TOPOLOGY OF COMPLEX PROJECTIVE HYPERSURFACES AND OF THEIR COMPLEMENTS

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ABSTRACT. We explore how the presence of singularities affects the geometry and topology of complex projective hypersurfaces and of their complements.

CONTENTS

1. PROJECTIVE HYPERSURFACES AND THEIR COMPLEMENTS

1.1. **Introduction. Generalities.** Let $\mathbb{C}P^{n+1}$ be the complex projective space with homogeneous coordinates $[x_0 : x_1 : \ldots : x_{n+1}]$. We consider $\mathbb{C}P^{n+1}$ with its complex topology. A homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ defines a projective hypersurface

$$V(f) = \{x \in \mathbb{C}P^{n+1} \mid f(x) = 0\}.$$

A point $x \in V(f)$ is called *singular* if the tangent space of $V(f)$ at x is not defined. Formally, *the singular locus* of $V(f)$ is:

$$\mathrm{Sing}(V(f)) = \{x \in V(f) \mid f_1(x) = \cdots = f_{n+1}(x) = 0\}$$

where $f_i = \frac{\partial f}{\partial x_i}$.

We are interested in the topology of $V = V(f)$, i.e., its *shape*, reflected in the computation of various topological invariants like fundamental group, Betti numbers or Euler characteristic. For instance, let $S = S^{2n+3}$ be the unit sphere in $\mathbb{C}^{n+2}$ and let $K_V = S \cap \widehat{V}$ be the link of f

Date: May 21, 2021.

at the origin in $\mathbb{C}^{n+2}$, where we denote by $\widehat{V} = \{x \in \mathbb{C}^{n+2} \mid f(x) = 0\}$ the affine cone on V. Restricting the Hopf bundle

$$S^1 \hookrightarrow S^{2n+3} \to \mathbb{C}P^{n+1}$$

to V, we get the *Hopf bundle of the hypersurface* V, namely

$$S^1 \hookrightarrow K_V \to V.$$

Using the homotopy sequence of a fibration and the fact that K_V is $(n-1)$-connected, yields:

Proposition 1.1. *The complex projective hypersurface $V \subset \mathbb{C}P^{n+1}$ is simply-connected for $n \geq 2$ and connected for $n = 1$.*

We also mention here the following classical result.

Theorem 1.2 (Lefschetz). *Let $V \subset \mathbb{C}P^{n+1}$ be a complex projective hypersurface. The inclusion $j\colon V \hookrightarrow \mathbb{C}P^{n+1}$ induces cohomology isomorphisms*

$$j^*\colon H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \xrightarrow{\cong} H^k(V;\mathbb{Z}) \text{ for all } k < n, \tag{1}$$

and a monomorphism for $k = n$ (regardless of the singularities of V).

Proof. Let $U = \mathbb{C}P^{n+1} \setminus V$. The cohomology long exact sequence for the pair $(\mathbb{C}P^{n+1}, V)$ and the Alexander duality isomorphism

$$H^k(\mathbb{C}P^{n+1}, V;\mathbb{Z}) \cong H_{2n+2-k}(U;\mathbb{Z})$$

show that is suffices to prove that

(i) $H_i(U;\mathbb{Z}) \cong 0$ for $i > n+1$,
(ii) $H_{n+1}(U;\mathbb{Z})$ is torsion free.

These are both consequences of the fact that U is affine of complex dimension $n+1$, and hence, by a result of Hamm, U has the homotopy type of a CW complex of real dimension $n+1$. □

Remark 1.3. In fact, it can be shown that the inclusion $j : V \hookrightarrow \mathbb{C}P^{n+1}$ is an n-homotopy equivalence. A similar statement holds for complete intersections.

As we will see later on, the structure of cohomology groups $H^i(V;\mathbb{Z})$, for $i \geq n$, can be very different from that of the projective space.

We also investigate the topology of the complement

$$U(f) := \mathbb{C}P^{n+1} \setminus V(f),$$

i.e., the *view* of $V(f)$ *from the outside*. This is an idea motivated by the classical Knot Theory, where one studies embeddings of S^1 into S^3. The image of such an embedding is a knot K, and $S^3 \setminus K$ is a $K(\pi,1)$-space, i.e., all topological information about it is contained in $\pi_1(S^3 \setminus K)$. The homology carries no useful information about the view of K from the outside, as $H_1(S^3 \setminus K;\mathbb{Z}) = \mathbb{Z}$. On the other hand, it is known that $\pi_1(S^3 \setminus K) \cong \mathbb{Z}$ iff K is the unknot (i.e., isotopic to the linear embedding of the circle). Knots are studied, e.g., by considering Alexander-type invariants of the complement $S^3 \setminus K$, which are associated to the $\mathbb{Z}$-fold covering defined by the abelianization map $\pi_1(S^3 \setminus K) \to H_1(S^3 \setminus K;\mathbb{Z}) \cong \mathbb{Z}$.

The precise connection between Knot Theory and Hypersurface Singularities can be seen as follows: let $n = 1$, and x be a point on the curve $V(f)$, that is assumed to be unibranched at x. A small enough closed ball B in $\mathbb{C}^2$, centered at x, has boundary homeomorphic to S^3. Moreover $V(f) \cap \partial B$ is homeomorphic to S^1. The corresponding knot is the unknot iff x is a nonsingular point. If x is singular, the knotting can be studied by knot theory techniques (e.g., Alexander polynomial) via the *local Milnor fibration of f at x*.

The "local picture" of a hypersurface singularity germ is a higher-dimensional analogue of a knot/link in S^3, and is classically described by the *Milnor fibration* (Milnor '68): if $(X,0)$ is a hypersurface singularity germ defined at $0 \in \mathbb{C}^{n+1}$ by a reduced analytic function germ g, then for B_ε a small enough ball around $0 \in \mathbb{C}^{n+1}$, with boundary S_ε, $X \cap B_\varepsilon$ is a cone over the *link* $K = X \cap S_\varepsilon$. Moreover, K is $(n-2)$-connected (no matter how bad the singularity at 0 is), and for all $0 < \delta \ll \varepsilon$, there is a fibration

$$F = B_\varepsilon \cap g^{-1}(\delta) \hookrightarrow S_\varepsilon \setminus K \to S^1$$

with F a $(n-s-1)$-connected manifold, where $s = \dim_{\mathbb{C}} \mathrm{Sing}(X,0)$. Milnor showed that F has the homotopy type of a finite CW complex of real dimension n. For example, in the case of an isolated hypersurface singularity, the Milnor fibre F has the homotopy type of a bouquet of $\mu(g)$ n-spheres, where $\mu(g)$ is called the *Milnor number of g*. In this case, the Milnor fiber can be regarded as a "smoothing" of X in a neighborhood of the singular point. Finally, if $g : \mathbb{C}^{n+1} \to \mathbb{C}$ is homogeneous, there is a *global Milnor fibration* $F = \{g = 1\} \hookrightarrow \mathbb{C}^{n+1} \setminus X(g) \to \mathbb{C}^*$, where $X(g) = \{x \in \mathbb{C}^{n+1} \mid g(x) = 0\}$, and it is easy to see that F is homotopy equivalent to the Milnor fiber associated to the germ of g at the origin.

1.2. **Topology of smooth complex projective hypersurfaces.**

1.2.1. *Diffeomorphism type.* The following result states that the shape and view from outside of a smooth projective hypersurface in $\mathbb{C}P^{n+1}$ are completely determined by its degree.

Theorem 1.4. *Let $f, g \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ be two homogeneous polynomials of the same degree d, such that the corresponding projective hypersurfaces $V(f)$ and $V(g)$ are smooth. Then:*

(i) *The hypersurfaces $V(f)$ and $V(g)$ are diffeomorphic.*
(ii) *The complements $U(f)$ and $U(g)$ are diffeomorphic.*

Sketch of proof. The assertion follows from the fact that, given any two nonsingular degree d hypersurfaces in $\mathbb{C}P^{n+1}$, there exists a diffeomorphism $\mathbb{C}P^{n+1} \to \mathbb{C}P^{n+1}$ isotopic to the identity that restricts to a diffeomorphism of the two hypersurfaces. □

Example 1.5. The topology of a smooth projective curve ($n = 1$) is determined by its degree, or equivalently, by its *genus*. In fact, topologically, any such curve is obtained from S^2 by attaching a number of "handles". This number of handles is the genus. By the *genus-degree formula* discussed in Example 1.8 below, we see that for $d = 1$ and $d = 2$ we get the sphere $S^2 = \mathbb{C}P^1$, and for $d = 3$ we get an elliptic curve which is diffeomorphic to the torus $S^1 \times S^1$.

Remark 1.6. The assertion of Theorem 1.4 is not valid for *real* projective hypersurfaces, nor for complex *affine* hypersurfaces. For instance, a smooth real projective curve is a collection of circles, but their exact numbers and relative position depends on the coefficients of the defining polynomial. As an exercise, show that the real curves in $\mathbb{R}P^2$ defined by $f = x_0^2 + x_1^2 + x_2^2$ and $g = x_0^2 - x_1^2 + x_2^2$ are not diffeomorphic. Similarly, consider the complex affine curves in $\mathbb{C}^2$ given by $f = x^3 + y^3 - 1$ and $g = x + x^2 y - 1$. Then $V(f)$ is homeomorphic to a torus with 3 deleted points, while $V(g)$ is a punctured plane; hence $b_1(V(f)) = 4 \neq 1 = b_1(V(g))$.

1.2.2. *Euler characteristic.*

Proposition 1.7. *Let $V \subset \mathbb{C}P^{n+1}$ be a degree d smooth complex projective hypersurface. Then the Euler characteristic of V is given by the formula:*

$$\chi(V) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\}. \tag{2}$$

Proof. Since the diffeomorphism type of a smooth complex projective hypersurface is determined only by its degree and dimension, one can assume without any loss of generality that V is defined by the degree d homogeneous polynomial: $f = \sum_{i=0}^{n+1} x_i^d$.

The affine cone $\widehat{V} = \{f = 0\} \subset \mathbb{C}^{n+2}$ on V has an isolated singularity at the cone point $0 \in \mathbb{C}^{n+2}$. Consider the affine Milnor fibration

$$F = \{f = 1\} \hookrightarrow \mathbb{C}^{n+2} \setminus \widehat{V} \xrightarrow{f} \mathbb{C}^*,$$

whose fiber F is homotopy equivalent to a bouquet of μ $(n+1)$-dimensional spheres, where μ is the Milnor number of f at the origin. Moreover, the map $F \to \mathbb{C}P^{n+1} \setminus V$ defined by

$$(x_0, \dots, x_{n+1}) \mapsto [x_0 : \dots : x_{n+1}]$$

is a d-fold cover of $\mathbb{C}P^{n+1} \setminus V$, so

$$\chi(F) = d \cdot \chi(\mathbb{C}P^{n+1} \setminus V) = d \cdot (\chi(\mathbb{C}P^{n+1}) - \chi(V)). \tag{3}$$

Finally, the Milnor number of f at the origin in $\mathbb{C}^{n+2}$ is given by $\mu = (d-1)^{n+2}$, hence

$$\chi(F) = 1 + (-1)^{n+1}(d-1)^{n+2}. \tag{4}$$

The desired expression for $\chi(V)$ follows from (3) and (4). □

Example 1.8. Assume $n = 1$, so V is smooth complex projective curve, i.e., a Riemann surface. Topologically, such V is obtained from S^2 by attaching a number of "handles". This number is called the it genus $g(V)$ of V, and $\chi(V) = 2 - 2g(V)$. Together with (2), this yields the celebrated *genus-degree formula*:

$$g(V) = \frac{(d-1)(d-2)}{2}.$$

1.2.3. *Integral (co)homology. Betti numbers.* Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface of degree d. If the hypersurface $V \subset \mathbb{C}P^{n+1}$ is moreover *smooth*, then one gets by Theorem 1.2 and Poincaré duality that $H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^n;\mathbb{Z})$ for all $k \neq n$. The Universal Coefficient Theorem also yields in this case that $H^n(V;\mathbb{Z})$ is free abelian, and its rank $b_n(V)$ can be easily computed from formula (2) for the Euler characteristic of V. Hence:

Theorem 1.9. *Let $V \subset \mathbb{C}P^{n+1}$ be a smooth hypersurface of degree d. Then the integral (co)homology of V is torsion free, and the corresponding Betti numbers are given as follows:*

1. $b_i(V) = 0$ *for $i \neq n$ odd or $i \notin [0, 2n]$.*
2. $b_i(V) = 1$ *for $i \neq n$ even and $i \in [0, 2n]$.*
3. $b_n(V) = \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + \frac{3(-1)^n + 1}{2}$.

Exercise 1.10. Show that the Betti numbers of a smooth cubic in $\mathbb{C}P^3$ are $1, 0, 7, 0, 1$.

1.3. **Kato's theorem.** Assume now that $V(f) \subset \mathbb{C}P^{n+1}$ is a reduced degree d hypersurface. Let $f = f_1 \cdots f_r$ be a square-free (irreducible) decomposition of f. Let $V_i = \{f_i = 0\}$, $i = 1, \dots, r$, be the irreducible components of V. Let $d_i = \deg(f_i)$, hence $d = \sum_i d_i$.

Using Lefschetz duality, one gets that

$$H^{2n}(V;\mathbb{Z}) \cong H^{2n}(V_1;\mathbb{Z}) \oplus \cdots \oplus H^{2n}(V_r;\mathbb{Z}) = \mathbb{Z}^r. \tag{5}$$

In fact, $H^{2n}(V;\mathbb{Z}) \cong H^{2n}(V, \operatorname{Sing}(V);\mathbb{Z}) \cong H_0(V \setminus \operatorname{Sing}(V);\mathbb{Z}) \cong \mathbb{Z}^r$, since $V \setminus \operatorname{Sing}(V)$ has exactly r path-connected components, one for each irreducible component of V.

Moreover, the inclusion $j : V \hookrightarrow \mathbb{C}P^{n+1}$ induces in degree $2n$-cohomology the morphism

$$j^{2n} : H^{2n}(\mathbb{C}P^{n+1};\mathbb{Z}) \to H^{2n}(V;\mathbb{Z}) \ , \quad a \mapsto (d_1 a, \dots, d_r a). \tag{6}$$

This can be seen from the fact that a generic line in $\mathbb{C}P^{n+1}$ intersects the hypersurface V_i in exactly d_i points ($i = 1, \dots, r$).

The following result complements Lefschetz's Theorem 1.2.

Theorem 1.11 (Kato)**.** *Let $V \subset \mathbb{C}P^{n+1}$ be a reduced degree d complex projective hypersurface with $s = \dim_{\mathbb{C}} \operatorname{Sing}(V)$ the complex dimension of its singular locus. (By convention, we set $s = -1$ if V is nonsingular.) Then*

$$H^k(V;\mathbb{Z}) \cong H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \ \text{for all} \ \ n+s+2 \le k \le 2n. \tag{7}$$

Moreover, if $j : V \hookrightarrow \mathbb{C}P^{n+1}$ denotes the inclusion, the induced cohomology homomorphisms

$$j^k : H^k(\mathbb{C}P^{n+1};\mathbb{Z}) \longrightarrow H^k(V;\mathbb{Z}), \ \ n+s+2 \le k \le 2n, \tag{8}$$

are given by multiplication by d if k is even.

Proof. The statement of the theorem is valid only if $n \ge s+2$, so in particular we can assume that V is irreducible and hence $H^{2n}(V;\mathbb{Z}) \cong \mathbb{Z}$. Moreover, the fact that j^{2n} is multiplication by $d = \deg(V)$ is true regardless of the dimension of singular locus, see (6). If $n = s+2$ there is nothing else to prove, so we may assume (without any loss of generality) that $n \ge s+3$.

Let $S := S^{2n+3}$ be a small enough sphere at the origin in $\mathbb{C}^{n+2}$, and let $K_V := S \cap \widehat{V}$ be the link at the origin of the affine cone $\widehat{V} = \{f = 0\} \subset \mathbb{C}^{n+2}$ on V. The fiber F of the Milnor fibration

$$F \hookrightarrow S \setminus K_V \xrightarrow{f} S^1$$

of the singularity of $\widehat{V}$ at $0 \in \mathbb{C}^{n+2}$ is $(n-s-1)$-connected (since the dimension of the singularity of $\widehat{V}$ at 0 is $(s+1)$-dimensional). It then follows from the Wang sequence of the Milnor fibration, i.e.,

$$\to H_{k+1}(S \setminus K_V;\mathbb{Z}) \to H_k(F;\mathbb{Z}) \overset{h_* - id}{\to} H_k(F;\mathbb{Z}) \to H_k(S \setminus K_V;\mathbb{Z}) \to$$

that $H_k(S \setminus K_V;\mathbb{Z}) = 0$ for $2 \le k \le n-s-1$. By Alexander duality, for k in the same range we get

$$H^{2n+2-k}(K_V;\mathbb{Z}) \cong H^{2n+3-k}(S, K_V;\mathbb{Z}) \cong 0.$$

Equivalently,

$$H^k(K_V;\mathbb{Z}) = 0 \ \ \text{for} \ \ n+s+3 \le k \le 2n. \tag{9}$$

The cohomology Gysin sequences for the diagram of fibrations

$$\begin{array}{ccc} S & \longrightarrow & \mathbb{C}P^{n+1} \\ \uparrow & & \uparrow \\ K_V & \longrightarrow & V \end{array}$$

yield commutative diagrams (with $\mathbb{Z}$-coefficients):

$$\begin{array}{ccccccc} H^{2\ell+1}(S) & \longrightarrow & H^{2\ell}(\mathbb{C}P^{n+1}) & \xrightarrow[\cong]{\psi} & H^{2\ell+2}(\mathbb{C}P^{n+1}) & \longrightarrow & H^{2\ell+2}(S) \\ \downarrow & & \downarrow j^{2\ell} & & \downarrow j^{2\ell+2} & & \downarrow \\ H^{2\ell+1}(K_V) & \longrightarrow & H^{2\ell}(V) & \xrightarrow[\cong]{\psi_V} & H^{2\ell+2}(V) & \longrightarrow & H^{2\ell+2}(K_V) \end{array} \tag{10}$$

Here, ψ is the cup product with the cohomology generator $a \in H^2(\mathbb{C}P^{n+1};\mathbb{Z})$, and similarly, ψ_V is the cup product with $j^2(a)$. For $n+s+2 \leq 2\ell \leq 2n-2$, it follows from (9) that both ψ and ψ_V are isomorphisms. Once we show that $H^{2n-1}(V;\mathbb{Z}) = 0$, the assertion about j^k follows by decreasing induction on ℓ, using the fact mentioned at the beginning of the proof that j^{2n} is given by multiplication by d. To show $H^{2n-1}(V;\mathbb{Z}) = 0$, use the above Gysin sequence to get

$$0 = H^{2n}(K_V;\mathbb{Z}) \longrightarrow H^{2n-1}(V;\mathbb{Z}) \xrightarrow{\psi_V} H^{2n+1}(V;\mathbb{Z}) = 0,$$

thus completing the proof. □

Corollary 1.12. *Let $V \subset \mathbb{C}P^{n+1}$ be a projective hypersurface which has the same $\mathbb{Z}$-cohomology algebra as $\mathbb{C}P^n$. If $n \geq 2$, then V is isomorphic as a variety to $\mathbb{C}P^n$.*

Example 1.13. Consider the cuspidal curve $C = x^2y - z^3 = 0$ in $\mathbb{C}P^2$. The projection of C from the singular point $[0:1:0]$ onto $\mathbb{C}P^1$ is a homeomorphism, so C and $\mathbb{C}P^1$ have the same cohomology algebra (but of course C is not isomorphic as a variety to $\mathbb{C}P^1$). This shows that the assumption $n \geq 2$ in the above corollary is essential.

As we will see later on, the structure of cohomology groups $H^i(V;\mathbb{Z})$, for $i = n, \dots, n+s+1$, can be very different from that of the projective space. Furthermore, as already observed by Zariski in 1930s, the Betti numbers of $V(f)$ depend on the *position of singularities*.

Example 1.14. Let

$$V_6 = \{f(x,y,z) + w^6 = 0\} \subset \mathbb{C}P^3$$

be a sextic surface, so that f defines a plane sextic C_6 with only six cusp singular points. If the six cusps of C_6 are situated on a conic in $\mathbb{C}P^2$, e.g., $f(x,y,z) = (x^2+y^2)^3 + (y^3+z^3)^2$, then $b_2(V_6) = 2$. Otherwise, $b_2(V_6) = 0$. This phenomenon is explained by the fact that, while the two types of sextic curves are homeomorphic, they cannot be deformed one into the other.

1.4. **Topology of projective hypersurface complements: computation of first homotopy and homology group, relation to Milnor fiber.**

1.4.1. *Fundamental group.* If $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ is a hypersurface with complement

$$U = \mathbb{C}P^{n+1} \setminus V,$$

let us assume that $s = \dim \mathrm{Sing}(V) \leq n-2$. Then the Milnor fiber F of f at the origin in $\mathbb{C}^{n+2}$ is simply-connected, so the d-fold covering $F \to U$ is the universal covering for U. It follows that

$$\pi_1(U) \cong \mathbb{Z}/d\mathbb{Z}$$

and

$$\pi_i(U) = 0 \quad \text{for} \ \ i = 0 \ \text{ or } \ 2 \leq i \leq n-s-1.$$

1.4.2. *First homology group.* Assume now that $V(f) \subset \mathbb{C}P^{n+1}$ is reduced, i.e., $f = f_1 \cdots f_r$ is a square-free polynomial, with irreducible components $V_i = \{f_i = 0\}$, $i = 1, \ldots, r$. Let $d_i = \deg(f_i)$, $i = 1, \ldots, r$. Let $U = \mathbb{C}P^{n+1} \setminus V$. Then

Proposition 1.15.

$$H_1(U;Z) \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/\mathrm{gcd}(d_1, \ldots, d_r)\mathbb{Z}.$$

Proof. By Alexander duality, there is an isomorphism

$$H_1(U;Z) \cong H^{2n+1}(\mathbb{C}P^{n+1}, V; \mathbb{Z}).$$

The cohomology long exact sequence of the pair $(\mathbb{C}P^{n+1}, V)$ yields

$$H^{2n}(\mathbb{C}P^{n+1}, \mathbb{Z}) \stackrel{j^{2n}}{\to} H^{2n}(V, \mathbb{Z}) \to H^{2n+1}(\mathbb{C}P^{n+1}, V; \mathbb{Z}) \to 0,$$

with $j : V \hookrightarrow \mathbb{C}P^{n+1}$ the inclusion map. The assertion follows now from (5) and (6), i.e.,

$$H^{2n}(V;\mathbb{Z}) \cong H^{2n}(V_1;\mathbb{Z}) \oplus \cdots \oplus H^{2n}(V_r;\mathbb{Z}) = \mathbb{Z}^r,$$

and the projection of j^{2n} on $H^{2n}(V_i;\mathbb{Z})$ is given by multiplication by d_i. □

Corollary 1.16. *If $X \subset \mathbb{C}^{n+1}$ is an affine hypersurface with r irreducible components, then*

$$H_1(\mathbb{C}^{n+1} \setminus X; \mathbb{Z}) \cong \mathbb{Z}^r.$$

Proof. Apply the previous result to $V = \bar{X} \cup H_\infty \subset \mathbb{C}P^{n+1}$, where $\bar{X}$ is the projective completion of X and $H_\infty = \mathbb{C}P^{n+1} \setminus \mathbb{C}^{n+1}$ is the hyperplane at infinity. □

1.4.3. *Further relation to Milnor fiber.* We need the following:

Lemma 1.17. *If $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ is a degree d reduced projective hypersurface, the inclusion map $j{:}V \hookrightarrow \mathbb{C}P^{n+1}$ induces momomorphisms*

$$j^k{:}H^k(\mathbb{C}P^{n+1};\mathbb{C}) \rightarrowtail H^k(V;\mathbb{C}) \ \ \textit{for all k with} \ \ 0 \leq k \leq 2n. \tag{11}$$

Proof. Exercise. □

In particular, the long exact sequence for the cohomology of $(\mathbb{C}P^{n+1}, V)$ breaks into short exact sequences:

$$0 \longrightarrow H^k(\mathbb{C}P^{n+1};\mathbb{C}) \longrightarrow H^k(V;\mathbb{C}) \longrightarrow H^{k+1}(\mathbb{C}P^{n+1}, V;\mathbb{C}) \to 0. \tag{12}$$

On the other hand, the Alexander duality yields isomorphisms:

$$H^{k+1}(\mathbb{C}P^{n+1}, V;\mathbb{C}) \cong H_{2n+1-k}(U;\mathbb{C}). \tag{13}$$

If $F = \{f = 1\}$ is the affine Milnor fiber of the homogeneous polynomial f, with the corresponding monodromy homeomorphism h (which is given by multiplication by a primitive dth root of unity), then one has as above the identification $U = F/\langle h \rangle$, and hence

$$H_*(U;\mathbb{C}) \cong H_*(F;\mathbb{C})^{h_*}, \tag{14}$$

the fixed part under the homology monodromy operator. Combining (12), (13) and (14), one gets the following useful consequence.

Corollary 1.18. *A hypersurface $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ has the same $\mathbb{C}$-cohomology as $\mathbb{C}P^n$ if and only if the monodromy operator*

$$h_* : \widetilde{H}_*(F;\mathbb{C}) \longrightarrow \widetilde{H}_*(F;\mathbb{C})$$

acting on the reduced $\mathbb{C}$-homology of the corresponding affine Milnor fiber $F = \{f = 1\}$, has no eigenvalue equal to 1.

Example 1.19. The hypersurface $V_n = \{x_0 x_1 \cdots x_n + x_{n+1}^{n+1} = 0\}$ has the same $\mathbb{C}$-cohomology as $\mathbb{C}P^n$. However, it can be shown that the $\mathbb{Z}$-cohomology groups of V_n may contain torsion.

2. Vanishing cycles and applications

2.1. **Nearby and vanishing cycles. Specialization.** Let $f : X \to D \subset \mathbb{C}$ be a *proper* holomorphic map defined on a complex analytic variety X, where D is a small disc at the origin. Let $X_t = f^{-1}(t)$ be the fiber over $t \in D$. For $x \in X_0$, let $\mathring{B}_{\varepsilon,x}$ be an open ball of small enough radius ε in X, centered at x. (If X is singular, such a ball is defined by using an embedding of the germ (X,x) in a complex affine space.) Then for $|t|$ non-zero and sufficiently small, $F_x = \mathring{B}_{\varepsilon,x} \cap X_t$ is a (local) Milnor fiber of f at x.

This local Milnor information at points in $X_0 = f^{-1}(0)$ has been *sheafified* by Grothendieck and Deligne, who defined *nearby and vanishing cycle complexes of sheaves* $\psi_f \underline{A}_X$, resp., $\varphi_f \underline{A}_X$ (where A is a ring of coefficients, e.g., $\mathbb{Z}$ or a field, and $\underline{A}_X$ is the constant sheaf with stalk A on X). More precisely, the stalk at $x \in X_0$ of the cohomology sheaves of these complexes is computed as:

$$\mathscr{H}^k(\psi_f \underline{A}_X)_x \cong H^k(F_x;A) \;\text{ and }\; \mathscr{H}^k(\varphi_f \underline{A}_X)_x \cong \widetilde{H}^k(F_x;\mathbb{Z}).$$

If, moreover, X is smooth, then since the Milnor fiber at a smooth point of X_0 is contractible, the vanishing cycle complex is supported only on $\mathrm{Sing}(X_0)$.

Since f is proper, the (hyper)cohomology groups of these complexes fit into the following *specialization sequence*:

$$\cdots \longrightarrow H^k(X_0;A) \longrightarrow H^k(X_t;A) \longrightarrow H^k(X_0;\varphi_f \underline{A}_X) \longrightarrow \cdots \tag{15}$$

for $t \in D^*$. So, just like in the local case, the vanishing cycle complex measures the change in topology under the specialization map $sp : X_t \to X_0$. Moreover, if A is a field, using the fact that the fibers of f are compact, the corresponding Euler characteristics are well defined and one gets

$$\chi(X_t) = \chi(X_0) + \chi(X_0, \varphi_f \underline{Z}_X), \tag{16}$$

with

$$\chi(X_0, \varphi_f \underline{A}_X) := \chi\left(H^*(X_0; \varphi_f \underline{A}_X)\right).$$

Assume next that the fibers of f are complex algebraic varieties, like in the situations considered below. Then $\chi(X_0,\varphi_f\underline{A}_X)$ can be computed in terms of a stratification of X_0, by using the additivity and multiplicativity properties of the Euler characteristic. For instance, if X is nonsingular and $\mathscr{S}$ is a stratification of X_0 such that $\varphi_f\underline{A}_X$ is $\mathscr{S}$-constructible, one gets:

Lemma 2.1.

$$\chi(X_0,\varphi_f\underline{A}_X)=\sum_{S\in\mathscr{S}}\chi(S)\cdot\mu_S, \tag{17}$$

where

$$\mu_S:=\chi\left(\mathscr{H}^*(\varphi_f\underline{A}_X)_{x_S}\right)=\chi\left(\widetilde{H}^*(F_{x_S};A)\right)$$

is the Euler characteristic of the reduced cohomology of the Milnor fiber F_{x_S} of f at some point $x_S\in S$.

Example 2.2 (Isolated singularities)**.** In the above notations, assume moreover that X is nonsingular and the zero-fiber X_0 has *only isolated singularities*.

Assume $\dim_{\mathbb{C}}X=n+1$. Then, for $x\in\mathrm{Sing}(X_0)$, the corresponding Milnor fiber $F_x\simeq\bigvee_{\mu_x}S^n$ is up to homotopy a bouquet of n-spheres, and the stalk calculation for vanishing cycles yields:

$$H^k(X_0;\varphi_f\underline{A}_X)\cong\bigoplus_{x\in\mathrm{Sing}(X_0)}\mathscr{H}^k(\varphi_f\underline{A}_X)_x=\begin{cases}0, & k\neq n,\\ \bigoplus_{x\in\mathrm{Sing}(X_0)}\widetilde{H}^n(F_x;A), & k=n.\end{cases}$$

Then the long exact sequence (15) becomes the following *specialization sequence*:

$$0\longrightarrow H^n(X_0;A)\longrightarrow H^n(X_t;A)\longrightarrow\bigoplus_{x\in\mathrm{Sing}(X_0)}\widetilde{H}^n(F_x;A)$$
$$\longrightarrow H^{n+1}(X_0;A)\longrightarrow H^{n+1}(X_t;A)\longrightarrow 0,$$

for $t\in D^*$, together with isomorphisms

$$H^k(X_0;A)\cong H^k(X_t;A)\ ,\ \text{for } k\neq n,n+1.$$

Taking Euler characteristics, one gets for $t\in D^*$ the identity:

$$\chi(X_t)=\chi(X_0)+\sum_{x\in\mathrm{Sing}(X_0)}\chi(\widetilde{H}^*(F_x;A))=\chi(X_0)+(-1)^n\sum_{x\in\mathrm{Sing}(X_0)}\mu_x$$

or, equivalently,

$$\chi(X_0)=\chi(X_t)+(-1)^{n+1}\sum_{x\in\mathrm{Sing}(X_0)}\mu_x. \tag{18}$$

2.2. **Vanishing cycles for a family of complex projective hypersurfaces and applications.** Let $V=\{f=0\}\subset\mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface of degree d. Fix a Whitney stratification $\mathscr{S}$ of V and consider a *one-parameter smoothing* of degree d, namely

$$V_t:=\{f_t=f-tg=0\}\subset\mathbb{C}P^{n+1}\ \ (t\in\mathbb{C}),$$

for g a general polynomial of degree d. Note that, for $t\neq 0$ small enough, V_t is smooth and tranversal to the stratification $\mathscr{S}$. Let

$$B=\{f=g=0\}$$

be the *base locus* of the pencil. Consider the *incidence variety*

$$V_D := \{(x,t) \in \mathbb{C}P^{n+1} \times D \mid x \in V_t\},$$

with D a small disc centered at $0 \in \mathbb{C}$ so that V_t is smooth for all $t \in D^* := D \setminus \{0\}$. Denote by

$$\pi \colon V_D \to D$$

the proper projection map, and note that $V = V_0 = \pi^{-1}(0)$ and $V_t = \pi^{-1}(t)$ for all $t \in D^*$. In what follows we will write V for V_0 and use V_t for a "smoothing" of V.

By definition, the incidence variety V_D is a complete intersection of pure complex dimension $n+1$. It is nonsingular if $V = V_0$ has only isolated singularities, but otherwise it has singularities where the base locus B of the pencil $\{f_t\}_{t\in D}$ intersects the singular locus $\Sigma := \mathrm{Sing}(V)$ of V.

In what follows we give applications of the vanishing cycle complex $\varphi_\pi \underline{A}_{V_D}$ associated to the projection π.

2.2.1. *Euler characteristic of an arbitrary complex projective hypersurface.* Consider the specialization sequence (15) for π, namely:

$$\cdots \longrightarrow H^k(V;A) \xrightarrow{sp^k} H^k(V_t;A) \xrightarrow{can^k} H^k(V;\varphi_\pi \underline{A}_{V_D}) \longrightarrow H^{k+1}(V;A) \xrightarrow{sp^{k+1}} \cdots \tag{19}$$

Here, the maps sp^k are the specialization morphisms in cohomology, while the maps can^k are called "canonical" morphisms.

Recall that the stalk of the cohomology sheaves of $\varphi_\pi \underline{A}_{V_D}$ at a point $x \in V$ are computed by:

$$\mathscr{H}^j(\varphi_\pi \underline{A}_{V_D})_x \cong \widetilde{H}^j(B_x \cap V_t;A),$$

where B_x denotes the intersection of V_D with a sufficiently small ball in some chosen affine chart $\mathbb{C}^{n+1} \times D$ of the ambient space $\mathbb{C}P^{n+1} \times D$ (hence B_x is contractible). Here $B_x \cap V_t = F_{\pi,x}$ is the Milnor fiber of π at x. Let us now consider the function

$$h := f/g \colon \mathbb{C}P^{n+1} \setminus W \to \mathbb{C}$$

where $W := \{g = 0\}$, and note that $h^{-1}(0) = V \setminus B$ with $B = V \cap W$ the base locus of the pencil. If $x \in V \setminus B$, then in a neighborhood of x one can describe V_t ($t \in D^*$) as

$$\{x \mid f_t(x) = 0\} = \{x \mid h(x) = t\},$$

that is, as the Milnor fiber of h at x. Note also that h defines V in a neighborhood of $x \notin B$. Since the Milnor fiber of a complex hypersurface singularity germ does not depend on the choice of a local equation, we can therefore use h or a local representative of f when considering Milnor fibers of π at points in $V \setminus B$. We will therefore use the notation F_x for the Milnor fiber of the hypersurface singularity germ (V,x).

It is a well known fact that the projection π has no vanishing cycles along the base locus B (in fact, by integrating a controlled vector field, it can be shown that the Milnor fiber of π at a point in B is contractible.) In view of the above discussion, we get from (16):

$$\chi(V_t) = \chi(V) + \chi(V \setminus B, \varphi_h \underline{A}_{V_D}). \tag{20}$$

Therefore, Lemma 2.1 yields the following result.

Theorem 2.3 (Parusiński-Pragacz). *Let $V = \{f = 0\} \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface of degree d, and fix a Whitney stratification $\mathscr{S}$ of V. Let $W = \{g = 0\} \subset \mathbb{C}P^{n+1}$ be a smooth degree d projective hypersurface which is transversal to $\mathscr{S}$. Then*

$$\chi(V) = \chi(W) - \sum_{S\in\mathscr{S}} \chi(S \setminus W) \cdot \mu_S, \tag{21}$$

where

$$\mu_S := \chi\left(\widetilde{H}^*(F_{x_S};A)\right)$$

is the Euler characteristic of the reduced cohomology of the Milnor fiber F_{x_S} of V at some point $x_S \in S$.

Example 2.4 (Isolated singularities)**.** If the degree d hypersurface $V \subset \mathbb{C}P^{n+1}$ has only isolated singularities, one gets by (21) and Proposition 1.7 the following formula for the Euler characteristic of V:

$$\chi(V) = (n+2) - \frac{1}{d}\{1 + (-1)^{n+1}(d-1)^{n+2}\} + (-1)^{n+1} \sum_{x\in\mathrm{Sing}(V)} \mu_x. \tag{22}$$

In particular, if V is a projective *curve* (i.e., $n = 1$), then the Betti numbers of V are computed as: $b_0(V) = 1$; $b_2(V) = r$, with r denoting the number of irreducible components of V (e.g., see (5)); finally, (22) yields that

$$b_1(V) = r + 1 + d^2 - 3d - \sum_{x\in\mathrm{Sing}(V)} \mu_x. \tag{23}$$

2.2.2. *Vanishing (co)homology and integral (co)homology of projective hypersurfaces; Betti numbers estimates (in the range not covered by Kato's theorem).* For a singular degree d reduced projective hypersurface V, consider a one-parameter smoothing V_t together with the incidence variety V_D and projection map $\pi\colon V_D \to D$, as in the previous section. We note that since the incidence variety $V_D = \pi^{-1}(D)$ deformation retracts to $V = \pi^{-1}(0)$, it follows readily that

$$H^k(V;\varphi_\pi \underline{A}_{V_D}) \cong H^{k+1}(V_D,V_t;A).$$

These groups are call the *vanishing cohomology* groups of V, and they will be denoted by $H^k_\varphi(V)$. In particular, the groups $H^k_\varphi(V)$ are the cohomological version of the *vanishing homology groups*

$$H^\curlyvee_k(V) := H_k(V_D,V_t;\mathbb{Z})$$

introduced and studied by Siersma-Tibăr.

Properties of vanishing cycles together with vanishing results of Artin type can be used to prove the following result, which generalizes the situation of Example 2.2 as well as results of Siersma-Tibăr for 1-dimensional singularities.

Theorem 2.5 (M.-Tibăr-Păunescu). *Let $V \subset \mathbb{C}P^{n+1}$ be a degree d reduced projective hypersurface with $s = \dim_{\mathbb{C}} \mathrm{Sing}(V)$. Then*

$$H^k_\varphi(V) \cong 0 \quad \textit{for all integers } k \notin [n, n+s]. \tag{24}$$

Moreover, $H^n_\varphi(V)$ is a free abelian group.

By the Universal Coefficient Theorem, we get the concentration degrees of the vanishing homology groups $H_k^{\curlyvee}(V)$ of a projective hypersurface in terms of the dimension of its singular locus (proved by Siersma-Tibăr for 1-dimensional singularities):

Corollary 2.6. *With the above notations and assumptions, we have that*

$$H_k^{\curlyvee}(V) \cong 0 \quad \text{for all integers } k \notin [n+1, n+s+1]. \tag{25}$$

Moreover, $H_{n+s+1}^{\curlyvee}(V)$ *is free.*

An immediate consequence of Theorem 2.5 and of the specialization sequence (19) is the following result on the integral cohomology of a complex projective hypersurface.

Corollary 2.7. *Let* $V \subset \mathbb{C}P^{n+1}$ *be a degree d reduced projective hypersurface with a singular locus of complex dimension s. Then:*

(i) $H^k(V;\mathbb{Z}) \cong H^k(V_t;\mathbb{Z}) \cong H^k(\mathbb{C}P^n;\mathbb{Z})$ *for all integers* $k \notin [n, n+s+1]$.
(ii) $H^n(V;\mathbb{Z}) \cong \text{Ker}\,(can^n)$ *is free.*
(iii) $H^{n+s+1}(V;\mathbb{Z}) \cong H^{n+s+1}(\mathbb{C}P^n;\mathbb{Z}) \oplus \text{Coker}\,(can^{n+s})$.
(iv) $H^k(V;\mathbb{Z}) \cong \text{Ker}\,(can^k) \oplus \text{Coker}\,(can^{k-1})$ *for all integers* $k \in [n+1, n+s]$, $s \geq 1$.

In particular,

$$b_n(V) \leq b_n(V_t) = \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + \frac{3(-1)^n + 1}{2},$$

and

$$b_k(V) \leq \text{rk}\, H_{\varphi}^{k-1}(V) + b_k(\mathbb{C}P^n) \quad \text{for all integers } k \in [n+1, n+s+1],\ s \geq 0.$$

Remark 2.8. One can easily formulate the homological counterpart of the above corollary, which in particular yields that $H_{n+s+1}(V;\mathbb{Z})$ is free. Note also that, since $H^k(V_t;\mathbb{Z})$ is free for all k, Ker (can^k) is also free. So the torsion in $H^k(V;\mathbb{Z})$ for $k \in [n+1, n+s+1]$ may also come from the summand Coker (can^{k-1}).

The ranks of the (possibly non-trivial) vanishing (co)homology groups can be estimated in terms of the local topology of singular strata and of their generic transversal types by making use of homological algebra techniques (e.g., the hypercohomology spectral sequence). Such estimates can be made precise for hypersurfaces with low-dimensional singular loci. Concretely, as special cases of Corollary 2.7, one recasts Siersma-Tibăr's result for $s \leq 1$, and in particular Dimca's computation for $s = 0$. Concerning the estimation of the rank of the highest interesting (co)homology group, we get:

Theorem 2.9 (M.-Tibăr-Păunescu)**.** *Let* $V \subset \mathbb{C}P^{n+1}$ *be a degree d reduced projective hypersurface with a singular locus of complex dimension s. For each connected stratum* $S_i \subseteq \text{Sing}(V)$ *of top dimension s in a Whitney stratification of V, let* $F_i^{\pitchfork}$ *denote its transversal Milnor fiber with corresponding Milnor number* $\mu_i^{\pitchfork}$*. Then:*

$$b_{n+s+1}(V) \leq 1 + \sum_i \mu_i^{\pitchfork}, \tag{26}$$

and the inequality is strict for $n+s$ *even.*

In fact, the inequality in (26) is deduced from

$$b_{n+s+1}(V) \leq 1 + \text{rk}\, H^{n+s}_{\varphi}(V), \tag{27}$$

together with

$$\text{rk}\, H^{n+s}_{\varphi}(V) \leq \sum_i \mu_i^{\pitchfork}, \tag{28}$$

and the inequality (27) is strict for $n+s$ even. Note that if $s=0$, i.e., V has only isolated singularities, then $\mu_i^{\pitchfork}$ is just the usual Milnor number of such a singularity of V.

Example 2.10 (Singular quadrics)**.** Let n and q be integers satisfying $4 \leq q \leq n+1$, and let

$$f_q(x_0, \dots x_{n+1}) = \sum_{0 \leq i,j \leq n+1} q_{ij} x_i x_j$$

be a quadric of rank $q := \text{rk}(Q)$ with $Q = (q_{ij})$. The singular locus Σ of the quadric hypersurface $V_q = \{f_q = 0\} \subset \mathbb{C}P^{n+1}$ is a linear space of complex dimension $s = n+1-q$ satisfying $0 \leq s \leq n-3$. The generic transversal type for $\Sigma = \mathbb{C}P^s$ is an A_1-singularity, so $\mu^{\pitchfork} = 1$. A direct calculation shows that if the rank q is even (i.e., $n+s+1$ is even), then $b_{n+s+1}(V_q) = 2$, and hence the upper bound in (26) is sharp.

Exercise 2.11. Compute the integral cohomology of the quadric hypersurface $V_q = \{f_q = 0\} \subset \mathbb{C}P^{n+1}$ from the previous example.

Let us remark that if the projective hypersurface $V \subset \mathbb{C}P^{n+1}$ has singularities in codimension 1, i.e., $s = n-1$, then using (5) we get $b_{n+s+1}(V) = b_{2n}(V) = r$, where r denotes the number of irreducible components of V. In particular, Theorem 2.9 yields the following:

Corollary 2.12. *If the reduced projective hypersurface $V \subset \mathbb{C}P^{n+1}$ has singularities in codimension 1, then the number r of irreducible components of V satisfies the inequality:*

$$r \leq 1 + \sum_i \mu_i^{\pitchfork}. \tag{29}$$

Remark 2.13. Note that if the projective hypersurface $V \subset \mathbb{C}P^{n+1}$ is a $\mathbb{Q}$-homology manifold, then the Lefschetz isomorphism (3.4) and Poincaré duality over the rationals yield that $b_i(V) = b_i(\mathbb{C}P^n)$ for all $i \neq n$. Moreover, $b_n(V)$ can be deduced by computing the Euler characteristic of V, as in Theorem 2.3.

The computation of Betti numbers of a projective hypersurface which is a rational homology manifold can be deduced without appealing to Poincaré duality by using the vanishing cohomology instead, as the next result shows:

Proposition 2.14 (M.-Tibăr-Păunescu)**.** *If the projective hypersurface $V \subset \mathbb{C}P^{n+1}$ is a $\mathbb{Q}$-homology manifold, then $H^k_{\varphi}(V) \otimes \mathbb{Q} \cong 0$ for all $k \neq n$. In particular, in this case one gets: $b_i(V) = b_i(V_t) = b_i(\mathbb{C}P^n)$ for all $i \neq n$, and $b_n(V) = b_n(V_t) + \text{rk} H^n_{\varphi}(V)$.*

Example 2.15. Let $V = \{f = 0\} \subset \mathbb{C}P^4$ be the 3-fold in homogeneous coordinates $[x : y : z : t : v]$, defined by

$$f = y^2 z + x^3 + tx^2 + v^3.$$

The singular locus of V is the projective line $\Sigma = \{[0:0:z:t:0] \mid z,t \in \mathbb{C}\}$. By (3.4), we get: $b_0(V) = 1$, $b_1(V) = 0$, $b_2(V) = 1$. Since V is irreducible, (5) yields: $b_6(V) = 1$. We are therefore interested to understand the Betti numbers $b_3(V)$, $b_4(V)$ and $b_5(V)$.

V has a Whitney stratification with strata:

$$S_3 := V \setminus \Sigma, \;\; S_1 := \Sigma \setminus [0:0:0:1:0], \;\; S_0 := [0:0:0:1:0],$$

giving V a two-step filtration $V \supset \Sigma \supset [0:0:0:1:0]$.

The transversal singularity for the top singular stratum S_1 is the Brieskorn type singularity $y^2 + x^3 + v^3 = 0$ at the origin of $\mathbb{C}^3$ (in a normal slice to S_1), with corresponding transversal Milnor number $\mu_1^{\pitchfork} = 4$. So Theorem 2.9 yields that $b_5(V) \leq 5$, while Corollary 2.7 gives $b_3(V) \leq 10$. As we will indicate below, the actual values of $b_3(V)$ and $b_5(V)$ are zero.

It can in fact be shown that the hypersurface V is in fact a $\mathbb{Q}$-homology manifold, so it satisfies Poincaré duality over the rationals. In particular, $b_5(V) = b_1(V) = 0$ and $b_4(V) = b_2(V) = 1$. To determine $b_3(V)$, it suffices to compute the Euler characteristic of V, since $\chi(V) = 4 - b_3(V)$. Let us denote by $Y \subset \mathbb{C}P^4$ a smooth 3-fold which intersects the Whitney stratification of V transversally. Then (2) yields that $\chi(Y) = -6$ and we have by Theorem 2.3 that

$$\chi(V) = \chi(Y) - \chi(S_1 \setminus Y) \cdot \mu_1^{\pitchfork} - \chi(S_0) \cdot (\chi(F_0) - 1), \tag{30}$$

where F_0 denotes the Milnor fiber of V at the singular point S_0. By local inspection it can be shown that $F_0 \simeq S^3 \vee S^3$. So, using the fact that the general 3-fold Y intersects S_1 at 3 points, we get from (30) that $\chi(V) = 4$. Hence $b_3(V) = 0$. Moreover, since $H^3(V;\mathbb{Z})$ is free, this also shows that in fact $H^3(V;\mathbb{Z}) \cong 0$.

Exercise 2.16. Compute the integral cohomology and vanishing cohomology groups of the projective cone on the projective curve $C = \{xyz = 0\} \subset \mathbb{C}P^2$.

At this end, we mention here the following supplement to the Lefschetz hyperplane section theorem for hypersurfaces, which can be used to give a new (inductive) proof of Kato's Theorem 1.11 (without using the connectivity of the Milnor fiber):

Theorem 2.17 (M.-Tibăr-Păunescu)**.** *Let $V \subset \mathbb{C}P^{n+1}$ be a reduced complex projective hypersurface with $s = \dim \mathrm{Sing}(V)$ the complex dimension of its singular locus. (By convention, we set $s = -1$ if V is nonsingular.) Let $H \subset \mathbb{C}P^{n+1}$ be a generic hyperplane (i.e., transversal to a Whitney stratification of V), and denote by $V_H := V \cap H$ the corresponding hyperplane section of V. Then*

$$H^k(V, V_H; \mathbb{Z}) = 0 \;\; \text{for} \;\; k < n \;\; \text{and} \;\; n+s+1 < k < 2n. \tag{31}$$

Moreover, $H^{2n}(V, V_H; \mathbb{Z}) \cong \mathbb{Z}^r$, where r is the number of irreducible components of V, and $H^n(V, V_H; \mathbb{Z})$ is (torsion-)free.

Proof. The long exact sequence for the cohomology of the pair (V, V_H) together with (5) yield that:

$$H^{2n}(V, V_H; \mathbb{Z}) \cong H^{2n}(V; \mathbb{Z}) \cong \mathbb{Z}^r.$$

Moreover, we have isomorphisms:

$$H^k(V, V_H; \mathbb{Z}) \cong H_c^k(V^a; \mathbb{Z}),$$

where $V^a := V \setminus V_H$. Therefore, the vanishing in (31) for $k < n$ is a consequence of the Artin vanishing theorem for the n-dimensional affine variety V^a. Note that vanishing in this range is equivalent to the classical Lefschetz hyperplane section theorem.

Since V is reduced, we have that $s < n$. If $n = s+1$ then $n+s+1 = 2n$ and there is nothing else to prove in (31). So let us now assume that $n > s+1$. For $n+s+1 < k < 2n$, we have the following sequence of isomorphisms:

$$\begin{aligned} H^k(V,V_H;\mathbb{Z}) &\cong H^k(V\cup H, H;\mathbb{Z}) \\ &\cong H_{2n+2-k}(\mathbb{C}P^{n+1}\setminus H, \mathbb{C}P^{n+1}\setminus (V\cup H);\mathbb{Z}) \\ &\cong H_{2n+1-k}(\mathbb{C}P^{n+1}\setminus (V\cup H);\mathbb{Z}), \end{aligned} \tag{32}$$

where the first isomorphism follows by excision, the second is an application of the Poincaré-Alexander-Lefschetz duality, and the third follows from the cohomology long exact sequence of a pair. Set

$$M = \mathbb{C}P^{n+1}\setminus (V\cup H),$$

and let $L = \mathbb{C}P^{n-s}$ be a generic linear subspace (i.e., transversal to both V and H). Then, by transversality, $L\cap V$ is a nonsingular hypersurface in L, transversal to the hyperplane at infinity $L\cap H$ in L. Therefore, $M\cap L = L\setminus (V\cup H)\cap L$ has the homotopy type of a wedge

$$M\cap L \simeq S^1 \vee S^{n-s} \vee \ldots \vee S^{n-s}.$$

Thus, by the Lefschetz hyperplane section theorem (applied $s+1$ times), we obtain:

$$H_i(M;\mathbb{Z}) \cong H_i(M\cap L;\mathbb{Z}) \cong 0$$

for all integers i in the range $1 < i < n-s$. Substituting $i = 2n+1-k$ in (32), we get that $H^k(V,V_H;\mathbb{Z}) \cong 0$ for all integers k in the range $n+s+1 < k < 2n$. □

Remark 2.18. An interesting fact about the shape of projective hypersurfaces is the following result of Dimca-Papadima. Let H be a hyperplane in $\mathbb{C}P^{n+1}$ that is transversal to $V(f)$. Then Dimca-Papadima showed that the affine part $V^a(f) = V(f)\setminus H$ is homotopy equivalent to a bouquet of n-spheres. The number of spheres in this bouquet depends only on V (and not on the defining polynomial f), and is called the *polar degree of* f. It was originally introduced as the topological degree of the Gauss map

$$\text{grad} : \mathbb{C}P^{n+1}\setminus \text{Sing}(V) \to \mathbb{C}P^{n+1}$$

and conjectured (by Dolgachev) to to a topological invariant of V.

Of course, the difficult problem (as indicated by the above theorem) is to glue the information on $V^a(f)$ and $V(f)\cap H$ in order to obtain useful information about $V(f)$.

3. Alexander modules of hypersurface complements

In this section, we give a brief overview of global analogues of the Milnor fibration, together with a local-to-global analysis.

3.1. **Definition. Preliminaries.** Let $X = \{f = f_1 \cdots f_r = 0\} \subset \mathbb{C}^{n+1}$ be a reduced, degree d hypersurface *in general position at infinity* (i.e., the hyperplane H_∞ at infinity in $\mathbb{C}P^{n+1}$ is transversal in the stratified sense on the projective completion $\bar{X}$ of X). Set

$$M = \mathbb{C}^{n+1} \setminus X = \mathbb{C}P^{n+1} \setminus (\bar{X} \cup H_\infty).$$

Then $H_1(M) \cong \mathbb{Z}^r$ (cf. Corollary 1.16), generated by meridian loops γ_i about the non-singular part of each irreducible component X_i of X. (Note the similarity with link theory.)

The homotopy type of M can described as in Sect. 1.4.1 (or by using the Lefschetz hyperplane section theorem) as follows.

Proposition 3.1 (Libgober)**.** *With the above notations and assumptions, if X has no codimension* 1 *singularities, then:*

$$\begin{cases} \pi_1(M) = \mathbb{Z} \\ \pi_i(M) = 0, \text{ for } 1 < i \leq n-s-1, \quad s = \dim_{\mathbb{C}} \mathrm{Sing}(X). \end{cases} \tag{33}$$

Let M^c be the infinite cyclic cover of M corresponding to the kernel of the *total linking number homomorphism* $lk : \pi_1(M) \to \mathbb{Z}$, $lk(\gamma_i) = 1$, $i = 1, \ldots, r$. Under the action of the deck group $\mathbb{Z}$, each $H_i(M^c;\mathbb{Q})$ becomes a $\Gamma := \mathbb{Q}[t,t^{-1}]$-module, called the *i-th Alexander modules of the hypersurface complement*. One should think of M^c as a global counterpart of the Milnor fibration.

Since Γ is PID, torsion Γ-modules have well-defined associated polynomials (also called *orders*).

Since M is affine of complex dimension $n+1$, it is homotopy equivalent to a finite CW-complex of dimension $n+1$. This implies that each $H_i(M^c;\mathbb{Q})$ is of finite type over Γ (but not over $\mathbb{Q}$). Moreover,

$$\begin{cases} H_i(M^c;\mathbb{Q}) \cong 0 \ \text{ for } i > n+1 \\ H_{n+1}(M^c;\mathbb{Q}) \text{ is free over } \Gamma \end{cases} \tag{34}$$

The goal is to get interesting information about $H_i(M^c;\mathbb{Q})$, $i \leq n$.

If X is non-singular, then (33) yields that M^c is the universal cover of M and

$$\widetilde{H}_i(M^c;\mathbb{Z}) \cong 0 \ \text{ for } \ i < n+1$$

The case of a hypersurface X with *only isolated singularities* (including at infinity, though we restrict here to transversality at infinity), was considered by Libgober, following a suggestion by Mumford. Libgober proved that the only interesting Alexander module, $H_n(M^c;\mathbb{Q})$, is a torsion Γ-module. Moreover, if

$$\Delta_n(t) := \mathrm{order} H_n(M^c;\mathbb{Q}),$$

then $\Delta_n(t)$ has all zeros among roots of unity of order d, and it divides (up to a factor $(t-1)^{r-1}$) the product:

$$\prod_{x \in \mathrm{Sing}(X)} \Delta_x(t)$$

of the local Alexander polynomials associated to the isolated singularities of X. So Δ_n depends on the local type of singularities. The fact that $\Delta_n(t)$ is cyclotomic already imposes strong restrictions on the type of groups that can be fundamental groups of hypersurface complements, given partial answers to a question of Serre. For instance, the group of the figure eight knot (with $\Delta(t) = t^2 - 3t + 1$) cannot be of the form $\pi_1(\mathbb{C}^2 \setminus X)$.

Example 3.2. Let $\bar{C} \subset \mathbb{C}P^2$ be a degree d irreducible curve having only nodes and cusps as singular points. Let $C = \bar{C} \setminus L$ for L a generic line. If $d \neq 0 \pmod 6$, the above divisibility result implies that $\Delta_C(t) = 1$.

Example 3.3 (Zariski-Libgober)**.** If $\bar{C} \subset \mathbb{C}P^2$ is an irreducible sextic having only cusps singularities, then the global Alexander polynomial $\Delta_C(t)$ of the curve C is either 1 or a power of $t^2 - t + 1$. There are two distinct cases:

(1) if $\bar{C}$ is in "special *position*", i.e., the 6 cusps are on a conic, then $\Delta_C(t) = t^2 - t + 1$ (and $\pi_1(\mathbb{C}^2 \setminus C) = B_3$).
(2) if $\bar{C}$ is in "general position", i.e., the cusps are not on a conic, then $\Delta_C(t) = 1$ (and $\pi_1(\mathbb{C}^2 \setminus C)$ is abelian).

So the global Alexander polynomials also *depend on the position of singularities*.

3.2. **Non-isolated singularities.** Libgober's work has been generalized to the non-isolated singularities case by the lecturer, using intersection homology, perverse sheaves, etc. Follow up work was done by Dimca-Libgober, Liu, etc.

Theorem 3.4 (M.)**.** *For $i \leq n$, $H_i(M^c;\mathbb{C})$ is a finitely generated semi-simple torsion $\mathbb{C}[t,t^{-1}]$-module (hence a finite dimensional $\mathbb{C}$-vector space), which is annihilated by $t^d - 1$. (Recall that $d = \deg f$.)*

Proof. Let S_∞ be a $(2n+1)$-sphere in $\mathbb{C}^{n+1}$ of a sufficiently large radius (i.e., the boundary of a small tubular neighborhood in $\mathbb{C}P^{n+1}$ of the hyperplane H_∞ at infinity). Denote by $X_\infty := S_\infty \cap X$ the *link of X at infinity*, and by $M_\infty = S_\infty \setminus X_\infty$ the corresponding complement. Let M^c_∞ be the infinite cyclic cover of M_∞ defined by the composition

$$lk_\infty : \pi_1(M_\infty) \to \pi_1(M) \overset{lk}{\to} \mathbb{Z}.$$

Note that M_∞ is homotopy equivalent to $T(H_\infty) \setminus (\bar{X} \cup H_\infty)$, where $T(H_\infty)$ is the tubular neighborhood of H_∞ in $\mathbb{C}P^{n+1}$ for which S_∞ is the boundary. Then a classical argument based on the Lefschetz hyperplane theorem yields that the homomorphism $\pi_i(M_\infty) \to \pi_i(M)$ induced by inclusion is an isomorphism for $i < n$ and it is surjective for $i = n$. It follows that

$$\pi_i(M, M_\infty) = 0 \ \text{ for } i \leq n,$$

hence M has the homotopy type of a CW complex obtained from M_∞ by adding cells of dimension $\geq n+1$. So the same is true for any covering, and in particular for the corresponding infinite cyclic coverings. So the group homomorphisms

$$H_i(M^c_\infty;\mathbb{C}) \to H_i(M^c;\mathbb{C}) \tag{35}$$

are isomorphisms for $i < n$ and surjective for $i = n$. Since these group homomorphisms are induced by an inclusion map, they are in fact $\mathbb{C}[t,t^{-1}]$-module homomorphisms.

In view of (35), it suffices to prove that the *Alexander modules at infinity* $H_i(M^c_\infty;\mathbb{C})$ $(i \leq n)$ are torsion semi-simple $\mathbb{C}[t,t^{-1}]$-modules, which are annihilated by $t^d - 1$. For this we note that M_∞ is a circle fibration over $H \setminus \bar{X} \cap H$, which is homotopy equivalent to the complement in $\mathbb{C}^{n+1}$ to the affine cone over the projective hypersurface $\bar{X} \cap H = \{f_d = 0\}$, where f_d is the top degree monomial (of degree d) of f. Hence the infinite cyclic cover M^c_∞ is homotopy equivalent to the Milnor fiber $\{f_d = 1\}$ of f_d, so $H_i(M^c_\infty;\mathbb{C})$ $(i \leq n)$ is a torsion finitely generated $\mathbb{C}[t,t^{-1}]$-module. Moreover, since the monodromy of the Milnor fibration mentioned above is of finite order d (and hence semisimple), it also follows that these Alexander modules at infinity in the range $i \leq n$ are semi-simple and annihilated by $t^d - 1$. □

Definition 3.5. For $i \leq n$, the order, $\Delta_i(t)$ of $H_i(M^c;\mathbb{Q})$ is the characteristic polynomial of the generating covering transformation, and is called *the i-th global Alexander polynomial of the hypersurface X*.

Corollary 3.6. *$rank_\Gamma H_{n+1}(M^c;\mathbb{Q}) = (-1)^{n+1}\chi(M) = \mathrm{pol}(\bar{X})$, where $\mathrm{pol}(\bar{X})$ is the polar degree of the projectivisation of X.*

As a consequence of Theorem 3.4, we get.

Corollary 3.7. *The zeros of $\Delta_i(t)$, $i \leq n$, are roots of unity of order d.*

Remark 3.8 (Affine cone on projective hypersurface arrangements in $\mathbb{C}P^n$)**.** If X is defined by a degree d *homogeneous* polynomial f, then there is a Γ-module isomorphism:

$$H_i(M^c;\mathbb{Q}) \cong H_i(F;\mathbb{Q}),$$

where $F = f^{-1}(1)$ is the fiber of the global Milnor fibration $M = \mathbb{C}^{n+1} - f^{-1}(0) \xrightarrow{f} \mathbb{C}^*$ associated to the homogeneous polynomial f, and the module structure on $H_i(F;\mathbb{Q})$ is induced by the monodromy action. Hence zeros of the global Alexander polynomials of X coincide with the eigenvalues of the monodromy operators acting on the homology of F. Since the monodromy homeomorphism has finite order d, all these eigenvalues are roots of unity of order d. So *a polynomial in general position at infinity behaves much like a homogeneous polynomial*.

Since $H_i(M^c;\mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-vector space for $i \leq n$, it is natural to investigate the existence of mixed Hodge structures on these Alexander modules. We have:

Theorem 3.9 (Dimca-Libgober, Liu)**.** *For $i \leq n$, there is a mixed Hodge structure on $H_i(M^c;\mathbb{Q})$.*

3.2.1. *Divisibility.* In this section we explain how the global Alexander invariants of a hypersurface can be understood in terms of local topological information around the singularities.

Let $\mathscr{S}$ be a Whitney stratification of X, i.e. a decomposition of X into disjoint connected non-singular subvarieties $\{S_\alpha\}$, called *strata*, such that X is *uniformly singular* along each stratum. This yields a Whitney stratification of the pair $(\mathbb{C}^{n+1}, X)$, with $\mathscr{S}$ the set of singular strata. Fix $S \in \mathscr{S}$ a k-dimensional stratum of $(\mathbb{C}^{n+1}, X)$. A point $p \in S$ has an associated *link pair* $(S^{2n-2k+1}(p), K(p))$, defined as before in a normal slice to the stratum. The link pair has constant topological type along S, denoted by $(S^{2n-2k+1}, K)$. This is a singular algebraic link, and has an associated local Milnor fibration:

$$F^k \hookrightarrow S^{2n-2k+1} \setminus K \to S^1$$

with fibre F^k and monodromy $h^k : F^k \to F^k$. Let

$$\Delta^k_j(t) = \det(tI - (h^k)_* : H_j(F^k;\mathbb{Q}) \to H_j(F^k;\mathbb{Q}))$$

be the j-th *local Alexander polynomial* associated to S.

Theorem 3.10 (M.)**.** *Fix an arbitrary irreducible component of X, say X_1, and fix $i \leq n$. Then the zeros of $\Delta_i(t)$ are among the zeros of polynomials $\Delta_j^k(t)$, associated to links of strata $S \subset X_1$, such that $n-i \leq k = dim S \leq n$, and j is in the range $2n-2k-i \leq j \leq n-k$. Moreover, if X has no codimension 1 singularities and is a rational homology manifold, then $\Delta_i(1) \neq 0$.*

Remark 3.11. 0-dim strata of X only contribute to Δ_n, 1-dim strata contribute to Δ_n and Δ_{n-1} and so on.

Corollary 3.12 (Vanishing of Alexander polynomials)**.** *Let X be a degree d hypersurface in general position at infinity, which is rationally smooth and has no codimension 1 singularities. Assume that the local monodromies of link pairs of strata contained in some irreducible component X_1 of X have orders which are relatively prime to d (e.g., the transversal singularities along strata of X_1 are Brieskorn-type singularities, having all exponents relatively prime to d). Then $\Delta_i(t) \sim 1$, for $1 \leq i \leq n$.*

Exercise 3.13. Compute the Alexander polynomials for $X = \{y^2z + x^3 + x^2 + v^3 = 0\} \subset \mathbb{C}^4$.

Remark 3.14 (Projective hypersurface arrangements)**.** Applying the divisibility result of Theorem 3.10 to the case when X is the affine cone over a projective hypersurface arrangement in $\mathbb{C}P^n$ (i.e., X is defined by a homogeneous polynomial), we get a similar result for the characteristic polynomials $P_q(t)$ ($q \leq n-1$) of monodromy operators of the Milnor fiber of the arrangement, thus upper bounds for the multiplicities of eigenvalues of the monodromy operators.

Corollary 3.15 (Triviality of monodromy)**.** *If $\lambda \neq 1$ is a d-th root of unity such that λ is not an eigenvalue of any of the local monodromies corresponding to link pairs of singular strata of Y_1 in a stratification of the pair $(\mathbb{C}P^n, Y)$, then λ is not an eigenvalue of the monodromy operators acting on $H_q(F;\mathbb{Q})$ for $q \leq n-1$.*

Remark 3.16. Let $P_q(t)$ be the characteristic polynomial of the monodromy operator $h_q : H_q(F;\mathbb{Q}) \to H_q(F;\mathbb{Q})$, with F the Milnor fiber of the homogeneous polynomial f. The polynomials $P_q(t)$, $q = 0, \cdots, n$, are related by the formula:

$$\prod_{q=0}^{n} P_q(t)^{(-1)^{q+1}} = (1-t^d)^{-\chi(F)/d}$$

where $\chi(F)$ is the Euler characteristic of the Milnor fiber. Therefore, it suffices to compute only the polynomials $P_0(t), \cdots, P_{n-1}(t)$ and the Euler characteristic of F.

Example 3.17. If $\bigcup_{i=1}^{s} Y_i$ is a normal crossing divisor at any point $x \in Y_1$, the monodromy action on $H_q(F;\mathbb{Q})$ is trivial for $q \leq n-1$.

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