MATH 853: TOPICS IN ALGEBRAIC TOPOLOGY

LAURENTIU MAXIM

Introduction to Intersection Homology and applications to Singularity Theory

Description. The motivation for the development of intersection homology is that the main results and properties of smooth manifolds fail to be true for singular varieties when considering "ordinary" homology. Intersection homology was introduced by M. Goresky and R. MacPherson in 1974 for the purpose of recovering such properties and results when dealing with singular varieties. That is the case of Poincaré duality, Poincaré-Lefschetz duality, de Rham theorem, Morse theory, Hodge theory, etc..

The aim of the course is to provide an elementary introduction of intersection homology, firstly by recalling the results and main properties in the smooth case, then by showing to what extend the intersection homology allows to recover these properties in the singular case. The guiding principle of the course is to provide an explicit and geometric introduction of the mathematical objects which are defined in this context, as well as the most significative examples.

The basic idea of intersection homology is that if one wants to recover classical properties of homology in the case of singular varieties, one has to consider only cycles which meet the singular part with a controlled defect of transversality. The behavior of intersection homology relatively to classical properties of usual homology (Mayer-Vietoris, Excision, Künneth, etc.) will be discussed. The most important property which differs from usual homology theory is the local calculus. In fact this is the property which distinguishes intersection homology theory from classical homology theory in the sense that intersection homology is not a homotopy invariant. The local calculus also provides the transition to sheaf theory, and motivates the second part of the course, which is described below.

A second definition of intersection homology is due to Deligne, and makes use of sheaf theory. I will develop the necessary background needed for defining Deligne's intersection sheaf complex, whose (hyper)cohomology computes the intersection homology groups. This complex of sheaves can be described axiomatically in a way that is independent of the stratification, leading to a proof of the topological invariance of intersection homology groups.

Deligne's intersection sheaf complex is a basic example of the notion of perverse sheaf. I will give a simple introduction to the deep theory of perverse sheaves, by providing basic geometric examples. I will then state one of the most profound result of the theory, namely the Beilinson-Bernstein-Deligne decomposition theorem, which describes the behavior of perverse sheaves with respect to algebraic morphisms, and discuss some of its immediate consequences.

In the last part of the course I will introduce the nearby and vanishing cycle functors, and discuss several applications to the study of topology of complex hypersurface singularities.

Prerequisites. Basic notions of Algebraic Topology and Algebraic Geometry.

Suggested Reading. There is no prefered textbook, though you can find the topics covered in this course in one of the following:

- Alexandru Dimca: "Sheaves in Topology", Springer, Universitext.
- Markus Banagl: "Topological Invariants of Stratified Spaces" (Springer Monographs in Mathematics).
- Borel et all: "Intersection cohomology".
- F. Kirwan, J. Woolf: "An Introduction to Intersection Homology", 2nd ed.

The following article explains the history of intersection homology and its connections with various problems in mathematics:

- Steven L. Kleiman: "The Development of Intersection Homology Theory", math.HO/0701462

L. Maxim : Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison WI 53706-1388

E-mail address: maxim@math.wisc.edu