cdot \prod_{\rho \in \sigma_E(1)} \frac{\partial}{\partial h_\rho} \right) \left(\int_{P(h)} p_{\sigma_E}(\partial_i) f(m)\, dm \right)_{|h=0} \\
&= \sum_{E \preceq P} \left(\int_{E(h)} p_{\sigma_E}(\partial_i) f(m)\, dm \right)_{|h=0} \\
&= \sum_{E \preceq P} \int_E p_{\sigma_E}(\partial_i) f(m)\, dm\,.
\end{aligned} \tag{270}$$

Here the first equality follows from the definition (252) of $p(\partial_i, \frac{\partial}{\partial h}) \left(\int_{P(h)} f(m)\, dm \right)_{|h=0}$ and the choice (268) of $p(t_i, x_\rho)$. Then the second equality follows from (261), and the final equality is just continuity in h. □

Remark 7.17. In the classical case $\mathscr{F} := \mathscr{O}_X$, this is just Cappell-Shaneson's recipe for the definition of the differential operators $p_{\sigma_E}(\partial_i)$, described here geometrically in terms of the equivariant Todd class $\mathrm{td}_*^{\mathbb{T}}(X) := \mathrm{td}_*^{\mathbb{T}}([\mathscr{O}_X]) \in \widehat{H}_{\mathbb{T}}^*(X; \mathbb{Q})$ (see [18][Theorem 2]). More precisely, the differential operator $\mathscr{E} = \sum_{E \preceq P} P_E \cdot U_E \in \mathbb{Q}[[\partial_1, \ldots, \partial_n]][U_F]$ used in [43][6.2, page 622] and [18][page 888] maps under the homomorphism (250) to a presentation of the equivariant Todd class $\mathrm{td}_*^{\mathbb{T}}(X) = \sum_{E \preceq P} p_{\sigma_E}(t_i)[V_{\sigma_E}]_{\mathbb{T}} \in \widehat{H}_{\mathbb{T}}^*(X; \mathbb{Q})$. Indeed, the classes $[V_{\sigma_E}]_{\mathbb{T}}$ are the images of the elements $U_E \in \mathbb{Q}[[\partial_1, \ldots, \partial_n]][U_F]$ used in loc.cit., and the operators $P_E \in \mathbb{Q}[[\partial_1, \ldots, \partial_n]]$ are deduced in [18, 43] from a representative $T(E) \in \mathbb{Q}[[\partial_1, \ldots, \partial_n]][U_F]$ mapping to the equivariant Todd class $\mathrm{td}_*^{\mathbb{T}}(X)$ calculated via the equivariant version of Theorem 2.6. So we can choose p_{σ_E} to be the images of these P_E in $(\widehat{\Lambda}_{\mathbb{T}})_{\mathbb{Q}}$ under (250). Hence, the presentation

$$p(t_i, x_\rho) := \sum_{E \preceq P} \left(\mathrm{mult}(\sigma_E) \cdot \prod_{\rho \in \sigma_E(1)} x_\rho \right) \cdot p_{\sigma_E}(t_i) \in \mathbb{Q}[[t_1, \ldots, t_n, x_\rho \mid \rho \in \Sigma(1)]]$$

chosen here is exactly the image of $\mathscr{E}$ in $\mathbb{Q}[[t_1, \ldots, t_n]][[x_\rho]]$.

In this case, when $\mathscr{F} := \mathscr{O}_X$, the right hand side of (269) becomes $\sum_{m\in P\cap M} f(m)$ by Example 5.23. Specializing further to $f = 1$ yields then the classical formula (see, e.g., [27][page 112]):

$$|P\cap M| = \sum_{E\preceq P} r_E \cdot vol(E)\,, \tag{271}$$

with $r_E = p_{\sigma_E}(0)$ the constant coefficient of p_{σ_E} and

$$\mathrm{td}_*(X) = \sum_{E\preceq P} r_E[V_\sigma] \in H^*(X;\mathbb{Q}) \tag{272}$$

the non-equivariant version of

$$\mathrm{td}^{\mathbb{T}}_*([\mathscr{O}_X]) = \sum_{E\preceq P} p_{\sigma_E}(t_i)[V_{\sigma_E}]_{\mathbb{T}} \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})\,. \tag{273}$$

When $\mathscr{F} := \omega_X$, the right hand side of (269) becomes $\sum_{m\in \mathrm{Int}(P)\cap M} f(m)$ by Example 5.24. Moreover, here we can choose

$$\mathrm{td}^{\mathbb{T}}_*([\omega_X]) = \sum_{E\preceq P} p_{\sigma_E}(-t_i)\cdot(-1)^{\mathrm{codim}(E)}[V_{\sigma_E}]_{\mathbb{T}} \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})\,, \tag{274}$$

by Remark 3.26, with $p_{\sigma_E}(t_i)$ as in (273). Specializing further to $f = 1$ yields the formula

$$|\mathrm{Int}(P)\cap M| = \sum_{E\preceq P} (-1)^{\mathrm{codim}(E)}\cdot r_E \cdot vol(E)\,, \tag{275}$$

with r_E as in (272).

Example 7.18. Here we list further specializations of formula (269) for appropriate choices of $[\mathscr{F}] \in K^{\mathbb{T}}_0(X)$, resp., $[\mathscr{F}] \in K^{\mathbb{T}}_0(X)[y]$, with $\mathrm{td}^{\mathbb{T}}_*([\mathscr{F}]) = \sum_{E\preceq P} p_{\sigma_E}(t_i)[V_{\sigma_E}]_{\mathbb{T}} \in \widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})$, resp., $\widehat{H}^*_{\mathbb{T}}(X;\mathbb{Q})[y]$ for some elements $p_{\sigma_E}(t_i) \in \widehat{H}^*_{\mathbb{T}}(pt;\mathbb{Q})$, resp., $\widehat{H}^*_{\mathbb{T}}(pt;\mathbb{Q})[y]$.

(1) For $[\mathscr{F}] = mC^{\mathbb{T}}_y(X) \in K^{\mathbb{T}}_0(X)[y]$, (269) becomes by equation (230):

$$\sum_{E\preceq P}\int_E p_{\sigma_E}(\partial_i) f(m)\, dm = \sum_{E''\preceq P} (1+y)^{\dim(E'')} \sum_{m\in \mathrm{Relint}(E'')\cap M} f(m)\,.$$

(2) For $[\mathscr{F}] = mC^{\mathbb{T}}_0([U \hookrightarrow X]) \in K^{\mathbb{T}}_0(X)$, with $U = X\setminus D_K$ the open complement of the divisor $D_K = \bigcup_{i\in K} D_i$ and $D_i := D_{\rho_i}$, (269) becomes by equation (210):

$$\sum_{E\preceq P}\int_E p_{\sigma_E}(\partial_i) f(m)\, dm = \sum_{m\in P^K\cap M} f(m)\,.$$

Here P^K is the set obtained from P by removing the facets F_i for $i\in K$. More generally, for $[\mathscr{F}] = mC^{\mathbb{T}}_y([U \hookrightarrow X]) \in K^{\mathbb{T}}_0(X)[y]$, (269) becomes by equation (234):

$$\sum_{E\preceq P}\int_E p_{\sigma_E}(\partial_i) f(m)\, dm = \sum_{E''\preceq P^K} (1+y)^{\dim(E'')}\cdot \sum_{m\in \mathrm{Relint}(E'')\cap M} f(m)\,.$$

(3) For $[\mathscr{F}] = [(i_\sigma)_*\omega_{V_\sigma}] \in K^{\mathbb{T}}_0(X)$, with $V_\sigma = V_{\sigma_{E'}}$ the closure of the orbit of σ in $X = X_P$ corresponding to the face E' of P, (269) becomes by equation (214):

$$\sum_{E\preceq P}\int_E p_{\sigma_E}(\partial_i) f(m)\, dm = \sum_{m\in \mathrm{Relint}(E')\cap M} f(m)\,.$$

(4) For $[\mathscr{F}] = [(i_\sigma)_* \mathscr{O}_{V_\sigma}] \in K_0^{\mathbb{T}}(X)$, with $V_\sigma = V_{\sigma_{E'}}$ the closure of the orbit of σ in $X = X_P$ corresponding to the face E' of P, (269) becomes by equation (218):

$$\sum_{E \preceq P} \int_E p_{\sigma_E}(\partial_i) f(m)\, dm = \sum_{m \in E' \cap M} f(m)\,.$$

More generally, for $[\mathscr{F}] = (i_\sigma)_* mC_y^{\mathbb{T}}(V_\sigma) \in K_0^{\mathbb{T}}(X)[y]$, (269) becomes by equation (239):

$$\sum_{E \preceq P} \int_E p_{\sigma_E}(\partial_i) f(m)\, dm = \sum_{E'' \preceq E'} (1+y)^{\dim(E'')} \cdot \sum_{m \in \mathrm{Relint}(E'') \cap M} f(m)\,.$$

□

7.3. **Generalized Reciprocity for Dedekind Sums via Euler-Maclaurin formulae.** We conclude this paper with the following application of formula (269), with D replaced by $(1+y)D$.

Corollary 7.19. *In the context of Theorem 7.16, we have an equality of polynomials in $1+y$,*

(276)
$$\sum_{E_y \preceq P_y} \int_{E_y} p_{\sigma_{E_y}}(\partial_i) f(m)\, dm = \sum_{m \in M} \left(\sum_{i=0}^{n} (-1)^i \cdot \dim_{\mathbb{C}} H^i(X; \mathscr{O}_X((1+y)D) \otimes \mathscr{F})_{\chi^{-m}} \right) \cdot f(m)\,.$$

When evaluating these polynomials at zero (i.e., for $y = -1$), one gets the following identity:

$$\sum_{v \in P} (p_{\sigma_v}(\partial_i) f)(0) = \sum_{m \in M} \left(\sum_{i=0}^{n} (-1)^i \cdot \dim_{\mathbb{C}} H^i(X; \mathscr{F})_{\chi^{-m}} \right) \cdot f(m). \tag{277}$$

where the left hand sum is over the vertices of P.

Proof. The right hand sides of the above expressions are calculated in Corollary 5.4. For the left hand side of (276), note that for a given face E of P, if f is a homogeneous polynomial of degree d_f, and $p(\partial_i)$ is a homogeneous differential operator of degree $d \leq d_f$, then

$$\int_{E_y} p(\partial_i) f(m)\, dm = (1+y)^{\dim(E) + d_f - d} \int_E p(\partial_i) f(m)\, dm. \tag{278}$$

This explains the polynomial behavior of the left hand side of (276), and it yields (277) upon evaluation at zero (since by letting $y = -1$, all terms on the right hand side of (278) vanish except for $\dim(E) = 0$ and $d = d_f$). □

Example 7.20. If $\mathscr{F} = \mathscr{O}_X$ in (277), one gets by Example 3.18 the following identity:

$$\sum_{v \in P} (p_{\sigma_v}(\partial_i) f)(0) = f(0). \tag{279}$$

For instance, in the case of lattice polygones, this formula yields generalizations of reciprocity laws for classical *Dedekind sums* (using, e.g., the explicit description of the operators $p_{\sigma_v}(\partial_i)$ from [18][page 889]). Details will be explained in forthcoming work. □

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