Measuring the complexity of hypersurface singularities in algebraic geometry and topology

LAURENTIU MAXIM
University of Wisconsin-Madison

Arapura60, Madison, WI
September 13, 20019

I. *Characteristic classes*

Virtual tangent bundle of a hypersurface

- Let $X \overset{i}{\hookrightarrow} Y$ be a complex algebraic *hypersurface* (or *lci*) in a complex algebraic manifold Y, with *normal bundle* $N_X Y$.
- The *virtual tangent bundle* of X is:

$$T_X^{\mathrm{vir}} := [T_Y|_X] - [N_X Y] \in K^0(X)$$

- T_X^{vir} is *independent of the embedding* in Y, so it is a well-defined element in $K^0(X)$, the Grothendieck group of algebraic vector bundles on X.
- If X is *smooth*: $T_X^{\mathrm{vir}} = [T_X] \in K^0(X)$.

Characteristic classes

- Let R be a commutative ring with unit, and

$$cl^* : \left(K^0(X), \oplus\right) \to \left(H^*(X) \otimes R, \cup\right)$$

a *multiplicative characteristic class theory* of complex algebraic vector bundles, with $H^*(X) = H^{2*}(X;\mathbb{Z})$.

- Associate to a hypersurface (or lci) X an *intrinsic* homology class (i.e., independent of the embedding $X \hookrightarrow Y$):

$$cl_*^{\mathrm{vir}}(X) := cl^*(T_X^{\mathrm{vir}}) \cap [X] \in H_*(X) \otimes R,$$

with $[X] \in H_*(X)$ the fundamental class of X in a suitable homology theory $H_*(X)$ (e.g., $H_{2*}^{BM}(X)$=Borel-Moore homology).

- Assume $cl_*(-)$ is a *homology characteristic class theory* for complex algebraic varieties, so that if X smooth:

$$cl_*(X) = cl^*(T_X) \cap [X] \quad \text{(normalization)}$$

- If X is *smooth*:

$$cl_*^{\mathrm{vir}}(X) \overset{\mathrm{def}}{:=} cl^*(T_X^{\mathrm{vir}}) \cap [X] \overset{\mathrm{sm}}{=} cl^*(T_X) \cap [X] \overset{nor}{=} cl_*(X) .$$

- If X is *singular*, the difference

$$\mathcal{M}cl_*(X) := cl_*^{\mathrm{vir}}(X) - cl_*(X)$$

depends in general on the singularities of X.

- If $k : X_{\mathrm{sing}} \hookrightarrow X$, then

$$\mathcal{M}cl_*(X) \in \mathrm{Image}(k_*),$$

so $\mathcal{M}cl_*(X)$ *measures the complexity of singularities of X*.

- **Corollary**: $cl_k^{\mathrm{vir}}(X) = cl_k(X)$, for $k > \dim X_{\mathrm{sing}}$.

Problem: *Describe* $\mathcal{M}cl_*(X) = cl_*^{\mathrm{vir}}(X) - cl_*(X)$ *in terms of the geometry of the singular locus* $\Sigma := X_{\mathrm{sing}}$ *of* X*.*

Byproduct: Compute the (very) *complicated* "actual" homology class $cl_*(X)$ in terms of the simpler (cohomological) virtual class and invariants of the singularities of X.

Chern and Milnor classes of singular hypersurfaces

- $cl^* = c^* =$ *Chern class*
- *Virtual Chern (or Fulton-Johnson) class* of X:

$$c_*^{\mathrm{vir}}(X) := c^*(T_X^{\mathrm{vir}}) \cap [X].$$

- $cl_* = c_* =$ *Chern class transformation of MacPherson*,

$$c_* : K_0(D_c^b(X)) \overset{\chi_{st}}{\to} F(X) \overset{c_*}{\to} H_*(X),$$

with

$$c_*(X) := c_*([\mathbb{Q}_X]) = c_*(1_X).$$

(Here $F(X)$ is the group of *constructible functions* on X.)

- *Gauss-Bonnet-Chern*: if X is compact: $\chi(X) = \int_{[X]} c_*(X)$.
- *Milnor class* of X: $\mathcal{M}_*(X) := c_*^{\mathrm{vir}}(X) - c_*(X)$.

Example (Reason for terminology)

If X^n is a hypersurface with only *isolated* singularities, then

$$\mathcal{M}_*(X) = \sum_{x \in X_{\mathrm{sing}}} \chi\left(\widetilde{H}^*(F_x;\mathbb{Q})\right) = \sum_{x \in X_{\mathrm{sing}}} (-1)^n \mu_x,$$

where F_x and μ_x are the the *Milnor fiber* and Milnor number of the IHS germ $(X,x) \subset (\mathbb{C}^{n+1},0)$.

Milnor fibration, nearby and vanishing cycles

- $X^n = f^{-1}(0)$, $f : Y^{n+1} \to \mathbb{C}$ regular, Y-complex manifold
- For $x \in X_{\rm sing}$ and $0 < \delta \ll \epsilon$, there is a *Milnor fibration*:

$$B_\epsilon(x) \cap f^{-1}(D^*_\delta) \xrightarrow{f} D^*_\delta,$$

  whose *Milnor fiber* F_x is a local smoothing of X near x.
- if $x \in X_{\rm sing}$ is isolated, then $F_x \simeq \bigvee_{\mu_x} S^n$, with μ_x the *Milnor number* of f at x; the S^n's are called *vanishing cycles* at x.
- Deligne: there exist *nearby* and resp. *vanishing cycle* functors $\psi_f, \varphi_f : D^b_c(Y) \to D^b_c(X)$, so that

$$\mathcal{H}^k(\psi_f \mathbb{Q}_Y)_x \simeq H^k(F_x;\mathbb{Q}) \ , \ \ \mathcal{H}^k(\varphi_f \mathbb{Q}_Y)_x \simeq \widetilde{H}^k(F_x;\mathbb{Q})$$

- If $x \in X_{\rm reg}$, then F_x is contractible, so $\mathrm{Supp}(\varphi_f \mathbb{Q}_Y) \subseteq X_{\rm sing}$.

Verdier specialization for MacPherson-Chern classes yields:

$$c_*^{\mathrm{vir}}(X) = c_*(\psi_f(\mathbb{Q}_Y))$$

So the *Milnor class* of X is computed by:

$$\mathcal{M}_*(X) := c_*^{\mathrm{vir}}(X) - c_*(X) = c_*(\varphi_f(\mathbb{Q}_Y)) \in H_*(X_{\mathrm{sing}})$$

Vast literature on Milnor classes: Aluffi, Yokura, Ohmoto, Brasselet-Lehmann-Seade-Suwa, Parusiński-Pragacz, Schürmann, M., Fulwood, Callejas-Bedregal-Morgado-Seade, etc.

Hodge polynomials and Hirzebruch classes

- *Hodge polynomial* $\chi_y(X)$ of a complex algebraic variety X is:

$$\sum_j (-1)^j \chi_y(H^j(X;\mathbb{Q})) := \sum_{j,p} (-1)^j \dim \mathrm{Gr}_F^p H^j(X;\mathbb{C}) \cdot (-y)^p$$

- *generalized Hirzebruch-Riemann-Roch theorem*: if X is *smooth* and *compact*, then

$$\chi_y(X) = \int_{[X]} T_y^*(X) \cap [X],$$

  with $T_y^*(X) := T_y^*(T_X)$ the *cohomology Hirzebruch class* of X.

- $T_{-1}^*(X) = c^*(X)$, just as $\chi_{-1} = \chi$.

- if X is singular, then

$$\chi_y(X) = \int_{[X]} T_{y*}(X),$$

where $T_{y*}(X) := T_{y*}([\mathbb{Q}_X^H)]$ is the homology Hirzebruch class of X, for

$$T_{y*} : K_0(\mathrm{MHM}(X)) \to H_*(X) \otimes \mathbb{Q}[y^{\pm 1}]$$

the Brasselet-Schürmann-Yokura Hirzebruch class transformation (2005).

- if X is *smooth*, then

$$T_{y*}(X) = T_y^*(T_X) \cap [X] \quad \text{(normalization)}$$

- $$T_{-1*}(X) = c_*(X) \in H_*(X) \otimes \mathbb{Q},$$

(a class version of $\chi_{-1} = \chi$.)

Milnor-Hirzebruch classes of complex hypersurfaces

Definition

The *Milnor-Hirzebruch class* of a complex algebraic hypersurface X in the complex algebraic manifold Y is defined as:

$$\mathcal{M}_{y*}(X) := T^{\mathrm{vir}}_{y*}(X) - T_{y_*}(X),$$

where

$$T^{\mathrm{vir}}_{y*}(X) := T^*_y(T^{\mathrm{vir}}_X) \cap [X] \in H_*(X) \otimes \mathbb{Q}[y].$$

Remark

- $\mathcal{M}_{-1*}(X) = \mathcal{M}_*(X) \otimes \mathbb{Q}$

Milnor-Hirzebruch classes of (global) hypersurfaces

- Let Y^{n+1} be a complex algebraic manifold, and $f : Y \to \mathbb{C}$ an algebraic function, with $X := \{f = 0\}$.
- Deligne's *nearby* and resp. *vanishing cycle* functors $\psi_f, \varphi_f : D^b_c(Y) \to D^b_c(X)$ admit lifts to Saito's *mixed Hodge modules*.
- In particular, the stalk cohomologies

$$\mathcal{H}^k(\psi_f \mathbb{Q}_Y)_x \simeq H^k(F_x; \mathbb{Q}) \;, \;\; \mathcal{H}^k(\varphi_f \mathbb{Q}_Y)_x \simeq \widetilde{H}^k(F_x; \mathbb{Q})$$

carry $\mathbb{Q}$-mixed Hodge structures.

Specialization of Hirzebruch classes (Schürmann) yields:

Corollary

1. $T^{\rm vir}_{y*}(X) := T^*_y(T^{\rm vir}_X) \cap [X] = T_{y*}(\psi_f([\mathbb{Q}^H_Y]))$
2. $\mathcal{M}_{y*}(X) := T^{\rm vir}_{y*}(X) - T_{y_*}(X) = T_{y_*}(\varphi_f([\mathbb{Q}^H_Y]))$

Example (Isolated singularities)

If the n-dimensional hypersurface X has *only isolated singularities*, then

$$\mathcal{M}_{y*}(X) = \sum_{x \in X_{\rm sing}} (-1)^n \chi_y([\widetilde{H}^n(F_x;\mathbb{Q})]),$$

where F_x is the Milnor fiber of the IHS (X, x).

Many computations in literature: Cappell, M., Saito, Schürmann, Shaneson, Yokura, etc. Applications to *Donaldson-Thomas theory*.

Sample calculation

Theorem (Cappell-M.-Schürmann-Shaneson, 2010)

If $\Sigma = X_{\mathrm{sing}}$ has dimension r, then:

$$\mathcal{M}_{y_*}(X) = (-1)^{n-r}\chi_y([\widetilde{H}^{n-r}(F_{N,x};\mathbb{Q})]) \cdot [\Sigma] + \ell.o.t$$

where $F_{N,x}$ is the transversal Milnor fiber at $x \in \Sigma_{\mathrm{reg}}$, i.e., the Milnor fiber of the isolated singularity germ $(X \cap N, x)$ defined (locally in the analytic topology) by restricting f to a normal slice N at a regular point $x \in \Sigma_{\mathrm{reg}}$.

The nearby and vanishing cycle functors ψ_f, φ_f come equipped with *monodromy actions* compatible with the local monodromies of the Milnor fibrations. By using the *semi-simple* part of the local monodromy action on $\widetilde{H}^*(F_x; \mathbb{Q})$ and the corresponding eigenspace decomposition, Steenbrink-Varchenko defined the (local) *Hodge spectrum* of the IHS germ (X, x).

Spectral Hirzebruch and Milnor-Hirzebruch classes

M.-Saito-Schürmann: *spectral Hirzebruch class*

$$T^{sp}_{t*} : K_0^{mon}(\mathrm{MHM}(X)) \to \bigcup_{n\geq 1} H_*(X) \otimes \mathbb{Q}[t^{\frac{1}{n}}, t^{-\frac{1}{n}}],$$

which is a characteristic class version of the Hodge spectrum

$$hsp : K_0^{mon}(\mathrm{mHs}) \to \bigcup_{n\geq 1} \mathbb{Z}[t^{\frac{1}{n}}, t^{-\frac{1}{n}}].$$

Here $K_0^{mon}(\mathrm{mHs})$ is the Grothendieck group of $\mathbb{Q}$-mixed Hodge structures with a finite order automorphism.

$$hsp(H,T) := \sum_{\alpha\in\mathbb{Q}\cap(0,1)} t^{\alpha}\Big(\sum_{p\in\mathbb{Z}} \dim Gr^p_F H_{\mathbb{C},\alpha}\cdot t^p\Big) \in \mathbb{Z}[t^{\pm 1/\mathrm{ord}(T)}],$$

where $H_{\mathbb{C},\alpha}$ is the $\exp(2\pi i\alpha)$-eigenspace of $H_{\mathbb{C}} := H\otimes\mathbb{C}$.

Spectral Hirzebruch class transformation:

$$T^{sp}_{t*} : K^{mon}_0(\mathrm{MHM}(X)) \to \bigcup_{n\geq 1} H_*(X) \otimes \mathbb{Q}[t^{\frac{1}{n}}, t^{-\frac{1}{n}}],$$

where $K^{mon}_0(\mathrm{MHM}(X))$ is the Grothendieck group of algebraic mixed Hodge modules with a finite order automorphism, e.g., the semi-simple part h_s of the monodromy acting on ψ_f, φ_f.

The *spectral classes* $T^{sp}_{t*}(M, T)$ are *refined* versions (for $t = -y$ and forgetting the action) of the Hirzebruch classes $T_{y*}(M)$.

Moreover, if X is *compact*:

$$\int_{[X]} T^{sp}_{t*}(M, T) = hsp([H^\bullet(M), T^\bullet]) := \sum_j (-1)^j hsp([H^j(M), T^j]).$$

If $X = f^{-1}(0)$, with $f : Y \to \mathbb{C}$ as before, we define the *spectral Milnor-Hirzebruch class of X* by:

$$\mathcal{M}^{sp}_{t*}(X) := T^{sp}_{t*}(\varphi_f \mathbb{Q}_Y, h_s) \in H_*(X_{\mathrm{sing}})[t^{1/ord(h_s)}].$$

II. *Multiplier ideals and jumping coefficients*

Multiplier ideals and jumping coefficients

$X := f^{-1}(0)$ reduced hypersurface in a complex manifold Y.
The *multiplier ideal* of X, *with coefficient* $\alpha \in \mathbb{Q}$, is:

$$\mathcal{J}(\alpha X) := \{g \in \mathcal{O}_Y, \ \frac{|g|^2}{|f|^{2\alpha}} \text{ is locally integrable}\}.$$

The $\mathcal{J}(\alpha X)$ form a decreasing sequence of ideal sheaves of $\mathcal{O}_Y$ satisfying:

$$\mathcal{J}(\alpha X) = \mathcal{O}_Y \ \ (\alpha \leq 0), \quad \mathcal{J}((\alpha+1)X) = f\mathcal{J}(\alpha X) \ \ (\alpha \geq 0)$$

(*smaller multiplier ideals* $\rightsquigarrow$ *worse singularities*.)
$\mathcal{J}(\alpha X)$ satisfies *right-continuity in* α:

$$\mathcal{J}(\alpha X) = \mathcal{J}((\alpha+\varepsilon)X), \quad 0 < \varepsilon \ll 1.$$

Jumping coefficients of f (or X) are defined by:

$$JC(X) := \{\alpha \in \mathbb{Q} \mid \mathcal{J}((\alpha-\varepsilon)X)/\mathcal{J}(\alpha X) \neq 0\}.$$

log canonical threshold

log canonical threshold of f

$$lct(f) := \min\{\alpha \in JC(X)\}.$$

smaller lct $\rightsquigarrow$ *worse singularities.*

Example

- $f(x,y) = x^2 - y^2 : \mathbb{C}^2 \to \mathbb{C}$, $lct(f) = 1$.
- $f(x,y) = x^2 - y^3 : \mathbb{C}^2 \to \mathbb{C}$, $lct(f) = 5/6$.

Remark

- $1 \in JC(f)$ *(from the smooth points of* X*).*
- $lct(f) = 1 \iff X$ *has Du Bois/log canonical singularities.*
- $JC(X) = \big(JC(X) \cap (0,1]\big) + \mathbb{N}$, *so can restrict to* $\alpha \in \mathbb{Q} \cap (0,1]$.

III. *Use of characteristic classes in birational geometry*

Let $\mathcal{M}^{sp}_{t*}(X)|_{t^\alpha} \in H_*(X_{\text{sing}})$ be the *coefficient of* t^α in $\mathcal{M}^{sp}_{t*}(X)$.

Theorem (M.-Saito-Schürmann)

If $\alpha \in (0,1) \cap \mathbb{Q}$ is <u>*not*</u> *a jumping coefficient for f, then $\mathcal{M}^{sp}_{t*}(X)|_{t^\alpha} = 0$. The converse holds if X_{sing} is projective.*

Theorem (M.-Saito-Schürmann)

$$\mathcal{M}_{y*}(X)|_{y=0} = \bigoplus_{\alpha \in \mathbb{Q} \cap (0,1)} \mathcal{M}^{sp}_{t*}(X)|_{t^\alpha} \in H_*(X_{\text{sing}})$$

Theorem (M.-Saito-Schürmann)

Assume X_{sing} is projective. Then:

$$X \text{ has only Du Bois singularities} \iff \mathcal{M}_{y*}(X)|_{y=0} = 0.$$

Ishii (1985) proved the isolated singularities case.

What's behind these results?

Multiplier ideals are essentially the same as the *V-filtration* on $\mathcal{O}_Y$.
In fact, Budur-Saito showed:

$$\alpha \notin JC(f) \iff Gr_V^{\alpha}\mathcal{O}_Y = 0.$$

The vanishing cycle complex and its eigenspaces (hence also the spectral classes) have a V-filtration description.

THANK YOU !!!

Thom-Sebastiani for spectral Milnor-Hirzebruch classes

Theorem (M.-Saito-Schürmann)

Let $X_i = f_i^{-1}(0)$*, for* $f_i : Y_i \to \mathbb{C}$ *a non-constant function on a connected complex manifold* Y_i*, and* $\Sigma_i := \mathrm{Sing}(X_i)$*,* $i = 1, 2$*. Let* $X := f^{-1}(0) \subset Y := Y_1 \times Y_2$*, with* $f := f_1 + f_2$ *and* $\Sigma := \mathrm{Sing}(X)$*. Then:*

$$\mathcal{M}^{sp}_{t*}(X) = -\mathcal{M}^{sp}_{t*}(X_1) \boxtimes \mathcal{M}^{sp}_{t*}(X_2) \in H_*(\Sigma)[t^{1/ord(h_s)}],$$

after replacing Y_i *by an open neighborhood of* X_i *(*$i = 1, 2$*) if necessary (to get* $\Sigma = \Sigma_1 \times \Sigma_2$*).*

Remark

If X_i *(*$i = 1, 2$*) has only isolated singularities, the theorem reduces to the Thom-Sebastiani formula for the Hodge spectrum (Scherk-Steenbrink, Varchenko).*

Thom-Sebastiani for multiplier ideals

In the notations of the Thom-Sebastiani theorem:

Theorem (M.-Saito-Schürmann)

We have the equality for any $\alpha \in (0,1)$*:*

$$\mathcal{J}(\alpha X) = \sum_{\alpha_1+\alpha_2=\alpha} \mathcal{J}(\alpha_1 X_1) \boxtimes \mathcal{J}(\alpha_2 X_2) \subset \mathcal{O}_Y = \mathcal{O}_{Y_1} \boxtimes \mathcal{O}_{Y_2},$$

and

$$JC(f) \cap (0,1) = \big(JC(f_1) + JC(f_2)\big) \cap (0,1),$$

$$lct(f) = \min\{1, lct(f_1) + lct(f_2)\}.$$

Definition of Hirzebruch classes of singular varieties

- $$T_{y_*} : K_0(\mathrm{MHM}(X)) \to H_{2*}^{BM}(X) \otimes \mathbb{Q}[y^{\pm 1}],$$

  is the *Brasselet-Schürmann-Yokura transformation* (2005):

  $$T_{y_*}(M) := td_{(1+y)*} DR_y[M]$$

  where $td_{(1+y)*}$ is a twisted BFM Todd class transformation.
- If X *smooth*,

  $$DR_y[M] := \sum_{i,p} (-1)^{i-n} [Gr_F^p \mathcal{M} \otimes \Omega_X^i] \cdot (-y)^{p+i} \in K_0(X)[y^{\pm 1}]$$

  with $(\mathcal{M}, F)$ the underlying filtered left $\mathcal{D}_X$-modules of M.
- In X *singular*, use local embeddings into smooth varieties.
- Set:

  $$T_{y_*}(X) := T_{y_*}([\mathbb{Q}_X]).$$