LAURENTIU MAXIM

UNIVERSITY OF WISCONSIN-MADISON

LECTURE NOTES ON HOMOTOPY THEORY AND APPLICATIONS

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Basics of Homotopy Theory

1.1 Homotopy Groups

Definition 1.1.1. *For each $n \geq 0$ and X a topological space with $x_0 \in X$, the n-th homotopy group of X is defined as*

$$\pi_n(X, x_0) = \{f : (I^n, \partial I^n) \to (X, x_0)\} / \sim$$

where $I = [0, 1]$ and $\sim$ is the usual homotopy of maps.

Remark 1.1.2. Note that we have the following diagram of sets:

$$\begin{tikzcd} (I^n, \partial I^n) \arrow[rr, "f"] \arrow[dr] & & (X, x_0) \\ & (I^n/\partial I^n, \partial I^n/\partial I^n) \arrow[ur, "g"] & \end{tikzcd}$$

with $(I^n/\partial I^n, \partial I^n/\partial I^n) \simeq (S^n, s_0)$. So we can also define

$$\pi_n(X, x_0) = \{g : (S^n, s_0) \to (X, x_0)\} / \sim .$$

Remark 1.1.3. If $n = 0$, then $\pi_0(X)$ is the set of connected components of X. Indeed, we have $I^0 = \text{pt}$ and $\partial I^0 = \emptyset$, so $\pi_0(X)$ consists of homotopy classes of maps from a point into the space X.

Now we will prove several results analogous to the case $n = 1$, which corresponds to the fundamental group.

Proposition 1.1.4. *If $n \geq 1$, then $\pi_n(X, x_0)$ is a group with respect to the operation $+$ defined as:*

$$(f + g)(s_1, s_2, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, \ldots, s_n) & \frac{1}{2} \leq s_1 \leq 1. \end{cases}$$

(Note that if $n = 1$, this is the usual concatenation of paths/loops.)

Proof. First note that since only the first coordinate is involved in this operation, the same argument used to prove that π_1 is a group is valid

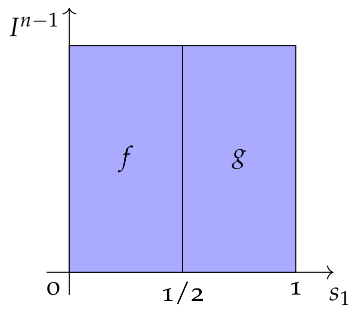


Figure 1.1: $f+g$

here as well. Then the identity element is the constant map taking all of I^n to x_0 and the inverse element is given by

$$-f(s_1, s_2, \ldots, s_n) = f(1 - s_1, s_2, \ldots, s_n).$$

□

Proposition 1.1.5. *If $n \geq 2$, then $\pi_n(X, x_0)$ is abelian.*

Intuitively, since the $+$ operation only involves the first coordinate, if $n \geq 2$, there is enough space to "slide f past g".

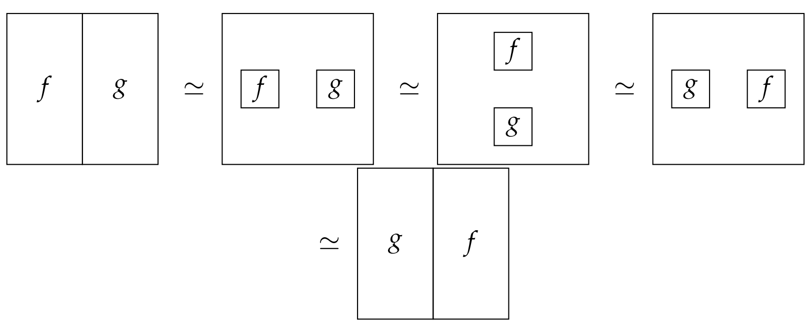


Figure 1.2: $f+g \simeq g+f$

Proof. Let $n \geq 2$ and let $f, g \in \pi_n(X, x_0)$. We wish to show that $f + g \simeq g + f$. We first shrink the domains of f and g to smaller cubes inside I^n and map the remaining region to the base point x_0. Note that this is possible since both f and g map to x_0 on the boundaries, so the resulting map is continuous. Then there is enough room to slide f past g inside I^n. We then enlarge the domains of f and g back to their original size and get $g + f$. So we have "constructed" a homotopy between $f + g$ and $g + f$, and hence $\pi_n(X, x_0)$ is abelian. □

Remark 1.1.6. If we view $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, s_0) \to (X, x_0)$, then we have the following visual representation of $f + g$ (one can see this by collapsing boundaries in the above cube interpretation).

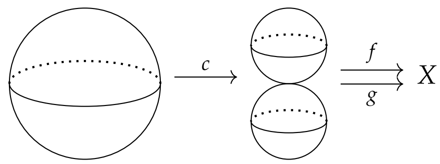


Figure 1.3: $f+g$, revisited

Next recall that if X is path-connected and $x_0, x_1 \in X$, then there is an isomorphism
$$\beta_\gamma : \pi_1(X, x_1) \to \pi_1(X, x_0)$$
where γ is a path from x_1 to x_0, i.e., $\gamma : [0,1] \to X$ with $\gamma(0) = x_1$ and $\gamma(1) = x_0$. The isomorphism β_γ is given by
$$\beta_\gamma([f]) = [\bar{\gamma} * f * \gamma]$$
for any $[f] \in \pi_1(X, x_1)$, where $\bar{\gamma} = \gamma^{-1}$ and $*$ denotes path concatanation. We next show a similar fact holds for all $n \geq 1$.

Proposition 1.1.7. *If $n \geq 1$ and X is path-connected, then there is an isomorphism $\beta_\gamma : \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by*
$$\beta_\gamma([f]) = [\gamma \cdot f],$$
where γ is a path in X from x_1 to x_0, and $\gamma \cdot f$ is constructed by first shrinking the domain of f to a smaller cube inside I^n, and then inserting the path γ radially from x_1 to x_0 on the boundaries of these cubes.

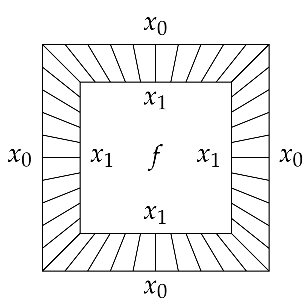


Figure 1.4: β_γ

Proof. It is easy to check that the following properties hold:

1. $\gamma \cdot (f+g) \simeq \gamma \cdot f + \gamma \cdot g$
2. $(\gamma \cdot \eta) \cdot f \simeq \gamma \cdot (\eta \cdot f)$, for η a path from x_0 to x_1
3. $c_{x_0} \cdot f \simeq f$, where c_{x_0} denotes the constant path based at x_0.
4. β_γ is well-defined with respect to homotopies of γ or f.

Note that (1) implies that β_γ is a group homomorphism, while (2) and (3) show that β_γ is invertible. Indeed, if $\overline{\gamma}(t) = \gamma(1-t)$, then $\beta_\gamma^{-1} = \beta_{\overline{\gamma}}$. $\square$

So, as in the case $n = 1$, if the space X is path-connected, then π_n is independent of the choice of base point. Further, if $x_0 = x_1$, then (2) and (3) also imply that $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$ as:

$$\pi_1 \times \pi_n \to \pi_n$$
$$(\gamma, [f]) \mapsto [\gamma \cdot f]$$

Definition 1.1.8. *We say X is an abelian space if π_1 acts trivially on π_n for all $n \geq 1$.*

In particular, this implies that π_1 is abelian, since the action of π_1 on π_1 is by inner-automorphisms, which must all be trivial.

We next show that π_n is a functor.

Proposition 1.1.9. *A map $\phi : X \to Y$ induces group homomorphisms $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, \phi(x_0))$ given by $[f] \mapsto [\phi \circ f]$, for all $n \geq 1$.*

Proof. First note that, if $f \simeq g$, then $\phi \circ f \simeq \phi \circ g$. Indeed, if ψ_t is a homotopy between f and g, then $\phi \circ \psi_t$ is a homotopy between $\phi \circ f$ and $\phi \circ g$. So ϕ_* is well-defined. Moreover, from the definition of the group operation on π_n, it is clear that we have $\phi \circ (f + g) = (\phi \circ f) + (\phi \circ g)$. So $\phi_*([f + g]) = \phi_*([f]) + \phi_*([g])$. Hence ϕ_* is a group homomorphism. $\square$

The following is a consequence of the definition of the above induced homomorphisms:

Proposition 1.1.10. *The homomorphisms induced by $\phi : X \to Y$ on higher homotopy groups satisfy the following two properties:*

1. $(\phi \circ \psi)_* = \phi_* \circ \psi_*$.
2. $(id_X)_* = id_{\pi_n(X, x_0)}$.

We thus have the following important consequence:

Corollary 1.1.11. *If $\phi : (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, \phi(x_0))$ is an isomorphism, for all $n \geq 1$.*

Example 1.1.12. Consider $\mathbb{R}^n$ (or any contractible space). We have $\pi_i(\mathbb{R}^n) = 0$ for all $i \geq 1$, since $\mathbb{R}^n$ is homotopy equivalent to a point.

The following result is very useful for computations:

Proposition 1.1.13. *If $p : \widetilde{X} \to X$ is a covering map, then $p_* : \pi_n(\widetilde{X}, \widetilde{x}) \to \pi_n(X, p(\widetilde{x}))$ is an isomorphism for all $n \geq 2$.*

Proof. First we show that p_* is surjective. Let $x = p(\widetilde{x})$ and consider $f : (S^n, s_0) \to (X, x)$. Since $n \geq 2$, we have that $\pi_1(S^n) = 0$, so

$f_*(\pi_1(S^n, s_0)) = 0 \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}))$. So f admits a lift to $\widetilde{X}$, i.e., there exists $\widetilde{f} : (S^n, s_0) \to (\widetilde{X}, \widetilde{x})$ such that $p \circ \widetilde{f} = f$. Then $[f] = [p \circ \widetilde{f}] = p_*([\widetilde{f}])$. So p_* is surjective.

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}) \arrow[d, "p"] \\ (S^n, s_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & (X, x) \end{tikzcd}$$

Next, we show that p_* is injective. Suppose $[\widetilde{f}] \in \ker p_*$. So $p_*([\widetilde{f}]) = [p \circ \widetilde{f}] = 0$. Let $p \circ \widetilde{f} = f$. Then $f \simeq c_x$ via some homotopy $\phi_t : (S^n, s_0) \to (X, x_0)$ with $\phi_1 = f$ and $\phi_0 = c_x$. Again, by the lifting criterion, there is a unique $\widetilde{\phi}_t : (S^n, s_0) \to (\widetilde{X}, \widetilde{x})$ with $p \circ \widetilde{\phi}_t = \phi_t$.

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}) \arrow[d, "p"] \\ (S^n, s_0) \arrow[ur, "\widetilde{\phi}_t"] \arrow[r, "\phi_t"] & (X, x) \end{tikzcd}$$

Then we have $p \circ \widetilde{\phi}_1 = \phi_1 = f$ and $p \circ \widetilde{\phi}_0 = \phi_0 = c_x$, so by the uniqueness of lifts, we must have $\widetilde{\phi}_1 = \widetilde{f}$ and $\widetilde{\phi}_0 = c_{\widetilde{x}}$. Then $\widetilde{\phi}_t$ is a homotopy between $\widetilde{f}$ and $c_{\widetilde{x}}$. So $[\widetilde{f}] = 0$. Thus p_* is injective. $\square$

Example 1.1.14. Consider S^1 with its universal covering map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi i t}$. We already know that $\pi_1(S^1) = \mathbb{Z}$. If $n \geq 2$, Proposition 1.1.13 yields that $\pi_n(S^1) = \pi_n(\mathbb{R}) = 0$.

Example 1.1.15. Consider $T^n = S^1 \times S^1 \times \cdots \times S^1$, the n-torus. We have $\pi_1(T^n) = \mathbb{Z}^n$. By using the universal covering map $p : \mathbb{R}^n \to T^n$, we have by Proposition 1.1.13 that $\pi_i(T^n) = \pi_i(\mathbb{R}^n) = 0$ for $i \geq 2$.

Definition 1.1.16. *If $\pi_n(X) = 0$ for all $n \geq 2$, the space X is called aspherical.*

Remark 1.1.17. As a side remark, the celebrated *Singer-Hopf conjecture* asserts that if X is a smooth closed aspherical manifold of dimension $2k$, then $(-1)^k \cdot \chi(X) \geq 0$, where χ denotes the Euler characteristic.

Proposition 1.1.18. *Let $\{X_\alpha\}_\alpha$ be a collection of path-connected spaces. Then*

$$\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n(X_\alpha)$$

for all n.

Proof. First note that a map $f : Y \to \prod_\alpha X_\alpha$ is a collection of maps $f_\alpha : Y \to X_\alpha$. For elements of π_n, take $Y = S^n$ (note that since all spaces are path-connected, we may drop the reference to base points). For homotopies, take $Y = S^n \times I$. $\square$

Example 1.1.19. A natural question to ask is if there exist spaces X and Y such that $\pi_n(X) \cong \pi_n(Y)$ for all n, but with X and Y not homotopy equivalent. Whitehead's Theorem (to be discussed later on) states that if a map of CW complexes $f : X \to Y$ induces isomorphisms on all π_n, then f is a homotopy equivalence. So for the above question to have a positive answer, we must find X and Y so that there is no continuous map $f : X \to Y$ inducing the isomorphisms on π_n's. Consider

$$X = S^2 \times \mathbb{R}P^3 \text{ and } Y = \mathbb{R}P^2 \times S^3.$$

Then $\pi_n(X) = \pi_n(S^2 \times \mathbb{R}P^3) = \pi_n(S^2) \times \pi_n(\mathbb{R}P^3)$. Since S^3 is a covering of $\mathbb{R}P^3$, for all $n \geq 2$ we have that $\pi_n(X) = \pi_n(S^2) \times \pi_n(S^3)$. We also have $\pi_1(X) = \pi_1(S^2) \times \pi_1(\mathbb{R}P^3) = \mathbb{Z}/2$. Similarly, we have $\pi_n(Y) = \pi_n(\mathbb{R}P^2 \times S^3) = \pi_n(\mathbb{R}P^2) \times \pi_n(S^3)$. And since S^2 is a covering of $\mathbb{R}P^2$, for $n \geq 2$ we have that $\pi_n(Y) = \pi_n(S^2) \times \pi_n(S^3)$. Finally, $\pi_1(Y) = \pi_1(\mathbb{R}P^2) \times \pi_1(S^3) = \mathbb{Z}/2$. So

$$\pi_n(X) = \pi_n(Y) \text{ for all } n.$$

By considering homology groups, however, we see that X and Y are not homotopy equivalent. Indeed, by the Künneth formula, we get that $H_5(X) = \mathbb{Z}$ while $H_5(Y) = 0$ (since $\mathbb{R}P^3$ is oriented while $\mathbb{R}P^2$ is not).

Just like there is a homomorphism $\pi_1(X) \longrightarrow H_1(X)$, we can also construct *Hurewicz homomorphisms*

$$h_X : \pi_n(X) \longrightarrow H_n(X)$$

defined by

$$[f : S^n \to X] \mapsto f_*[S^n],$$

where $[S^n]$ is the fundamental class of S^n. A very important result in homotopy theory is the following:

Theorem 1.1.20. *(Hurewicz)*
If $n \geq 2$ and $\pi_i(X) = 0$ for all $i < n$, then $H_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$.

Moreover, there is also a relative version of the Hurewicz theorem (see the next section for a definition of the relative homotopy groups), which can be used to prove the following:

Corollary 1.1.21. *If X and Y are CW complexes with $\pi_1(X) = \pi_1(Y) = 0$, and a map $f : X \to Y$ induces isomorphisms on all integral homology groups H_n, then f is a homotopy equivalence.*

We'll discuss all of these in the subsequent sections.

1.2 Relative Homotopy Groups

Given a triple (X, A, x_0) where $x_0 \in A \subseteq X$, we define relative homotopy groups as follows:

Definition 1.2.1. *Let X be a space and let $A \subseteq X$ and $x_0 \in A$. Let*

$$I^{n-1} = \{(s_1, \dots, s_n) \in I^n | \ s_n = 0\}$$

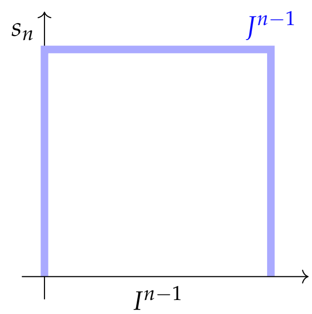
and set

$$J^{n-1} = \overline{\partial I^n \backslash I^{n-1}}.$$

Then define the n-th homotopy group of the pair (X, A) with basepoint x_0 as:

$$\pi_n(X, A, x_0) = \big\{ f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0) \big\} / \sim$$

where, as before, $\sim$ is the homotopy equivalence relation.



Alternatively, by collapsing J^{n-1} to a point, we obtain a commutative diagram

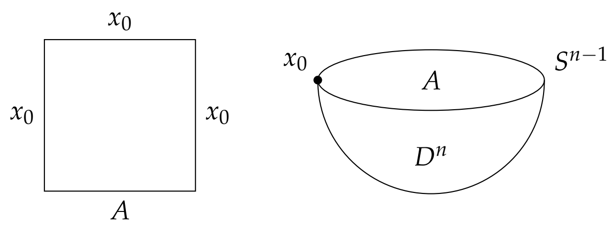
$$\begin{tikzcd}
(I^n, \partial I^n, J^{n-1}) \arrow[rr, "f"] \arrow[dr, "g"'] & & (X, A, x_0) \\
& (D^n, S^{n-1}, s_0) \arrow[ur] &
\end{tikzcd}$$

where g is obtained by collapsing J^{n-1}. So we can take

$$\pi_n(X, A, x_0) = \big\{ g : (D^n, S^{n-1}, s_0) \to (X, A, x_0) \big\} / \sim .$$

A sum operation is defined on $\pi_n(X, A, x_0)$ by the same formulas as for $\pi_n(X, x_0)$, except that the coordinate s_n now plays a special role and is no longer available for the sum operation. Thus, we have:

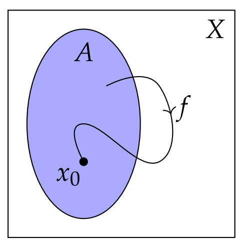
Proposition 1.2.2. *If $n \geq 2$, then $\pi_n(X, A, x_0)$ forms a group under the usual sum operation. Further, if $n \geq 3$, then $\pi_n(X, A, x_0)$ is abelian.*



Remark 1.2.3. Note that the proposition fails in the case $n = 1$. Indeed, we have that

$$\pi_1(X, A, x_0) = \{f : (I, \{0, 1\}, \{1\}) \to (X, A, x_0)\} / \sim .$$

Then $\pi_1(X, A, x_0)$ consists of homotopy classes of paths starting anywhere A and ending at x_0, so we cannot always concatenate two paths.



Just as in the absolute case, a map of pairs $\phi : (X, A, x_0) \to (Y, B, y_0)$ induces homomorphisms $\phi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ for all $n \geq 2$.

A very important feature of the relative homotopy groups is the following:

Proposition 1.2.4. *The relative homotopy groups of (X, A, x_0) fit into a long exact sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots$$
$$\cdots \to \pi_0(X, x_0) \to 0,$$

where the map ∂_n is defined by $\partial_n[f] = [f|_{I^{n-1}}]$ and all others are induced by inclusions.

Remark 1.2.5. Near the end of the above sequence, where group structures are not defined, exactness still makes sense: the image of one map is the kernel of the next, which consists of those elements mapping to the homotopy class of the constant map.

Example 1.2.6. Let X be a path-connected space, and

$$CX := X \times [0, 1] / X \times \{0\}$$

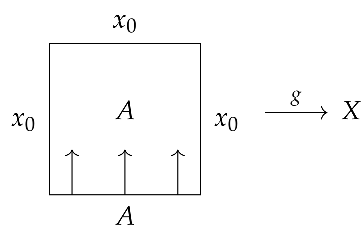
be the cone on X. We can regard X as a subspace of CX via $X \times \{1\} \subset CX$. Since CX is contractible, the long exact sequence of homotopy groups gives isomorphisms

$$\pi_n(CX, X, x_0) \cong \pi_{n-1}(X, x_0).$$

In what follows, it will be important to have a good description of the zero element $0 \in \pi_n(X, A, x_0)$.

Lemma 1.2.7. *Let $[f] \in \pi_n(X, A, x_0)$. Then $[f] = 0$ if, and only if, $f \simeq g$ for some map g with image contained in A.*

Proof. ($\Leftarrow$) Suppose $f \simeq g$ for some g with Image $g \subset A$.



Then we can deform I^n to J^{n-1} as indicated in the above picture, and so $g \simeq c_{x_0}$. Since homotopy is a transitive relation, we then get that $f \simeq c_{x_0}$.
($\Rightarrow$) Suppose $[f] = 0$ in $\pi_n(X, A, x_0)$. So $f \simeq c_{x_0}$. Take $g = c_{x_0}$. $\square$

Recall that if X is path-connected, then $\pi_n(X, x_0)$ is independent of our choice of base point, and $\pi_1(X)$ acts on $\pi_n(X)$ for all $n \geq 1$. In the relative case, we have:

Lemma 1.2.8. *If A is path-connected, then $\beta_\gamma : \pi_n(X, A, x_1) \to \pi_n(X, A, x_0)$ is an isomorphism, where γ is a path in A from x_1 to x_0.*

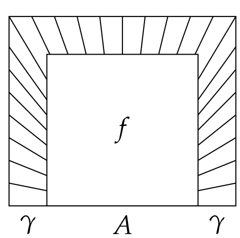


Figure 1.5: relative β_γ

Remark 1.2.9. In particular, if $x_0 = x_1$, we get an action of $\pi_1(A)$ on $\pi_n(X, A)$.

It is easy to see that the following three conditions are equivalent:

1. every map $S^i \to X$ is homotopic to a constant map,
2. every map $S^i \to X$ extends to a map $D^{i+1} \to X$, with $S^i = \partial D^{i+1}$,
3. $\pi_i(X, x_0) = 0$ for all $x_0 \in X$.

In the relative setting, the following are equivalent for any $i > 0$:

1. every map $(D^i, \partial D^i) \to (X, A)$ is homotopic rel. ∂D^i to a map $D^i \to A$,
2. every map $(D^i, \partial D^i) \to (X, A)$ is homotopic through such maps to a map $D^i \to A$,
3. every map $(D^i, \partial D^i) \to (X, A)$ is homotopic through such maps to a constant map $D^i \to A$,
4. $\pi_i(X, A, x_0) = 0$ for all $x_0 \in A$.

Remark 1.2.10. As seen above, if $\alpha : S^n = \partial e^{n+1} \to X$ represents an element $[\alpha] \in \pi_n(X, x_0)$, then $[\alpha] = 0$ if and only if α extends to a map $e^{n+1} \to X$. Thus if we enlarge X to a space $X' = X \cup_\alpha e^{n+1}$ by adjoining an $(n+1)$-cell e^{n+1} with α as attaching map, then the inclusion $j : X \hookrightarrow X'$ induces a homomorphism $j_* : \pi_n(X, x_0) \to \pi_n(X', x_0)$ with $j_*[\alpha] = 0$. We say that $[\alpha]$ "has been killed".

The following is left as an exercise:

Lemma 1.2.11. *Let (X, x_0) be a space with a basepoint, and let $X' = X \cup_\alpha e^{n+1}$ be obtained from X by adjoining an $(n+1)$-cell. Then the inclusion $j : X \hookrightarrow X'$ induces a homomorphism $j_* : \pi_i(X, x_0) \to \pi_i(X', x_0)$, which is an isomorphism for $i < n$ and surjective for $i = n$.*

Definition 1.2.12. *We say that the pair (X, A) is n-connected if $\pi_i(X, A) = 0$ for $i \leq n$ and X is n-connected if $\pi_i(X) = 0$ for $i \leq n$.*

In particular, X is 0-connected if and only if X is connected. Moreover, X is 1-connected if and only if X is simply-connected.

1.3 Homotopy Extension Property

Definition 1.3.1 (Homotopy Extension Property)**.** *Given a pair (X, A), a map $F_0 : X \to Y$, and a homotopy $f_t : A \to Y$ such that $f_0 = F_0|_A$, we say that (X, A) satisfies the homotopy extension property (HEP) if there is a homotopy $F_t : X \to Y$ extending f_t and F_0. In other words, (X, A) has homotopy extension property if any map $X \times \{0\} \cup A \times I \to Y$ extends to a map $X \times I \to Y$.*

Proposition 1.3.2. *Any CW pair has the homotopy extension property. In fact, for every CW pair (X, A), there is a deformation retract $r : X \times I \to X \times \{0\} \cup A \times I$, hence $X \times I \to Y$ can be defined by the composition $X \times I \xrightarrow{r} X \times \{0\} \cup A \times I \to Y$.*

Proof. We have an obvious deformation retract $D^n \times I \xrightarrow{r} D^n \times \{0\} \cup S^{n-1} \times I$. For every n, consider the pair $(X_n, A_n \cup X_{n-1})$, where X_n denotes the n-skeleton of X. Then

$$X_n \times I = \left[X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I\right] \cup D^n \times I,$$

where the cylinders $D^n \times I$ corresponding to n-cells D^n in $X \backslash A$ are glued along $D^n \times \{0\} \cup S^{n-1} \times I$ to the pieces $X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I$. By deforming these cylinders $D^n \times I$ we get a deformation retraction

$$r_n : X_n \times I \to X_n \times \{0\} \cup (A_n \cup X_{n-1}) \times I.$$

Concatenating these deformation retractions by performing r_n over $\left[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right]$, we get a deformation retraction of $X \times I$ onto $X \times \{0\} \cup A \times I$. Continuity follows since CW complexes have the weak topology with respect to their skeleta, so a map of CW complexes is continuous if and only if its restriction to each skeleton is continuous. $\square$

1.4 Cellular Approximation

All maps are assumed to be continuous.

Definition 1.4.1. *Let X and Y be CW-complexes. A map $f : X \to Y$ is called cellular if $f(X_n) \subset Y_n$ for all n, where X_n denotes the n-skeleton of X and similarly for Y.*

Definition 1.4.2. *Let $f : X \to Y$ be a map of CW complexes. A map $f' : X \to Y$ is a cellular approximation of f if f' is cellular and f is homotopic to f'.*

Theorem 1.4.3 (Cellular Aproximation Theorem)**.** *Any map $f : X \to Y$ between CW-complexes has a cellular approximation $f' : X \to Y$. Moreover, if f is already cellular on a subcomplex $A \subseteq X$, we can take $f'|_A = f|_A$.*

The proof of Theorem 1.4.3 uses the following key technical result.

Lemma 1.4.4. *Let $f : X \cup e^n \to Y \cup e^k$ be a map of CW complexes, with e^n, e^k denoting an n-cell and, resp., k-cell attached to X and, resp., Y. Assume that $f(X) \subseteq Y$, $f|_X$ is cellular, and $n < k$. Then $f \overset{h.e.}{\simeq} f'$ (rel. X) , with* $\text{Image}(f') \subseteq Y$.

Remark 1.4.5. If in the statement of Lemma 1.4.4 we assume that X and Y are points, then we get that the inclusion $S^n \hookrightarrow S^k$ $(n < k)$ is homotopic to the constant map $S^n \to \{s_0\}$ for some point $s_0 \in S^k$.

Lemma 1.4.4 is used along with induction on skeleta to prove the cellular approximation theorem as follows.

Proof of Theorem 1.4.3. Suppose $f|_{X_n}$ is cellular, and let e^n be an (open) n-cell of X. Since $\bar{e^n}$ is compact, $f(\bar{e^n})$ (hence also $f(e^n)$) meets only finitely many open cells of Y. Let e^k be an open cell of maximal dimension in Y which meets $f(e^n)$. If $k \leq n$, f is already cellular on e_n. If $n < k$, Lemma 1.4.4 can be used to homotop $f|_{X_{n-1} \cup e^n}$ (rel. X_{n-1}) to a map whose image on e^n misses e^k. By finitely many iterations of this process, we eventually homotop $f|_{X_{n-1} \cup e^n}$ (rel. X_{n-1}) to a map $f' : X_{n-1} \cup e^n \to Y_n$, i.e., whose image on e^n misses all cells in Y of dimension $> n$. Doing this for all n-cells of X, staying fixed on n-cells in A where f is already cellular, we obtain a homotopy of $f|_{X_n}$ (rel. $X_{n-1} \cup A_n$) to a cellular map. By the homotopy extension property 1.3.2, we can extend this homotopy (together with the constant homotopy on A) to a homotopy defined on all of X. This completes the induction step.

For varying $n \to \infty$, we concatenate the above homotopies to define a homotopy from f to a cellular map f' (rel. A) by performing the above construction (i.e., the n-th homotopy) on the t-interval $[1 - 1/2^n, 1 - 1/2^{n+1}]$. □

We also have the following relative version of Theorem 1.4.3:

Theorem 1.4.6 (Relative cellular approximation)**.** *Any map $f : (X, A) \to (Y, B)$ of CW pairs has a cellular approximation by a homotopy through such maps of pairs.*

Proof. First we use the cellular approximation for $f|_A : A \to B$. Let $f' : A \to B$ be a cellular map, homotopic to $f|_A$ via a homotopy H. By the Homotopy Extension Property 1.3.2, we can regard H as a homotopy on all of X, so we get a map $f' : X \to Y$ such that $f'|_A$ is a cellular map. By the second part of the cellular approximation theorem 1.4.3, $f' \overset{h.e.}{\simeq} f''$, with $f'' : X \to Y$ a cellular map satisfying $f'|_A = f''|_A$. The map f'' provides the required cellular approximation of f. □

Corollary 1.4.7. *Let $A \subset X$ be CW complexes and suppose that all cells of $X \setminus A$ have dimension $> n$. Then $\pi_i(X, A) = 0$ for $i \leq n$.*

Proof. Let $[f] \in \pi_i(X, A)$. By the relative version of the cellular approximation, the map of pairs $f : (D^i, S^{i-1}) \to (X, A)$ is homotopic to a map g with $g(D^i) \subset X_i$. But for $i \leq n$, we have that $X_i \subset A$, so Image $g \subset A$. Therefore, by Lemma 1.2.7, $[f] = [g] = 0$. □

Corollary 1.4.8. *If X is a CW complex, then $\pi_i(X, X_n) = 0$ for all $i \leq n$.*

Therefore, the long exact sequence for the homotopy groups of the pair (X, X_n) yields the following:

Corollary 1.4.9. *Let X be a CW complex. For $i < n$, we have $\pi_i(X) \cong \pi_i(X_n)$.*

1.5 Excision for homotopy groups. The Suspension Theorem

We state here the following useful result without proof:

Theorem 1.5.1 (Excision). *Let X be a CW complex which is a union of subcomplexes A and B, such that $C = A \cap B$ is path connected. Assume that (A, C) is m-connected and (B, C) is n-connected, with $m, n \geq 1$. Then the map $\pi_i(A, C) \longrightarrow \pi_i(X, B)$ induced by inclusion is an isomorphism if $i < m + n$ and a surjection for $i = m + n$.*

The following consequence is very useful for itering homotopy groups of spheres:

Theorem 1.5.2 (Freudenthal Suspension Theorem). *Let X be an $(n-1)$-connected CW complex. For any map $f : S^i \to X$, consider its suspension,*

$$\Sigma f : \Sigma S^i = S^{i+1} \to \Sigma X.$$

The assignment

$$[f] \in \pi_i(X) \ \mapsto [\Sigma f] \in \pi_{i+1}(\Sigma X)$$

defines a homomorphism $\pi_i(X) \to \pi_{i+1}(\Sigma X)$, which is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$.

Proof. Decompose the suspension ΣX as the union of two cones $C_+ X$ and $C_- X$ intersecting in a copy of X. By using long exact sequences of pairs and the fact that the cones $C_+ X$ and $C_- X$ are contractible, the suspension map can be written as a composition:

$$\pi_i(X) \cong \pi_{i+1}(C_+, X) \longrightarrow \pi_{i+1}(\Sigma X, C_- X) \cong \pi_{i+1}(\Sigma X),$$

with the middle map induced by inclusion.

Since X is $(n-1)$-connected, from the long exact sequence of $(C_\pm X, X)$, we see that the pairs $(C_\pm X, X)$ are n-connected. Therefore, the Excision Theorem 1.5.1 yields that $\pi_{i+1}(C_+, X) \longrightarrow \pi_{i+1}(\Sigma X, C_- X)$ is an isomorphism for $i + 1 < 2n$ and it is surjective for $i + 1 = 2n$. $\square$

1.6 Homotopy Groups of Spheres

We now turn our attention to computing (some of) the homotopy groups $\pi_i(S^n)$. For $i \leq n, i = n+1, n+2, n+3$ and a few more cases, these homotopy groups are known (and we will work them out later on). In general, however, this is a very difficult problem. For $i = n$, we would expect to have $\pi_n(S^n) = \mathbb{Z}$ by associating to each (homotopy class of a) map $f : S^n \to S^n$ its degree. For $i < n$, we will show that $\pi_i(S^n) = 0$. Note that if $f : S^i \to S^n$ is not surjective, i.e., there is $y \in S^n \backslash f(S^i)$, then f factors through $\mathbb{R}^n$, which is contractible. By composing f with the retraction $\mathbb{R}^n \to x_0$ we get that $f \simeq c_{x_0}$. However, there are surjective maps $S^i \to S^n$ for $i < n$, in which case the above "proof" fails. To make things work, we "alter" f to make it cellular, so the following holds.

Proposition 1.6.1. *For $i < n$, we have $\pi_i(S^n) = 0$.*

Proof. Choose the standard CW-structure on S^i and S^n. For $[f] \in \pi_i(S^n)$, we may assume by Theorem 1.4.3 that $f : S^i \to S^n$ is cellular. Then $f(S^i) \subset (S^n)_i$. But $(S^n)_i$ is a point, so f is a constant map. □

Recall now the following special case of the Suspension Theorem 1.5.2 for $X = S^n$:

Theorem 1.6.2. *Let $f : S^i \to S^n$ be a map, and consider its suspension,*

$$\Sigma f : \Sigma S^i = S^{i+1} \to \Sigma S^n = S^{n+1}.$$

The assignment

$$[f] \in \pi_i(S^n) \ \mapsto [\Sigma f] \in \pi_{i+1}(S^{n+1})$$

defines a homomorphism $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$, which is an isomorphism $\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$ for $i < 2n - 1$ and a surjection for $i = 2n - 1$.

Corollary 1.6.3. *$\pi_n(S^n)$ is either $\mathbb{Z}$ or a finite quotient of $\mathbb{Z}$ (for $n \geq 2$), generated by the degree map.*

Proof. By the Suspension Theorem 1.6.2, we have the following:

$$\mathbb{Z} \cong \pi_1(S^1) \twoheadrightarrow \pi_2(S^2) \cong \pi_3(S^3) \cong \pi_4(S^4) \cong \cdots$$

□

To show that $\pi_1(S^1) \cong \pi_2(S^2)$, we can use *the long exact sequence for the homotopy groups of a fibration*, see Theorem 1.11.8 below. For any fibration (e.g., a covering map)

$$F \hookrightarrow E \longrightarrow B$$

there is a long exact sequence

$$\cdots \longrightarrow \pi_i(F) \longrightarrow \pi_i(E) \longrightarrow \pi_i(B) \longrightarrow \pi_{i-1}(F) \longrightarrow \cdots \qquad (1.6.1)$$

Applying the above long exact sequence to the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$, we obtain:

$$\cdots \longrightarrow \pi_2(S^1) \longrightarrow \pi_2(S^3) \longrightarrow \pi_2(S^2) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(S^3) \longrightarrow \cdots$$

Using the fact that $\pi_2(S^3) = 0$ and $\pi_1(S^3) = 0$, we therefore have an isomorphism:

$$\pi_2(S^2) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Note that by using the vanishing of the higher homotopy groups of S^1, the long exact sequence (1.11.8) also yields that

$$\pi_3(S^2) \cong \pi_2(S^2) \cong \mathbb{Z}.$$

Remark 1.6.4. Unlike the homology and cohomology groups, the homotopy groups of a finite CW-complex can be infinitely generated. This fact is discussed in the next example.

Example 1.6.5. For $n \geq 2$, consider the finite CW complex $S^1 \vee S^n$. We then have that

$$\pi_n(S^1 \vee S^n) = \pi_n(\widetilde{S^1 \vee S^n}),$$

where $\widetilde{S^1 \vee S^n}$ is the universal cover of $S^1 \vee S^n$, as depicted in the attached figure. By contracting the segments between consecutive

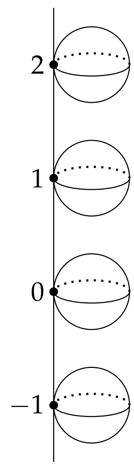


Figure 1.6: universal cover of $S^1 \vee S^n$

integers, we have that

$$\widetilde{S^1 \vee S^n} \simeq \bigvee_{k \in \mathbb{Z}} S^n_k,$$

with S^n_k denoting the n-sphere corresponding to the integer k. So for any $n \geq 2$, we have:

$$\pi_n(S^1 \vee S^n) = \pi_n(\bigvee_{k \in \mathbb{Z}} S^n_k),$$

which is the free abelian group generated by the inclusions $S^n_k \hookrightarrow \bigvee_{k \in \mathbb{Z}} S^n_k$. Indeed, we have the following:

Lemma 1.6.6. *$\pi_n(\bigvee_\alpha S^n_\alpha)$ is free abelian and generated by the inclusions of factors.*

Proof. Suppose first that there are only finitely many S^n_α's in the wedge $\bigvee_\alpha S^n_\alpha$. Then we can regard $\bigvee_\alpha S^n_\alpha$ as the n-skeleton of $\prod_\alpha S^n_\alpha$. The cell structure of a particular S^n_α consists of a single 0-cell e^0_α and a single n-cell, e^n_α. Thus, in the product $\prod_\alpha S^n_\alpha$ there is one 0-cell $e^0 = \prod_\alpha e^0_\alpha$, which, together with the n-cells

$$\bigcup_\alpha (\prod_{\beta \neq \alpha} e^0_\beta) \times e^n_\alpha,$$

form the n-skeleton $\bigvee_\alpha S^n_\alpha$. Hence $\prod_\alpha S^n_\alpha \setminus \bigvee_\alpha S^n_\alpha$ has only cells of dimension at least $2n$, which by Corollary 1.4.8 yields that the pair $(\prod_\alpha S^n_\alpha, \bigvee_\alpha S^n_\alpha)$ is $(2n-1)$-connected. In particular, as $n \geq 2$, we get:

$$\pi_n(\bigvee_\alpha S^n_\alpha) \cong \pi_n\Big(\prod_\alpha S^n_\alpha\Big) \cong \prod_\alpha \pi_n(S^n_\alpha) = \bigoplus_\alpha \pi_n(S^n_\alpha) = \bigoplus_\alpha \mathbb{Z}.$$

To reduce the case of infinitely many summands S^n_α to the finite case, consider the homomorphism $\Phi : \bigoplus_\alpha \pi_n(S^n_\alpha) \longrightarrow \pi_n(\bigvee_\alpha S^n_\alpha)$ induced by the inclusions $S^n_\alpha \hookrightarrow \bigvee_\alpha S^n_\alpha$. Then Φ is onto since any map $f : S^n \to \bigvee_\alpha S^n_\alpha$ has compact image contained in the wedge sum of finitely many S^n_α's, so by the above finite case, $[f]$ is in the image of Φ. Moreover, a nullhomotopy of f has compact image contained in the wedge sum of finitely many S^n_α's, so by the above finite case we have that Φ is also injective. $\square$

To conclude our example, we showed that $\pi_n(S^1 \vee S^n) \cong \pi_n(\bigvee_{k\in\mathbb{Z}} S^n_k)$, and $\pi_n(\bigvee_{k\in\mathbb{Z}} S^n_k)$ is free abelian generated by the inclusion of each of the infinite number of n-spheres. Therefore, $\pi_n(S^1 \vee S^n)$ is infinitely generated.

Remark 1.6.7. Under the action of π_1 on π_n, we can regard π_n as a $\mathbb{Z}[\pi_1]$-module. Here $\mathbb{Z}[\pi_1]$ is the group ring of π_1 with $\mathbb{Z}$-coefficients, whose elements are of the form $\sum_\alpha n_\alpha \gamma_\alpha$, with $n_\alpha \in \mathbb{Z}$ and only finitely many non-zero, and $\gamma_\alpha \in \pi_1$. Since all the n-spheres S^n_k in the universal cover $\bigvee_{k\in\mathbb{Z}} S^n_k$ are identified under the π_1-action, π_n is a free $\mathbb{Z}[\pi_1]$-module of rank 1, i.e.,

$$\begin{aligned} \pi_n \cong \mathbb{Z}[\pi_1] \cong \mathbb{Z}[\mathbb{Z}] &\cong \mathbb{Z}[t, t^{-1}], \\ 1 &\mapsto t \\ -1 &\mapsto t^{-1} \\ n &\mapsto t^n, \end{aligned}$$

which is infinitely generated (by the powers of t) over $\mathbb{Z}$ (i.e., as an abelian group).

Remark 1.6.8. If we consider the class of spaces for which π_1 acts trivially on all of π_n's, a result of Serre asserts that the homotopy groups of such spaces are finitely generated if and only if homology groups are finitely generated.

1.7 *Whitehead's Theorem*

Definition 1.7.1. *A map $f : X \to Y$ is a weak homotopy equivalence if it induces isomorphisms on all homotopy groups π_n.*

Notice that a homotopy equivalence is a weak homotopy equivalence. The following important result provides a converse to this fact in the world of CW complexes.

Theorem 1.7.2 (Whitehead)**.** *If X and Y are CW complexes and $f : X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence. Moreover, if X is a subcomplex of Y, and f is the inclusion map, then X is a deformation retract of Y.*

The following consequence is very useful in practice:

Corollary 1.7.3. *If X and Y are CW complexes with $\pi_1(X) = \pi_1(Y) = 0$, and $f : X \to Y$ induces isomorphisms on homology groups H_n for all n, then f is a homotopy equivalence.*

The above corollary follows from Whitehead's theorem and the following relative version of the Hurewicz Theorem 1.10.1 (to be discussed later on):

Theorem 1.7.4 (Hurewicz)**.** *If $n \geq 2$, and $\pi_i(X, A) = 0$ for $i < n$, with A simply-connected and non-empty, then $H_i(X, A) = 0$ for $i < n$ and $\pi_n(X, A) \cong H_n(X, A)$.*

Before discussing the proof of Whitehead's theorem, let us give an example that shows that having induced isomorphisms on all homology groups is not sufficient for having a homotopy equivalence (so the simply-connectedness assumption in Corollary 1.7.3 cannot be dropped):

Example 1.7.5. Let

$$f : X = S^1 \hookrightarrow (S^1 \vee S^n) \cup e^{n+1} = Y \qquad (n \geq 2)$$

be the inclusion map, with the attaching map for the $(n+1)$-cell of Y described below. We know from Example 1.6.5 that $\pi_n(S^1 \vee S^n) \cong \mathbb{Z}[t, t^{-1}]$. We define Y by attaching the $(n+1)$-cell e^{n+1} to $S^1 \vee S^n$ by a map $g : S^n = \partial e^{n+1} \to S^1 \vee S^n$ so that $[g] \in \pi_n(S^1 \vee S^n)$ corresponds to the element $2t - 1 \in \mathbb{Z}[t, t^{-1}]$. We then see that

$$\pi_n(Y) = \mathbb{Z}[t, t^{-1}]/(2t-1) \neq 0 = \pi_n(X),$$

since by setting $t = \frac{1}{2}$ we get that $\mathbb{Z}[t, t^{-1}]/(2t-1) \cong \mathbb{Z}[\frac{1}{2}] = \{\frac{a}{2^k} \mid k \in \mathbb{Z}_{\geq 0}\} \subset \mathbb{Q}$. In particular, f is not a homotopy equivalence. Moreover, from the long exact sequence of homotopy groups for the $(n-1)$-connected pair (Y, X), the inclusion $X \hookrightarrow Y$ induces an isomorphism on homotopy groups π_i for $i < n$. Finally, this inclusion map also induces isomorphisms on all homology groups, $H_n(X) \cong H_n(Y)$ for all n, as can be seen from cellular homology. Indeed, the cellular boundary map

$$H_{n+1}(Y_{n+1}, Y_n) \to H_n(Y_n, Y_{n-1})$$

is an isomorphism since the degree of the composition of the attaching map $S^n \to S^1 \vee S^n$ of e^{n+1} with the collapse map $S^1 \vee S^n \to S^n$ is $2 - 1 = 1$.

Let us now get back to the proof of Whitehead's Theorem 1.7.2. To prove Whitehead's theorem, we will use the following:

Lemma 1.7.6 (Compression Lemma)**.** *Let (X, A) be a CW pair, and (Y, B) be a pair with Y path-connected and $B \neq \emptyset$. Suppose that for each $n > 0$ for which $X \setminus A$ has cells of dimension n, $\pi_n(Y, B, b_0) = 0$ for all $b_0 \in B$. Then any map $f : (X, A) \to (Y, B)$ is homotopic to some map $f' : X \to B$ fixing A (i.e., with $f'|_A = f|_A$).*

Proof. Assume inductively that $f(X_{k-1} \cup A) \subseteq B$. Let e^k be a k-cell in $X \backslash A$, with characteristic map $\alpha : (D^k, S^{k-1}) \to X$. Ignoring basepoints, we regard α as an element $[\alpha] \in \pi_k(X, X_{k-1} \cup A)$. Then $f_*[\alpha] = [f \circ \alpha] \in \pi_k(Y, B) = 0$ by our hypothesis, since e^k is a k-cell in $X \backslash A$. By Lemma 1.2.7, there is a homotopy $H : (D^k, S^{k-1}) \times I \to (Y, B)$ such that $H_0 = f \circ \alpha$ and $\text{Image}(H_1) \subseteq B$.

Performing this process for all k-cells in $X \backslash A$ simultaneously, we get a homotopy from f to f' such that $f'(X_k \cup A) \subseteq B$. Using the homotopy extension property 1.3.2, we can regard this as a homotopy on all of X, i.e., $f \simeq f'$ as maps $X \to Y$, so the induction step is completed.

Finitely many applications of the induction step finish the proof if the cells of $X \setminus A$ are of bounded dimension. In general, we have

$$f \underset{H_1}{\simeq} f_1, \text{ with } f_1(X_1 \cup A) \subseteq B,$$

$$f_1 \underset{H_2}{\simeq} f_2, \text{ with } f_2(X_2 \cup A) \subseteq B,$$

$$\cdots$$

$$f_{n-1} \underset{H_n}{\simeq} f_n, \text{ with } f_n(X_n \cup A) \subseteq B,$$

and so on. Any finite skeleton is eventually fixed under these homotopies.

Define a homotopy $H : X \times I \to Y$ as

$$H = H_i \text{ on } \left[1 - \tfrac{1}{2^{i-1}}, 1 - \tfrac{1}{2^i}\right].$$

Note that H is continuous by CW topology, so it gives the required homotopy. □

Proof of Whitehead's theorem. We can split the proof of Theorem 1.7.2 into two cases:

<u>Case 1</u>: If f is an inclusion $X \hookrightarrow Y$, since $\pi_n(X) = \pi_n(Y)$ for all n, we get by the long exact sequence for the homotopy groups of the pair (Y, X) that $\pi_n(Y, X) = 0$ for all n. Applying the above compression

lemma 1.7.6 to the identity map $id : (Y, X) \to (Y, X)$ yields a deformation retraction $r : Y \to X$ of Y onto X.
Case 2: The general case of a map $f : X \to Y$ can be reduced to the above case of an inclusion by using the *mapping cylinder* of f, i.e.,

$$M_f := (X \times I) \sqcup Y/(x,1) \sim f(x).$$

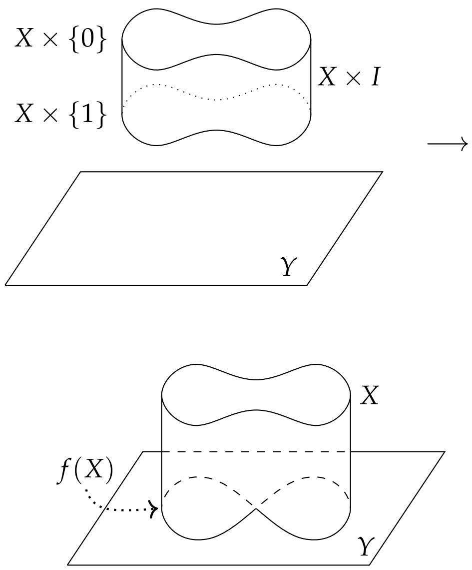


Figure 1.7: The mapping cylinder M_f of f

Note that M_f contains both $X = X \times \{0\}$ and Y as subspaces, and M_f deformation retracts onto Y. Moreover, the map f can be written as the composition of the inclusion i of X into M_f, and the retraction r from M_f to Y:

$$f : X = X \times \{0\} \overset{i}{\hookrightarrow} M_f \xrightarrow{r} Y.$$

Since M_f is homotopy equivalent to Y via r, it suffices to show that M_f deformation retracts onto X, so we can replace f with the inclusion map i. If f is a cellular map, then M_f is a CW complex having X as a subcomplex. So we can apply Case 1. If f is not cellular, than f is homotopic to some cellular map g, so we may work with g and the mapping cylinder M_g and again reduce to Case 1. $\square$

We can now prove Corollary 1.7.3:

Proof. After replacing Y by the mapping cylinder M_f, we may assume that f is an inclusion $X \hookrightarrow Y$. As $H_n(X) \cong H_n(Y)$ for all n, we have by the long exact sequence for the homology groups of the pair (Y, X) that $H_n(Y, X) = 0$ for all n.

Since X and Y are simply-connected, we have $\pi_1(Y, X) = 0$. So by the relative Hurewicz Theorem 1.10.1, the first non-zero $\pi_n(Y, X)$ is isomorphic to the first non-zero $H_n(Y, X)$. So $\pi_n(Y, X) = 0$ for all n.

Then, by the homotopy long exact sequence for the pair (Y, X), we get that

$$\pi_n(X) \cong \pi_n(Y)$$

for all n, with isomorphisms induced by the inclusion map f. Finally, Whitehead's theorem 1.7.2 yields that f is a homotopy equivalence. $\square$

Example 1.7.7. Let $X = \mathbb{R}P^2$ and $Y = S^2 \times \mathbb{R}P^\infty$. First note that $\pi_1(X) = \pi_1(Y) \cong \mathbb{Z}/2$. Also, since S^2 is a covering of $\mathbb{R}P^2$, we have that

$$\pi_i(X) \cong \pi_i(S^2), \quad i \geq 2.$$

Moreover, $\pi_i(Y) \cong \pi_i(S^2) \times \pi_i(\mathbb{R}P^\infty)$, and as $\mathbb{R}P^\infty$ is covered by $S^\infty = \bigcup_{n\geq 0} S^n$, we get that

$$\pi_i(Y) \cong \pi_i(S^2) \times \pi_i(S^\infty), \quad i \geq 2.$$

To calculate $\pi_i(S^\infty)$, we use cellular approximation. More precisely, we can approximate any $f : S^i \to S^\infty$ by a cellular map g so that Image $g \subset S^n$ for $i \ll n$. Thus, $[f] = [g] \in \pi_i(S^n) = 0$, and we see that

$$\pi_i(X) \cong \pi_i(S^2) \cong \pi_i(Y), \quad i \geq 2.$$

Altogether, we have shown that X and Y have the same homotopy groups. However, as can be easily seen by considering homology groups, X and Y are not homotopy equivalent. In particular, by Whitehead's theorem, there cannot exist a map $f : \mathbb{R}P^2 \to S^2 \times \mathbb{R}P^\infty$ inducing isomorphisms on π_n for all n. (If such a map existed, it would have to be a homotopy equivalence.)

Example 1.7.8. As we will see later on, the CW complexes S^2 and $S^3 \times \mathbb{C}P^\infty$ have isomorphic homotopy groups, but they are not homotopy equivalent.

1.8 CW approximation

Recall that map $f : X \to Y$ is a *weak homotopy equivalence* if it induces isomorphisms on all homotopy groups π_n. As seen in Theorem 1.10.3, a weak homotopy equivalence induces isomorphisms on all homology and cohomology groups. Furthermore, Whitehead's Theorem 1.7.2 shows that a weak homotopy equivalence of CW complexes is a homotopy equivalence.

In this section we show that given any space X, there exists a (unique up to homotopy) CW complex Z and a weak homotopy equivalence $f : Z \to X$. Such a map $f : Z \to X$ is called a *CW approximation* of X.

Definition 1.8.1. *Given a pair (X, A), with $\emptyset \neq A$ a CW complex, an n-connected CW model of (X, A) is an n-connected CW pair (Z, A), together* *with a map $f : Z \to X$ with $f|_A = id_A$, so that $f_* : \pi_i(Z) \to \pi_i(X)$ is an isomorphism for $i > n$ and an injection for $i = n$ (for any choice of basepoint).*

Remark 1.8.2. If such models exist, by letting A consist of one point in each path-component of X and $n = 0$, we get a CW approximation Z of X.

Theorem 1.8.3. *For any pair (X, A) with A a nonempty CW complex such n-connected models (Z, A) exist. Moreover, Z can be built from A by attaching cells of dimension greater than n. (Note that by cellular approximation this implies that $\pi_i(Z, A) = 0$ for $i \leq n$).*

We will prove this theorem after discussing the following consequences:

Corollary 1.8.4. *Any pair of spaces (X, X_0) has a CW approximation (Z, Z_0).*

Proof. Let $f_0 : Z_0 \to X_0$ be a CW approximation of X_0, and consider the map $g : Z_0 \to X$ defined by the composition of f_0 and the inclusion map $X_0 \hookrightarrow X$. Let M_g be the mapping cylinder of g. Hence we get the sequence of maps $Z_0 \hookrightarrow M_g \to X$, where the map $r : M_g \to X$ is a deformation retract.

Now, let (Z, Z_0) be a 0-connected CW model of (M_g, Z_0). Consider the composition:

$$(f, f_0) : (Z, Z_0) \longrightarrow (M_g, Z_0) \xrightarrow{(r, f_0)} (X, X_0)$$

So the map $f : Z \to X$ is obtained by composing the weak homotopy equivalence $Z \to M_g$ and the deformation retract (hence homotopy equivalence) $M_g \to X$. In other words, f is a weak homotopy equivalence and $f|_{Z_0} = f_0$, thus proving the result. □

Corollary 1.8.5. *For each n-connected CW pair (X, A) there is a CW pair (Z, A) that is homotopy equivalent to (X, A) relative to A, and such that Z is built from A by attaching cells of dimension $> n$.*

Proof. Let (Z, A) be an n-connected CW model of (X, A). By Theorem 1.8.3, Z is built from A by attaching cells of dimension $> n$. We claim that $Z \overset{h.e.}{\simeq} X$ (rel. A). First, by definition, the map $f : Z \to X$ has the property that f_* is an isomorphism on π_i for $i > n$ and an injection on π_n. For $i < n$, by the n-connectedness of the given model, $\pi_i(X) \cong \pi_i(A) \cong \pi_i(Z)$ where the isomorphisms are induced by f since the following diagram commutes,

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ A & \xrightarrow{id} & A \end{array}$$

(with $A \hookrightarrow Z$ and $A \hookrightarrow X$ the inclusion maps.) For $i = n$, by n-connectedness of (X, A) the composition

$$\pi_n(A) \twoheadrightarrow \pi_n(Z) \rightarrowtail \pi_n(X)$$

is onto. So the induced map $f_* : \pi_n(Z) \to \pi_n(X)$ is surjective. Altogether, f_* induces isomorphisms on all π_i, so by Whitehead's Theorem we conclude that $f : Z \to X$ is a homotopy equivalence.

We make f stationary on A as follows. Define the quotient space

$$W_f := M_f / \{\{a\} \times I \sim \text{pt}, \forall a \in A\}$$

of the mapping cylinder M_f obtained by collapsing each segment $\{a\} \times I$ to a point, for any $a \in A$. Assuming f has been made cellular, W_f is a CW complex containing X and Z as subcomplexes, and W_f deformation retracts onto X just as M_f does.

Consider the map $h : Z \to X$ given by the composition $Z \hookrightarrow W_f \to X$, where $W_f \to X$ is the deformation retract. We claim that Z is a deformation retract of W_f, thus giving us that h is a homotopy equivalence relative to A. Indeed, $\pi_i(W_f) \cong \pi_i(X)$ (since W_f is a deformation retract of X) and $\pi_i(X) \cong \pi_i(Z)$ since X is homotopy equivalent to Z. Using Whitehead's theorem, we conclude that Z is a deformation retract of W_f. $\square$

Proof of Theorem 1.8.3. We will construct Z as a union of subcomplexes

$$A = Z_n \subseteq Z_{n+1} \subseteq \cdots$$

such that for each $k \geq n+1$, Z_k is obtained from Z_{k-1} by attaching k-cells.

We will show by induction that we can construct Z_k together with a map $f_k : Z_k \to X$ such that $f_k|_A = id_A$ and f_{k*} is injective on π_i for $n \leq i < k$ and onto on π_i for $n < i \leq k$. We start the induction at $k = n$, with $Z_n = A$, in which case the conditions on π_i are void.

For the induction step, $k \to k+1$, consider the set $\{\phi_\alpha\}_\alpha$ of generators $\phi_\alpha : S^k \to Z_k$ of $\ker(f_{k*} : \pi_k(Z_k) \to \pi_k(X))$. Define

$$Y_{k+1} := Z_k \cup_\alpha \cup_{\phi_\alpha} e_\alpha^{k+1},$$

where e_α^{k+1} is a $(k+1)$-cell attached to Z_k along ϕ_α.

Then $f_k : Z_k \to X$ extends to Y_{k+1}. Indeed, $f_k \circ \phi_\alpha : S^k \to Z_k \to X$ is nullhomotopic, since $[f_k \circ \phi_\alpha] = f_{k*}[\phi_\alpha] = 0$. So we get a map $g : Y_{k+1} \to X$. It is easy to check that the g_* is injective on π_i for $n \leq i \leq k$, and onto on π_k. In fact, since we extend f_k on $(k+1)$-cells, we only need to check the effect on π_k. The elements of $\ker(g_*)$ on π_k are represented by nullhomotopic maps (by construction) $S^k \to Z_k \subset Y_{k+1} \to X$. So g_* is one-to-one on π_k. Moreover, g_* is onto on π_k since, by hypothesis, the composition $\pi_k(Z_k) \to \pi_k(Y_{k+1}) \to \pi_k(X)$ is onto.

Let $\{\phi_\beta : S^{k+1} \to X\}$ be a set of generators of $\pi_{k+1}(X, x_0)$ and let $Z_{k+1} = Y_{k+1} \bigvee_\beta S^{k+1}_\beta$. We extend g to a map $f_{k+1} : Z_{k+1} \to X$ by defining $f_{k+1}|_{S^{k+1}_\beta} = \phi_\beta$. This implies that f_{k+1} induces an epimorphism on π_{k+1}. The remaining conditions on homotopy groups are easy to check. □

Remark 1.8.6. If X is path-connected and A is a point, the construction of a CW model for (X, A) gives a CW approximation of X with a single 0-cell. In particular, by Whitehead's Theorem 1.7.2, *any connected CW complex is homotopy equivalent to a CW complex with a single* 0*-cell.*

Proposition 1.8.7. *Let $g : (X, A) \to (X', A')$ be a map of pairs, where A, A' are nonempty CW complexes. Let (Z, A) be an n-connected CW model of (X, A) with associated map $f : (Z, A) \to (X, A)$, and let (Z', A') be an n'-connected model of (X', A') with associated map $f' : (Z', A') \to (X', A')$. Assume that $n \geq n'$. Then there exists a map $h : Z \to Z'$, unique up to homotopy, such that $h|_A = g|_A$ and,*

$$\begin{array}{ccc} (Z, A) & \xrightarrow{f} & (X, A) \\ h\big\downarrow & & g\big\downarrow \\ (Z', A') & \xrightarrow{f'} & (X', A') \end{array}$$

commutes up to homotopy.

Proof. The proof is a standard induction on skeleta. We begin with the map $g : A \to A' \subseteq Z'$, and recall that Z is obtained from A by attaching cells of dimension $> n$. Let k be the smallest dimension of such a cell, thus $(A \cup Z_k, A)$ has a k-connected model, $f_k : (Z^k, A) \to (A \cup Z_k, A)$ such that $f_k|_A = id_A$. Composing this new map with g allows us to consider g as having been extended to the k skeleton of Z. Iterating this process produces our map. □

Corollary 1.8.8. *CW-approximations are unique up to homotopy equivalence. More generally, n-connected models of a pair (X, A) are unique up to homotopy relative to A.*

Proof. Assume that $f : (Z, A) \to (X, A)$ and $f' : (Z', A) \to (X, A)$ are two n-connected models of (X, A). Then we may take $(X, A) = (X', A')$ and $g = id$ in the above lemma twice, and conclude that there are two maps $h_0 : Z \to Z'$ and $h_1 : Z' \to Z$, such that $f \circ h_1 \simeq f'$ (rel. A) and $f' \circ h_0 \simeq f$ (rel. A). In particular, $f \circ (h_1 \circ h_0) \simeq f$ (rel. A) and $f' \circ (h_0 \circ h_1) \simeq f'$ (rel. A). The uniqueness in Proposition 1.8.7 then implies that $h_1 \circ h_0$ and $h_0 \circ h_1$ are homotopic to the respective identity maps (rel. A). □

Remark 1.8.9. By taking $n = n'$ is Proposition 1.8.7, we get a functoriality property for n-connected CW models. For example, a map $X \to X'$ of spaces induces a map of CW approximations $Z \to Z'$.

Remark 1.8.10. By letting n vary, and by letting (Z^n, A) be an n-connected CW model for (X, A), then Proposition 1.8.7 gives a *tower* of CW models

$$\begin{tikzcd} & \arrow[d] & \\ & Z^2 \arrow[d] \arrow[ddr] & \\ & Z^1 \arrow[d] \arrow[dr] & \\ A \arrow[uur] \arrow[ur] \arrow[r] & Z^0 \arrow[r] & X \end{tikzcd}$$

with commutative triangle on the left, and homotopy-commutative triangles on the right.

Example 1.8.11 (Whitehead towers)**.** Assume X is an arbitrary CW complex with $A \subset X$ a point. Then the resulting tower of n-connected CW modules of (X, A) amounts to a sequence of maps

$$\cdots \longrightarrow Z^2 \longrightarrow Z^1 \longrightarrow Z^0 \longrightarrow X$$

with Z^n n-connected and the map $Z^n \to X$ inducing isomorphisms on all homotopy groups π_i with $i > n$. The space Z^0 is path-connected and homotopy equivalent to the component of X containing A, so one may assume that Z^0 equals this component. The space Z^1 is simply-connected, and the map $Z^1 \to X$ has the homotopy properties of the universal cover of the component Z^0 of X. In general, if X is connected the map $Z^n \to X$ has the homotopy properties of an n-connected cover of X. An example of a 2-connected cover of S^2 is the Hopf map $S^3 \to S^2$.

Example 1.8.12 (Postnikov towers)**.** If X is a connected CW complex, the tower of n-connected models for the pair (CX, X), with CX the cone on X, is called the *Postnikov tower* of X. Relabeling Z^n as X^{n-1}, the Postnikov tower gives a commutative diagram

$$\begin{tikzcd} & \arrow[d] \\ & X^3 \arrow[d] \\ & X^2 \arrow[d] \\ X \arrow[uur] \arrow[ur] \arrow[r] & X^1 \end{tikzcd}$$

where the induced homomorphism $\pi_i(X) \to \pi_i(X^n)$ is an isomorphism for $i \leq n$ and $\pi_i(X^n) = 0$ if $i > n$. Indeed, by Definition 1.8.1 we get $\pi_i(X^n) = \pi_i(Z^{n+1}) \cong \pi_i(CX) = 0$ for $i \geq n+1$.

1.9 Eilenberg-MacLane spaces

Definition 1.9.1. *A space X having just one nontrivial homotopy group $\pi_n(X) = G$ is called an Eilenberg-MacLane space $K(G, n)$.*

Example 1.9.2. We have already seen that S^1 is a $K(\mathbb{Z}, 1)$ space, and $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$ space. The fact that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$ space will be discussed in Example 1.11.16 by making use of fibrations and the associated long exact sequence of homotopy groups.

Lemma 1.9.3. *If a CW-pair (X, A) is r-connected ($r \geq 1$) and A is s-connected ($s \geq 0$), then the map $\pi_i(X, A) \to \pi_i(X/A)$ induced by the quotient map $X \to X/A$ is an isomorphism if $i \leq r + s$ and onto if $i = r + s + 1$.*

Proof. Let CA be the cone on A and consider the complex

$$Y = X \cup_A CA$$

obtained from X by attaching the cone CA along $A \subseteq X$. Since CA is a contactible subcomplex of Y, the quotient map

$$q : Y \longrightarrow Y/CA = X/A$$

is obtained by deforming CA to the cone point inside Y, so it is a homotopy equivalence. So we have a sequence of homomorphisms

$$\pi_i(X, A) \longrightarrow \pi_i(Y, CA) \xleftarrow{\cong} \pi_i(Y) \xrightarrow{\cong} \pi_i(X/A),$$

where the first and second maps are induced by the inclusion of pairs, the second map is an isomorphism by the long exact sequence of the pair (Y, CA)

$$0 = \pi_i(CA) \to \pi_i(Y) \to \pi_i(Y, CA) \to \pi_{i-1}(CA) = 0,$$

and the third map is the isomorphism q_*. It therefore remains to investigate the map $\pi_i(X, A) \longrightarrow \pi_i(Y, CA)$. We know that (X, A) is r-connected and (CA, A) is $(s + 1)$-connected. The second fact once again follows from the long exact sequence of the pair and the fact that A is s-connected. Using the Excision Theorem 1.5.1, the desired result follows immediately. □

Lemma 1.9.4. *Assume $n \geq 2$. If $X = (\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta e^{n+1}_\beta$ is obtained from $\bigvee_\alpha S^n_\alpha$ by attaching $(n+1)$-cells e^{n+1}_β via basepoint-preserving maps $\phi_\beta : S^n_\beta \to \bigvee_\alpha S^n_\alpha$, then*

$$\pi_n(X) = \pi_n(\bigvee_\alpha S^n_\alpha)/\langle \phi_\beta \rangle = (\bigoplus_\alpha \mathbb{Z})/\langle \phi_\beta \rangle.$$

Proof. Consider the following portion of the long exact sequence for the homotopy groups of the n-connected pair $(X, \bigvee_\alpha S^n_\alpha)$:

$$\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \xrightarrow{\partial} \pi_n(\bigvee_\alpha S^n_\alpha) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, \bigvee_\alpha S^n_\alpha) = 0,$$

where the fact that $\pi_n(X, \bigvee_\alpha S^n_\alpha) = 0$ follows by Corollary 1.4.8 of the Cellular Approximation theorem. So $\pi_n(X) \cong \pi_n(\bigvee_\alpha S^n_\alpha)/\mathrm{Image}(\partial)$.

We have the identification $X/\bigvee_\alpha S^n_\alpha \simeq \bigvee_\beta S^{n+1}_\beta$, so by Lemma 1.9.3 and Lemma 1.6.6 we get that $\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \cong \pi_{n+1}(\bigvee_\beta S^{n+1}_\beta)$ is free with a basis consisting of the characteristic maps Φ_β of the cells e^{n+1}_β. Since $\partial([\Phi_\beta]) = [\phi_\beta]$, the claim follows. □

Example 1.9.5. Any abelian group G can be realized as $\pi_n(X)$ with $n \geq 2$ for some space X. In fact, given a presentation $G = \langle g_\alpha \mid r_\beta \rangle$, we can can take

$$X = \Big(\bigvee_\alpha S^n_\alpha\Big) \cup \bigcup_\beta e^{n+1}_\beta,$$

with the S^n_α's corresponding to the generators of G, and with e^{n+1}_β attached to $\bigvee_\alpha S^n_\alpha$ by a map $f : S^n_\beta \to \bigvee_\alpha S^n_\alpha$ satisfying $[f] = r_\beta$. Note also that by cellular approximation, $\pi_i(X) = 0$ for $i < n$, but nothing can be said about $\pi_i(X)$ with $i > n$.

Theorem 1.9.6. *For any $n \geq 1$ and any group G (which is assumed abelian if $n \geq 2$) there exists an Eilenberg-MacLane space $K(G, n)$.*

Proof. Let $X_{n+1} = (\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta e^{n+1}_\beta$ be the $(n-1)$-connected CW complex of dimension $n+1$ with $\pi_n(X_{n+1}) = G$, as constructed in Example 1.9.5. Enlarge X_{n+1} to a CW complex X_{n+2} obtained from X_{n+1} by attaching $(n+2)$-cells e^{n+2}_γ via maps representing some set of generators of $\pi_{n+1}(X_{n+1})$. Since (X_{n+2}, X_{n+1}) is $(n+1)$-connected (by Corollary 1.4.8), the long exact sequence for the homotopy groups of the pair (X_{n+2}, X_{n+1}) yields isomorphisms $\pi_i(X_{n+2}) = \pi_i(X_{n+1})$ for $i \leq n$, together with the exact sequence

$$\cdots \to \pi_{n+2}(X_{n+2}, X_{n+1}) \xrightarrow{\partial} \pi_{n+1}(X_{n+1}) \to \pi_{n+1}(X_{n+2}) \to 0.$$

Next note that ∂ is an isomorphism by construction and Lemma 1.9.3. Indeed, Lemma 1.9.3 yields that the quotient map $X_{n+2} \to X_{n+2}/X_{n+1}$ induces an epimorphism

$$\pi_{n+2}(X_{n+2}, X_{n+1}) \to \pi_{n+2}(X_{n+2}/X_{n+1}) \cong \pi_{n+2}(\bigvee_\gamma S^{n+2}_\gamma),$$

which is an isomorphism for $n \geq 2$. Moreover, we also have an epimorphism $\pi_{n+2}(\bigvee_\gamma S^{n+2}_\gamma) \to \pi_{n+1}(X_{n+1})$ by our construction of X_{n+2}. As ∂ is onto, we then get that $\pi_{n+1}(X_{n+2}) = 0$.

Repeat this construction inductively, at the k-th stage attaching $(n+k+1)$-cells to X_{n+k} to create a CW complex X_{n+k+1} with vanishing π_{n+k} and without changing the lower homotopy groups. The union of this increasing sequence of CW complexes is then a $K(G,n)$ space. $\square$

Corollary 1.9.7. *For any sequence of groups $\{G_n\}_{n\in\mathbb{N}}$, with G_n abelian for $n\geq 2$, there exists a space X such that $\pi_n(X)\cong G_n$ for any n.*

Proof. Call $X^n = K(G_n, n)$. Then $X = \prod_n X^n$ has the desired prescribed homotopy groups. $\square$

Lemma 1.9.8. *Let X be a CW complex of the form $(\bigvee_\alpha S^n_\alpha)\cup\bigcup_\beta e^{n+1}_\beta$ for some $n\geq 1$. Then for every homomorphism $\psi:\pi_n(X)\to\pi_n(Y)$ with Y a path-connected space, there exists a map $f: X\to Y$ such that $f_*=\psi$ on π_n.*

Proof. Recall from Lemma 1.9.4 that $\pi_n(X)$ is generated by the inclusions $i_\alpha : S^n_\alpha \hookrightarrow X$. Let f send the wedge point of X to a basepoint of Y, and extend f onto S^n_α by choosing a fixed representative for $\psi([i_\alpha])\in\pi_n(Y)$. This then allows us to define f on the n-skeleton $X_n=\bigvee_\alpha S^n_\alpha$ of X, and we notice that, by construction of $f: X_n\to Y$, we have that

$$f_*([i_\alpha]) = [f\circ i_\alpha] = [f|_{S^n_\alpha}] = \psi([i_\alpha]).$$

Because the i_α generate $\pi_n(X_n)$, we then get that $f_*=\psi$.

To extend f over a cell e^{n+1}_β, we need to show that the composition of the attaching map $\phi_\beta : S^n\to X_n$ for this cell with f is nullhomotopic in Y. We have $[f\circ\phi_\beta] = f_*([\phi_\beta]) = \psi([\phi_\beta]) = 0$, as the ϕ_β are precisely the relators in $\pi_n(X)$ by Example 1.9.5. Thus we obtain an extension $f: X\to Y$. Moreover, $f_*=\psi$ since the elements $[i_\alpha]$ generate $\pi_n(X_n) = \pi_n(X)$. $\square$

Proposition 1.9.9. *The homotopy type of a CW complex $K(G,n)$ is uniquely determined by G and n.*

Proof. Let K and K' be $K(G,n)$ CW complexes, and assume without loss of generality (since homotopy equivalence is an equivalence relation) that K is the particular $K(G,n)$ constructed in Theorem 1.9.6, i.e., built from a space X as in Lemma 1.9.8 by attaching cells of dimension $n+2$ and higher. Since $X = K_{n+1}$, we have that $\pi_n(X)=\pi_n(K)=\pi_n(K')$, and call the composition of these isomorphisms $\psi:\pi_n(X)\to\pi_n(K')$. By Lemma 1.9.8, there is a map $f: X\to K'$ inducing ψ on π_n. To extend this map over K, we proceed inductively, first extending it over the $(n+2)$-cells, than over the $(n+3)$-cells, and so on.

Let e^{n+2}_γ be an $(n+2)$-cell of K, with attaching map $\phi_\gamma : S^{n+1}\to X$. Then $f\circ\phi_\gamma : S^{n+1}\to K'$ is nullhomotopic since $\pi_{n+1}(K')=0$. Therefore, f extends over e^{n+2}_γ. Proceed similarly for higher dimensional cells of K to get a map $f : K \to K'$ which is a weak homotopy equivalence. By Whitehead's Theorem 1.7.2, we conclude that f is a homotopy equivalence. $\square$

1.10 Hurewicz Theorem

Theorem 1.10.1 (Hurewicz)**.** *If a space X is $(n-1)$-connected and $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$. Moreover, if a pair (X, A) is $(n-1)$-connected with $n \geq 2$, and $\pi_1(A) = 0$, then $H_i(X, A) = 0$ for all $i < n$ and $\pi_n(X, A) \cong H_n(X, A)$.*

Proof. First, since all hypotheses and assertions in the statement deal with homology and homotopy groups, if we prove the statement for a CW approximation of X (or (X, A)) then the results will also hold for the original space (or pair). Hence, we assume without loss of generality that X is a CW complex and (X, A) is a CW-pair.

Secondly, the relative case can be reduced to the absolute case. Indeed, since (X, A) is $(n-1)$-connected and that A is 1-connected, Lemma 1.9.3 implies that $\pi_i(X, A) = \pi_i(X/A)$ for $i \leq n$, while $H_i(X, A) = \widetilde{H}_i(X/A)$ always holds for CW-pairs.

In order to prove the absolute case of the theorem, let x_0 be a 0-cell in X. Since X, hence also (X, x_0), is $(n-1)$-connected, Corollary 1.8.5 tells us that we can replace X by a homotopy equivalent CW complex with $(n-1)$-skeleton a point, i.e., $X_{n-1} = x_0$. In particular, $\widetilde{H}_i(X) = 0$ for $i < n$. For showing that $\pi_n(X) \cong H_n(X)$, we may disregard any cells of dimension greater than $n+1$ since these have no effect on π_n or H_n. Thus we may assume that X has the form $(\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta e^{n+1}_\beta$. By Lemma 1.9.4, we then have that $\pi_n(X) \cong (\bigoplus_\alpha \mathbb{Z})/\langle \phi_\beta \rangle$. On the other hand, cellular homology yields the same calculation for $H_n(X)$, so we are done. $\square$

Remark 1.10.2. One cannot expect any sort of relationship between $\pi_i(X)$ and $H_i(X)$ beyond n. For example, S^n has trivial homology in degrees $> n$, but many nontrivial homotopy groups in this range, if $n \geq 2$. On the other hand, $\mathbb{C}P^\infty$ has trivial higher homotopy groups in the range > 2 (as a $K(\mathbb{Z}, 2)$ space), but many nontrivial homology groups in this range.

Recall the Hurewicz Theorem has been already used for proving the important Corollary 1.7.3. Here we give another important application of Theorem 1.10.1:

Theorem 1.10.3. *If $f : X \to Y$ induces isomorphisms on homotopy groups π_n for all n, then it induces isomorphisms on homology and cohomology groups with G coefficients, for any group G.*

Proof. By the universal coefficient theorems, it suffices to show that f induces isomorphisms on integral homology groups $H_*(-;\mathbb{Z})$.

We only prove here the assertion under the extra condition that X is simply connected (the general case follows easily from spectral sequence theory, and it will be dealt with later on). As before, after replacing Y with the homotopy equivalent space defined by the mapping cylinder M_f of f, we can assume that f is an inclusion. Since by the hypothesis, $\pi_n(X) \cong \pi_n(Y)$ for all n, with isomorphisms induced by the inclusion f, the homotopy long exact sequence of the pair (Y, X) yields that $\pi_n(Y, X) = 0$ for all n. By the relative Hurewicz theorem (as $\pi_1(X) = 0$), this gives that $H_n(Y, X) = 0$ for all n. Hence, by the long exact sequence for homology, $H_n(X) \cong H_n(Y)$ for all n, and the proof is complete. $\square$

Example 1.10.4. Take $X = \mathbb{R}P^2 \times S^3$ and $Y = S^2 \times \mathbb{R}P^3$. As seen in Example 1.1.19, X and Y have isomorphic homotopy groups π_n for all n, but $H_5(X) \not\cong H_5(Y)$. So there cannot exist a map $f : X \to Y$ inducing the isomorphisms on the π_n.

1.11 Fibrations. Fiber bundles

Definition 1.11.1 (Homotopy Lifting Property)**.** *A map $p : E \to B$ has the homotopy lifting property (HLP) with respect to a space X if, given a homotopy $g_t : X \to B$, and a lift $\widetilde{g}_0 : X \to E$ of g_0, there exists a homotopy $\widetilde{g}_t : X \to E$ lifting g_t and extending $\widetilde{g}_0$.*

$$\begin{tikzcd} X \arrow[r, "\widetilde{g}_0"] \arrow[d, equal] & E \arrow[d, "p"] \\ X \arrow[r, "g_t"'] \arrow[ur, dashed, "\widetilde{g}_t"] & B \end{tikzcd}$$

Definition 1.11.2 (Lift Extension Property)**.** *A map $p : E \to B$ has the lift extension property (LEP) with respect to a pair (Z, A) if for all maps $f : Z \to B$ and $g : A \to E$, there exists a lift $\widetilde{f} : Z \to E$ of f extending g.*

$$\begin{tikzcd} & & E \arrow[d, "p"] \\ A \arrow[r, hook] \arrow[urr, "g"] & Z \arrow[r, "f"'] \arrow[ur, dashed, "\widetilde{f}"] & B \end{tikzcd}$$

Remark 1.11.3. (HLP) is a special case of (LEP), with $Z = X \times [0, 1]$, and $A = X \times \{0\}$.

Definition 1.11.4. *A fibration $p : E \to B$ is a map having the homotopy lifting property with respect to all spaces X.*

Definition 1.11.5 (Homotopy Lifting Property with respect to a pair)**.** *A map $p : E \to B$ has the homotopy lifting property with respect to a pair (X, A) if each homotopy $g_t : X \to B$ lifts to a homotopy $\widetilde{g}_t : X \to E$ starting with a given lift $\widetilde{g}_0$ and extending a given lift $\widetilde{g}_t : A \to E$.*

Remark 1.11.6. The homotopy lifting property with respect to the pair (X, A) is the lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$.

Remark 1.11.7. The homotopy lifting property with respect to a disk D^n is equivalent to the homotopy lifting property with respect to the pair $(D^n, \partial D^n)$, since the pairs $(D^n \times I, D^n \times \{0\})$ and $(D^n \times I, D^n \times \{0\} \cup \partial D^n \times I)$ are homeomorphic. This implies that *a fibration has the homotopy lifting property with respect to all CW pairs (X, A)*. Indeed, the homotopy lifting property for disks is in fact equivalent to the homotopy lifting property with respect to all CW pairs (X, A). This can be easily seen by induction over the skeleta of X, so it suffices to construct a lifting $\widetilde{g}_t$ one cell of $X \setminus A$ at a time. Composing with the characteristic map $D^n \to X$ of a cell then gives the reduction to the case $(X, A) = (D^n, \partial D^n)$.

Theorem 1.11.8 (Long exact sequence for homotopy groups of a fibration)**.** *Given a fibration $p : E \to B$, points $b \in B$ and $e \in F := p^{-1}(b)$, there is an isomorphism $p_* : \pi_n(E, F, e) \xrightarrow{\cong} \pi_n(B, b)$ for all $n \geq 1$. Hence, if B is path-connected, there is a long exact sequence of homotopy groups:*

$$\cdots \longrightarrow \pi_n(F, e) \longrightarrow \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \longrightarrow \pi_{n-1}(F, e) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_0(E, e) \longrightarrow 0$$

Proof. To show that p_* is onto, represent an element of $\pi_n(B, b)$ by a map $f : (I^n, \partial I^n) \to (B, b)$, and note that the constant map to e is a lift of f to E over $J^{n-1} \subset I^n$. The homotopy lifting property for the pair $(I^{n-1}, \partial I^{n-1})$ extends this to a lift $\widetilde{f} : I^n \to E$. This lift satisfies $\widetilde{f}(\partial I^n) \subset F$ since $f(\partial I^n) = b$. So $\widetilde{f}$ represents an element of $\pi_n(E, F, e)$ with $p_*([\widetilde{f}]) = [f]$ since $p\widetilde{f} = f$.

To show the injectivity of p_*, let $\widetilde{f}_0, \widetilde{f}_1 : (I^n, \partial I^n, J^{n-1}) \to (E, F, e)$ be so that $p_*(\widetilde{f}_0) = p_*(\widetilde{f}_1)$. Let $H : (I^n \times I, \partial I^n \times I) \to (B, b)$ be a homotopy from $p\widetilde{f}_0$ to $p\widetilde{f}_1$. We have a partial lift given by $\widetilde{f}_0$ on $I^n \times \{0\}$, $\widetilde{f}_1$ on $I^n \times \{1\}$ and the constant map to e on $J^{n-1} \times I$. The homotopy lifting property for CW pairs extends this to a lift $\widetilde{H} : I^n \times I \to E$ giving a homotopy $\widetilde{f}_t : (I^n, \partial I^n, J^{n-1}) \to (E, F, e)$ from $\widetilde{f}_0$ to $\widetilde{f}_1$.

Finally, the long exact sequence of the fibration follows by plugging $\pi_n(B, b)$ in for $\pi_n(E, F, e)$ in the long exact sequence for the pair (E, F). The map $\pi_n(E, e) \to \pi_n(E, F, e)$ in the latter sequence becomes the composition $\pi_n(E,e) \to \pi_n(E,F,e) \xrightarrow{p_*} \pi_n(B,b)$, which is exactly $p_* : \pi_n(E,e) \to \pi_n(B,b)$. The surjectivity of $\pi_0(F,e) \to \pi_0(E,e)$ follows from the path-connectedness of B, since a path in E from an arbitrary point $x \in E$ to F can be obtained by lifting a path in B from $p(x)$ to b. $\square$

Definition 1.11.9. *Given two fibrations $p_i : E_i \to B$, $i = 1,2$, a map $f : E_1 \to E_2$ is fiber-preserving if the diagram*

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

commutes. Such a map f is called a fiber homotopy equivalence if f is both fiber-preserving and a homotopy equivalence, i.e., there is a map $g : E_2 \to E_1$ such that f and g are fiber-preserving and $f \circ g$ and $g \circ f$ are homotopic to the identity maps by fiber-preserving maps.

Definition 1.11.10 (Fiber Bundle)**.** *A map $p : E \to B$ is a fiber bundle with fiber F if, for any point $b \in B$, there exists a neighborhood U_b of b with a homeomorphism $h : p^{-1}(U_b) \to U_b \times F$ so that the following diagram commutes:*

$$\begin{tikzcd} p^{-1}(U_b) \arrow[rr, "h"] \arrow[dr, "p"'] & & U_b \times F \arrow[dl, "pr"] \\ & U_b & \end{tikzcd}$$

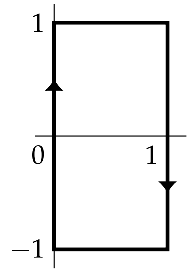
Remark 1.11.11. Fibers of fibrations are homotopy equivalent, while fibers of fiber bundles are homeomorphic.

Theorem 1.11.12 (Hurewicz)**.** *Fiber bundles over paracompact spaces are fibrations.*

Here are some easy examples of fiber bundles.

Example 1.11.13. If F is discrete, a fiber bundle with fiber F is a covering map. Moreover, the long exact sequence for the homotopy groups yields that $p_* : \pi_i(E) \to \pi_i(B)$ is an isomorphism if $i \geq 2$ and a monomorphism for $i = 1$.

Example 1.11.14. The Möbius band $I \times [-1,1] \big/ (0,y) \sim (1,-y) \longrightarrow S^1$ is a fiber bundle with fiber $[-1,1]$, induced from the projection map $I \times [-1,1] \to I$.



Example 1.11.15. By glueing the unlabeled edges of a Möbius band, we get $K \to S^1$ (where K is the Klein bottle), a fiber bundle with fiber S^1.

Example 1.11.16. The following is a fiber bundle with fiber S^1:

$$\begin{aligned} S^1 \hookrightarrow S^{2n+1}(\subset \mathbb{C}^{n+1}) &\longrightarrow \mathbb{C}P^n \\ (z_0, \ldots, z_n) &\mapsto [z_0 : \ldots : z_n] = [\underline{z}] \end{aligned}$$

For $[\underline{z}] \in \mathbb{C}P^n$, there is an i such that $z_i \neq 0$. Then we have a neighborhood

$$U_{[\underline{z}]} = \{[z_0 : \ldots : 1 : \ldots : z_n]\} \cong \mathbb{C}^n$$

(with the entry 1 in place of the ith coordinate) of $[\underline{z}]$, with a homeomorphism

$$\begin{aligned} p^{-1}(U_{[\underline{z}]}) &\longrightarrow U_{[\underline{z}]} \times S^1 \\ (z_0, \ldots, z_n) &\mapsto ([z_0 : \ldots : z_n], z_i/|z_i|). \end{aligned}$$

By letting n go to infinity, we get a diagram of fibrations

$$\begin{tikzcd}
S^1 \arrow[dr, hook] & = & S^1 \arrow[dr, hook] & = & \cdots & = & S^1 \arrow[dr, hook] & \\
& S^{2n+1} \arrow[d] & \subset & S^{2n+3} \arrow[d] & \subset & \cdots & \subset & S^\infty \arrow[d] \\
& \mathbb{C}P^n & \subset & \mathbb{C}P^{n+1} & \subset & \cdots & \subset & \mathbb{C}P^\infty
\end{tikzcd}$$

In particular, from the long exact sequence of the fibration

$$S^1 \hookrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$$

with S^∞ contactible, we obtain that

$$\pi_i(\mathbb{C}P^\infty) \cong \pi_{i-1}(S^1) = \begin{cases} \mathbb{Z} & i = 2 \\ 0 & i \neq 2 \end{cases}$$

i.e.,

$$\mathbb{C}P^\infty = K(\mathbb{Z}, 2),$$

as already mentioned in our discussion about Eilenberg-MacLane spaces.

Remark 1.11.17. As we will see later on, for any topological group G there exists a "universal fiber bundle" $G \hookrightarrow EG \xrightarrow{\pi_G} BG$ with EG contractible, classifying the space of (principal) G-bundles. That is, any G-bundle $\pi : E \to B$ over a space B is determined by (the homotopy class of) a classifying map $f : B \to BG$ by pull-back: $\pi \cong f^*\pi_G$:

$$\begin{tikzcd} E \arrow[d, "\pi"'] & EG \arrow[d, "\pi_G"] & \simeq \{\mathrm{pt}\} \\ B \arrow[r, "f"'] & BG & \end{tikzcd}$$

From this point of view, $\mathbb{C}P^\infty$ can be identified with the clasifying space BS^1 of (principal) S^1-bundles.

Example 1.11.18. By letting $n = 1$ in the fibration of Example 1.11.16, the corresponding bundle

$$S^1 \hookrightarrow S^3 \longrightarrow \mathbb{C}P^1 \cong S^2 \tag{1.11.1}$$

is called the *Hopf fibration*. The long exact sequence of homotopy group for the Hopf fibration gives: $\pi_2(S^2) \cong \pi_1(S^1)$ and $\pi_n(S^3) \cong \pi_n(S^2)$ for all $n \geq 3$. Together with the fact that $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, this shows that S^2 and $S^3 \times \mathbb{C}P^\infty$ are simply-connected CW complexes with isomorphic homotopy groups, though they are not homotopy equivalent as can be easily seen from cellular homology.

Example 1.11.19. A fiber bundle similar to that of Example 1.11.16 can be obtained by replacing $\mathbb{C}$ with the quaternions $\mathbb{H}$, namely:

$$S^3 \hookrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n.$$

(Note that S^{4n+3} can be identified with the unit sphere in $\mathbb{H}^{n+1}$.) In particular, by letting $n = 1$ we get a second Hopf fiber bundle

$$S^3 \hookrightarrow S^7 \longrightarrow \mathbb{H}P^1 \cong S^4. \tag{1.11.2}$$

A third example of a Hopf bundle

$$S^7 \hookrightarrow S^{15} \longrightarrow S^8 \tag{1.11.3}$$

can be constructed by using the nonassociative 8-dimensional algebra $\mathbb{O}$ of Cayley octonions, whose elements are pair of quaternions (a_1, a_2) with multiplication defined by

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 - \bar{b}_2 a_2, a_2 \bar{b}_1 + b_2 a_1).$$

Here we regard S^{15} as the unit sphere in the 16-dimensional vector space $\mathbb{O}^2$, and the projection map $S^{15} \longrightarrow S^8 = \mathbb{O} \cup \{\infty\}$ is $(z_0, z_1) \mapsto z_0 z_1^{-1}$ (just like for the other Hopf bundles). There are no fiber bundles with fiber, total space and base spheres, other than those provided by the Hopf bundles of (1.11.1), (1.11.2) and (1.11.3). Finally, note that there is an "octonion projective plane" $\mathbb{O}P^2$ obtained by glueing a cell e^{16} to S^8 via the Hopf map $S^{15} \to S^8$; however, there is no octonion analogue of $\mathbb{R}P^n$, $\mathbb{C}P^n$ or $\mathbb{H}P^n$ for higher n, since the associativity of multiplication is needed for the relation $(z_0, \cdots, z_n) \sim \lambda(z_0, \cdots, z_n)$ to be an equivalence relation.

Example 1.11.20. Other examples of fiber bundles are provided by the orthogonal and unitary groups:

$$\begin{aligned} O(n-1) \hookrightarrow O(n) &\to S^{n-1} \\ A &\mapsto Ax, \end{aligned}$$

where x is a fixed unit vector in $\mathbb{R}^n$. Similarly, there is a fibration

$$\begin{aligned} U(n-1) \hookrightarrow U(n) &\to S^{2n-1} \\ A &\mapsto Ax, \end{aligned}$$

with x a fixed unit vector in $\mathbb{C}^n$. These examples will be discussed in some detail in the next section.

1.12 *More examples of fiber bundles*

Definition 1.12.1. *For $n \leq k$, the n-th Stiefel manifold associated to $\mathbb{R}^k$ is defined as*

$$V_n(\mathbb{R}^k) := \{n\text{-frames in } \mathbb{R}^k\},$$

where an n-frame in $\mathbb{R}^k$ is an n-tuple $\{v_1, \ldots, v_n\}$ of orthonormal vectors in $\mathbb{R}^k$, i.e., $v_1, \ldots, v_n$ are pairwise orthonormal: $\langle v_i, v_j \rangle = \delta_{ij}$.

We assign $V_n(\mathbb{R}^k)$ the subspace topology induced from

$$V_n(\mathbb{R}^k) \subset \underbrace{S^{k-1} \times \cdots \times S^{k-1}}_{n \text{ times}},$$

where $S^{k-1} \times \cdots \times S^{k-1}$ has the usual product topology.

Example 1.12.2. $V_1(\mathbb{R}^k) = S^{k-1}$.

Example 1.12.3. $V_n(\mathbb{R}^n) \cong O(n)$.

Definition 1.12.4. *The n-th Grassmann manifold associated to $\mathbb{R}^k$ is defined as:*

$$G_n(\mathbb{R}^k) := \{n\text{-dimensional vector subspaces in } \mathbb{R}^k\}.$$

Example 1.12.5. $G_1(\mathbb{R}^k) = \mathbb{R}P^{k-1}$

There is a natural surjection

$$p : V_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k)$$

given by

$$\{v_1, \ldots, v_n\} \mapsto \operatorname{span}\{v_1, \ldots, v_n\}.$$

The fact that p is onto follows by the Gram-Schmidt procedure. So $G_n(\mathbb{R}^k)$ is endowed with the quotient topology via p.

Lemma 1.12.6. *The projection p is a fiber bundle with fiber $V_n(\mathbb{R}^n) = O(n)$.*

Proof. Let $V \in G_n(\mathbb{R}^k)$ be fixed. The fiber $p^{-1}(V)$ consists on n-frames in $V \cong \mathbb{R}^n$, so it is homeomorphic to $V_n(\mathbb{R}^n)$. Let us now choose an orthonormal frame on V. By projection and Gram-Schmidt, we get orthonormal frames on all "nearby" (in some neighborhood U of V) vector subspaces V'. Indeed, by projecting the frame of V orthogonally onto V' we get a (non-orthonormal) basis for V', then apply the Gram-Schmidt process to this basis to make it orthonormal. This is a continuous process. The existence of such frames on all n-planes in U allows us to identify them with $\mathbb{R}^n$, so $p^{-1}(U)$ is identified with $U \times V_n(\mathbb{R}^n)$. □

To conclude this discussion, we have shown that for $k > n$, there are fiber bundles:

$$O(n) \hookrightarrow V_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k) \tag{1.12.1}$$

A similar method gives the following fiber bundle for all triples $m < n \leq k$:

$$\begin{array}{ccccc} V_{n-m}(\mathbb{R}^{k-m}) & \hookrightarrow & V_n(\mathbb{R}^k) & \xrightarrow{\quad p \quad} & V_m(\mathbb{R}^k) \\ & & \{v_1, \ldots, v_n\} & \longmapsto & \{v_1, \ldots, v_m\} \end{array} \tag{1.12.2}$$

Here, the projection p sends an n-frame onto the m-frame formed by its first m vectors, so the fiber consists of $(n-m)$-frames in the $(k-m)$-plane orthogonal to the given frame.

Example 1.12.7. If $k = n$ in the bundle (1.12.2), we get the fiber bundle

$$O(n-m) \hookrightarrow O(n) \longrightarrow V_m(\mathbb{R}^n). \tag{1.12.3}$$

Here, $O(n-m)$ is regarded as the subgroup of $O(n)$ fixing the first m standard basis vectors. So $V_m(\mathbb{R}^n)$ is identifiable with the coset space $O(n)/O(n-m)$, or the orbit space of the free action of $O(n-m)$ on $O(n)$ by right multiplication. Similarly,

$$G_m(\mathbb{R}^n) \cong O(n)/O(m) \times O(n-m),$$

where $O(m) \times O(n-m)$ consists of the orthogonal transformations of $\mathbb{R}^n$ taking the m-plane spanned by the first m standard basis vectors to itself.

If, moreover, we take $m = 1$ in (1.12.3), we get the fiber bundle

$$\begin{array}{ccccc} O(n-1) & \hookrightarrow & O(n) & \longrightarrow & S^{n-1} \\ A & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} & & \\ & & B & \longmapsto & Bu \end{array} \tag{1.12.4}$$

with $u \in S^{n-1}$ some fixed unit vector. In particular, this identifies S^{n-1} as an orbit (or homogeneous) space:

$$S^{n-1} \cong {}^{O(n)}\!/\!_{O(n-1)}.$$

Example 1.12.8. If $m = 1$ in the bundle (1.12.2), we get the fiber bundle

$$V_{n-1}(\mathbb{R}^{k-1}) \hookrightarrow V_n(\mathbb{R}^k) \longrightarrow S^{k-1}. \tag{1.12.5}$$

By using the long exact sequence for bundle (1.12.5) and induction on n, it follows readily that $V_n(\mathbb{R}^k)$ *is* $(k-n-1)$*-connected*.

Remark 1.12.9. The long exact sequence of homotopy groups for the bundle (1.12.4) shows that $\pi_i(O(n))$ is *independent of n for n large*. We call this *the stable homotopy group* $\pi_i(O)$. *Bott Periodicity* shows that $\pi_i(O)$ is periodic in i with period 8. Its values are:

i	1	2	3	4	5	6	7	8
$\pi_i(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Definition 1.12.10.

$$V_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} V_n(\mathbb{R}^k) \qquad G_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} G_n(\mathbb{R}^k)$$

The infinite grassmanian $G_n(\mathbb{R}^\infty)$ carries a lot of topological information. As we will see later on, the space $G_n(\mathbb{R}^\infty)$ is the classifying space for rank-n real vector bundles. In fact, we get a "limit" fiber bundle:

$$O(n) \hookrightarrow V_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty). \tag{1.12.6}$$

Moreover, we have the following:

Proposition 1.12.11. $V_n(\mathbb{R}^\infty)$ *is contractible.*

Proof. By using the bundle (1.12.5) for $k \to \infty$, we see that $\pi_i(V_n(\mathbb{R}^\infty)) = 0$ for all i. Using the CW structure and Whitehead's Theorem 1.7.2 shows that $V_n(\mathbb{R}^\infty)$ is contractible.

Alternatively, we can define an explicit homotopy $h_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$h_t(x_1, x_2, \dots) := (1-t)(x_1, x_2, \dots) + t(0, x_1, x_2, \dots).$$

Then h_t is linear for each t with $\ker h_t = \{0\}$. So h_t preserves independence of vectors. Applying h_t to an n-frame we get an n-tuple of independent vectors, which can be made orthonormal by the Gram-Schmidt (G-S, for short) process. We then get a deformation retraction of $V_n(\mathbb{R}^\infty)$ onto the subspace of n-frames with first coordinate zero. Repeating this procedure n times, we get a deformation of $V_n(\mathbb{R}^\infty)$ to the subspace of n-frames with first n coordinates zero.

Let $\{e_1, \dots, e_n\}$ be the standard n-frame in $\mathbb{R}^\infty$. For an n-frame $\{v_1, \dots, v_n\}$ of vectors with first n coordinates zero, define a homotopy $k_t : V_n(\mathbb{R}^\infty) \to V_n(\mathbb{R}^\infty)$ by

$$k_t\left(\{v_1, \dots, v_n\}\right) := \left[(1-t)\{v_1, \dots, v_n\} + t\{e_1, \dots, e_n\}\right] \circ (\mathrm{G-S}).$$

Then k_t preserves linear independence and orthonormality by Gram-Schmidt.

Composing h_t and k_t, any n-frame is moved continuously to the standard n-frame $\{e_1, \dots, e_n\}$. Thus $k_t \circ h_t$ is a contraction of $V_n(\mathbb{R}^\infty)$. $\square$

Similar considerations apply if we use $\mathbb{C}$ or $\mathbb{H}$ instead of $\mathbb{R}$, so we can define complex or quaternionic Stiefel and Grasmann manifolds, by using the usual hermitian inner products in $\mathbb{C}^k$ and $\mathbb{H}^k$, respectively. In particular, $O(n)$ gets replaced by $U(n)$ if $\mathbb{C}$ is used, and $Sp(n)$ is the quaternionic analog of this. Then similar fiber bundles can be constructed in the complex and quaternionic setting. For example, over $\mathbb{C}$ we get fiber bundles

$$U(n) \hookrightarrow V_n(\mathbb{C}^k) \xrightarrow{p} G_n(\mathbb{C}^k), \tag{1.12.7}$$

with $V_n(\mathbb{C}^k)$ a $(2k-2n)$-connected space. As $k \to \infty$, we get a fiber bundle

$$U(n) \hookrightarrow V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty), \tag{1.12.8}$$

with $V_n(\mathbb{C}^\infty)$ contractible. As we will see later on, this means that $V_n(\mathbb{C}^\infty)$ is the classifying space for rank-n complex vector bundles. We also have a fiber bundle similar to (1.12.4)

$$U(n-1) \hookrightarrow U(n) \longrightarrow S^{2n-1}, \tag{1.12.9}$$

whose long exact sequence of homotopy groups then shows that $\pi_i(U(n))$ is stable for large n. Bott periodicity shows that this stable group $\pi_i(U)$ repeats itself with period 2: the relevant groups are

0 for i even, and $\mathbb{Z}$ for i odd. Note that by (1.12.9), odd-dimensional spheres can be realized as complex homogeneous spaces via

$$S^{2n-1} \cong U(n)\big/U(n-1).$$

Many of these fiber bundles will become essential tools in the next chapter for computing (co)homology of matrix groups, with a view towards *classifying spaces* and *characteristic classes* of manifolds.

1.13 Turning maps into fibration

In this section, we show that *any map is homotopic to a fibration*.

Given a map $f : A \to B$, define

$$E_f := \{(a, \gamma) \mid a \in A,\ \gamma : [0,1] \to B \text{ with } \gamma(0) = f(a)\}.$$

E_f is a topological space with respect to the compact-open topology. Then A can be regarded as a subset of E_f, by mapping $a \in A$ to $(a, c_{f(a)})$, where $c_{f(a)}$ is the constant path based at the image of a under f. Define

$$\begin{aligned} E_f &\xrightarrow{p} B \\ (a,\gamma) &\mapsto \gamma(1) \end{aligned}$$

Then $p|_A = f$, so $f = p \circ i$ where i is the inclusion of A in E_f. Moreover, $i : A \longrightarrow E_f$ is a homotopy equivalence, and $p : E_f \longrightarrow B$ is a fibration with fiber A. So f can be factored as a composition of a homotopy equivalence and a fibration:

$$\begin{tikzcd} A \arrow[r, hook, "\text{h.e.}", "i"'] \arrow[rr, bend right, "f"'] & E_f \arrow[r, "\text{fibration}", "p"'] & B \end{tikzcd}$$

Example 1.13.1. If $A = \{b\} \hookrightarrow B$ and f is the inclusion of b in B, then $E_f =: PB$ is the contractible space of paths in B starting at b (called the *path-space* of B):

In this case, the above construction yields the *path fibration*

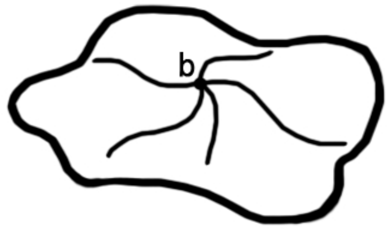
$$\Omega B = p^{-1}(b) \hookrightarrow PB \longrightarrow B,$$

where ΩB is the space of all loops in B based at b, and $PB \longrightarrow B$ is given by $\gamma \mapsto \gamma(1)$. Since PB is contractible, the associated long exact sequence of the fibration yields that

$$\pi_i(B) \cong \pi_{i-1}(\Omega B) \tag{1.13.1}$$

for all i.

The isomorphism (1.13.1) suggests that the Hurewicz Theorem 1.10.1 can also be proved by induction on the degree of connectivity. Indeed, if B is n-connected then ΩB is $(n-1)$-connected. We'll give the details of such an approach by using spectral sequences.



The following result is useful for computations:

Proposition 1.13.2 (Puppé sequence). *Given a fibration $F \hookrightarrow E \to B$, there is a sequence of maps*

$$\cdots \longrightarrow \Omega^2 B \longrightarrow \Omega F \longrightarrow \Omega E \longrightarrow \Omega B \longrightarrow F \longrightarrow E \longrightarrow B$$

with any two consecutive maps forming a fibration.

1.14 Exercises

1. Let $f : X \to Y$ be a homotopy equivalence. Let Z be any other space. Show that f induces bijections:

$$f_* : [Z, X] \to [Z, Y] \quad \text{and} \quad f^* : [Y, Z] \to [X, Z] \ ,$$

where $[A, B]$ denotes the set of homotopy classes of maps from the space A to B.

2. Find examples of spaces X and Y which have the same homology groups, cohomology groups, and cohomology rings, but with different homotopy groups.

3. Use homotopy groups in order to show that there is no retraction $\mathbb{RP}^n \to \mathbb{RP}^k$ if $n > k > 0$.

4. Show that an n-connected, n-dimensional CW complex is contractible.

5. (*Extension Lemma*)
Given a CW pair (X, A) and a map $f : A \to Y$ with Y path-connected, show that f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n.

6. Show that a CW complex retracts onto any contractible subcomplex. (Hint: Use the above extension lemma.)

7. If $p : (\tilde{X}, \tilde{A}, \tilde{x}_0) \to (X, A, x_0)$ is a covering space with $\tilde{A} = p^{-1}(A)$, show that the map $p_* : \pi_n(\tilde{X}, \tilde{A}, \tilde{x}_0) \to \pi_n(X, A, x_0)$ is an isomorphism for all $n > 1$.

8. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic. Conclude that S^∞ is contractible, and more generally, this is true for the infinite suspension $\Sigma^\infty(X) := \bigcup_{n \geq 0} \Sigma^n(X)$ of any CW complex X.

9. Use cellular approximation to show that the n-skeletons of homotopy equivalent CW complexes without cells of dimension $n+1$ are also homotopy equivalent.

10. Show that a closed simply-connected 3-manifold is homotopy equivalent to S^3. (Hint: Use Poincaré Duality, and also the fact that closed manifolds are homotopy equivalent to CW complexes.)

11. Show that a map $f : X \to Y$ of connected CW complexes is a homotopy equivalence if it induces an isomorphism on π_1 and if a lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ to the universal covers induces an isomorphism on homology.

12. Show that $\pi_7(S^4)$ is non-trivial. [Hint: It contains a $\mathbb{Z}$-summand.]

13. Prove that the space $SO(3)$ of orthogonal 3×3 matrices with determinant 1 is homeomorphic to $\mathbb{RP}^3$.

14. Show that if $S^k \to S^m \to S^n$ is a fiber bundle, then $k = n-1$ and $m = 2n-1$.

15. Show that if there were fiber bundles $S^{n-1} \to S^{2n-1} \to S^n$ for *all* n, then the groups $\pi_i(S^n)$ would be finitely generated free abelian groups computable by induction, and non-zero if $i \geq n \geq 2$.

16. Let $U(n)$ be the unitary group. Find $\pi_k(U(n))$ for $k = 1, 2, 3$ and $n \geq 2$.

17. If $p : E \to B$ is a fibration over a contractible space B, then p is fiber homotopy equivalent to the trivial fibration $B \times F \to B$.

2

Spectral Sequences. Applications

Most of our considerations involving spectral sequences will be applied to fibrations. If $F \hookrightarrow E \to B$ is such a fibration, then a spectral sequence can be regarded as a machine which takes as input the (co)homology of the base B and fiber F and outputs the (co)homology of the total space E. Our emphasis here is on applications of the theory of spectral sequences, and not so much on developing the theory itself.

2.1 Homological spectral sequences. Definitions

We begin with a discussion of homological spectral sequences.

Definition 2.1.1. *A (homological) spectral sequence is a sequence*

$$\{E^r_{*,*}, d^r_{*,*}\}_{r \geq 0}$$

of chain complexes of abelian groups, such that

$$E^{r+1}_{*,*} = H_*(E^r_{*,*}).$$

In more detail, we have abelian groups $\{E^r_{p,q}\}$ and maps (called "differentials")

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that $(d^r)^2 = 0$ and

$$E^{r+1}_{p,q} := \frac{\ker\left(d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\right)}{\operatorname{Image}\left(d^r_{p+r,q-r+1} : E^r_{p+r,q-r+1} \to E^r_{p,q}\right)}.$$

We will focus on the first quadrant spectral sequences, i.e., with $E^r_{p,q} = 0$ whenever $p < 0$ or $q < 0$. Hence, for any fixed (p,q) in the first quadrant and for sufficiently large r, the differentials $d^r_{p,q}$ and $d^r_{p+r,q-r+1}$ vanish, so that

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots = E^\infty_{p,q}.$$

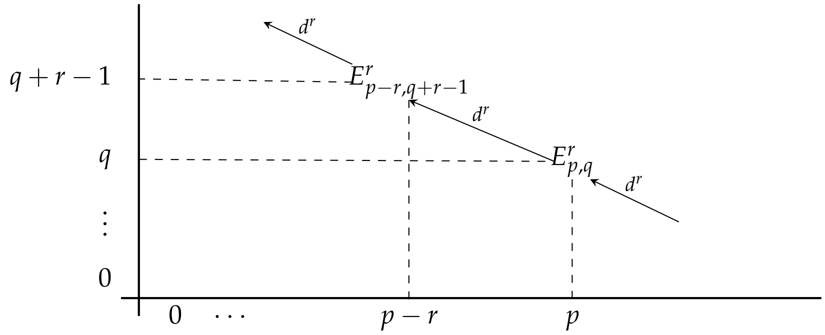


Figure 2.1: r-th page E^r

In this case we say that the spectral sequence *degenerates* at page E^r.

When it is clear from the context which differential we refer to, we will simply write d^r, instead of $d^r_{*,*}$.

Definition 2.1.2. *If $\{H_n\}_n$ are groups, we say the spectral sequence converges (or abuts) to H_*, and we write*

$$(E^r, d^r) \Rrightarrow H_*,$$

if for each n there is a filtration

$$H_n = D_{n,0} \supseteq D_{n-1,1} \supseteq \cdots \supseteq D_{1,n-1} \supseteq D_{0,n} \supseteq D_{-1,n+1} = 0$$

such that, for all p, q,

$$E^\infty_{p,q} = {D_{p,q}}\big/{D_{p-1,q+1}}.$$

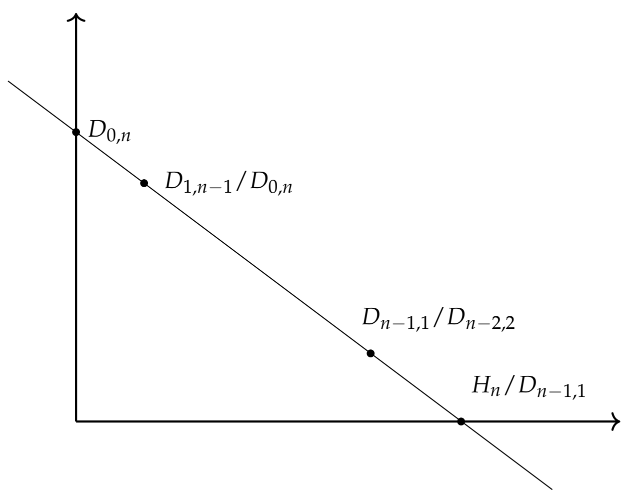


Figure 2.2: n-th diagonal of E^∞

To read off H_* from E^∞, we need to solve several extension problems. But if $E^r_{*,*}$ and H_* are vector spaces, then

$$H_n \cong \bigoplus_{p+q=n} E^\infty_{p,q},$$

since in this case all extension problems are trivial.

Remark 2.1.3. The following observation is very useful in practice:

- If $E^\infty_{p,q} = 0$, for all $p+q=n$, then $H_n = 0$.
- If $H_n = 0$, then $E^\infty_{p,q} = 0$ for all $p+q=n$.

Before explaining in more detail what is behind the theory of spectral sequences, we present the special case of a spectral sequence associated to fibrations, and discuss some immediate applications (including to Hurewicz theorem).

Theorem 2.1.4 (Serre). *If $\pi : E \to B$ is a fibration with fiber F, and with $\pi_1(B) = 0$ and $\pi_0(F) = 0$, then there is a first quadrant spectral sequence with*

$$E^2_{p,q} = H_p(B; H_q(F)) \Rrightarrow H_*(E) \tag{2.1.1}$$

converging to $H_*(E)$.

Remark 2.1.5. Fix some coefficient group $\mathbb{K}$. Then, since B and F are connected, we have:

- $E^2_{p,0} = H_p(B; H_0(F;\mathbb{K})) = H_p(B;\mathbb{K})$,
- $E^2_{0,q} = H_0(B; H_q(F;\mathbb{K})) = H_q(F;\mathbb{K})$

The remaining entries on the E^2-page are computed by the universal coefficient theorem.

Definition 2.1.6. *The spectral sequence of the above theorem shall be referred to as the Leray-Serre spectral sequence of a fibration, and any ring of coefficients can be used.*

Remark 2.1.7. If $\pi_1(B) \neq 0$, then the coefficients $H_q(F)$ on B are acted upon by $\pi_1(B)$, i.e., these coefficients are "twisted" by the monodromy of the fibration if it is not trivial. As we will see later on, in this case the E^2-page of the Leray-Serre spectral sequence is given by

$$E^2_{p,q} = H_p(B; \mathcal{H}_q(F)),$$

i.e., the homology of B with *local coefficients* $\mathcal{H}_q(F)$.

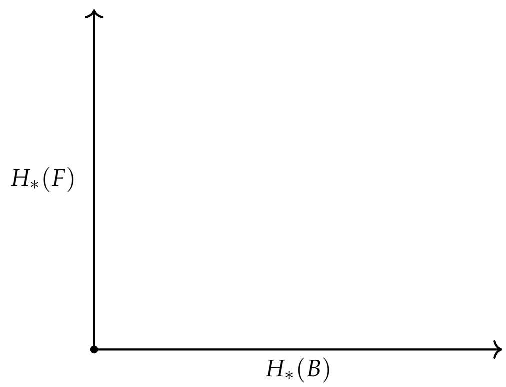


Figure 2.3: p-axis and q-axis of E^2

2.2 *Immediate Applications: Hurewicz Theorem Redux*

As a first application of the Lera-Serre spectral sequence, we can now give a new proof of the Hurewicz Theorem in the absolute case:

Theorem 2.2.1 (Hurewicz Theorem). *If X is $(n-1)$-connected, $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i \leq n-1$ and $\pi_n(X) \cong H_n(X)$.*

Proof. Consider the path fibration:

$$\Omega X \hookrightarrow PX \longrightarrow X, \tag{2.2.1}$$

and recall that the path space PX is contractible. Note that the loop space ΩX is connected, since $\pi_0(\Omega X) \cong \pi_1(X) = 0$. Moreover, since $\pi_1(X) = 0$, the Leray-Serre spectral sequence (2.1.1) for the path fibration has the E^2-page given by

$$E^2_{p,q} = H_p(X, H_q(\Omega X)) \Rightarrow H_*(PX).$$

We prove the statement of the theorem by induction on n. The induction starts at $n = 2$, in which case we clearly have $H_1(X) = 0$ since X is simply-connected. Moreover,

$$\pi_2(X) \cong \pi_1(\Omega X) \cong H_1(\Omega X),$$

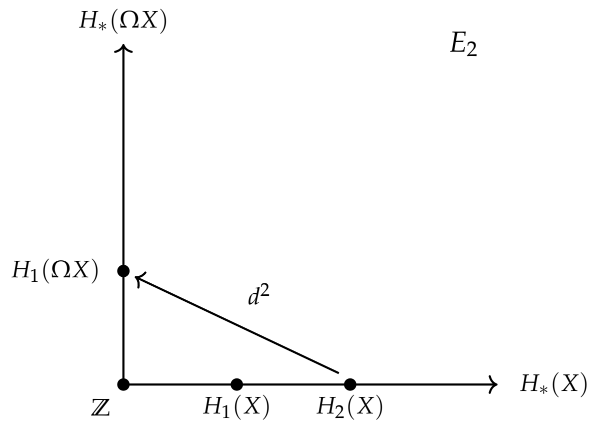
where the first isomorphism follows from the long exact sequence of homotopy groups for the path fibration, and the second isomorphism is the abelianization since $\pi_2(X)$, hence also $\pi_1(\Omega X)$, is abelian. So it remains to show that we have an isomorphism

$$H_1(\Omega X) \cong H_2(X). \tag{2.2.2}$$

Consider the E_2-page of the Leray-Serre spectral sequence for the path fibration. We need to show that

$$d^2 : E^2_{2,0} = H_2(X) \to E^2_{0,1} = H_1(\Omega X)$$

is an isomorphism.



Since $\{E^2_{p,q}\} \Rightarrow H_*(PX)$ and PX is contactible, we have by Remark 2.1.3 that $E^\infty_{p,q} = 0$ for all $p, q > 0$. Hence, if $d^2 : H_2(X) \to H_1(\Omega X)$ is not an isomorphism, then $E^3_{0,1} \neq 0$ and $E^3_{2,0} = \ker d^2 \neq 0$. But the differentials d^3 and higher will not affect $E^3_{0,1}$ and $E^3_{2,0}$. So these groups remain unchanged (hence non-zero) also on E^∞, contradicting the fact that $E^\infty = 0$ except for $(p, q) = (0, 0)$. This proves (2.2.2).

Now assume the statement of the theorem holds for $n-1$ and prove it for n. Since X is $(n-1)$-connected, we have by the homotopy long exact sequence of the path fibration that ΩX is $(n-2)$-connected. So by the induction hypothesis applied to ΩX (assuming now that $n.geq3$, as the case $n = 2$ has been dealt with earlier), we have that $\widetilde{H}_i(\Omega X) = 0$ for $i < n-1$, and $\pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X)$.

Therefore, we have isomorphisms:

$$\pi_n(X) \cong \pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X),$$

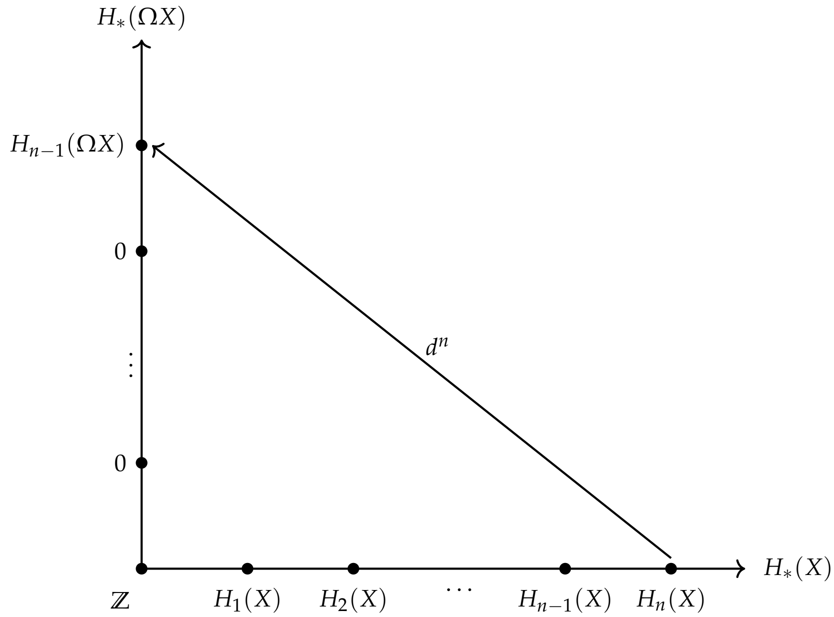
where the first isomorphism follows from the long exact sequence of homotopy groups for the path fibration, and the second is by the induction hypothesis, as already mentioned. So it suffices to show that we have an isomorphism

$$H_{n-1}(\Omega X) \cong H_n(X). \tag{2.2.3}$$

Consider the Leray-Serre spectral sequence for the path fibration. By using the universal coefficient theorem for homology, the terms on the E^2-page are given by

$$\begin{aligned} E^2_{p,q} &= H_p(X, H_q(\Omega X)) \\ &\cong H_p(X) \otimes H_q(\Omega X) \oplus \mathrm{Tor}(H_{p-1}(X), H_q(\Omega X)) \\ &= 0 \end{aligned}$$

for $0 < q < n-1$, by the induction hypothesis for ΩX.



Hence, the differentials $d^2, d^3 \cdots d^{n-1}$ acting on the entries on the p-axis for $p \leq n$, do not affect these entries. The entries $H_n(X)$ and $H_{n-1}(\Omega X)$ are affected only by the differential d^n. Also, higher differentials starting with d^{n+1} do not affect these entries. But since the spectral sequence converges to $H_*(PX)$ with PX contractible, all entries on the E^∞-page (except at the origin) must vanish. In particular, this implies that $H_i(X) = 0$ for $1 \leq i \leq n-1$, and $d^n : H_n(X) \to H_{n-1}(\Omega X)$ must be an isomorphism, thus proving (2.2.3). □

2.3 *Leray-Serre Spectral Sequence*

In this section, we give some more details about the Leray-Serre spectral sequence. We begin with some general considerations about spectral sequences.

Start off with a chain complex C_* with a bounded increasing filtration $F^\bullet C_*$, i.e., each $F^p C_*$ is a subcomplex of C_*, $F^{p-1}C_* \subseteq F^p C_*$ for any p, $F^p C_* = C_*$ for p very large, and $F^p C_* = 0$ for p very small. We get an induced filtration on the homology groups $H_i(C_*)$ by

$$F^p H_i(C_*) := \mathrm{Image}(H_i(F^p C_*) \to H_i(C_*)).$$

The general theory of spectral sequences (e.g., see Hatcher or Griffiths-Harris), asserts that there exists a homological spectral sequence with E^1-page given by:

$$E^1_{p,q} = H_{p+q}(F^p C_* / F^{p-1} C_*) \Rightarrow H_*(C_*)$$

and differential d^1 given by the connecting homomorphism in the long exact sequence of homology groups associated to the triple

$$(F^p C_*, F^{p-1} C_*, F^{p-2} C_*).$$

Moreover, we have

Theorem 2.3.1.

$$E^{\infty}_{p,q} = F^p H_{p+q}(C_*)/F^{p-1} H_{p+q}(C_*)$$

So to reconstruct $H_*(C_*)$ one needs to solve a collection of extension problems.

Back to the Leray-Serre spectral sequence, let $F \hookrightarrow E \stackrel{\pi}{\to} B$ be a fibration with B a simply-connected finite CW-complex. Let $C_*(E)$ be the singular chain complex of E, filtered by

$$F^p C_*(E) := C_*(\pi^{-1}(B_p)),$$

where B_p is the p-skeleton of B. Then,

$$\begin{aligned} F^p C_*(E)/F^{p-1} C_*(E) &= C_*(\pi^{-1}(B_p))/C_*(\pi^{-1}(B_{p-1})) \\ &= C_*(\pi^{-1}(B_p), \pi^{-1}(B_{p-1})). \end{aligned}$$

By excision,

$$H_*(F^p C_*(E)/F^{p-1} C_*(E)) = \bigoplus_{e_p} H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p))$$

where the direct sum is over the p-cells e^p in B. Since e^p is contractible, the fibration above it is trivial, so homotopy equivalent to $e^p \times F$. Thus,

$$\begin{aligned} H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p)) &\cong H_*(e^p \times F, \partial e_p \times F) \\ &\cong H_*(D^p \times F, S^{p-1} \times F) \\ &\cong H_{*-p}(F) \\ &\cong H_p(D^p, S^{p-1}; H_{*-p}(F)), \end{aligned}$$

where the third isomorphism follows by the Künneth formula. Altogether, there is a spectral sequence with E^1-page

$$E^1_{p,q} = H_{p+q}(F^p C_*(E)/F^{p-1} C_*(E)) \cong \bigoplus_{e_p} H_p(D^p, S^{p-1}; H_q(F)).$$

Here, d^1 takes $E^1_{p,q}$ to $\bigoplus_{e_{p-1}} H_{p-1}(D^{p-1}, S^{p-2}; H_q(F))$ by the boundary map of the long exact sequence of the triple (B_p, B_{p-1}, B_{p-2}). By cellular homology, this is exactly a description of the boundary map of the CW-chain complex of B with coefficients in $H_q(F)$, hence

$$E^2_{p,q} = H_p(B, H_q(F)).$$

Remark 2.3.2. If the base B of the fibration is not simply-connected, then the coefficients $H_q(F)$ on B in E^2 are acted upon by $\pi_1(B)$, i.e., these coefficients are "twisted" by the monodromy of the fibration if it is not trivial, so taking the homology of the E^1-page yields

$$E^2_{p,q} = H_p(B; \mathcal{H}_q(F)),$$

regarded now as the homology of B with local coefficients $\mathcal{H}_q(F)$.

The above considerations yield Serre's theorem:

Theorem 2.3.3. *Let $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ be a fibration with $\pi_1(B) = 0$ (or $\pi_1(B)$ acts trivially on $H_*(F)$) and $\pi_0(E) = 0$. Then, there is a first quadrant spectral sequence with E^2-page*

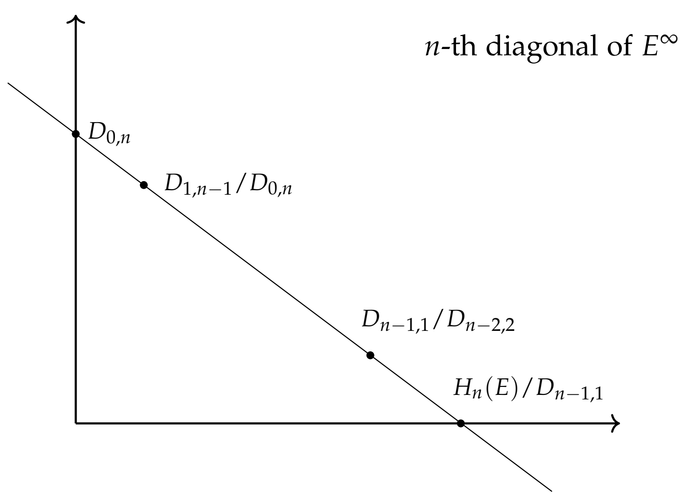
$$E^2_{p,q} = H_p(B, H_q(F))$$

which converges to $H_*(E)$.

Therefore, there exists a filtration

$$H_n(E) = D_{n,0} \supseteq D_{n-1,1} \supseteq \ldots \supseteq D_{0,n} \supseteq D_{-1,n+1} = 0$$

such that $E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}$.



(a) We have the following diagram of groups and homomorphisms:

$$\begin{tikzcd}
H_p(B) = E^2_{p,0} & \supseteq \ker d^2_{p,0} = E^3_{p,0} \supseteq \ker d^3_{p,0} = E^4_{p,0} \supseteq \ldots \supseteq & \ker d^p_{p,0} = E^{p+1}_{p,0} \\
 & & \vdots \arrow[u, "="'] \\
 & & E^\infty_{p,0} \arrow[u, "="'] \\
 & & H_p(E)/D_{p-1,1}H_p(E) \arrow[u, "="'] \\
 & & H_p(E) \arrow[u, two heads, "onto"'] \arrow[uuuull, bend left, "\pi_*"]
\end{tikzcd}$$

Moreover, the above diagram commutes, i.e., the composition

$$H_p(E) \twoheadrightarrow E^\infty_{p,0} \subseteq E^2_{p,0} = H_p(B), \tag{2.3.1}$$

which is also called the *edge homomorphism*, coincides with $\pi_* : H_p(E) \to H_p(B)$.

(b) We have the following diagram of groups and homomorphisms:

$$\begin{tikzcd}
H_q(F) = E^2_{0,q} \arrow[r, two heads] \arrow[ddddr, bend right, "i_*"'] & E^3_{0,q} = H_q(F)/\mathrm{Image}(d^2) \arrow[r, two heads] & \ldots \arrow[r, two heads] & E^{q+2}_{0,q} \arrow[d, "="] \\
 & & & \vdots \arrow[d, "="] \\
 & & & E^\infty_{0,q} \arrow[d, "="] \\
 & & & D_{0,q}H_q(E) \arrow[d, hook] \\
 & & & H_q(E)
\end{tikzcd}$$

Furthermore, this diagram commutes.

(c)

Theorem 2.3.4. *The image of the Hurewicz map $h^n_B : \pi_n(B) \to H_n(B)$ is contained in $E^n_{n,0}$, which is called the group of transgression elements.*

Furthermore, the following diagram commutes:

$$\begin{tikzcd}
\pi_n(B) \arrow[r, "h_B^n"] \arrow[d, "\partial", "\text{l.e.s.}"'] & H_n(B) = E^2_{n,0} \supseteq \dots \supseteq E^n_{n,0} \arrow[d, "d^n"] \\
\pi_{n-1}(F) \arrow[r, "h_F^{n-1}"] & H_{n-1}(F) = E^2_{0,n-1} \twoheadrightarrow \dots \twoheadrightarrow E^n_{0,n-1}
\end{tikzcd}$$

2.4 Hurewicz Theorem, continued

Under the assumptions of the Hurewicz theorem, consider the following transgression diagram of Theorem 2.3.4:

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "h_X^n"] \arrow[d, "\partial", "\cong"'] & H_n(X) = E^2_{n,0} = \dots = E^n_{n,0} \arrow[d, "d^n", "\cong"'] \\
\pi_{n-1}(\Omega X) \arrow[r, "\cong", "h_{\Omega X}^{n-1}"'] & H_{n-1}(\Omega X) = E^2_{0,n-1} = \dots = E^n_{0,n-1}
\end{tikzcd}$$

The Hurewicz homomorphism $h_{\Omega X}^{n-1}$ is an isomorphism by the inductive hypothesis, ∂ is an isomorphism by the homotopy long exact sequence associated to the path fibration for X, and d^n is an isomorphism by the spectral sequence argument used in the proof of the Hurewicz theorem. Therefore, $h_X^n : \pi_n(X) \to H_n(X)$ is an isomorphism since the diagram commutes.

Remark 2.4.1. It can also be shown inductively that under the assumptions of the Hurewicz theorem,

$$h_X^{n+1} : \pi_{n+1}(X) \longrightarrow H_{n+1}(X)$$

is an epimorphism.

In what follows we give more general versions of the Hurewicz theorem. Recall that even if X is a finite CW-complex the homotopy groups $\pi_i(X)$ are not necessarily finitely generated. However, we have the following result:

Theorem 2.4.2 (Serre). *If X is a finite CW-complex with $\pi_1(X) = 0$ (or more generally if X is abelian), then the homotopy groups $\pi_i(X)$ are finitely generated abelian groups for $i \geq 2$.*

Definition 2.4.3. *Let $\mathcal{C}$ be a category of abelian groups which is closed under extension, i.e., whenever*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of abelian groups with two of A, B, C contained in $\mathcal{C}$, then so is the third. A homomorphism $\varphi : A \to B$ is called a

- *monomorphism mod $\mathcal{C}$ if* $\ker\varphi \in \mathcal{C}$;
- *epimorphism mod $\mathcal{C}$ if* $\operatorname{coker}\varphi \in \mathcal{C}$;
- *isomorphism mod $\mathcal{C}$ if* $\ker\varphi, \operatorname{coker}\varphi \in \mathcal{C}$.

Example 2.4.4. Natural examples of categories $\mathcal{C}$ as above include {finite abelian groups}, {finitely generated abelian groups}, as well as $\{p$-groups$\}$.

We then have the following:

Theorem 2.4.5 (Hurewicz mod $\mathcal{C}$)**.** *Given $n \geq 2$, if $\pi_i(X) \in \mathcal{C}$ for $1 \leq i \leq n-1$, then $\widetilde{H}_i(X) \in \mathcal{C}$ for $i \leq n-1$, $h^n_X : \pi_n(X) \to H_n(X)$ is an isomorphism mod $\mathcal{C}$, and $h^{n+1}_X : \pi_{n+1}(X) \to H_{n+1}(X)$ is an epimorphism mod $\mathcal{C}$.*

We need the following easy fact which guarantees that in the Leray-Serre spectral sequence of the path fibration we have $E^n_{p,q} \in \mathcal{C}$.

Lemma 2.4.6. *If $G \in \mathcal{C}$ and X is a finite CW-complex, then $H_i(X;G) \in \mathcal{C}$ for any i. More generally (even if X is not a CW complex), if $A, B \in \mathcal{C}$, then* $\operatorname{Tor}(A,B) \in \mathcal{C}$.

Then the proof of Theorem 2.4.5 is the same as that of the classical Hurewicz theorem, after replacing "$\cong$" by "$\cong$ mod $\mathcal{C}$", and "0" by "$\mathcal{C}$":

$$\begin{tikzcd}
\pi_n(X) \arrow[rr, "h^n_X"] \arrow[d, "\partial", "\cong"'] & & H_n(X) = E^2_{n,0} = \ldots = E^n_{n,0} \arrow[d, "d^n", "\cong \bmod \mathcal{C}"'] \\
\pi_{n-1}(\Omega X) \arrow[r, "\cong \bmod \mathcal{C}", "h^{n-1}_{\Omega X}"'] & H_{n-1}(\Omega X) = E^2_{0,n-1} = \ldots & = E^n_{0,n-1}
\end{tikzcd}$$

Specifically, $h^{n-1}_{\Omega X}$ is an isomorphism mod $\mathcal{C}$ by the inductive hypothesis, ∂ is an isomorphism by the long exact sequence associated to the path fibration, and d^n is an isomorphism mod $\mathcal{C}$ by a spectral sequence argument similar to the one used in the proof of the Hurewicz theorem. Therefore, h^n_X is an isomorphism mod $\mathcal{C}$ since the diagram commutes.

Proof of Serre's Theorem 2.4.2. Let

$$\mathcal{C} = \{\text{finitely generated abelian groups}\}.$$

Then, $\widetilde{H}_i(X) \in \mathcal{C}$ since X is a finite CW-complex. By Theorem 2.4.5, we have $\pi_i(X) \in \mathcal{C}$ for $i \geq 2$. □

As another application, we can now prove the following result:

Theorem 2.4.7. *Let X and Y be any connected spaces and $f : X \to Y$ a weak homotopy equivalence (i.e., f induces isomorphisms on homotopy groups). Then f induces isomorphisms on (co)homology groups with any coefficients.*

Proof. By universal coefficient theorems, it suffices to show that f induces isomorphisms on integral homology. As such, we can assume that f is a fibration, and let F denote its fiber.

Since f is a weak homotopy equivalence, the long exact sequence of the fibration yields that $\pi_i(F) = 0$ for all $i \geq 0$. Hence, by the Hurewicz theorem, $\widetilde{H}_i(F) = 0$, for all $i \geq 0$. Also, $H_0(F) = \mathbb{Z}$, since F is connected.

Consider now the Leray-Serre spectral sequence associated to the fibration f, with E^2-page given by (see Remark 2.1.7):

$$E^2_{p,q} = H_p(Y, \mathcal{H}_q(F)) \Rightarrow H_*(X),$$

where $\mathcal{H}_q(F)$ is a local coefficient system (i.e., locally constant sheaf) on Y with stalk $H_q(F)$. Since F has no homology, except in degree zero (where $\mathcal{H}_0(F) = H_0(F)$ is always the trivial local system when F is path-connected), we get:

$$E^2_{p,q} = 0 \text{ for } q > 0,$$

and

$$E^2_{p,0} = H_p(Y).$$

Therefore, all differentials in the spectral sequence vanish, so

$$E^2 = \cdots = E^\infty.$$

Recall now that

$$H_n(X) = D_{n,0} \supseteq D_{n-1,1} \supseteq \cdots \supseteq 0$$

and $E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}$. So if $q > 0$, then $D_{p,q} = D_{p-1,q+1}$ since $E^\infty_{p,q} = 0$. In particular, $D_{n-1,1} = \cdots = D_{0,n} = D_{-1,n+1} = 0$. Therefore,

$$H_n(X) = E^\infty_{n,0} = E^2_{n,0} = H_n(Y)$$

and, by our remarks on the Leray-Serre spectral sequence (and edge homomorphism), the above composition of isomorphisms coincides with f_*, thus proving the claim. □

2.5 *Gysin and Wang sequences*

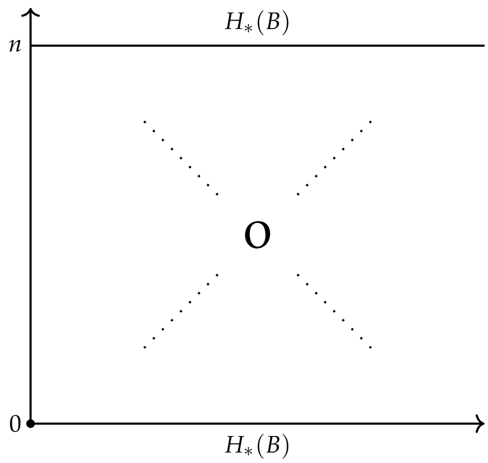
As another application of the Leray-Serre spectral sequence, we discuss the Gysin and Wang sequences.

Theorem 2.5.1 (Gysin sequence)**.** *Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibration, and suppose that F is a homology n-sphere. Assume that $\pi_1(B)$ acts trivially on $H_n(F)$, e.g., $\pi_1(B) = 0$. Then there exists an exact sequence*

$$\cdots \to H_i(E) \xrightarrow{\pi_*} H_i(B) \to H_{i-n-1}(B) \to H_{i-1}(E) \xrightarrow{\pi_*} H_{i-1}(B) \to \cdots$$

Proof. The Leray-Serre spectral sequence of the fibration has

$$E^2_{p,q} = H_p(B; H_q(F)) = \begin{cases} H_p(B) & , q = 0, n \\ 0 & , otherwise. \end{cases}$$



$$E^2 = \cdots = E^{n+1}$$

Thus the only possibly nonzero differentials are:

$$d^{n+1} : E^{n+1}_{p,0} \longrightarrow E^{n+1}_{p-n-1,n}.$$

In particular,

$$E^{n+1}_{p,q} = \cdots = E^2_{p,q}$$

for any (p, q), and

$$E^\infty_{p,q} = \begin{cases} 0 & , q \neq 0, n \\ \ker(d^{n+1} : E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}) & , q = 0 \\ \operatorname{coker}(d^{n+1} : E^{n+1}_{p+n+1,0} \to E^{n+1}_{p-n-1,n}) & , q = n. \end{cases} \tag{2.5.1}$$

The above calculations yield the exact sequences

$$0 \longrightarrow E^\infty_{p,0} \longrightarrow E^{n+1}_{p,0} \xrightarrow{d^{n+1}} E^{n+1}_{p-n-1,n} \longrightarrow E^\infty_{p-n-1,n} \longrightarrow 0.$$

The filtration on $H_i(E)$ reduces to

$$0 \subset E^\infty_{i-n,n} = D_{i-n,n} \subset D_{i,0} = H_i(E)$$

and so the sequences

$$0 \longrightarrow E^\infty_{i-n,n} \longrightarrow H_i(E) \longrightarrow E^\infty_{i,0} \longrightarrow 0 \tag{2.5.2}$$

are exact for each i.

The desired exact sequence follows by combining (2.5.1), (2.5.2) and the edge isomorphism (2.3.1). □

Theorem 2.5.2 (Wang)**.** *If $F \hookrightarrow E \to S^n$ is a fibration, then there is an exact sequence:*

$$\cdots \longrightarrow H_i(F) \longrightarrow H_i(E) \longrightarrow H_{i-n}(F) \longrightarrow H_{i-1}(F) \longrightarrow \cdots$$

Proof. Exercise. □

2.6 *Suspension Theorem for Homotopy Groups of Spheres*

We first need to compute the homology of the loop space ΩS^n for $n > 1$.

Proposition 2.6.1. *If $n > 1$, we have:*

$$H_*(\Omega S^n) = \begin{cases} \mathbb{Z} & , * = a(n-1), a \in \mathbb{N} \\ 0 & , \textit{otherwise} \end{cases}$$

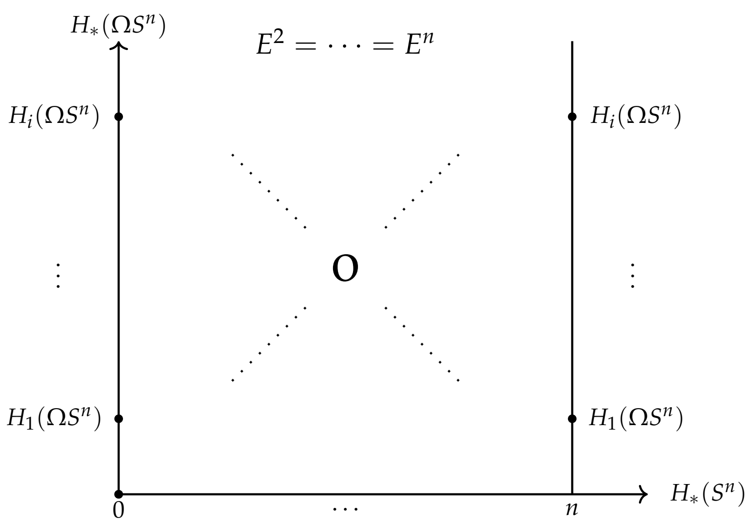
Proof. Consider the Leray-Serre spectral sequence for the path fibration (with $\pi_1(S^n) = \pi_0(\Omega S^n) = 0$)

$$\Omega S^n \hookrightarrow PS^n \simeq * \to S^n,$$

with E^2-page

$$E^2_{p,q} = H_p(S^n; H_q(\Omega S^n)) = \begin{cases} H_q(\Omega S^n) & , p = 0, n \\ 0 & , \text{otherwise} \end{cases}$$

which converges to $H_*(PS^n) = H_*(point)$. In particular, $E^\infty_{p,q} = 0$ for all $(p,q) \neq (0,0)$.



First note that we have $H_0(\Omega S^n) = \mathbb{Z}$ since $\pi_0(\Omega S^n) = \pi_1(S^n) = 0$. Moreover, $H_i(\Omega S^n) = E^2_{0,i} = E^3_{0,i} = E^\infty_{0,i} = 0$ for $0 < i < n-1$, since these entries are not affected by any differential. Furthermore, $d^2 = d^3 = \ldots = d^{n-1} = 0$ since these differential are too short to alter any of the entries they act on. So

$$E^2 = \ldots = E^n.$$

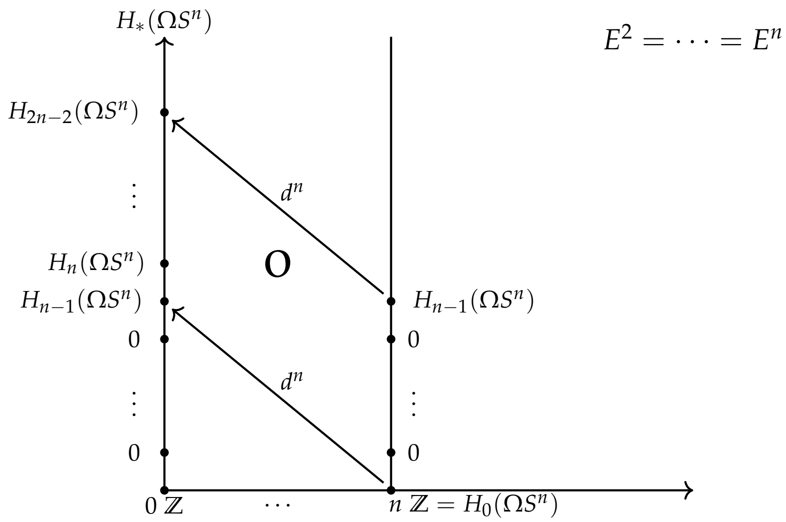
Similarly, we have $d^{n+1} = d^{n+2} = \ldots = 0$, as these differentials are too long, and so

$$E^{n+1} = E^{n+2} = \ldots = E^\infty.$$

Since $E^\infty_{p,q} = 0$ for all $(p,q) \neq (0,0)$, all nonzero entries in E^n (except at the origin) have to be killed in E^{n+1}. In particular,

$$d^n_{n,q} : E^n_{n,q} \longrightarrow E^n_{0,q+n-1}$$

are isomorphisms.



For instance, $d^n : \mathbb{Z} = H_0(\Omega S^n) = E^n_{n,0} \longrightarrow E^n_{0,n-1} = H_{n-1}(\Omega S^n)$ is an isomorphism, hence $H_{n-1}(\Omega S^n) = \mathbb{Z}$. More generally, we get isomorphisms

$$H_q(\Omega S^n) \cong H_{q+n-1}(\Omega S^n)$$

for any $q \geq 0$. Since $H_0(\Omega S^n) \cong \mathbb{Z}$ and $H_i(\Omega S^n) = 0$ for $0 < i < n-1$, this gives:

$$H_*(\Omega S^n) = \begin{cases} \mathbb{Z} & , * = a(n-1), a \in \mathbb{N} \\ 0 & , \text{otherwise} \end{cases}$$

as desired. $\square$

We can now give a new proof of the Suspension Theorem for homotopy groups.

Theorem 2.6.2. *If $n \geq 3$, there are isomorphisms $\pi_i(S^{n-1}) \cong \pi_{i+1}(S^n)$, for $i \leq 2n-4$, and we have an exact sequence:*

$$\mathbb{Z} \to \pi_{2n-3}(S^{n-1}) \to \pi_{2n-2}(S^n) \to 0.$$

Proof. We have $\mathbb{Z} \cong \pi_n(S^n) \cong \pi_{n-1}(\Omega S^n)$. Let $g : S^{n-1} \to \Omega S^n$ be a generator of $\pi_{n-1}(\Omega S^n)$. First, we claim that

$$g_* \text{ is an isomorphism on } H_i(-) \text{ for all } i < 2n-2.$$

This is clear if $i = 0$, since ΩS^n is connected. Given our calculation for $H_i(\Omega S^n)$ in Proposition 2.6.1, it suffices to prove the claim for $i = n-1$. We have a commutative diagram:

$$\begin{tikzcd}
g_* : H_{n-1}(S^{n-1}) \arrow[r] & H_{n-1}(\Omega S^n) \\
\pi_{n-1}(S^{n-1}) \arrow[u, "h"] \arrow[r, "g_*"] \arrow[ur, phantom, "\circlearrowright"] & \pi_{n-1}(\Omega S^n) \arrow[u, "h"'] \\
{[id]} \arrow[r, phantom, "\mapsto"] & {[g \circ id] = [g]}
\end{tikzcd}$$

where h is the Hurewicz map. The bottom arrow g_* is an isomorphism on π_{n-1} by our choice of g. The two vertical arrows are isomorphisms by the Hurewicz theorem (recall that $n \geq 3$, so both S^{n-1} and ΩS^n are simply-connected). By the commutativity of the diagram we get the isomorphism on the top horizontal arrow, thus proving the claim.

Since we deal only with homotopy and homology groups, we can moreover assume that g is an inclusion. Then the homology long exact sequence for the pair $(\Omega S^n, S^{n-1})$ reads as:

$$\cdots \to H_i(S^{n-1}) \xrightarrow{g_*} H_i(\Omega S^n) \to H_i(\Omega S^n, S^{n-1}) \to$$
$$\to H_{i-1}(S^{n-1}) \xrightarrow{g_*} H_{i-1}(\Omega S^n) \to \cdots$$

From the above claim, we obtain that $H_i(\Omega S^n, S^{n-1}) = 0$, for $i < 2n-2$, together with the exact sequence

$$0 \to \mathbb{Z} = H_{2n-2}(\Omega S^n) \xrightarrow{\cong} H_{2n-2}(\Omega S^n, S^{n-1}) \to 0$$

Since S^{n-1} is simply-connected (as $n-1 \geq 2$), by the relative Hurewicz theorem, we get that $\pi_i(\Omega S^n, S^{n-1}) = 0$ for $i < 2n-2$, and

$$\pi_{2n-2}(\Omega S^n, S^{n-1}) \cong H_{2n-2}(\Omega S^n, S^{n-1}) \cong \mathbb{Z}.$$

From the homotopy long exact sequence of the pair $(\Omega S^n, S^{n-1})$, we then get $\pi_i(\Omega S^n) \cong \pi_i(S^{n-1})$ for $i < 2n-3$ and the exact sequence

$$\cdots \to \mathbb{Z} \to \pi_{2n-3}(S^{n-1}) \to \pi_{2n-3}(\Omega S^n) \to 0$$

Finally, using the fact that $\pi_i(\Omega S^n) \cong \pi_{i+1}(S^n)$, we get the desired result. $\square$

By taking $i = 4$ and $n = 4$, we get the first isomorphism in the following:

Corollary 2.6.3. $\pi_4(S^3) \cong \pi_5(S^4) \cong \ldots \cong \pi_{n+1}(S^n)$

2.7 Cohomology Spectral Sequences

Let us now turn our attention to spectral sequences computing cohomology. In the case of a fibration, we have the following *Leray-Serre cohomology spectral sequence*:

Theorem 2.7.1 (Serre). *Let $F \hookrightarrow E \to B$ be a fibration, with $\pi_1(B) = 0$ (or $\pi_1(B)$ acting trivially on fiber cohomology) and $\pi_0(F) = 0$. Then there exists a cohomology spectral sequence with E_2-page*

$$E_2^{p,q} = H^p(B, H^q(F))$$

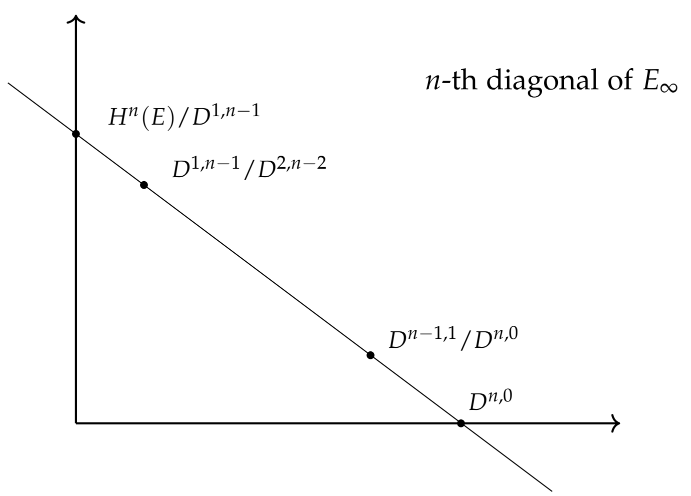
converging to $H^*(E)$. This means that, for each n, $H^n(E)$ admits a filtration

$$H^n(E) = D^{0,n} \supseteq D^{1,n-1} \supseteq \ldots \supseteq D^{n,0} \supseteq D^{n+1,-1} = 0$$

so that

$$E_\infty^{p,q} = D^{p,q} \big/ D^{p+1,q-1}.$$

Moreover, the differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfies $(d_r)^2 = 0$, and $E_{r+1} = H^*(E_r, d_r)$.



The corresponding statements analogous to those of Remarks 2.1.3 and 2.1.5 also apply to the spectral sequence of Theorem 2.7.1.

The Leray-Serre cohomology spectral sequence comes endowed with the structure of a *product* on each page E_r, which is induced from a product on E_2, i.e., there is a map

$$\bullet : E_r^{p,q} \times E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$$

satisfying the Leibnitz condition

$$d_r(x \bullet y) = d_r(x) \bullet y + (-1)^{\deg(x)} x \bullet d_r(y)$$

where $\deg(x) = p + q$. On the E_2-page this product is the cup product induced from

$$\begin{array}{rcl} H^p(B, H^q(F)) \times H^{p'}(B, H^{q'}(F)) & \longrightarrow & H^{p+p'}(B, H^{q+q'}(F)) \\ m \cdot \gamma \times n \cdot \nu & \mapsto & (m \cup n) \cdot (\gamma \cup \nu) \end{array}$$

with $m \in H^q(F)$, $n \in H^{q'}(F)$, $\gamma \in C^p(B)$ and $\nu \in C^{p'}(B)$, so that $m \cup n \in H^{q+q'}(F)$ and $\gamma \cup \nu \in C^{p+p'}(B)$.

As it is the case for homology, the cohomology Leray-Serre spectral sequence satisfies the following property:

Theorem 2.7.2. *Given a fibration $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ with F connected and $\pi_1(B) = 0$ (or $\pi_1(B)$ acts trivially on the fiber cohomology), the compositions*

$$H^q(B) = E_2^{q,0} \twoheadrightarrow E_3^{q,0} \twoheadrightarrow \cdots \twoheadrightarrow E_q^{q,0} \twoheadrightarrow E_{q+1}^{q,0} = E_\infty^{q,0} \subset H^q(E) \tag{2.7.1}$$

and

$$H^q(E) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \subset E_q^{0,q} \subset \cdots \subset E_2^{0,q} = H^q(F) \tag{2.7.2}$$

are the homomorphisms $\pi^* : H^q(B) \to H^q(E)$ and $i^* : H^q(E) \to H^q(F)$, respectively.

Recall that for a space of finite type, the (co)homology groups are finitely generated. By using the universal coefficient theorem in cohomology, we have the following useful result:

Proposition 2.7.3. *Suppose that $F \hookrightarrow E \to B$ is a fibration with F connected and assume that $\pi_1(B) = 0$ (or $\pi_1(B)$ acts trivially on the fiber cohomology). If B and F are spaces of finite type (e.g., finite CW complexes), then for a field $\mathbb{K}$ of coefficients we have:*

$$E_2^{p,q} = H^p(B; \mathbb{K}) \otimes_{\mathbb{K}} H^q(F; \mathbb{K}).$$

Sufficient conditions for the cohomology of the total space of a fibration to be the tensor product of the cohomology of the fiber and that of the base space are given by the following result.

Theorem 2.7.4 (Leray-Hirsch)**.** *Suppose $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ is a fibration, with B and F of finite type, $\pi_1(B) = 0$ and $\pi_0(F) = 0$, and let $\mathbb{K}$ be a field of coefficients. Assume that $i^* \colon H^*(E; \mathbb{K}) \to H^*(F; \mathbb{K})$ is onto. Then*

$$H^*(E; \mathbb{K}) \cong H^*(B; \mathbb{K}) \otimes_{\mathbb{K}} H^*(F; \mathbb{K}).$$

Proof. Consider the Leray-Serre cohomology spectral sequence

$$E_2^{p,q} = H^p(B; H^q(F; \mathbb{K})) \Rrightarrow H^*(E; \mathbb{K})$$

of the fibration $F \hookrightarrow E \to B$. By Proposition 2.7.3, we have:

$$E_2^{p,q} = H^p(B;\mathbb{K}) \otimes_{\mathbb{K}} H^q(F;\mathbb{K}).$$

In order to prove the theorem, it suffices to show that

$$E_2 = \cdots = E_\infty,$$

i.e., that all differentials d_2, d_3, etc., vanish. Indeed, since we work with field coefficients, all extension problems encountered in passing from E_∞ to $H^*(E;\mathbb{K})$ are trivial, i.e.,

$$H^n(E;\mathbb{K}) \cong \bigoplus_{p+q=n} E_\infty^{p,q}.$$

Recall from Theorem 2.7.2 that the composite

$$H^q(E;\mathbb{K}) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \subset E_q^{0,q} \subset \cdots \subset E_2^{0,q} = H^q(F;\mathbb{K})$$

is the homomorphism $i^* : H^q(E;\mathbb{K}) \to H^q(F;\mathbb{K})$. Since i^* is assumed onto, all these inclusions must be equalities. So all d_r, when restricted to the q-axis, must vanish. On the other hand, at E_2 we have

$$E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q} \tag{2.7.3}$$

since $\mathbb{K}$ is a field, and d_2 is already zero on $E_2^{p,0}$ since we work with a first quadrant spectral sequence. Since d_2 is a derivation with respect to (2.7.3), we conclude that $d_2 = 0$ and $E_3 = E_2$. The same argument applies to d_3 and, continuing in this fashion, we see that the spectral sequence collapses (degenerates) at E_2, as desired. □

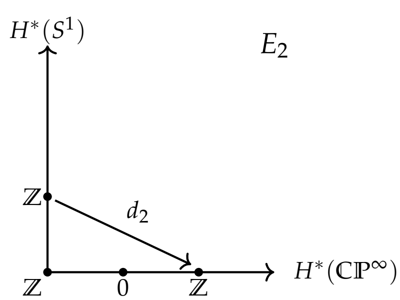
2.8 *Elementary computations*

Example 2.8.1. As a first example of the use of the Leray-Serre cohomology spectral sequence, we compute here the cohomology ring $H^*(\mathbb{C}P^\infty)$ of $\mathbb{C}P^\infty$.

Consider the fibration

$$S^1 \hookrightarrow S^\infty \simeq * \to \mathbb{C}P^\infty.$$

The E_2-page of the associated Leray-Serre cohomology spectral sequence starts with:



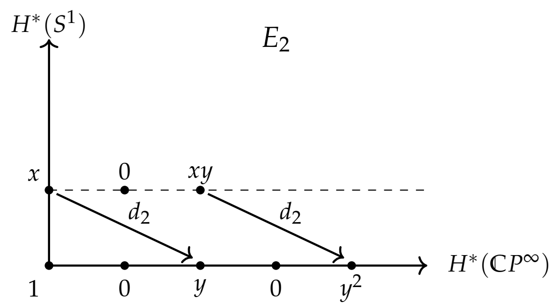
Here, $H^1(\mathbb{C}P^\infty) = E_2^{1,0} = 0$ since it is not affected by any differential d_r, and the E_∞-page has only zero entries except at the origin. Moreover, since the cohomology of the fiber is torsion-free, we get by the universal coefficient theorem in cohomology that

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty, H^q(S^1)) = H^p(\mathbb{C}P^\infty) \otimes H^q(S^1).$$

In particular, we have $E_2^{1,1} = 0$ and $E_2^{0,1} = H^1(S^1) = \mathbb{Z}$.

Since S^∞ has no positive cohomology, hence the E_∞-page has only zero entries except at the origin, it is easy to see that $d_2 : E_2^{0,1} \to E_2^{2,0}$ has to be an isomorphism, since these entries are not affected by any other differential. Hence we have $H^2(\mathbb{C}P^\infty) = E_2^{2,0} \cong \mathbb{Z}$. Since all entries on the E_2-page are concentrated at $q = 0$ and $q = 1$, the only differential which can affect these entries is d_2. A similar argument then shows that $d_2 : E_2^{p,1} \to E_2^{p+2,0}$ is an isomorphism for any $p \geq 0$. This yields that $H^{even}(\mathbb{C}P^\infty) = \mathbb{Z}$ and $H^{odd}(\mathbb{C}P^\infty) = 0$.

Let $\mathbb{Z} = \langle x \rangle = H^1(S^1)$. Let $y = d_2(x)$ be a generator of $H^2(\mathbb{C}P^\infty)$.



Then, after noting that $xy = (1 \otimes x)(y \otimes 1)$ is a generator of $\mathbb{Z} = E_2^{2,1}$, we have:

$$d_2(xy) = d_2(x)y + (-1)^{\deg(x)} x d_2(y) = y^2,$$

Therefore, $H^4(\mathbb{C}P^\infty) = \mathbb{Z} = \langle y^2 \rangle$, since the d_2 that hits y^2 is an isomorphism. By induction, we get that $d_2(xy^{n-1}) = y^n$ is a generator of $H^{2n}(\mathbb{C}P^\infty)$. Altogether, $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y]$, with $\deg(y) = 2$.

Example 2.8.2 (Cohomology groups of lens spaces)**.** In this example we compute the cohomology groups of lens spaces. Let us first recall the relevant definitions.

Assume $n \geq 1$. Consider the scaling action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$, and the induced S^1-action on S^{2n+1}. By identifying $\mathbb{Z}/r$ with the group of r^{th} roots of unity in $\mathbb{C}^*$, we get (by restriction) an action of $\mathbb{Z}/r$ on S^{2n+1}. The quotient

$$L(n, r) := {}^{S^{2n+1}}\!\big/_{\mathbb{Z}/r}$$

is called a *lens space*.

The action of $\mathbb{Z}/r$ on S^{2n+1} is clearly free, so the quotient map $S^{2n+1} \to L(n,r)$ is a covering map with deck group $\mathbb{Z}/r$. Since S^{2n+1} is simply-connected, it is the universal cover of $L(n,r)$. This yields that $\pi_1(L(n,r)) = \mathbb{Z}/r$ and all higher homotopy groups of $L(n,r)$ agree with those of the sphere S^{2n+1}.

By a telescoping construction, which amounts to letting $n \to \infty$, we get a covering map $S^\infty \to L(\infty,r) := {}^{S^\infty}\!/_{\mathbb{Z}/r}$ with contractible total space. In particular,

$$L(\infty,r) = K(\mathbb{Z}/r,1).$$

To compute the cohomology of $L(n,r)$, one may be tempted to use the Leray-Serre spectral sequence for the covering map $\mathbb{Z}/r \hookrightarrow S^{2n+1} \to L(n,r)$. However, since $L(n,r)$ is not simply-connected, computations may be tedious. Instead, we consider the fibration

$$S^1 \hookrightarrow L(n,r) \to \mathbb{C}P^n \tag{2.8.1}$$

whose base space is simply-connected. This fibration is obtained by noting that the action of S^1 on S^{2n+1} descends to an action of $S^1 = S^1/(\mathbb{Z}/r)$ on $L(n,r)$, with orbit space $\mathbb{C}P^n$.

Consider now the Leray-Serre cohomology spectral sequence for the fibration (2.8.1):

$$E_2^{p,q} = H^p(\mathbb{C}P^n, H^q(S^1;\mathbb{Z})) \Rightarrow H^{p+q}(L(n,r);\mathbb{Z})$$

and note that $E_2^{p,q} = 0$ for $q \neq 0,1$. This implies that all differentials d_3 and higher vanish, so

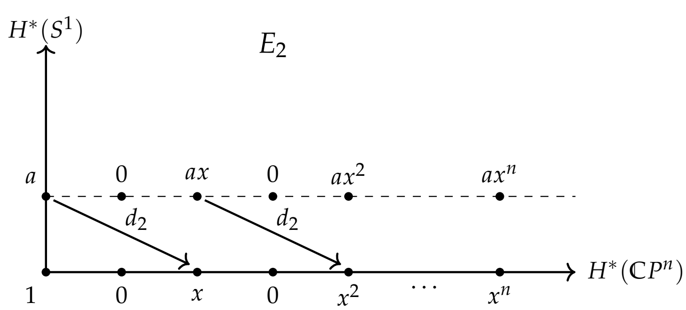
$$E_3 = \cdots = E_\infty.$$

On the E_2-page, we have by the universal coefficient theorem in cohomology that:

$$E_2^{p,q} = H^p(\mathbb{C}P^n;\mathbb{Z}) \otimes H^q(S^1;\mathbb{Z}).$$

Let a be a generator of $\mathbb{Z} = E_2^{0,1} \cong H^1(S^1;\mathbb{Z})$, and let x be a generator of $\mathbb{Z} = E_2^{2,0} \cong H^2(\mathbb{C}P^n;\mathbb{Z})$. We claim that

$$d_2(a) = rx. \tag{2.8.2}$$



To find d_2, it suffices to compute $H^2(L(n,r);\mathbb{Z})$. Indeed, by looking at the entries of the second diagonal of $E_\infty = \cdots = E_3$, we have: $H^2(L(n,r);\mathbb{Z}) = D^{0,2}$, $E_\infty^{0,2} = {D^{0,2}}/{D^{1,1}} = 0$, $E_\infty^{1,1} = {D^{1,1}}/{D^{2,0}} = 0$, and $E_\infty^{2,0} = D^{2,0} = {\mathbb{Z}}/{\mathrm{Image}(d_2)}$. In particular,

$$H^2(L(n,r);\mathbb{Z}) = D^{0,2} = D^{1,1} = D^{2,0} = {\mathbb{Z}}/{\mathrm{Image}(d_2)}. \tag{2.8.3}$$

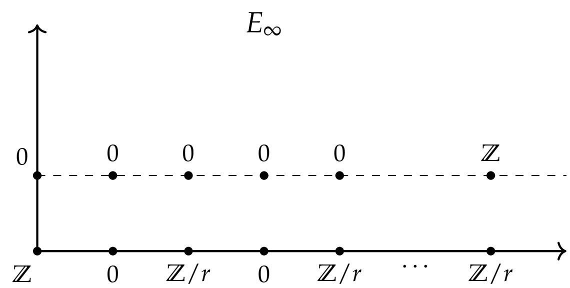
On the other hand, since $H_1(L(n,r);\mathbb{Z}) = \pi_1(L(n,r)) = \mathbb{Z}/r$, we get by the universal coefficient theorem that

$$H^2(L(n,r);\mathbb{Z}) = (\text{free part}) \oplus \mathbb{Z}/r. \tag{2.8.4}$$

By comparing (2.8.3) and (2.8.4), we conclude that $d_2(a) = rx$ and $H^2(L(n,r);\mathbb{Z}) = \mathbb{Z}/r$.

By using the Künneth formula and the ring structure of $H^*(\mathbb{C}P^n;\mathbb{Z})$, it follows from the Leibnitz formula and induction that $d_2(ax^{k-1}) = rx^k$ for $1 \leq k \leq n$, and we also have $d_2(ax^n) = 0$. In particular, all the nontrivial differentials labelled by d_2 are given by multiplication by r.

Since multiplication by r is injective, the $E_3 = \cdots = E_\infty$-page is given by



The extension problems for going from E_∞ to the cohomology of the total space $L(n,r)$ are in this case trivial, since every diagonal of E_∞ contains at most one nontrivial entry. We conclude that

$$H^i(L(n,r);\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/r & i = 2,4,\cdots,2n \\ \mathbb{Z} & i = 2n+1 \\ 0 & \textit{otherwise.} \end{cases}$$

By letting $n \to \infty$, we obtain similarly that

$$H^i(K(\mathbb{Z}/r,1);\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/r & i = 2k, k \geq 1 \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, if $r = 2$, this computes the cohomology of $\mathbb{R}P^\infty$.

2.9 Computation of $\pi_{n+1}(S^n)$

In this section we prove the following result:

Theorem 2.9.1. *If $n \geq 3$,*

$$\pi_{n+1}(S^n) = \mathbb{Z}/2.$$

Theorem 2.9.1 follows from the Suspension Theorem (see Corollary 2.6.3), together with the following explicit calculation:

Theorem 2.9.2.

$$\pi_4(S^3) = \mathbb{Z}/2.$$

The proof of Theorem 2.9.2 given here uses the Postnikov tower approximation of S^3, whose construction we recall here. (A different proof of this fact will be given in the next section, by using Whitehead towers.)

Lemma 2.9.3 (Postnikov approximation)**.** *Let X be a CW complex with $\pi_k := \pi_k(X)$. For any n, there is a sequence of fibrations*

$$K(\pi_k, k) \hookrightarrow Y_k \to Y_{k-1}$$

and maps $X \to Y_k$ with a commuting diagram

$$\begin{tikzcd}
Y_1 & Y_2 \arrow[l] & \cdots Y_{n-1} \arrow[l] & Y_n \arrow[l] \\
 & & & X \arrow[u] \arrow[ul] \arrow[ulll] \arrow[ull]
\end{tikzcd}$$

such that $X \to Y_k$ induces isomorphisms $\pi_i(X) \cong \pi_i(Y_k)$ for $i \leq k$, and $\pi_i(Y_k) = 0$ for $i > k$.

Proof. To construct Y_n we kill off the homotopy groups of X in degrees $\geq n+1$ by attaching cells of dimension $\geq n+2$. We then have $\pi_i(Y_n) = \pi_i(X)$ for $i \leq n$ and $\pi_i(Y_n) = 0$ if $i > n$. Having constructed Y_n, the space Y_{n-1} is obtained from Y_n by killing the homotopy groups of Y_n in degrees $\geq n$, which is done by attaching cells of dimension $\geq n+1$. Repeating this procedure, we get inclusions

$$X \subset Y_n \subset Y_{n-1} \subset \cdots \subset Y_1 = K(\pi_1, 1),$$

which we convert to fibrations. From the homotopy long exact sequence for each of these fibrations, we see that the fiber of $Y_k \to Y_{k-1}$ is a $K(\pi_k, k)$-space. $\square$

Proof of Theorem 2.9.2. We consider the Postnikov tower construction in the case $n = 4$, $X = S^3$, to obtain a fibration

$$K(\pi_4, 4) \hookrightarrow Y_4 \to Y_3 = K(\mathbb{Z}, 3), \tag{2.9.1}$$

where $\pi_4 = \pi_4(S^3) = \pi_4(Y_4)$. Here, $Y_3 = K(\mathbb{Z}, 3)$ since to get Y_3 we kill off all higher homotopy groups of S^3 starting at π_4. Since Y^4 is obtained from S^3 by attaching cells of dimension ≥ 6, it doesn't have cells of dimensions 4 and 5, thus

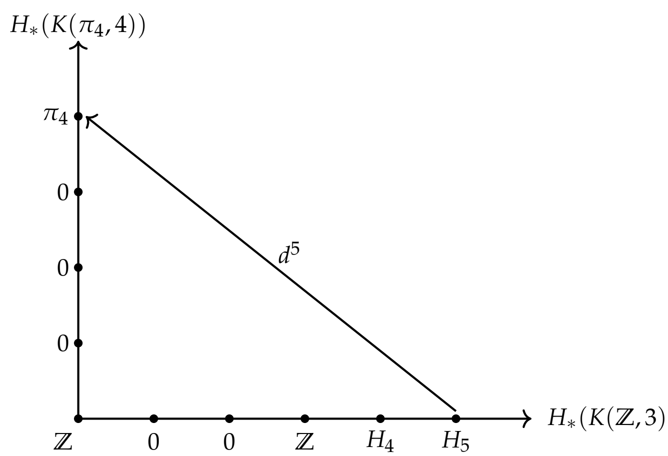
$$H_4(Y_4) = H_5(Y_4) = 0.$$

Let us now consider the homology spectral sequence for the fibration (2.9.1). By the Hurewicz theorem,

$$H_p(K(\mathbb{Z}, 3); \mathbb{Z}) = \begin{cases} 0 & p = 1, 2 \\ \mathbb{Z} & p = 3 \end{cases}$$

$$H_q(K(\pi_4, 4); \mathbb{Z}) = \begin{cases} 0 & q = 1, 2, 3 \\ \pi_4(S^3) & q = 4. \end{cases}$$

So the E^2-page looks like



Since $H_4(Y_4) = 0 = H_5(Y_4)$, all entries on the fourth and fifth diagonals of E^∞ are zero. The only differential that can affect $\pi_4(S^3) = E^2_{0,4} = \cdots = E^5_{0,4}$ is

$$d^5 : H_5(K(\mathbb{Z}, 3), \mathbb{Z}) \longrightarrow \pi_4(S^3),$$

and by the previous remark, this map has to be an isomorphism (note also that $E^2_{5,0} = H_5(K(\mathbb{Z}, 3), \mathbb{Z})$ can be affected only by d^5, and this element too has to be killed at E^∞). Hence

$$\pi_4(S^3) \cong H_5(K(\mathbb{Z}, 3), \mathbb{Z}). \tag{2.9.2}$$

In order to compute $H_5(K(\mathbb{Z}, 3), \mathbb{Z})$, we use the cohomology Leray-Serre spectral squence associated to the path fibration for $K(\mathbb{Z}, 3)$, namely

$$\Omega K(\mathbb{Z}, 3) \hookrightarrow PK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3),$$

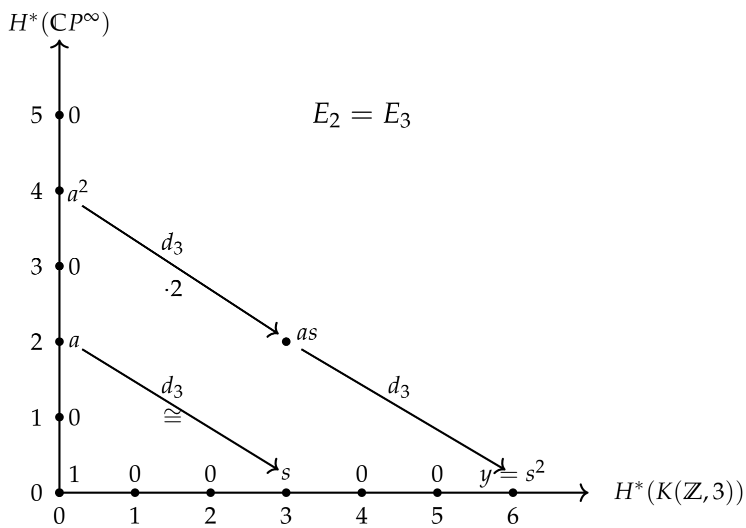
and note that, since $PK(\mathbb{Z},3)$ is contractible, we have $\pi_i(\Omega K(\mathbb{Z},3)) \cong \pi_{i+1}(K(\mathbb{Z},3))$, i.e., $\Omega K(\mathbb{Z},3) \simeq K(\mathbb{Z},2) = \mathbb{C}P^\infty$. Since each $H^j(\mathbb{C}P^\infty)$ is a finitely generated free abelian group, the universal coefficient theorem yields that

$$E_2^{p,q} = H^p(K(\mathbb{Z},3); H^q(\mathbb{C}P^\infty)) \cong H^p(K(\mathbb{Z},3)) \otimes H^q(\mathbb{C}P^\infty), \quad (2.9.3)$$

and the product structure on E_2 is that of the tensor product of $H^*(K(\mathbb{Z},3))$ and $H^*(\mathbb{C}P^\infty)$.

Since $E_2^{p,q} = 0$ for q odd, we have $d_2 = 0$, so $E_2 = E_3$. Similarly, all the even differentials d_{2n} are zero, so $E_{2n} = E_{2n+1}$, for all $n \geq 1$. Since the total space of the fibration is contractible, we have that $E_\infty^{p,q} = 0$ for all $(p,q) \neq (0,0)$, so every non-zero entry on the E_2-page (except at the origin) must be killed on subsequent pages.

Let $a \in H^2(\mathbb{C}P^\infty) \cong \mathbb{Z}$ be a generator. So a^k is a generator of $H^{2k}(\mathbb{C}P^\infty) = E_2^{0,2k}$, for any $k \geq 1$. We *create* elements on $E_2^{*,0}$, which will sooner or later kill off all the non-zero elements in the spectral sequence.



Note that $E_3^{1,0} = E_2^{1,0} = H^1(K(\mathbb{Z},3))$ is never touched by any differential, so

$$H^1(K(\mathbb{Z},3)) = E_\infty^{1,0} = 0.$$

Moreover, since $d_2 = 0$, we also have that

$$H^2(K(\mathbb{Z},3)) = E_2^{2,0} = E_3^{2,0} = E_\infty^{2,0} = 0.$$

The only differential that can affect $\langle a \rangle = E_2^{0,2} = E_3^{0,2}$ is $d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}$, so there must be an element $s \in E_3^{3,0}$ that kills off a, i.e., $d_3(a) = s$. On the other hand, since $E_3^{3,0}$ is only affected by d_3 and it must be killed off at infinity, we must have that $d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}$ is an isomorphism, so s generates

$$\mathbb{Z} = E_3^{3,0} = E_2^{3,0} = H^3(K(\mathbb{Z}, 3)).$$

By (2.9.3), we also have that $E_3^{3,2} = E_2^{3,2} = \mathbb{Z}$, generated by as. Note that

$$d_3(a^2) = 2ad_3(a) = 2as,$$

so $d_3^{0,4} : E_3^{0,4} \to E_3^{3,2}$ is given by multiplication by 2. In particular, $E_4^{0,4} = 0$. Next notice that $H^4(K(\mathbb{Z}, 3)) = E_3^{4,0}$ and $H^5(K(\mathbb{Z}, 3)) = E_3^{5,0}$ can only be touched by the differentials d_3, d_4, or d_5, but all of these are trivial maps because their domains are zero. Thus, as $H^4(K(\mathbb{Z}, 3))$ and $H^5(K(\mathbb{Z}, 3))$ can not killed by any differential, we have

$$H^4(K(\mathbb{Z}, 3)) = H^5(K(\mathbb{Z}, 3)) = 0.$$

Similarly, $H^6(K(\mathbb{Z}, 3)) = E_3^{6,0}$ and $\langle as \rangle = E_3^{3,2}$ are only affected by d_3. Since $d_3(a^2) = 2as$, we have $\ker(d_3 \colon \langle as \rangle = E_3^{3,2} \to E_3^{6,0}) = \mathrm{Image}(d_3 \colon E_3^{0,4} \to E_3^{3,2} = \langle as \rangle) = \langle 2as \rangle \subseteq \langle as \rangle$, and hence $H^6(K(\mathbb{Z}, 3)) = \mathrm{Image}(d_3 \colon E_3^{3,2} \to E_3^{6,0}) \cong \langle as \rangle / \langle 2as \rangle = \mathbb{Z}/2$.

In view of the above calculations, we get by the universal coefficient theorem that

$$H_5(K(\mathbb{Z}, 3)) = \mathbb{Z}/2. \tag{2.9.4}$$

The assertion of the theorem then follows by combining (2.9.2) and (2.9.4). □

Corollary 2.9.4.

$$\pi_4(S^2) = \mathbb{Z}/2.$$

Proof. This follows from Theorem 2.9.2 and the long exact sequence of homotopy groups for the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. □

2.10 *Whitehead tower approximation and $\pi_5(S^3)$*

In order to compute $\pi_5(S^3)$ we make use of the Whitehead tower approximation. We recall here the construction.

Whitehead tower

Let X be a connected CW complex, with $\pi_q = \pi_q(X)$ for any $q \geq 0$.

Definition 2.10.1. *A Whitehead tower of X is a sequence of fibrations*

$$\cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \to X_0 = X$$

such that

(a) X_n is n-connected

(b) $\pi_q(X_n) = \pi_q(X)$ *for* $q \geq n+1$

(c) the fiber of $X_n \to X_{n-1}$ *is a* $K(\pi_n, n-1)$*-space.*

Lemma 2.10.2. *For X a CW complex, Whitehead towers exist.*

Proof. We construct X_n inductively. Suppose that X_{n-1} has already been defined. Add cells to X_{n-1} to kill off $\pi_q(X_{n-1})$ for $q \geq n+1$. So we get a space Y which, by construction, is a $K(\pi_n, n)$-space. Now define the space

$$X_n := P_* X_{n-1} := \{f : I \to Y, f(0) = *, f(1) \in X_{n-1}\}$$

consisting of of paths in Y beginning at a basepoint $* \in X_{n-1}$ and ending somewhere in X_{n-1}. Endow X_n with the compact-open topology. As in the case of the path fibration, the map $\pi : X_n \to X_{n-1}$ defined by $\gamma \to \gamma(1)$ is a fibration with fiber $\Omega Y = K(\pi_n, n-1)$.

From the long exact sequence of homotopy groups associated to the fibration

$$K(\pi_n, n-1) \hookrightarrow X_n \to X_{n-1}$$

we get that $\pi_q(X_n) = \pi_q(X_{n-1})$ for $q \geq n+1$, and $\pi_q(X_n) = 0$ for $q \leq n-2$. Furthermore, the sequence

$$0 \longrightarrow \pi_n(X_n) \longrightarrow \pi_n(X_{n-1}) \longrightarrow \pi_{n-1}(K(\pi_n, n-1)) \longrightarrow \pi_{n-1}(X_n) \longrightarrow 0$$

is exact. So we are done if we show that the boundary homomorphism $\partial : \pi_n(X_{n-1}) \longrightarrow \pi_{n-1}(K(\pi_n, n-1))$ of the long exact sequence is an isomorphism. For this, note that the inclusion $X_{n-1} \subset Y = K(\pi_n, n) = X_{n-1} \cup \{\text{cells of dimension } \geq \text{n}+2\}$ induces an isomorphism $\pi_n(X_{n-1}) \cong \pi_n K(\pi_n, n) \cong \pi_{n-1}(K(\pi_n, n-1))$, which is precisely the above boundary map ∂. □

Calculation of $\pi_4(S^3)$ and $\pi_5(S^3)$

In this section we use the Whitehead tower for $X = S^3$ to compute $\pi_5(S^3)$.

Theorem 2.10.3.

$$\pi_5(S^3) \cong \mathbb{Z}/2.$$

Proof. Consider the Whitehead tower for $X = S^3$. Since S^3 is 2-connected, we have in the notation of Definition 2.10.1 that $X = X_1 = X_2$. Let $\pi_i := \pi_i(S^3)$, for any $i \geq 0$. We have fibrations

$$\begin{tikzcd} K(\pi_4, 3) \arrow[r] & X_4 \arrow[d] \\ K(\pi_3, 2) \arrow[r] & X_3 \arrow[d] \\ & S^3 \end{tikzcd}$$

Since $\pi_3 = \mathbb{Z}$, we have $K(\pi_3, 2) = \mathbb{C}P^\infty$. Moreover, since X_4 is 4-connected, we get by definition and Hurewicz that

$$\pi_5(S^3) \cong \pi_5(X_4) \cong H_5(X_4).$$

Similarly,

$$\pi_4(S^3) \cong \pi_4(X_3) \cong H_4(X_3).$$

Once again we are reduced to computing homology groups. Using the universal coefficient theorem, we will deduce the homology groups from cohomology.

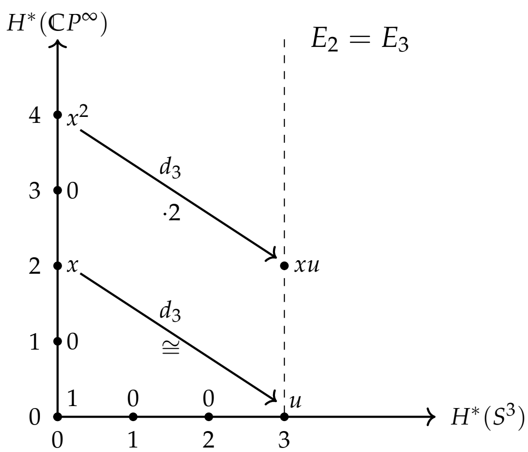
Consider now the cohomology spectral sequence for the fibration

$$\mathbb{C}P^\infty \hookrightarrow X_3 \to S^3.$$

The E_2-page is given by

$$E_2^{p,q} = H^p(S^3, H^q(\mathbb{C}P^\infty, \mathbb{Z})) = H^p(S^3) \otimes H^q(\mathbb{C}P^\infty) \Rightarrow H^*(X_3).$$

In particular, $E_2^{p,q} = 0$ unless $p = 0, 3$ and q is even.



Since $E_2^{p,q} = 0$ for q odd, we have $d_2 = 0$, so $E_2 = E_3$. In addition, for $r \geq 4$, $d_r = 0$. So $E_4 = E_\infty$.

Since X_3 is 3-connected, we have by Hurewicz that $H^2(X_3) = H^3(X_3) = 0$, so all entries on the second and third diagonals of $E_\infty = E_4$ are 0. This implies that $d_3^{0,2} : E_3^{0,2} = \mathbb{Z} \to E_3^{3,0} = \mathbb{Z}$ is an isomorphism. Let $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$ with x of degree 2, and let u be a generator of $H^3(S^3)$. Then we have $d_3(x) = u$. By the Leibnitz rule, $d_3 x^n = n x^{n-1} dx = n x^{n-1} u$, and since x^n generates $E_3^{0,2n}$ and $x^{n-1}u$ generates $E_3^{3,2n-2}$, the differential $d_3^{0,2n}$ is given by multiplication by n. This completely determines $E_4 = E_\infty$, hence the integral cohomology and (by the universal coefficient theorem) homology of X_3 is easily computed as:

q	0	1	2	3	4	5	6	7	$\cdots$	$2k$	$2k+1$	$\cdots$
$H^q(X_3)$	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$	$\cdots$	0	$\mathbb{Z}/k$	$\cdots$
$H_q(X_3)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$	0	$\cdots$	$\mathbb{Z}/k$	0	$\cdots$

In particular, $\pi_4 = H_4(X_3) = \mathbb{Z}/2$, which reproves Theorem 2.9.1.

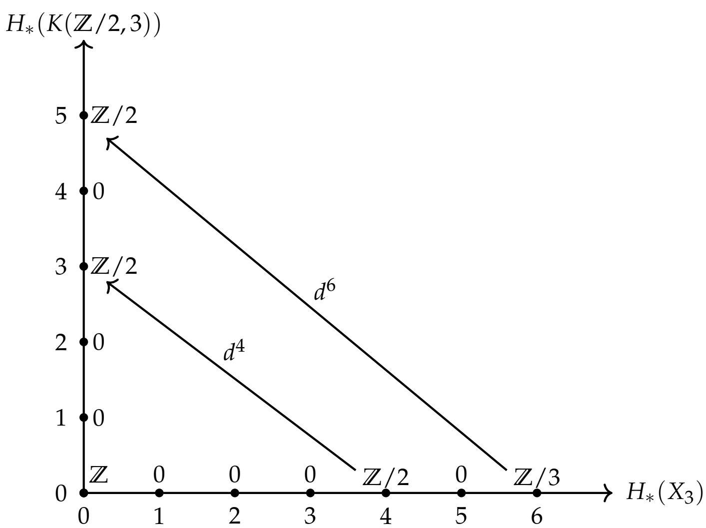
In order to compute $\pi_5(S^3) \cong H_5(X_4)$, we use the *homology* spectral sequence for the fibration

$$K(\pi_4, 3) \hookrightarrow X_4 \to X_3,$$

with E^2-page

$$E^2_{p,q} = H_p(X_3; H_q(K(\mathbb{Z}/2, 3))) \Rightarrow H_*(X_4).$$

Note that, by the Hurewicz theorem, we have: $H_i(K(\pi_4, 3)) = 0$ for $i = 1, 2$ and $H_3(K(\pi_4, 3)) = \pi_4 = \mathbb{Z}/2$. So $E^2_{p,q} = 0$ for $q = 1, 2$. Also, $E^2_{p,0} = H_p(X_3)$, whose values are computed in the above table.



Since X_4 is 4-connected, we have by Hurewicz that $H_3(X_4) = H_4(X_4) = 0$, so all entries on the third and fourth diagonal of E^∞ are zero. Since the first and second row of E^2 are zero, this forces $d^4 : E^4_{4,0} = E^2_{4,0} \to E^4_{0,3} = E^2_{0,3}$ to be an isomorphism (thus recovering the fact that $\pi_4 \cong \mathbb{Z}/2$), and

$$H_4(K(\mathbb{Z}/2, 3)) = E^2_{0,4} = E^\infty_{0,4} = 0.$$

Moreover, by a spectral sequence argument for the path fibration of $K(\mathbb{Z}/2, 3)$, we obtain (see Exercise 6)

$$E^2_{0,5} = H_5(K(\mathbb{Z}/2, 3)) = \mathbb{Z}/2,$$

and this entry can only be affected by $d^6 : E^6_{6,0} \cong \mathbb{Z}/3 \to E^6_{0,5} = E^2_{0,5} \cong \mathbb{Z}/2$, which is the zero map, so $E^\infty_{0,5} = \mathbb{Z}/2$. Thus, on the fifth diagonal of E^∞, all entries are zero except $E^\infty_{0,5} = \mathbb{Z}/2$, which yields $H_5(X_4) = \mathbb{Z}/2$, i.e., $\pi_5(S^3) = \mathbb{Z}/2$. $\square$

2.11 Serre's theorem on finiteness of homotopy groups of spheres

In this section we prove the following result:

Theorem 2.11.1 (Serre)**.**

(a) $\pi_i(S^{2k+1})$ *is finite for* $i > 2k+1$.

(b) $\pi_i(S^{2k})$ *is finite for* $i > 2k$, $i \neq 4k-1$, *and*

$$\pi_{4k-1}(S^{2k}) = \mathbb{Z} \oplus \{\text{finite abelian group}\}.$$

Proof of part (a). The case $k = 0$ is easy since $\pi_i(S^1)$ is in fact trivial for $i > 1$. For $k > 0$, recall Serre's theorem 2.4.2, according to which a simply-connected finite CW complex has finitely generated homotopy groups. In particular, the groups $\pi_i(S^{2k+1})$ are finitely generated abelian for all $i > 1$. Therefore, $\pi_i(S^{2k+1})$ $(i > 1)$ is finite if it is a torsion group.

In what follows we show that

$$\pi_i(S^{2k-1}) \cong \pi_{i+2}(S^{2k+1}) \text{ mod torsion}, \tag{2.11.1}$$

and part (a) of the theorem follows then by induction. The key to proving the isomorphism (2.11.1) is the fact that

$$\pi_{2k-1}(\Omega^2 S^{2k+1}) \cong \pi_{2k+1}(S^{2k+1}) = \mathbb{Z}.$$

Letting $\beta \colon S^{2k-1} \to \Omega^2 S^{2k+1}$ be a generator of $\pi_{2k-1}(\Omega^2 S^{2k+1})$, we will show that β induces an isomorphism mod torsion on H_* (i.e., an isomorphism on $H_*(-;\mathbb{Q})$). Let us assume this fact for now. WLOG, we assume that β is an inclusion, and then the homology long exact sequence of the pair $(\Omega^2 S^{2k+1}, S^{2k-1})$ yields that

$$H_*(\Omega^2 S^{2k+1}, S^{2k-1}) = 0 \text{ mod torsion}.$$

The relative version of the Hurewicz mod torsion Theorem 2.4.5 then tells us that

$$\pi_i(\Omega^2 S^{2k+1}, S^{2k-1}) = 0 \text{ mod torsion}$$

for all i, so again by the homotopy long exact sequence of the pair we get that $\pi_i(S^{2k-1}) \cong \pi_i(\Omega^2 S^{2k+1}) \cong \pi_{i+2}(S^{2k+1})$ mod torsion, as desired.

Thus, it remains to show that the generator $\beta \colon S^{2k-1} \to \Omega^2 S^{2k+1}$ of $\pi_{2k-1}\Omega^2(S^{2k+1})$ induces an isomorphism on $H_*(-;\mathbb{Q})$. The bulk of the argument amounts to showing that $H_i(\Omega^2(S^{2k+1});\mathbb{Q}) = 0$ for $i \neq 2k-1$, which we do by computing $H_i(\Omega^2(S^{2k+1});\mathbb{Q})^\vee = H^i(\Omega^2(S^{2k+1});\mathbb{Q})$ with the help of the cohomology spectral sequence for the path fibration $\Omega^2 S^{2k+1} \hookrightarrow * \to \Omega S^{2k+1}$. The E_2-page is given by

$$E_2^{p,q} = H^p(\Omega S^{2k+1}; H^q(\Omega^2 S^{2k+1};\mathbb{Q})) \Rightarrow H^*(*;\mathbb{Q}),$$

and since the total space of the fibration is contractible, we have $E_\infty^{p,q} = 0$ unless $p = q = 0$, in which case $E_\infty^{0,0} \cong \mathbb{Z}$.

It is a simple exercise (using the path fibration $\Omega S^{2k+1} \hookrightarrow * \to S^{2k+1}$) to show that

$$H^*(\Omega S^{2k+1}; \mathbb{Q}) \cong \mathbb{Q}[e], \ \deg e = 2k.$$

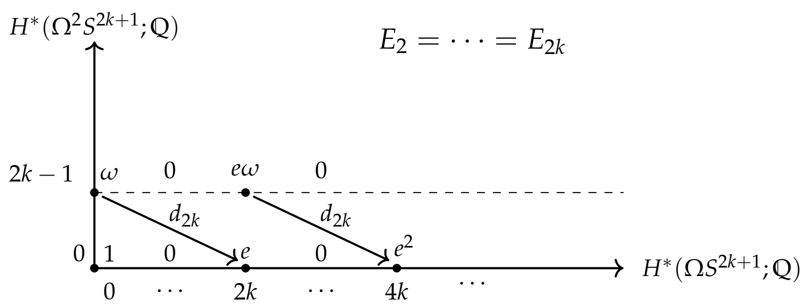
Hence,

$$\begin{aligned} E_2^{p,q} &= H^p(\Omega S^{2k+1}; H^q(\Omega^2 S^{2k+1}; \mathbb{Q})) \\ &\cong H^p(\Omega S^{2k+1}; \mathbb{Q}) \otimes_{\mathbb{Q}} H^q(\Omega^2 S^{2k+1}; \mathbb{Q}) \end{aligned}$$

has possibly non-trivial columns only at multiples p of $2k$, with $E_2^{2k,0} \cong \mathbb{Q} = \langle e^k \rangle$. This implies that $d_2, d_3, \ldots, d_{2k-1}$ are all zero, hence $E_2 = E_{2k}$. Furthermore, since the first non-trivial homotopy group $\pi_q(\Omega^2 S^{2k+1}) \cong \pi_{q+2}(S^{2k+1})$ appears at $q = 2k-1$, it follows by Hurewicz that

$$H^q(\Omega^2 S^{2k+1}; \mathbb{Q}) = 0, \text{ for } 0 < q < 2k-1.$$

Therefore, $E_2^{p,q} = 0$ for $0 < q < 2k-1$.



Since $E_{2k}^{2k,0} \cong H^{2k}(\Omega S^{2k+1}) = \langle e \rangle$ and $E_{2k}^{0,2k-1} \cong H^{2k-1}(\Omega^2 S^{2k+1})$ are only affected by $d_{2k}^{0,2k-1}\colon E_{2k}^{0,2k-1} \to E_{2k}^{2k,0}$, we must have that $d_{2k}^{0,2k-1}$ is an isomorphism in order for $E_{2k+1}^{2k,0} = E_\infty^{2k,0}$ and $E_{2k+1}^{0,2k-1} = E_\infty^{0,2k-1}$ to be zero. So $H^{2k-1}(\Omega^2 S^{2k+1}) \cong \mathbb{Q} = \langle \omega \rangle$, with $d_{2k}(\omega) = e$. As a consequence,

$$E_{2k}^{2jk,2k-1} = H^{2jk}(\Omega S^{2k+1}; \mathbb{Q}) \otimes_{\mathbb{Q}} H^{2k-1}(\Omega^2 S^{2k+1}) = \langle e^j \rangle \otimes_{\mathbb{Q}} \langle \omega \rangle = \langle e^j \omega \rangle$$

and $d_{2k}^{2jk,2k-1}\colon E_{2k}^{2jk,2k-1} \to E_{2k}^{2jk+2k,0}$ are isomorphisms since $d_{2k}(e^j\omega) = jd_{2k}(e)\omega + e^j d_{2k}(\omega) = e^{j+1}$. This implies that, except for $q \in \{0, 2k-1\}$, $E_{2k}^{p,q}$ is always trivial, and in particular that $H^i(\Omega^2 S^{2k+1}; \mathbb{Q}) = E_{2k}^{0,i}$ is trivial for $i \neq 0, 2k-1$. (If there was anything else in $H^*(\Omega^2 S^{2k+1}; \mathbb{Q})$, it would have to also be present at infinity.)

Next note that S^{2k-1} and $\Omega^2 S^{2k+1}$ are $(2k-2)$-connected, so by the Hurewicz theorem, their rational cohomology vanishes in degrees $i < 2k-1$. Hence, $\beta\colon S^{2k-1} \to \Omega^2 S^{2k+1}$ induces isomorphisms on

$H^i(-;\mathbb{Q})$ if $i \neq 2k-1$. In order to show that β induces an isomorphism on $H_{2k-1}(-;\mathbb{Q})$, recall the commutative diagram:

$$\begin{tikzcd} H_{2k-1}(S^{2k-1}) \arrow[r, "\beta_*"] & H_{2k-1}(\Omega^2 S^{2k+1}) \\ \pi_{2k-1}(S^{2k-1}) \arrow[r, "\beta_*"'] \arrow[u, "h", "\cong"'] & \pi_{2k-1}(\Omega^2 S^{2k+1}) \arrow[u, "h", "\cong"'] \end{tikzcd}$$

where the lower horizontal β_* is an isomorphism since β is the generator of $\pi_{2k-1}(\Omega^2 S^{2k+1})$, and the vertical arrows are isomorphisms by Hurewicz. Since the diagram commutes, the top horozontal map labelled β_* is an isomorphism also, and the proof of part (a) is complete. □

Proof of part (b). We shall construct a fibration

$$S^{2k-1} \hookrightarrow E \xrightarrow{\pi} S^{2k}$$

such that

$$\pi_i(E) \cong \pi_i(S^{4k-1}) \text{ (mod torsion)}. \tag{2.11.2}$$

Assuming for now that such a fibration exists, then since by part (a) we have that

$$\pi_i(S^{4k-1}) = \begin{cases} \text{finite} & i \neq 4k-1 \\ \mathbb{Z} & i = 4k-1 \end{cases},$$

we deduce that

$$\pi_i(E) = \begin{cases} \text{finite} & i \neq 4k-1 \\ \mathbb{Z} \oplus \text{finite} & i = 4k-1. \end{cases}$$

The homotopy long exact sequence:

$$\cdots \longrightarrow \pi_i(S^{2k-1}) \longrightarrow \pi_i(E) \longrightarrow \pi_i(S^{2k}) \longrightarrow \pi_{i-1}(S^{2k-1}) \longrightarrow \cdots$$

together with that fact proved in part (a) that

$$\pi_i(S^{2k-1}) = \begin{cases} \text{finite} & i \neq 2k-1 \\ \mathbb{Z} & i = 2k-1 \end{cases},$$

then yields that

$$\pi_i(S^{2k}) = \begin{cases} \text{finite} & i \neq 2k, 4k-1 \\ \mathbb{Z} \oplus \text{finite} & i = 4k-1, \end{cases}$$

as desired.

Note that in order to have (2.11.2), it is sufficient for E to satisfy $H_i(E) \cong H_i(S^{4k-1})$ modulo torsion, i.e.,

$$H_i(E) = \begin{cases} \text{finite} & i \neq 0, 4k-1 \\ \mathbb{Z} \oplus \text{finite} & i = 4k-1. \end{cases}$$

Indeed, by Hurewicz mod torsion, we then have that $\pi_{4k-1}(E) \cong H_{4k-1}(E)$ mod torsion, and let $f\colon S^{4k-1} \to E$ be a generator of the $\mathbb{Z}$-summand of $\pi_{4k-1}(E)$. WLOG, we can assume that f is an inclusion. The homology long exact sequence of the pair (E, S^{4k-1}) then implies that $H_*(E, S^{4k-1}) = 0$ mod torsion. By Hurewicz mod torsion this yields $\pi_*(E, S^{4k-1}) = 0$ mod torsion. Finally, the homotopy long exact sequence gives $\pi_i(E) \cong \pi_i(S^{4k-1})$ mod torsion.

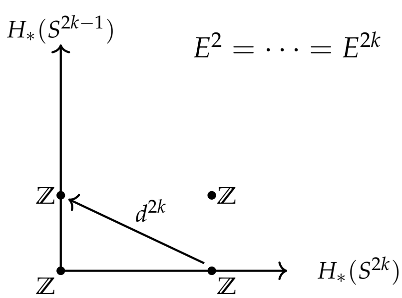
Back to the construction of the space E, we start with the tangent bundle $TS^{2k} \to S^{2k}$, and let $\pi : T_0S^{2k} \to S^{2k}$ be its restriction to the space of nonzero tangent vectors to S^{2k}. Then π is a fibration, since it is locally trivial, and its fiber is $\mathbb{R}^{2k} \setminus \{0\} \simeq S^{2k-1}$. We let

$$E = T_0S^{2k}.$$

Let us now consider the Leray-Serre homology spectral sequence of this fibration, with

$$E^2_{p,q} = H_p(S^{2k}; H_q(S^{2k-1})) = H_p(S^{2k}) \otimes H_q(S^{2k-1}) \Rightarrow H_*(E).$$

Therefore, the page E^2 has only four non-trivial entries at $(p,q) = (0,0)$, $(2k, 0)$, $(0, 2k-1)$, $(2k-1, 2k)$, and all these entries are isomorphic to $\mathbb{Z}$.



Clearly, the differentials $d^2, d^3, \ldots, d^{2k-1}$ are all zero, as are the differentials $d^{2k+1}, \ldots$. The only possibly non-zero differential in the spectral sequence is $d^{2k}_{2k,0}\colon E^{2k}_{2k,0} \to E^{2k}_{0,2k-1}$. Thus, $E^2 = \cdots = E^{2k}$ and $E^{2k+1} = \cdots = E^\infty$. Therefore, the space E has the desired homology if and only if

$$d^{2k}_{2k,0} \neq 0.$$

The map $d^{2k}_{2k,0}$ fits into a commutative diagram

$$\begin{tikzcd} \pi_{2k}(S^{2k}) \arrow[r, "\partial"] \arrow[d, "h"', "\cong"] & \pi_{2k-1}(S^{2k-1}) \arrow[d, "\cong"', "h"] \\ H_{2k}(S^{2k}) \arrow[r, "d_{2k}"] & H_{2k-1}(S^{2k-1}) \end{tikzcd}$$

where ∂ is the connecting homomorphism in the homotopy long exact sequence of the fibration, and h denotes the Hurewicz maps. Hence, $d_{2k} \neq 0$ if and only if $\partial \neq 0$. If, by contradiction, $\partial = 0$, then the homotopy long exact sequence of the fibration π contains the exact sequence

$$\pi_{2k}(E) \xrightarrow{\pi_*} \pi_{2k}(S^{2k}) \xrightarrow{\partial} 0.$$

In particular, there is $[\phi] \in \pi_{2k}(E)$ so that $\pi_*([\phi]) = [id]$, i.e., the diagram

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ S^{2k} \arrow[ur, "\phi"] \arrow[r, equal, "id"'] & S^{2k} \end{tikzcd}$$

commutes up to homotopy. By the homotopy lifting property of the fibration, there is then a map $\psi\colon S^{2k} \to E$ so that $\pi \circ \psi = id$. In other words, ψ is a section of the bundle π. This implies the existence of a nowhere-vanishing vector field on S^{2k}, which is a contradiction. $\square$

Remark 2.11.2. Serre's original proof of Theorem 2.11.1 used the Whitehead tower approximation of a sphere, together with the computation of the rational cohomology of $K(\mathbb{Z}, n)$ (see Exercise 13).

2.12 *Computing cohomology rings via spectral sequences*

The following computation will be useful when discussing about characteristic classes:

Example 2.12.1. In this example, we show that the cohomology ring $H^*(U(n);\mathbb{Z})$ is a free $\mathbb{Z}$-algebra on odd degree generators $x_1, \cdots, x_{2n-1}$, with $\deg(x_i) = i$, i.e.,

$$H^*(U(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_1, \cdots, x_{2n-1}].$$

We will prove this fact by induction on n, by using the Leray-Serre cohomology spectral sequence for the fibration

$$U(n-1) \hookrightarrow U(n) \to S^{2n-1}.$$

For the base case, note that $U(1) = S^1$, so $H^*(U(1)) = \Lambda_{\mathbb{Z}}[x_1]$ with $\deg(x_1) = 1$. For the induction step, we will show that

$$H^*(U(n)) = H^*(S^{2n-1}) \otimes H^*(U(n-1)). \tag{2.12.1}$$

Since $H^*(S^{2n-1}) = \Lambda_{\mathbb{Z}}[x_{2n-1}]$ with $\deg(x_{2n-1}) = 2n-1$, this will then give recursively that $H^*(U(n)) = \Lambda_{\mathbb{Z}}[x_1, \ldots, x_{2n-3}] \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}[x_{2n-1}] = \Lambda_{\mathbb{Z}}[x_1, \cdots, x_{2n-1}]$, with odd-degree generators $x_1, \cdots, x_{2n-1}$, with

$$\deg(x_i) = i.$$

Assume by induction that $H^*(U(n-1)) = \Lambda_{\mathbb{Z}}[x_1, \cdots, x_{2n-3}]$, with $\deg(x_i) = i$, and for $n \geq 2$ consider the cohomology spectral sequence

$$E_2^{p,q} = H^p(S^{2n-1}, H^q(U(n-1))) \Rightarrow H^*(U(n)).$$

By the universal coefficient theorem, we have that

$$E_2^{p,q} = H^p(S^{2n-1}) \otimes H^q(U(n-1)) = 0 \text{ if } p \neq 0, 2n-1.$$

So all the nonzero entries on the E_2-page are concentrated on the columns $p = 0$ (i.e., q-axis) and $p = 2n-1$. In particular,

$$d_1 = \cdots = d_{2n-2} = 0,$$

so

$$E_2 = \cdots = E_{2n-1}.$$

Furthermore, higher differentials starting with d_{2n} are also zero (since either their domain or target is zero), so

$$E_{2n} = \cdots = E_\infty.$$

Recall now that $x_1, \cdots, x_{2n-3}$ generate the cohomology of the fiber $U(n-1)$ and note that, due to their position on E_{2n-1}, we have that $d_{2n-1}(x_1) = \cdots = d_{2n-1}(x_{2n-3}) = 0$. Since $d_{2n-1}(x_{2n-1}) = 0$, we conclude by the Leibnitz rule that

$$d_{2n-1} = 0.$$

(Here, x_{2n-1} denotes the generator of $H^*(S^{2n-1})$.) Thus, $E_{2n-1} = E_{2n}$, so in fact the spectral sequence degenerates at the E_2-page, i.e.,

$$E_2 = \cdots = E_\infty.$$

Since the E_∞-term is a free, graded-commutative, bigraded algebra, it is a standard fact (e.g., see Example 1.K in McCleary's "A User's guide to spectral sequences") that the abutement $H^*(U(n))$ of the spectral sequence is also a free, graded commutative algebra isomorphic to the total complex associated to $E_\infty^{*,*}$, i.e.,

$$H^i(U(n)) \cong \bigoplus_{p+q=i} E_\infty^{p,q},$$

as desired.

Example 2.12.2. We can similarly compute $H^*(SU(n))$ either directly by induction from the fibration $SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}$ and the base case $SU(2) = S^3$, or by using our computation of $H^*(U(n))$ together with the diffeomorphism

$$U(n) \cong SU(n) \times S^1 \qquad (2.12.2)$$

given by $A \mapsto \left(\frac{1}{\sqrt[n]{\det A}} A, \det A \right)$. In particular, (2.12.2) yields by the Künneth formula:

$$H^*(U(n)) = H^*(SU(n)) \otimes H^*(S^1),$$

hence

$$H^*(SU(n)) = \Lambda_{\mathbb{Z}}[x_3, \ldots, x_{2n-1}]$$

with $\deg x_i = i$.

2.13 Exercises

1. Show that $\pi_i(\Sigma \mathbb{R}P^2)$ are finitely generated abelian groups for any $i \geq 0$. (Hint: Use Theorem 2.4.5, with $\mathcal{C}$ the category of finitely generated 2-groups.

2. Compute the homology of ΩS^1. (Hint: Use the fibration $\Omega S^1 \hookrightarrow \mathbb{Z} \to \mathbb{R}$ obtained by "looping" the covering $\mathbb{Z} \hookrightarrow \mathbb{R} \to S^1$, together with the Leray-Serre spectral sequence.)

3. Prove Wang's Theorem 2.5.2.

4. Let $\pi : E \to B$ be a fibration with fiber F, let $\mathbb{K}$ be a field, and assume that $\pi_1(B)$ acts trivially on $H_*(F; \mathbb{K})$. Assume that the Euler characteristics $\chi(B)$, $\chi(F)$ are defined (e.g., if B and F are finite CW complexes). Then $\chi(E)$ is defined and

$$\chi(E) = \chi(B) \cdot \chi(F).$$

5. Use a spectral sequence argument to show that $S^m \hookrightarrow S^n \to S^l$ is a fiber bundle, then $n = m + l$ and $l = m + 1$.

6. Prove that $H_5(K(\pi_4, 3)) = \mathbb{Z}/2$. (Hint: consider the two fibrations $K(\mathbb{Z}/2, 2) = \Omega K(\mathbb{Z}/2, 3) \hookrightarrow * \to K(\mathbb{Z}/2, 3)$, and $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1) \hookrightarrow * \to K(\mathbb{Z}/2, 2)$. Then compute $H_*(K(\mathbb{Z}/2, 2))$ via the spectral sequence of the second fibration, and use it in the spectral sequence of the first fibration to compute $H_*(K(\mathbb{Z}/2, 3))$.)

7. Compute the cohomology of the space of continuous maps $f : S^1 \to S^3$. (Hint: Let $X := \{f : S^1 \to S^3, f \text{ is continuous}\}$ and define

$\pi : X \to S^3$ by $f \mapsto f(1)$. Then π is a fibration with fiber ΩS^3. Apply the cohomology spectral sequence for the fibration $\Omega S^3 \hookrightarrow X \to S^3$ to conclude that $H^*(X) \cong H^*(S^3) \otimes H^*(\Omega S^3)$.)

8. Compute the cohomology of the space of continuous maps $f : S^1 \to S^2$.

9. Compute the cohomology of the space of continuous maps $f : S^1 \to \mathbb{C}P^n$.

10. Compute the cohomology ring $H^*(SO(n); \mathbb{Z}/2)$.

11. Compute the cohomology ring $H^*(V_k(\mathbb{C}^n); \mathbb{Z})$.

12. Show that $H^*(SO(4)) \cong H^*(S^3) \otimes H^*(\mathbb{R}P^3)$.

13. Show that

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \begin{cases} \mathbb{Q}[z_n] & \text{, if n is even} \\ \Lambda(z_n) & \text{, if n is odd,} \end{cases}$$

with $\deg(z_n) = n$. Here, $\Lambda(z_n) := \mathbb{Q}[z_n]/(z_n^2)$.
(Hint: Consider the spectral sequence for the path fibration

$$K(\mathbb{Z}, n-1) \hookrightarrow * \to K(\mathbb{Z}, n)$$

and induction.)

14. Compute the ring structure on $H^*(\Omega S^n)$.

15. Show that the p-torsion in $\pi_i(S^3)$ appears first for $i = 2p$, in which case it is $\mathbb{Z}/p$. (Hint: use the Whitehead tower of S^3, the homology spectral sequence of the relevant fibration, together with Hurewicz mod $\mathcal{C}_p$, where $\mathcal{C}_p$ is the class of torsion abelian groups whose p-primary subgroup is trivial.)

16. Where does the 7-torsion appear first in the homotopy groups of S^n?

3
Fiber bundles. Classifying spaces. Applications

3.1 Fiber bundles

Let G be a topological group (i.e., a topological space endowed with a group structure so that the group multiplication and the inversion map are continuous), acting continuously (on the left) on a topological space F. Concretely, such a continuous action is given by a continuous map $\rho\colon G \times F \to F$, $(g, m) \mapsto g \cdot m$, which satisfies the conditions $(gh) \cdot m = g \cdot (h \cdot m))$ and $e_G \cdot m = m$, for e_G the identity element of G.

Any continuous group action ρ induces a map

$$\mathrm{Ad}_\rho : G \longrightarrow \mathrm{Homeo}(F)$$

given by $g \mapsto (f \mapsto g \cdot f)$, with $g \in G$, $f \in F$. Note that Ad_ρ is a group homomorphism since

$$(\mathrm{Ad}\,\rho)(gh)(f) := (gh) \cdot f = g \cdot (h \cdot f) = \mathrm{Ad}_\rho(g)(\mathrm{Ad}_\rho(h)(f)).$$

Note that for nice spaces F (e.g., CW complexes), if we give $\mathrm{Homeo}(F)$ the compact-open topology, then $\mathrm{Ad}_\rho\colon G \to \mathrm{Homeo}(F)$ is a continuous group homomorphism, and any such continuous group homomorphism $G \to \mathrm{Homeo}(F)$ induces a continuous group action $G \times F \to F$.

We assume from now on that ρ is an *effective* action, i.e., that Ad_ρ is injective.

Definition 3.1.1 (Atlas for a fiber bundle with group G and fiber F). *Given a continuous map $\pi\colon E \to B$, an atlas for the structure of a fiber bundle with group G and fiber F on π consists of the following data:*

a) an open cover $\{U_\alpha\}_\alpha$ of B,

b) homeomorphisms $h_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$ (called trivializing charts or local trivializations) for each α so that the diagram

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "h_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\mathrm{pr}_1"] \\
& U_\alpha &
\end{tikzcd}$$

commutes,

c) *continuous maps (called transition functions)* $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ *so that the horizontal map in the commutative diagram*

$$\begin{tikzcd} & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "h_\alpha"'] \arrow[dr, "h_\beta"] & \\ (U_\alpha \cap U_\beta) \times F \arrow[rr, "h_\beta \circ h_\alpha^{-1}"'] & & (U_\alpha \cap U_\beta) \times F \end{tikzcd}$$

is given by

$$(x, m) \mapsto (x, g_{\beta\alpha}(x) \cdot m).$$

(By the effectivity of the action, if such maps $g_{\alpha\beta}$ *exist, they are unique.)*

Definition 3.1.2. *Two atlases* $\mathcal{A}$ *and* $\mathcal{B}$ *on* π *are compatible if* $\mathcal{A} \cup \mathcal{B}$ *is an atlas.*

Definition 3.1.3 (Fiber bundle with group G and fiber F)**.** *A structure of a fiber bundle with group G and fiber F on* $\pi\colon E \to B$ *is a maximal atlas for* $\pi\colon E \to B$.

Example 3.1.4.

1. When $G = \{e_G\}$ is the trivial group, $\pi\colon E \to B$ has the structure of a fiber bundle if and only if it is a trivial fiber bundle. Indeed, the local trivializations h_α of the atlas for the fiber bundle have to satisfy $h_\beta \circ h_\alpha^{-1}\colon (x, m) \mapsto (x, e_G \cdot m) = (x, m)$, which implies $h_\beta \circ h_\alpha^{-1} = id$, so $h_\beta = h_\alpha$ on $U_\alpha \cap U_\beta$. This allows us to glue all the local trivializations h_α together to obtain a global trivialization $h\colon \pi^{-1}(B) = E \cong B \times F$.

2. When F is discrete, $\mathrm{Homeo}(F)$ is also discrete, so G is discrete by the effectiveness assumption. So for the atlas of $\pi\colon E \to B$ we have $\pi^{-1}(U_\alpha) \cong U_\alpha \times F = \bigcup_{m\in F} U_\alpha \times \{m\}$, so π is in this case a covering map.

3. A locally trivial fiber bundle, as introduced in earlier chapters, is just a fiber bundle with structure group $\mathrm{Homeo}(F)$.

Lemma 3.1.5. *The transition functions* $g_{\alpha\beta}$ *satisfy the following properties:*

(a) $g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$, *for all* $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

(b) $g_{\beta\alpha}(x) = g_{\alpha\beta}^{-1}(x)$, *for all* $x \in U_\alpha \cap U_\beta$.

(c) $g_{\alpha\alpha}(x) = e_G$.

Proof. On $U_\alpha \cap U_\beta \cap U_\gamma$, we have: $(h_\alpha \circ h_\beta^{-1}) \circ (h_\beta \circ h_\gamma^{-1}) = h_\alpha \circ h_\gamma^{-1}$. Therefore, since Ad_ρ is injective (i.e., ρ is effective), we get that

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$$

for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Note that $(h_\alpha \circ h_\beta^{-1}) \circ (h_\beta \circ h_\alpha^{-1}) = id$, which translates into

$$(x, g_{\alpha\beta}(x) g_{\beta\alpha}(x) \cdot m) = (x, m).$$

So, by effectiveness, $g_{\alpha\beta}(x) g_{\beta\alpha}(x) = e_G$ for all $x \in U_\alpha \cap U_\beta$, whence $g_{\beta\alpha}(x) = g_{\alpha\beta}^{-1}(x)$.

Take $\gamma = \alpha$ in Property (a) to get $g_{\alpha\beta}(x) g_{\beta\alpha}(x) = g_{\alpha\alpha}(x)$. So by Property (b), we have $g_{\alpha\alpha}(x) = e_G$. □

Transition functions determine a fiber bundle in a unique way, in the sense of the following theorem.

Theorem 3.1.6. *Given an open cover $\{U_\alpha\}$ of B and continuous functions $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$ satisfying Properties (a)-(c), there is a unique structure of a fiber bundle over B with group G, given fiber F, and transition functions $\{g_{\alpha\beta}\}$.*

Proof Sketch. Let $\widetilde{E} = \bigsqcup_\alpha U_\alpha \times F \times \{\alpha\}$, and define an equivalence relation $\sim$ on $\widetilde{E}$ by

$$(x, m, \alpha) \sim (x, g_{\alpha\beta}(x) \cdot m, \beta),$$

for all $x \in U_\alpha \cap U_\beta$, and $m \in F$. Properties (a)-(c) of $\{g_{\alpha\beta}\}$ are used to show that $\sim$ is indeed an equivalence relation on $\widetilde{E}$. Specifically, symmetry is implied by property (b), reflexivity follows from (c) and transitivity is a consequence of the cycle property (a).

Let

$$E = \widetilde{E} / \sim$$

be the set of equivalence classes in E, and define $\pi : E \to B$ locally by $[(x, m, \alpha)] \mapsto x$ for $x \in U_\alpha$. Then it is clear that π is well-defined and continuos (in the quotient topology), and the fiber of π is F.

It remains to show the local triviality of π. Let $p : \widetilde{E} \to E$ be the quotient map, and let $p_\alpha := p|_{U_\alpha \times F \times \{\alpha\}} : U_\alpha \times F \times \{\alpha\} \to \pi^{-1}(U_\alpha)$. It is easy to see that p_α is a homeomorphism. We define the local trivializations of π by $h_\alpha := p_\alpha^{-1}$. □

Example 3.1.7.

1. Fiber bundles with fiber $F = \mathbb{R}^n$ and group $G = GL(n, \mathbb{R})$ are called *rank n real vector bundles*. For example, if M is a differentiable real n-manifold, and TM is the set of all tangent vectors to M, then $\pi : TM \to M$ is a real vector bundle on M of rank n. More precisely, if $\varphi_\alpha : U_\alpha \overset{\cong}{\to} \mathbb{R}^n$ are trivializing charts on M, the transition functions for TM are given by $g_{\alpha\beta}(x) = d(\varphi_\alpha \circ \varphi_\beta^{-1})_{\varphi_\beta(x)}$.

2. If $F = \mathbb{R}^n$ and $G = O(n)$, we get real vector bundles with a *Riemannian structure*.

3. Similarly, one can take $F = \mathbb{C}^n$ and $G = GL(n, \mathbb{C})$ to get *rank n complex vector bundles*. For example, if M is a complex manifold, the tangent bundle TM is a complex vector bundle.

4. If $F = \mathbb{C}^n$ and $G = U(n)$, we get real vector bundles with a *hermitian structure*.

We also mention here the following fact:

Theorem 3.1.8. *A fiber bundle has the homotopy lifting property with respect to all CW complexes (i.e., it is a Serre fibration). Moreover, fiber bundles over paracompact spaces are fibrations.*

Definition 3.1.9 (Bundle homomorphism). *Fix a topological group G acting effectively on a space F. A homomorphism between bundles $E' \xrightarrow{\pi'} B'$ and $E \xrightarrow{\pi} B$ with group G and fiber F is a pair $(f, \hat{f})$ of continuous maps, with $f : B' \to B$ and $\hat{f} : E' \to E$, such that:*

1. *the diagram*

$$\begin{tikzcd} E' \arrow[r, "\hat{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

commutes, i.e., $\pi \circ \hat{f} = f \circ \pi'$.

2. *if $\{(U_\alpha, h_\alpha)\}_\alpha$ is a trivializing atlas of π and $\{(V_\beta, H_\beta)\}_\beta$ is a trivializing atlas of π', then the following diagram commutes:*

$$\begin{tikzcd} (V_\beta \cap f^{-1}(U_\alpha)) \times F \arrow[dr, "\mathrm{pr}_1"'] & \pi'^{-1}(V_\beta \cap f^{-1}(U_\alpha)) \arrow[l, "H_\beta"'] \arrow[r, "\hat{f}"] \arrow[d, "\pi'"] & \pi^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[d, "\pi"] & U_\alpha \times F \arrow[dl, "\mathrm{pr}_1"] \\ & V_\beta \cap f^{-1}(U_\alpha) \arrow[r, "f"] & U_\alpha & \end{tikzcd}$$

and there exist functions $d_{\alpha\beta} : V_\beta \cap f^{-1}(U_\alpha) \to G$ such that for $x \in V_\beta \cap f^{-1}(U_\alpha)$ and $m \in F$ we have:

$$h_\alpha \circ \hat{f}_| \circ H_\beta^{-1}(x, m) = (f(x), d_{\alpha\beta}(x) \cdot m).$$

An isomorphism of fiber bundles is a bundle homomorphism $(f, \hat{f})$ which admits a map $(g, \hat{g})$ in the reverse direction so that both composites are the identity.

Remark 3.1.10. *Gauge transformations* of a bundle $\pi : E \to B$ are bundle maps from π to itself over the identity of the base, i.e., corresponding to continuous map $g : E \to E$ so that $\pi \circ g = \pi$. By definition, such g

restricts to an isomorphism given by the action of an element of the structure group on each fiber. The set of all gauge transformations forms a group.

Proposition 3.1.11. *Given functions $d_{\alpha\beta} : V_\beta \cap f^{-1}(U_\alpha) \to G$ and $d_{\alpha'\beta'} : V_{\beta'} \cap f^{-1}(U_{\alpha'}) \to G$ as in (2) above for different trivializing charts of π and resp. π', then for any $x \in V_\beta \cap V_{\beta'} \cap f^{-1}(U_\alpha \cap U_{\alpha'}) \neq \emptyset$, we have*

$$d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(f(x))\, d_{\alpha\beta}(x)\, g_{\beta\beta'}(x) \tag{3.1.1}$$

in G, where $g_{\alpha'\alpha}$ are transition functions for π and $g_{\beta\beta'}$ are transition functions for π',

Proof. Exercise. □

The functions $\{d_{\alpha\beta}\}$ determine bundle maps in the following sense:

Theorem 3.1.12. *Given a map $f : B' \to B$ and bundles $E \xrightarrow{\pi} B$, $E' \xrightarrow{\pi'} B'$, a map of bundles $(f, \hat{f}) : \pi' \to \pi$ exists if and only if there exist continuous maps $\{d_{\alpha\beta}\}$ as above, satisfying (3.1.1).*

Proof. Exercise. □

Theorem 3.1.13. *Every bundle map $\hat{f}$ over $f = \mathrm{id}_B$ is an isomorphism. In particular, gauge transformations are automorphisms.*

Proof Sketch. Let $d_{\alpha\beta} : V_\beta \cap U_\alpha \to G$ be the maps given by the bundle map $\hat{f} : E' \to E$. So, if $d_{\alpha'\beta'} : V_{\beta'} \cap U_{\alpha'} \to G$ is given by a different choice of trivializing charts, then (3.1.1) holds on $V_\beta \cap V_{\beta'} \cap U_\alpha \cap U_{\alpha'} \neq \emptyset$, i.e.,

$$d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(x)\, d_{\alpha\beta}(x)\, g_{\beta\beta'}(x) \tag{3.1.2}$$

in G, where $g_{\alpha'\alpha}$ are transition functions for π and $g_{\beta\beta'}$ are transition functions for π'. Let us now invert (3.1.2) in G, and set

$$\overline{d_{\beta\alpha}}(x) = d_{\alpha\beta}^{-1}(x)$$

to get:

$$\overline{d_{\beta'\alpha'}}(x) = g_{\beta'\beta}(x)\, \overline{d_{\beta\alpha}}(x)\, g_{\alpha\alpha'}(x).$$

So $\{\overline{d_{\beta\alpha}}\}$ are as in Definition 3.1.9 and satisfy (3.1.1). Theorem 3.1.12 implies that there exists a bundle map $\hat{g} : E \to E'$ over id_B.

We claim that $\hat{g}$ is the inverse $\hat{f}^{-1}$ of $\hat{f}$, and this can be checked locally as follows:

$$\begin{aligned}(x, m) \overset{\hat{f}}{\mapsto} (x, d_{\alpha\beta}(x) \cdot m) \overset{\hat{g}}{\mapsto} &(x, \overline{d_{\beta\alpha}}(x) \cdot (d_{\alpha\beta}(x) \cdot m)) \\ &= (x, \underbrace{\overline{d_{\beta\alpha}}(x) d_{\alpha\beta}(x)}_{e_G} \cdot m) \\ &= (x, m).\end{aligned}$$

So $\hat{g} \circ \hat{f} = id_{E'}$. Similarly, $\hat{f} \circ \hat{g} = id_E$ □

One way in which fiber bundle homomorphisms arise is from the pullback (or the induced bundle) construction.

Definition 3.1.14 (Induced Bundle)**.** *Given a bundle $E \xrightarrow{\pi} B$ with group G and fiber F, and a continuous map $f : X \to B$, we define*

$$f^*E := \{(x,e) \in X \times E \mid f(x) = \pi(e)\},$$

with projections $f^*\pi : f^*E \to X$, $(x,e) \mapsto x$, and $\hat{f} : f^*E \to E$, $(x,e) \mapsto e$, so that the following diagram commutes:

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] & e \arrow[dd, mapsto] \\ X \arrow[r, "f"'] & B & \\ x \arrow[rr, mapsto] & & f(x) \end{tikzcd}$$

$f^*\pi$ is called the induced bundle under f or the pullback of π by f, and as we show below it comes equipped with a bundle map $(f,\hat{f}) : f^*\pi \to \pi$.

The above definition is justified by the following result:

Theorem 3.1.15.

(a) $f^*\pi : f^*E \to X$ is a fiber bundle with group G and fiber F.

(b) $(f,\hat{f}) : f^*\pi \to \pi$ is a bundle map.

Proof Sketch. Let $\{(U_\alpha, h_\alpha)\}_\alpha$ be a trivializing atlas of π, and consider the following commutative diagram:

$$\begin{tikzcd} (f^*\pi)^{-1}(f^{-1}(U_\alpha)) \arrow[r] \arrow[d] & \pi^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[d] & U_\alpha \times F \arrow[dl] \\ f^{-1}(U_\alpha) \arrow[r, "f"'] & U_\alpha & \end{tikzcd}$$

We have

$$(f^*\pi)^{-1}(f^{-1}(U_\alpha)) = \{(x,e) \in f^{-1}(U_\alpha) \times \underbrace{\pi^{-1}(U_\alpha)}_{\cong U_\alpha \times F} \mid f(x) = \pi(e)\}.$$

Define

$$k_\alpha : (f^*\pi)^{-1}(f^{-1}(U_\alpha)) \longrightarrow f^{-1}(U_\alpha) \times F$$

by

$$(x,e) \mapsto (x, \mathrm{pr}_2(h_\alpha(e))).$$

Then it is easy to check that k_α is a homeomorphism (with inverse $k_\alpha^{-1}(x,m) = (x, h_\alpha^{-1}(f(x),m))$, and in fact the following assertions hold:

(i) $\{(f^{-1}(U_\alpha), k_\alpha)\}_\alpha$ is a trivializing atlas of $f^*\pi$.

(ii) the transition functions of $f^*\pi$ are $f^*g_{\alpha\beta} := g_{\alpha\beta} \circ f$, i.e., $f^*g_{\alpha\beta}(x) = g_{\alpha\beta}(f(x))$ for any $x \in f^{-1}(U_\alpha \cap U_\beta)$.

□

Remark 3.1.16. It is easy to see that $(f \circ g)^*\pi = g^*(f^*\pi)$ and $(id_B)^*\pi = \pi$. Moreover, the pullback of a trivial bundle is a trivial bundle.

As we shall see later on, the following important result holds:

Theorem 3.1.17. *Given a fibre bundle $\pi : E \to B$ with group G and fiber F, and two homotopic maps $f \simeq g : X \to B$, there is an isomorphism $f^*\pi \cong g^*\pi$ of bundles over X. (In short, induced bundles under homotopic maps are isomorphic.)*

As a consequence, we have:

Corollary 3.1.18. *A fiber bundle over a contractible space B is trivial.*

Proof. Since B is contractible, id_B is homotopic to the constant map ct. Let

$$b := \text{Image}(ct) \overset{i}{\hookrightarrow} B,$$

so $i \circ ct \simeq id_B$. We have a diagram of maps and induced bundles:

$$\begin{tikzcd}
ct^*i^*E \arrow[r] \arrow[d, "ct^*i^*\pi"'] & i^*E \arrow[r] \arrow[d, "i^*\pi"] & E \arrow[d, "\pi"] \\
B \arrow[r, "ct"] \arrow[rr, bend right, "id_B"'] & \{b\} \arrow[r, "i"] & B
\end{tikzcd}$$

Theorem 3.1.17 then yields:

$$\pi \cong (id_B)^*\pi \cong ct^*i^*\pi.$$

Since any fiber bundle over a point is trivial, we have that $i^*\pi \cong \{b\} \times F$ is trivial, hence $\pi \cong ct^*i^*\pi \cong B \times F$ is also trivial. □

Proposition 3.1.19. *If*

$$\begin{tikzcd}
E' \arrow[r, "\tilde{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\
B' \arrow[r, "f"] & B
\end{tikzcd}$$

is a bundle map, then $\pi' \cong f^*\pi$ as bundles over B'.

Proof. Define $h : E' \to f^*E$ by $e' \mapsto (\pi'(e'), \tilde{f}(e')) \in B' \times E$. This is well-defined, i.e., $h(e') \in f^*E$, since $f(\pi'(e')) = \pi(\tilde{f}(e'))$.

It is easy to check that h provides the desired bundle isomorphism over B'.

$$\begin{tikzcd}
E' \arrow[dr, "h"] \arrow[drr, "\tilde{f}"] \arrow[ddr, "\pi'"'] & & \\
& f^*E \arrow[r, "\tilde{f}"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\
& B' \arrow[r, "f"] & B
\end{tikzcd}$$

□

Example 3.1.20. We can now show that the set of isomorphism classes of bundles over S^n with group G and fiber F is isomorphic to $\pi_{n-1}(G)$. Indeed, let us cover S^n with two contractible sets U_+ and U_- obtained by removing the south, resp., north pole of S^n. Let $i_\pm : U_\pm \hookrightarrow S^n$ be the inclusions. Then any bundle π over S^n is trivial when restricted to $U_\pm$, that is, $i_\pm^*\pi \cong U_\pm \times F$. In particular, $U_\pm$ provides a trivializing cover (atlas) for π, and any such bundle π is completely determined by the transition function $g_\pm : U_+ \cap U_- \simeq S^{n-1} \to G$, i.e., by an element in $\pi_{n-1}(G)$.

More generally, we aim to "classify" fiber bundles on a given topological space. Let $\mathcal{B}(X, G, F, \rho)$ denote the isomorphism classes (over id_X) of fiber bundles on X with group G and fiber F, and G-action on F given by ρ. If $f : X' \to X$ is a continuous map, the pullback construction defines a map

$$f^* : \mathcal{B}(X, G, F, \rho) \longrightarrow \mathcal{B}(X', G, F, \rho)$$

so that $(id_X)^* = id$ and $(f \circ g)^* = g^* \circ f^*$.

3.2 *Principal Bundles*

As we will see later on, the fiber F doesn't play any essential role in the classification of fiber bundle, and in fact it is enough to understand the set

$$\mathcal{P}(X, G) := \mathcal{B}(X, G, G, m_G)$$

of fiber bundles with group G and fiber G, where the action of G on itself is given by the multiplication m_G of G. Elements of $\mathcal{P}(X, G)$ are called *principal G-bundles*. Of particular importance in the classification theory of such bundles is the *universal principal G-bundle* $G \hookrightarrow EG \to BG$, with contractible total space EG.

Example 3.2.1. Any regular cover $p : E \to X$ is a principal G-bundle, with group $G = {}^{\pi_1(X)}\!\big/_{p_*\pi_1(E)}$. Here G is given the discrete topology. In particular, the universal covering $\widetilde{X} \to X$ is a principal $\pi_1(X)$-bundle.

Example 3.2.2. Any free (right) action of a finite group G on a (Hausdorff) space E gives a regular cover and hence a principal G-bundle $E \to E/G$.

More generally, we have the following:

Theorem 3.2.3. *Let $\pi : E \to X$ be a principal G-bundle. Then G acts freely and transitively on the right of E so that ${}^{E}\!\big/_{G} \cong X$. In particular, π is the quotient (orbit) map.*

Proof. We will define the action locally over a trivializing chart for π. Let U_α be a trvializing open in X with trivializing homeomorphism $h_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times G$. We define a right action on G on $\pi^{-1}(U_\alpha)$ by

$$\begin{aligned}\pi^{-1}(U_\alpha) \times G &\to \pi^{-1}(U_\alpha) \cong U_\alpha \times G \\ (e, g) &\mapsto e \cdot g := h_\alpha^{-1}\left(\pi(e), \mathrm{pr}_2(h_\alpha(e)) \cdot g\right)\end{aligned}$$

Let us show that this action can be globalized, i.e., it is independent of the choice of the trivializing open U_α. If (U_β, h_β) is another trivializing chart in X so that $e \in \pi^{-1}(U_\alpha \cap U_\beta)$, we need to show that $e \cdot g = h_\beta^{-1}\left(\pi(e), \mathrm{pr}_2\left(h_\beta(e)\right) \cdot g\right)$, or equivalently,

$$h_\alpha^{-1}\left(\pi(e), \mathrm{pr}_2(h_\alpha(e)) \cdot g\right) = h_\beta^{-1}\left(\pi(e), \mathrm{pr}_2\left(h_\beta(e)\right) \cdot g\right). \tag{3.2.1}$$

After applying h_α and using the transition function $g_{\alpha\beta}$ for $\pi(e) \in U_\alpha \cap U_\beta$, (3.2.1) becomes

$$\begin{aligned}\left(\pi(e), \mathrm{pr}_2(h_\alpha(e)) \cdot g\right) &= h_\alpha h_\beta^{-1}\left(\pi(e), \mathrm{pr}_2\left(h_\beta(e)\right) \cdot g\right) \\ &= \left(\pi(e), g_{\alpha\beta}(\pi(e)) \cdot \left(\mathrm{pr}_2\left(h_\beta(e)\right) \cdot g\right)\right),\end{aligned}$$

which is guaranteed by the definition of an atlas for π.

It is easy to check locally that the action is free and transitive. Moreover, ${}^{E}\!\big/_{G}$ is locally given as ${}^{U_\alpha \times G}\!\big/_{G} \cong U_\alpha$, and this local quotient globalizes to X. □

The converse of the above theorem holds in some important cases.

Theorem 3.2.4. *Let E be a compact Hausdorff space and G a compact Lie group acting freely on E. Then the orbit map $E \to E/G$ is a principal G-bundle.*

Corollary 3.2.5. *Let G be a Lie group, and let $H < G$ be a compact subgroup. Then the projection onto the orbit space $\pi : G \to G/H$ is a principal H-bundle.*

Let us now fix a G-space F. We define a map

$$\mathcal{P}(X,G) \to \mathcal{B}(X,G,F,\rho)$$

as follows. Start with a principal G bundle $\pi : E \to X$, and recall from the previous theorem that G acts freely on the right on E. Since G acts on the left on F, we have a left G-action on $E \times F$ given by:

$$g \cdot (e,f) \mapsto (e \cdot g^{-1}, g \cdot f).$$

Let

$$E \times_G F := {}^{E \times F}\!/_{G}$$

be the corresponding orbit space, with projection map $\omega : E \times_G F \to {}^{E}\!/_{G} \cong X$ fitting into a commutative diagram

$$\begin{tikzcd}
 & E \times F \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr] & \\
E \arrow[dr, "\pi"'] & & {}^{E \times F}\!/_{G} \arrow[dl, "\omega"] \\
 & X &
\end{tikzcd} \tag{3.2.2}$$

Definition 3.2.6. *The projection $\omega := \pi \times_G F : E \times_G F \to X$ is called the associated bundle with fiber F.*

The terminology in the above definition is justified by the following result.

Theorem 3.2.7. *$\omega : E \times_G F \to X$ is a fiber bundle with group G, fiber F, and having the same transition functions as π. Moreover, the assignment $\pi \mapsto \omega := \pi \times_G F$ defines a one-to-one correspondence $\mathcal{P}(X,G) \to \mathcal{B}(X,G,F,\rho)$.*

Proof. Let $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ be a trivializing chart for π. Recall that for $e \in \pi^{-1}(U_\alpha)$, $f \in F$ and $g \in G$, if we set $h_\alpha(e) = (\pi(e), h) \in U_\alpha \times G$, then G acts on the right on $\pi^{-1}(U_\alpha)$ by acting on the right on $h = pr_2(h_\alpha(e))$. Then we have by the diagram (3.2.2) that

$$\begin{aligned}
\omega^{-1}(U_\alpha) &\cong {}^{\pi^{-1}(U_\alpha) \times F}\!/_{(e,f) \sim (e \cdot g^{-1}, g \cdot f)} \\
&\cong {}^{U_\alpha \times G \times F}\!/_{(u,h,f) \sim (u, hg^{-1}, g \cdot f)}.
\end{aligned}$$

Let us define

$$k_\alpha : \omega^{-1}(U_\alpha) \to U_\alpha \times F$$

by

$$[(u,h,f)] \mapsto (u, h \cdot f).$$

This is a well-defined map since

$$[(u, hg^{-1}, g \cdot f)] \mapsto (u, hg^{-1}g \cdot f) = (u, h \cdot f).$$

It is easy to check that k_α is a trivializing chart for ω with inverse induced by $U_\alpha \times F \to U_\alpha \times G \times F$, $(u, f) \mapsto (u, id_G, f)$. It is clear that ω and π have the same transition functions as they have the same trivializing opens. $\square$

The associated bundle construction is easily seen to be functorial in the following sense.

Proposition 3.2.8. *If*

$$\begin{tikzcd} E' \arrow[r, "\widehat{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

is a map of principal G-bundles (so $\widehat{f}$ is a G-equivariant map, i.e., $\widehat{f}(e \cdot g) = \widehat{f}(e) \cdot g$), then there is an induced map of associated bundles with fiber F,

$$\begin{tikzcd} E' \times_G F \arrow[r, "\widehat{f} \times_G id_F"] \arrow[d, "\pi'"'] & E \times_G F \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

Example 3.2.9. Let $\pi : S^1 \to S^1$, $z \mapsto z^2$ be regarded as a principal $\mathbb{Z}/2$-bundle, and let $F = [-1, 1]$. Let $\mathbb{Z}/2 = \{1, -1\}$ act on F by multiplication. Then the bundle associated to π with fiber $F = [-1, 1]$ is the Möbius strip $S^1 \times_{\mathbb{Z}/2} [-1, 1] = {}^{S^1 \times [-1,1]}\!\diagup_{(x,t) \sim (a(x), -t)}$, with $a : S^1 \to S^1$ denoting the antipodal map. Similarly, the bundle associated to π with fiber $F = S^1$ is the Klein bottle.

Let us now get back to proving the following important result.

Theorem 3.2.10. *Let $\pi : E \to Y$ be a fiber bundle with group G and fiber F, and let $f \simeq g : X \to Y$ be two homotopic maps. Then $f^*\pi \cong g^*\pi$ over id_X.*

It is of course enough to prove the theorem in the case of principal G-bundles. The idea of proof is to construct a bundle map over id_X between $f^*\pi$ and $g^*\pi$:

$$\begin{tikzcd} f^*E \arrow[rr, dotted, "?"] \arrow[dr] & & g^*E \arrow[dl] \\ & X & \end{tikzcd}$$

So we first need to understand maps of principal G-bundles, i.e., to solve the following problem: given two principal G-bundles bundles $E_1 \xrightarrow{\pi_1} X$ and $E_2 \xrightarrow{\pi_2} Y$, describe the set $maps(\pi_1, \pi_2)$ of bundle maps

$$\begin{tikzcd} E_1 \arrow[r, "\hat{f}"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Since G acts on the right of E_1 and E_2, we also get an action on the left of E_2 by $g \cdot e_2 := e_2 \cdot g^{-1}$. Then we get an associated bundle of π_1 with fiber E_2, namely

$$\omega := \pi_1 \times_G E_2 : E_1 \times_G E_2 \longrightarrow X.$$

We have the following result:

Theorem 3.2.11. *Bundle maps from π_1 to π_2 are in one-to-one correspondence to sections of ω.*

Proof. We work locally, so it suffices to consider only trivial bundles.

Given a bundle map $(f, \widehat{f}) : \pi_1 \mapsto \pi_2$, let $U \subset Y$ open, and $V \subset f^{-1}(U)$ open, so that the following diagram commutes (this is the bundle maps in trivializing charts)

$$\begin{tikzcd} V \times G \arrow[r, "\widehat{f}"] \arrow[d, "\pi_1"] & U \times G \arrow[d, "\pi_2"] \\ V \arrow[r, "f"] & U \end{tikzcd}$$

We define a section σ in

$$\begin{tikzcd} (V \times G) \times_G (U \times G) \arrow[d, "\omega"] \\ V \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

as follows. For $e_1 \in V \times G$, with $x = \pi_1(e_1) \in V$, we set

$$\sigma(x) = [e_1, \widehat{f}(e_1)].$$

This map is well-defined, since for any $g \in G$ we have:

$$[e_1 \cdot g, \widehat{f}(e_1 \cdot g)] = [e_1 \cdot g, \widehat{f}(e_1) \cdot g] = [e_1 \cdot g, g^{-1} \cdot \widehat{f}(e_1)] = [e_1, \widehat{f}(e_1)].$$

Now, it is an exercise in point-set topology (using the local definition of a bundle map) to show that σ is continuous.

Conversely, given a section of $E_1 \times_G E_2 \overset{\omega}{\mapsto} X$, we define a bundle by $(f, \widehat{f})$ by

$$\widehat{f}(e_1) = e_2,$$

where $\sigma(\pi_1(e_1)) = [(e_1, e_2)]$. Note that this is an equivariant map because

$$[e_1 \cdot g, e_2 \cdot g] = [e_1 \cdot g, g^{-1} \cdot e_2] = [e_1, e_2],$$

hence $\widehat{f}(e_1 \cdot g) = e_2 \cdot g = \widehat{f}(e_1) \cdot g$. Thus $\widehat{f}$ descends to a map $f : X \to Y$ on the orbit spaces. We leave it as an exercise to check that $(f, \widehat{f})$ is indeed a bundle map, i.e., to show that locally $\widehat{f}(v, g) = (f(v), d(v)g)$ with $d(v) \in G$ and $d : V \to G$ a continuous function. $\square$

The following result will be needed in the proof of Theorem 3.2.10.

Lemma 3.2.12. *Let $\pi : E \to X \times I$ be a bundle, and let $\pi_0 := i_0^* \pi : E_0 \to X$ be the pullback of π under $i_0 : X \to X \times I$, $x \mapsto (x, 0)$. Then $\pi \cong (pr_1)^* \pi_0 \cong \pi_0 \times id_I$, where $pr_1 : X \times I \to X$ is the projection map.*

Proof. It suffices to find a bundle map $(pr_1, \widehat{pr}_1)$ so that the following diagram commutes

$$\begin{tikzcd}
E_0 \arrow[r, "\widehat{i}_0"] \arrow[d, "\pi_0"'] & E \arrow[r, dotted, "\widehat{pr}_1"] \arrow[d, "\pi"] & E_0 \arrow[d, "\pi_0"] \\
X \arrow[r, hook, "i_0"] & X \times I \arrow[r, "\mathrm{pr}_1"] & X
\end{tikzcd}$$

By Theorem 3.2.11, this is equivilant to the existence of a section σ of $\omega : E \times_G E_0 \to X \times I$. Note that there exists a section σ_0 of $\omega_0 : E_0 \times_G E_0 \to X = X \times \{0\}$, corresponding to the bundle map $(id_X, id_{E_0}) : \pi_0 \to \pi_0$. Then composing σ_0 with the top inclusion arrow, we get the following diagram

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\sigma_0"] \arrow[d, hook] & E \times_G E_0 \arrow[d, "\omega"] \\
X \times I \arrow[r, "id"] \arrow[ur, dotted, "s"] & X \times I
\end{tikzcd}$$

Since ω is a fibration, by the homotopy lifting property one can extend $s\sigma_0$ to a section σ of ω. $\square$

We can now finish the proof of Theorem 3.2.10.

Proof of Theorem 3.2.10. Let $H : X \times I \to Y$ be a homotopy between f and g, with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Consider the induced bundle $H^* \pi$ over $X \times I$. Then we have the following diagram.

$$\begin{tikzcd}
f^*E \arrow[rr] \arrow[dd, "f^*\pi"'] & & H^*E \arrow[r, "\widehat{H}"] \arrow[dd, "H^*\pi"] & E \arrow[dd, "\pi"] \\
& g^*E \arrow[ur] \arrow[dd, "g^*\pi", near start] & & \\
X \times \{0\} \arrow[rr, hook, "i_0"] & & X \times I \arrow[r, "H"] \arrow[dr, "\mathrm{pr}_1"] & Y \\
& X \times \{1\} \arrow[ur, hook, "i_1"] & & X
\end{tikzcd}$$

Since $f = H(-,0)$, we get $f^*\pi = i_0^* H^*\pi$. By Lemma 3.2.12, $H^*\pi \cong pr_1^*(f^*\pi) \cong pr_1^*(g^*\pi)$, and thus $f^*\pi = i_0^* H^*\pi = i_0^* \,\mathrm{pr}_1^* \,g^*\pi = g^*\pi$. $\square$

We conclude this section with the following important consequence of Theorem 3.2.11

Corollary 3.2.13. *A principle G-bundle $\pi : E \to X$ is trivial if and only if π has a section.*

Proof. The bundle π is trivial if and only if $\pi = ct^*\pi'$, with $ct : X \to$ *point* the constant map, and $\pi' : G \to$ *point* the trvial bundle over a point space. This is equivalent to saying that there is a bundle map

$$\begin{tikzcd}
E \arrow[r, dotted] \arrow[d, "\pi"] & G \arrow[d, "\pi'"] \\
X \arrow[r, "ct"] & point
\end{tikzcd}$$

or, by Theorem 3.2.11, to the existence of a section of the bundle $\omega : E \times_G G \to X$. On the other hand, $\omega \cong \pi$, since $E \times_G G \to X$ looks locally like

$$\pi^{-1}(U_\alpha) \times G \big/ \sim \; \cong U_\alpha \times G \times G \big/ (u, g_1, g_2) \sim (u, g_1 g^{-1}, g g_2) \cong U_\alpha \times G,$$

with the last homeomorphism defined by $[(u, g_1, g_2)] \mapsto (u, g_1 g_2)$.

Altogether, π is trivial if and only if $\pi : E \mapsto X$ has a section. $\square$

3.3 *Classification of principal G-bundles*

Let us assume for now that there exists a principal G-bundle $\pi_G : EG \to BG$, with contractible total space EG. As we will see below, such a bundle plays an essential role in the classification theory of principal G-bundles. Its base space BG turns out to be unique up to homotopy, and it is called the *classifying space for principal G-bundles* due to the following fundamental result:

Theorem 3.3.1. *If X is a CW-complex, there exists a bijective correspondence*

$$\begin{aligned} \Phi : \mathcal{P}(X, G) &\xrightarrow{\cong} [X, BG] \\ f^*\pi_G &\leftarrow\!\shortmid f \end{aligned}$$

Proof. By Theorem 3.2.10, Φ is well-defined.

Let us next show that Φ is onto. Let $\pi \in \mathcal{P}(X, G)$, $\pi : E \to X$. We need to show that $\pi \cong f^*\pi_G$ for some map $f : X \to BG$, or equivalently, that there is a bundle map $(f, \widehat{f}) : \pi \to \pi_G$. By Theorem 3.2.11, this is equivalent to the existence of a section of the bundle $E \times_G EG \to X$ with fiber EG. Since EG is contractible, such a section exists by the following:

Lemma 3.3.2. *Let X be a CW complex, and $\pi : E \to X \in \mathcal{B}(X, G, F, \rho)$ with $\pi_i(F) = 0$ for all $i \geq 0$. If $A \subseteq X$ is a subcomplex, then every section of π over A extends to a section defined on all of X. In particular, π has a section. Moreover, any two sections of π are homotopic.*

Proof. Given a section $\sigma_0 : A \to E$ of π over A, we extend it to a section $\sigma : X \to E$ of π over X by using induction on the dimension of cells in $X - A$. So it suffices to assume that X has the form

$$X = A \cup_\phi e^n,$$

where e^n is an n-cell in $X - A$, with attaching map $\phi : \partial e^n \to A$. Since e^n is contractible, π is trivial over e^n, so we have a commutative diagram

$$\begin{tikzcd}
& \pi^{-1}(e^n) \arrow[r, "\cong", "h"'] \arrow[d, "\pi"] & e^n \times F \arrow[dl, "\mathrm{pr}_1"'] \\
\partial e^n \arrow[ur, "\sigma_0"] \arrow[r, hook] & e^n \arrow[ur, bend right, "\sigma"'] &
\end{tikzcd}$$

with $h : \pi^{-1}(e^n) \to e^n \times F$ the trivializing chart for π over e^n, and σ to be defined. After composing with h, we regard the restriction of σ_0 over ∂e^n as given by

$$\sigma_0(x) = (x, \tau_0(x)) \in e^n \times F,$$

with $\tau_0 : \partial e^n \cong S^{n-1} \to F$. Since $\pi_{n-1}(F) = 0$, τ_0 extends to a map $\tau : e^n \to F$ which can be used to extend σ_0 over e^n by setting

$$\sigma(x) = (x, \tau(x)).$$

After composing with h^{-1}, we get the desired extension of σ_0 over e^n.

Let us now assume that σ and σ' are two sections of π. To find a homotopy between σ and σ', it suffices to construct a section Σ of $\pi \times id_I : E \times I \to X \times I$. Indeed, if such Σ exists, then $\Sigma(x, t) = (\sigma_t(x), t)$, and σ_t provides the desired homotopy. Now, by regarding

σ as a section of $\pi \times id_I$ over $X \times \{0\}$, and σ' as a section of $\pi \times id_I$ over $X \times \{1\}$, the question reduces to constructing a section of $\pi \times id_I$, which extends the section over $X \times \{0,1\}$ defined by (σ, σ'). This can be done as in the first part of the proof. $\square$

In order to finish the proof of Theorem 3.3.1, it remains to show that Φ is a one-to-one map. If $\pi_0 = f^*\pi_G \cong g^*\pi_G = \pi_1$, we will show that $f \simeq g$. Note that we have the following commutative diagrams:

$$\begin{array}{ccc} E_0 = f^*E_G & \xrightarrow{\widehat{f}} & E_G \\ \Big\downarrow{\scriptstyle \pi_0} & & \Big\downarrow{\scriptstyle \pi_G} \\ X = X \times \{0\} & \xrightarrow{f} & B_G \end{array}$$

$$\begin{array}{ccc} E_0 \cong E_1 = g^*E_G & \xrightarrow{\widehat{g}} & E_G \\ \Big\downarrow{\scriptstyle \pi_0} & & \Big\downarrow{\scriptstyle \pi_G} \\ X = X \times \{1\} & \xrightarrow{g} & B_G \end{array}$$

where we regard $\widehat{g}$ as defined on E_0 via the isomorphism $\pi_0 \cong \pi_1$. By putting together the above diagrams, we have a commutative diagram

$$\begin{array}{ccccc} E_0 \times I & \xleftarrow{\hookleftarrow} & E_0 \times \{0,1\} & \xrightarrow{\widehat{\alpha} = (\widehat{f},0) \cup (\widehat{g},1)} & E_G \\ \Big\downarrow{\scriptstyle \pi_0 \times Id} & & \Big\downarrow{\scriptstyle \pi_0 \times \{0,1\}} & & \Big\downarrow{\scriptstyle \pi_G} \\ X \times I & \xleftarrow{\hookleftarrow} & X \times \{0,1\} & \xrightarrow{\alpha = (f,0) \cup (g,1)} & B_G \end{array}$$

Therefore, it suffices to extend $(\alpha, \widehat{\alpha})$ to a bundle map $(H, \widehat{H}) : \pi_0 \times Id \to \pi_G$, and then H will provide the desired homotopy $f \simeq g$.

By Theorem 3.2.11, such a bundle map $(H, \widehat{H})$ corresponds to a section σ of the fiber bundle

$$\omega : (E_0 \times I) \times_G E_G \to X \times I.$$

On the other hand, the bundle map $(\alpha, \widehat{\alpha})$ already gives a section σ_0 of the fiber bundle

$$\omega_0 : (E_0 \times \{0,1\}) \times_G E_G \to X \times \{0,1\},$$

which under the obvious inclusion $(E_0 \times \{0,1\}) \times_G E_G \subseteq (E_0 \times I) \times_G E_G$ can be regarded as a section of ω over the subcomplex $X \times \{0,1\}$. Since EG is contractible, Lemma 3.3.2 allows us to extend σ_0 to a section σ of ω defined on $X \times I$, as desired. $\square$

Example 3.3.3. We give here a more conceptual reasoning for the assertion of Example 3.1.20. By Theorem 3.3.1, we have

$$\mathcal{B}(S^n, G, F, \rho) \cong \mathcal{P}(S^n, G) \cong [S^n, BG] = \pi_n(BG) \cong \pi_{n-1}(G),$$

where the last isomorphism follows from the homotopy long exact sequence for π_G, since EG is contractible.

Back to the universal principal G-bundle, we have the following

Theorem 3.3.4. *Let G be a locally compact topological group. Then a universal principal G-bundle $\pi_G : EG \to BG$ exists (i.e., satisfying $\pi_i(EG) = 0$ for all $i \geq 0$), and the construction is functorial in the sense that a continuous group homomorphism $\mu : G \to H$ induces a bundle map $(B\mu, E\mu) : \pi_G \to \pi_H$. Moreover, the classifying space B_G is unique up to homotopy.*

Proof. To show that BG is unique up to homotopy, let us assume that $\pi_G : E_G \to B_G$ and $\pi'_G : E'_G \to B'_G$ are universal principal G-bundles. By regarding π_G as the universal principal G-bundle for π'_G, we get a map $f : B'_G \to B_G$ such that $\pi'_G = f^*\pi_G$, i.e., a bundle map:

$$\begin{tikzcd} E'_G \arrow[r, "\hat{f}"] \arrow[d, "\pi'_G"] & E_G \arrow[d, "\pi_G"] \\ B'_G \arrow[r, "f"] & B_G \end{tikzcd}$$

Similarly, y regarding π'_G as the universal principal G-bundle for π_G, there exists a map $g : B_G \to B'_G$ such that $\pi_G = g^*\pi'_G$. Therefore,

$$\pi_G = g^*\pi'_G = g^*f^*\pi_G = (f \circ g)^*\pi_G.$$

On the other hand, we have $\pi_G = (id_{B_G})^*\pi_G$, so by Theorem 3.3.1 we get that $f \circ g \simeq id_{B_G}$. Similarly, we get $g \circ f \simeq id_{B'_G}$, and hence $f : B'_G \to B_G$ is a homotopy equivalence.

We will not discuss the existence of the universal bundle here, instead we will indicate the universal G-bundle, as needed, in specific examples. □

Example 3.3.5. Recall from Section 1.12 that we have a fiber bundle

$$O(n) \hookrightarrow V_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty), \tag{3.3.1}$$

with $V_n(\mathbb{R}^\infty)$ contractible. In particular, the uniqueness part of Theorem 3.3.4 tells us that $BO(n) \simeq G_n(\mathbb{R}^\infty)$ is the classifying space for rank n real vector bundles. Similarly, there is a fiber bundle

$$U(n) \hookrightarrow V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty), \tag{3.3.2}$$

with $V_n(\mathbb{C}^\infty)$ contractible. Therefore, $BU(n) \simeq G_n(\mathbb{C}^\infty)$ is the classifying space for rank n complex vector bundles.

Before moving to the next example, let us mention here without proof the following useful result:

Theorem 3.3.6. *Let G be an abelian group, and let X be a CW complex. There is a natural bijection*

$$T : [X, K(G,n)] \longrightarrow H^n(X,G)$$

$$[f] \mapsto f^*(\alpha)$$

where $\alpha \in H^n(K(G,n),G) \cong \mathrm{Hom}(H_n(K(G,n),\mathbb{Z}),G)$ is given by the inverse of the Hurewicz isomorphism $G = \pi_n(K(G,n)) \to H_n(K(G,n),\mathbb{Z})$.

Example 3.3.7 (Classification of real line bundles)**.** Let $G = \mathbb{Z}/2$ and consider the principal $\mathbb{Z}/2$-bundle $\mathbb{Z}/2 \hookrightarrow S^\infty \to \mathbb{R}P^\infty$. Since S^∞ is contractible, the uniqueness of the universal bundle yields that $B\mathbb{Z}/2 \cong \mathbb{R}P^\infty$. In particular, we see that $\mathbb{R}P^\infty$ classifies the real line (i.e., rank-one) bundles. Since we also have that $\mathbb{R}P^\infty = K(\mathbb{Z}/2,1)$, we get:

$$\mathcal{P}(X,\mathbb{Z}/2) = [X, B\mathbb{Z}/2] = [X, K(\mathbb{Z}/2,1)] \cong H^1(X,\mathbb{Z}/2)$$

for any CW complex X, where the last identification follows from Theorem 3.3.6. Let now π be a real line bundle on a CW complex X, with classifying map $f_\pi : X \to \mathbb{R}P^\infty$. Since $H^*(\mathbb{R}P^\infty,\mathbb{Z}/2) \cong \mathbb{Z}/2[w]$, with w a generator of $H^1(\mathbb{R}P^\infty,\mathbb{Z}/2)$, we get a well-defined degree one cohomology class

$$w_1(\pi) := f_\pi^*(w)$$

called the *first Stiefel-Whitney class of* π. The bijection $\mathcal{P}(X,\mathbb{Z}/2) \xrightarrow{\cong} H^1(X,\mathbb{Z}/2)$ is then given by $\pi \mapsto w_1(\pi)$, so real line bundles on X are classified by their first Stiefel-Whitney classes.

Example 3.3.8 (Classification of complex line bundles)**.** Let $G = S^1$ and consider the principal S^1-bundle $S^1 \hookrightarrow S^\infty \to \mathbb{C}P^\infty$. Since S^∞ is contractible, the uniqueness of the universal bundle yields that $BS^1 \cong \mathbb{C}P^\infty$. In particular, as $S^1 = GL(1,\mathbb{C})$, we see that $\mathbb{C}P^\infty$ classifies the complex line (i.e., rank-one) bundles. Since we also have that $\mathbb{C}P^\infty = K(\mathbb{Z},2)$, we get:

$$\mathcal{P}(X,S^1) = [X, BS^1] = [X, K(\mathbb{Z},2)] \cong H^2(X,\mathbb{Z})$$

for any CW complex X, where the last identification follows from Theorem 3.3.6. Let now π be a complex line bundle on a CW complex X, with classifying map $f_\pi : X \to \mathbb{C}P^\infty$. Since $H^*(\mathbb{C}P^\infty,\mathbb{Z}) \cong \mathbb{Z}[c]$, with c a generator of $H^2(\mathbb{C}P^\infty,\mathbb{Z})$, we get a well-defined degree two cohomology class

$$c_1(\pi) := f_\pi^*(c)$$

called the *first Chern class of* π. The bijection $\mathcal{P}(X,S^1) \xrightarrow{\cong} H^2(X,\mathbb{Z})$ is then given by $\pi \mapsto c_1(\pi)$, so complex line bundles on X are classified by their first Chern classes.

Remark 3.3.9. If X is any orientable closed oriented surface, then $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$, so Example 3.3.8 shows that isomorphism classes of complex line bundles on X are in bijective correspondence with the set of integers. On the other hand, if X is a non-orientable closed surface, then $H^2(X,\mathbb{Z}) \cong \mathbb{Z}/2$, so there are only two isomorphism classes of complex line bundles on such a surface.

3.4 *Exercises*

1. Let $p : S^2 \to \mathbb{R}P^2$ be the (oriented) double cover of $\mathbb{R}P^2$. Since $\mathbb{R}P^2$ is a non-orientable surface, we know by Remark 3.3.9 that there are only two isomorphism classes of complex line bundles on $\mathbb{R}P^2$: the trivial one, and a non-trivial complex line bundle which we denote by $\pi : E \to \mathbb{R}P^2$. On the other hand, since S^2 is a closed orientable surface, the isomorphism classes of complex line bundles on S^2 are in bijection with $\mathbb{Z}$. Which integer corresponds to complex line bundle $p^*\pi : p^*E \to S^2$ on S^2?

2. Consider a locally trivial fiber bundle $S^2 \hookrightarrow E \overset{\pi}{\to} S^2$. Recall that such π can be regarded as a fiber bundle with structure group $G = Homeo(S^2) \cong SO(3)$. By the classification Theorem 3.3.1, $SO(3)$-bundles over S^2 correspond to elements in

$$[S^2, BSO(3)] = \pi_2(BSO(3)) \cong \pi_1(SO(3)).$$

(a) Show that $\pi_1(SO(3)) \cong \mathbb{Z}/2$. (Hint: Show that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.)

(b) What is the non-trivial $SO(3)$-bundle over S^2?

3. Let $\pi : E \to X$ be a principal S^1-bundle over the simply-connected space X. Let $a \in H^1(S^1,\mathbb{Z})$ be a generator. Show that

$$c_1(\pi) = d_2(a),$$

where d_2 is the differential on the E_2-page of the Leray-Serre spectral sequence associated to π, i.e., $E_2^{p,q} = H^p(X, H^q(S^1)) \Rightarrow H^{p+q}(E,\mathbb{Z})$.

4. By the classification Theorem 3.3.1, (isomorphism classes of) S^1-bundles over S^2 are given by

$$[S^2, BS^1] = \pi_2(BS^1) \cong \pi_1(S^1) \cong \mathbb{Z}$$

and this correspondence is realized by the first Chern class, i.e., $\pi \mapsto c_1(\pi)$.

(a) What is the first Chern class of the Hopf bundle $S^1 \hookrightarrow S^3 \to S^2$?

(b) What is the first Chern class of the sphere (or unit) bundle of the tangent bundle TS^2?

(c) Construct explicitely the S^1-bundle over S^2 corresponding to $n \in \mathbb{Z}$. (Hint: Think of lens spaces, and use the above Exercise 3 and Example 2.8.2.)

4
Vector Bundles. Characteristic classes. Cobordism. Applications.

4.1 Chern classes of complex vector bundles

We begin with the following

Proposition 4.1.1.

$$H^* (BU(n); \mathbb{Z}) \cong \mathbb{Z} [c_1, \cdots, c_n],$$

with $\deg c_i = 2i$

Proof. Recall from Example 2.12.1 that $H^*(U(n); \mathbb{Z})$ is a free $\mathbb{Z}$-algebra on odd degree generators $x_1, \cdots, x_{2n-1}$, with $\deg(x_i) = i$, i.e.,

$$H^*(U(n); \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[x_1, \cdots, x_{2n-1}].$$

Then using the Leray-Serre spectral sequence for the universal $U(n)$-bundle, and using the fact that $EU(n)$ is contractible, yields the desired result.

Alternatively, the functoriality of the universal bundle construction yields that for any subgroup $H < G$ of a topological group G, there is a fibration $G/H \hookrightarrow BH \to BG$. In our case, consider $U(n-1)$ as a subgroup of $U(n)$ via the identification $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Hence, there exists fibration

$$U(n)/U(n-1) \cong S^{2n-1} \hookrightarrow BU(n-1) \to BU(n).$$

Then the Leray-Serre spectral sequence and induction on n gives the desired result, where we use the fact that $BU(1) \simeq \mathbb{C}P^\infty$ and $H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[c]$ with $\deg c = 2$. $\square$

Definition 4.1.2. *The generators $c_1, \cdots, c_n$ of $H^* (BU(n); \mathbb{Z})$ are called the universal Chern classes of $U(n)$-bundles.*

Recall from the classification theorem 3.3.1, that given $\pi : E \to X$ a principal $U(n)$-bundle, there exists a "classifying map" $f_\pi : X \to BU(n)$ such that $\pi \cong f_\pi^* \pi_{U(n)}$.

Definition 4.1.3. *The i-th Chern class of the $U(n)$-bundle $\pi : E \to X$ with classifying map $f_\pi : X \to BU(n)$ is defined as*

$$c_i(\pi) := f_\pi^*(c_i) \in H^{2i}(X; \mathbb{Z}).$$

Remark 4.1.4. Note that if π is a $U(n)$-bundle, then by definition we have that $c_i(\pi) = 0$, if $i > n$.

Let us now discuss important properties of Chern classes.

Proposition 4.1.5. *If $\mathcal{E}$ denotes the trivial $U(n)$-bundle on a space X, then $c_i(\mathcal{E}) = 0$ for all $i > 0$.*

Proof. Indeed, the trivial bundle is classified by the constant map $ct : X \to BU(n)$, which induces trivial homomorphisms in positive degree cohomology. □

Proposition 4.1.6 (Functoriality of Chern classes). *If $f : Y \to X$ is a continuous map, and $\pi : E \to X$ is a $U(n)$-bundle, then $c_i(f^*\pi) = f^* c_i(\pi)$, for any i.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
f^*E \arrow[r, "\widehat{f}"] \arrow[d, "f^*\pi"] & E \arrow[r] \arrow[d, "\pi"] & EU(n) \arrow[d, "\pi_{U(n)}"] \\
Y \arrow[r, "f"] & X \arrow[r, "f_\pi"] & BU(n)
\end{tikzcd}$$

which shows that $f_\pi \circ f$ classifies the $U(n)$-bundle $f^*\pi$ on Y. Therefore,

$$\begin{aligned}
c_i(f^*\pi) &= (f_\pi \circ f)^* c_i \\
&= f^* \left(f_\pi^* c_i \right) \\
&= f^* c_i \left(\pi \right).
\end{aligned}$$

□

Definition 4.1.7. *The total Chern class of a $U(n)$-bundle $\pi : E \to X$ is defined by*

$$c(\pi) = c_0(\pi) + c_1(\pi) + \cdots c_n(\pi) = 1 + c_1(\pi) + \cdots c_n(\pi) \in H^*(X; \mathbb{Z}),$$

as an element in the cohomology ring of the base space.

Definition 4.1.8 (Whitney sum). *Let $\pi_1 \in \mathcal{P}(X, U(n))$, $\pi_2 \in \mathcal{P}(X, U(m))$. Consider the product bundle $\pi_1 \times \pi_2 \in \mathcal{P}(X \times X, U(n) \times U(m))$, which* *can be regarded as a $U(n+m)$-bundle via the canonical inclusion $U(n) \times U(m) \hookrightarrow U(n+m)$, $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. The Whitney sum of the bundles π_1 and π_2 is defined as:*

$$\pi_1 \oplus \pi_2 := \Delta^*(\pi_1 \times \pi_2),$$

where $\Delta : X \to X \times X$ is the diagonal map given by $x \mapsto (x, x)$.

Remark 4.1.9. The Whitney sum $\pi_1 \oplus \pi_2$ of π and π_s is the $U(n+m)$-bundle on X with transition functions (in a common refinement of the trvialization atlases for π_1 and π_2) given by $\left(\begin{array}{c|c} g^1_{\alpha\beta} & 0 \\ \hline 0 & g^2_{\alpha\beta} \end{array}\right)$ where $g^i_{\alpha\beta}$ are the transition function of π_i, $i = 1, 2$.

Proposition 4.1.10 (Whitney sum formula)**.** *If $\pi_1 \in \mathcal{P}(X, U(n))$ and $\pi_2 \in \mathcal{P}(X, U(m))$, then*

$$c(\pi_1 \oplus \pi_2) = c(\pi_1) \cup c(\pi_2).$$

Equivalently, $c_k(\pi_1 \oplus \pi_2) = \sum_{i+j=k} c_i(\pi_1) \cup c_j(\pi_2)$

Proof. First note that

$$B(U(n) \times U(m)) \simeq BU(n) \times BU(m). \tag{4.1.1}$$

Indeed, by taking the product of the universal bundles for $U(n)$ and $U(m)$, we get a $U(n) \times U(m)$-bundle over $BU(n) \times BU(m)$, with total space $EU(n) \times EU(m)$:

$$U(n) \times U(m) \hookrightarrow EU(n) \times EU(m) \to BU(n) \times BU(m). \tag{4.1.2}$$

Since $\pi_i(EU(n) \times EU(m)) \cong \pi_i(EU(n)) \times \pi_i(EU(m)) \cong 0$ for all i, it follows that (4.1.2) is the universal bundle for $U(n) \times U(m)$, thus proving (4.1.1).

Next, the inclusion $U(n) \times U(m) \hookrightarrow U(n+m)$ yields a map

$$\omega : B(U(n) \times U(m)) \simeq BU(n) \times BU(m) \longrightarrow BU(n+m).$$

By using the Künneth formula, one can show (e.g., see Milnor's book, p.164) that:

$$\omega^* c_k = \sum_{i+j=k} c_i \times c_j.$$

Therefore,

$$\begin{aligned} c_k(\pi_1 \oplus \pi_2) &= c_k(\Delta^*(\pi_1 \times \pi_2)) \\ &= \Delta^* c_k(\pi_1 \times \pi_2) \\ &= \Delta^*(f^*_{\pi_1 \times \pi_2}(c_k)) \end{aligned}$$

$$
\begin{aligned}
&= \Delta^*(f^*_{\pi_1} \times f^*_{\pi_2})(\omega^* c_k) \\
&= \sum_{i+j=k} \Delta^*(f^*_{\pi_1}(c_i) \times f^*_{\pi_2}(c_j)) \\
&= \sum_{i+j=k} \Delta^*(c_i(\pi_1) \times c_j(\pi_2)) \\
&= \sum_{i+j=k} c_i(\pi_1) \cup c_j(\pi_2).
\end{aligned}
$$

Here, we use the fact that the classifying map for $\pi_1 \times \pi_2$, regarded as a $U(n+m)$-bundle is $\omega \circ (f_{\pi_1} \times f_{\pi_2})$. $\square$

Since the trivial bundle has trivial Chern classes in positive degrees, we get

Corollary 4.1.11 (Stability of Chern classes)**.** *Let $\mathcal{E}^1$ be the trivial $U(1)$-bundle. Then*

$$c(\pi \oplus \mathcal{E}^1) = c(\pi).$$

4.2 *Stiefel-Whitney classes of real vector bundles*

As in Proposition 4.1.1, one easily obtains the following

Proposition 4.2.1.

$$H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2\,[w_1, \cdots, w_n]\,,$$

with $\deg w_i = i$.

Proof. This can be easily deduced by induction on n from the Leray-Serre spectral sequence of the fibration

$$O(n)/O(n-1) \cong S^{n-1} \hookrightarrow BO(n-1) \to BO(n),$$

by using the fact that $BO(1) \simeq \mathbb{R}P^\infty$ and $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1]$. $\square$

Definition 4.2.2. *The generators $w_1, \cdots, w_n$ of $H^*(BO(n); \mathbb{Z}/2)$ are called the universal Stiefel-Whitney classes of $O(n)$-bundles.*

Recall from the classification theorem 3.3.1 that, given $\pi : E \to X$ a principal $O(n)$-bundle, there exists a "classifying map" $f_\pi : X \to BO(n)$ such that $\pi \cong f^*_\pi \pi_{U(n)}$.

Definition 4.2.3. *The i-th Stiefel-Whitney class of the $O(n)$-bundle $\pi : E \to X$ with classifying map $f_\pi : X \to BO(n)$ is defined as*

$$w_i(\pi) := f^*_\pi(w_i) \in H^i(X; \mathbb{Z}/2).$$

The total Stiefel-Whitney class of π is defined by

$$w(\pi) = 1 + w_1(\pi) + \cdots w_n(\pi) \in H^*(X; \mathbb{Z}/2),$$

as an element in the cohomology ring with $\mathbb{Z}/2$-coefficients.

Remark 4.2.4. If π is a $O(n)$-bundle, then by definition we have that $w_i(\pi) = 0$, if $i > n$. Also, since the trivial bundle is classified by the constant map, it follows that the positive-degree Stiefel-Whitney classes of a trivial $O(n)$-bundle are all zero.

Stiefel-Whitney classes of $O(n)$-bundles enjoy similar properties as the Chern classes.

Proposition 4.2.5. *The Stiefel-Whitney classes satisfy the functoriality property and the Whitney sum formula.*

4.3 *Stiefel-Whitney classes of manifolds and applications*

If M is a smooth manifold, its tangent bundle TM can be regarded as an $O(n)$-bundle.

Definition 4.3.1. *The Stiefel-Whitney classes of a smooth manifold M are defined as*

$$w_i(M) := w_i(TM).$$

Theorem 4.3.2 (Wu)**.** *Stiefel-Whitney classes are homotopy invariants, i.e., if $h : M_1 \to M_2$ is a homotopy equivalence then $h^* w_i(M_2) = w_i(M_1)$, for any $i \geq 0$.*

Characteristic classes are particularly useful for solving a wide range of topological problems, including the following:

(a) Given an n-dimensional smooth manifold M, find the minimal integer k such that M can be embedded/immersed in $\mathbb{R}^{n+k}$.

(b) Given an n-dimensional smooth manifold M, is there an $(n+1)$-dimensional smooth manifold W such that $\partial W = M$?

(c) Given a topological manifold M, classify/find exotic smooth structures on M.

The embedding problem

Let $f : M^m \to N^{m+k}$ be an embedding of smooth manifolds. Then

$$f^* TN = TM \oplus \nu, \tag{4.3.1}$$

where ν is the normal bundle of M in N. In particular, ν is of rank k, hence $w_i(\nu) = 0$ for all $i > k$. The Whitney product formula for Stiefel-Whitney classes, together with (4.3.1), yields that

$$f^* w(N) = w(M) \cup w(\nu). \tag{4.3.2}$$

Note that $w(M) = 1 + w_1(M) + \cdots$ is invertible in $H^*(M; \mathbb{Z}/2)$, hence

$$w(\nu) = w(M)^{-1} \cup f^* w(N).$$

In particular, if $N = \mathbb{R}^{m+k}$, one gets $w(\nu) = w(M)^{-1}$.

The same considerations apply in the case when $f : M^m \to N^{m+k}$ is required to be only an immersion. In this case, the existence of the normal bundle ν is guaranteed by the following simple result:

Lemma 4.3.3. *Let*

$$\begin{tikzcd} E_1 \arrow[rr, "i"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

be a linear monomorphism of vector bundles, i.e., in local coordinates, i is given by $U \times \mathbb{R}^n \to U \times \mathbb{R}^m$ ($n \leq m$), $(u, v) \mapsto (u, \ell(u)v)$, where $\ell(u)$ is a linear map of rank n for all $u \in U$. Then there exists a vector bundle $\pi_1^\perp : E_1^\perp \to X$ so that $\pi_2 \cong \pi_1 \oplus \pi_1^\perp$.

To summarize, we showed that if $f : M^m \to N^{m+k}$ is an embedding or an immersion of smooth manifolds, than one can solve for $w(\nu)$ in (4.3.2), where ν is the normal bundle of M in N. Moreover, since ν has rank k, we must have that $w_i(\nu) = 0$ for all $i > k$.

The following result of Whitney states that one can always solve for $w(\nu)$ if the codimension k is large enough. More precisely, we have:

Theorem 4.3.4 (Whitney)**.** *Any smooth map $f : M^m \to N^{m+k}$ is homotopic to an embedding for $k \geq m + 1$.*

Let us now consider the problem of embedding (or immersing) $\mathbb{R}P^m$ into $\mathbb{R}^{m+k}$. If ν is the corresponding normal bundle of rank k, we have that $w(\nu) = w(\mathbb{R}P^m)^{-1}$.

We need the following calculation:

Theorem 4.3.5.

$$w(\mathbb{R}P^m) = (1 + x)^{m+1}, \tag{4.3.3}$$

where $x \in H^1(\mathbb{R}P^m; \mathbb{Z}/2)$ is a generator.

Before proving Theorem 4.3.5, let us discuss some examples.

Example 4.3.6. Let us investigate constraints on the codimension k of an embedding of $\mathbb{R}P^9$ into $\mathbb{R}^{9+k}$. By Theorem 4.3.5, we have:

$$w(\mathbb{R}P^9) = (1+x)^{10} = (1+x)^8(1+x)^2 = (1+x^8)(1+x^2) = 1+x^2+x^8,$$

since $x^{10} = 0$ in $H^*(\mathbb{R}P^9; \mathbb{Z}/2)$. Therefore,

$$w(\mathbb{R}P^9)^{-1} = 1 + x^2 + x^4 + x^6.$$

If an embedding (or immersion) f of $\mathbb{R}P^9$ into $\mathbb{R}^{9+k}$ exists, then $w(\nu) = w^{-1}(\mathbb{R}P^9)$, where ν is the corresponding rank k normal bundle. In particular, $w_6(\nu) \neq 0$. Since we must have $w_i(\nu) = 0$ for $i > k$, we conclude that $k \geq 6$. For example, this shows that $\mathbb{R}P^9$ cannot be embedded into $\mathbb{R}^{14}$.

Example 4.3.7. Similarly, if $m = 2^r$ then

$$w(\mathbb{R}P^{2^r}) = (1+x)^{2^r+1} = (1+x)^{2^r}(1+x) = 1 + x + x^{2^r}.$$

If there exists an embedding or immersion $\mathbb{R}P^{2^r} \hookrightarrow \mathbb{R}^{2^r+k}$ with normal bundle ν, then

$$w(\nu) = w(\mathbb{R}P^{2^r})^{-1} = 1 + x + x^2 + \cdots + x^{2^r-1},$$

hence $k \geq 2^r - 1 = m - 1$. In particular, $\mathbb{R}P^8$ cannot be immersed in $\mathbb{R}^{14}$. In this case, one can actually construct an immersion of $\mathbb{R}P^{2^r}$ into $\mathbb{R}^{2^r+k}$ for any $k \geq 2^r - 1$, due to the following result:

Theorem 4.3.8 (Whitney)**.** *An m-dimensional smooth manifold can be embedded in $\mathbb{R}^{2m}$ and immersed in $\mathbb{R}^{2m-1}$.*

Definition 4.3.9. *A smooth manifold is parallelizable if its tanget bundle TM is trivial.*

Example 4.3.10. Lie groups, hence in particular S^1, S^3 and S^7, are parallelizable.

Theorem 4.3.5 can be used to prove the following:

Theorem 4.3.11. *$w(\mathbb{R}P^m) = 1$ if and only if $m + 1 = 2^r$ for some r. In particular, if $\mathbb{R}P^m$ is parallelizable, then $m + 1 = 2^r$ for some r.*

Proof. Note that if $\mathbb{R}P^m$ is parallelizable, then $w(\mathbb{R}P^m) = 1$ since $T\mathbb{R}P^m$ is a trivial bundle. If $m + 1 = 2^r$, then $w(\mathbb{R}P^m) = (1+x)^{2^r} = 1 + x^{2^r} = 1 + x^{m+1} = 1$. On the other hand, if $m + 1 = 2^r k$, where $k > 1$ is an odd integer, we have

$$w(\mathbb{R}P^m) = [(1+x)^{2^r}]^k = (1 + x^{2^r})^k = 1 + kx^{2^r} + \cdots \neq 1,$$

since $x^{2^r} \neq 0$ (indeed, $2^r < 2^r k = m + 1$). $\square$

In fact, the following result holds:

Theorem 4.3.12 (Adams)**.** *$\mathbb{R}P^m$ is parallelizable if and only if $m \in \{1, 3, 7\}$.*

Let us now get back to the proof of Theorem 4.3.5

Proof of Theorem 4.3.5. The idea is to find a splitting of (a stabilization of) $T\mathbb{R}P^m$ into line bundles, then to apply the Whitney sum formula.

Recall that $O(1)$-bundles on $\mathbb{R}P^m$ are classified by

$$[\mathbb{R}P^m, BO(1)] = [\mathbb{R}P^m, K(\mathbb{Z}/2, 1)] \cong H^1(\mathbb{R}P^m; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

We'll denote by $\mathcal{E}^1$ the trivial $O(1)$-bundle, and let π be the non-trivial $O(1)$-bundle on $\mathbb{R}P^m$. Since $O(1) \cong \mathbb{Z}/2$, $O(1)$-bundles are regular double coverings. It is then clear that π corresponds to the 2-fold cover $S^m \to \mathbb{R}P^m$.

We have $w(\mathcal{E}^1) = 1 \in H^*(\mathbb{R}P^n; \mathbb{Z}/2)$. To calculate $w(\pi)$, we notice that the inclusion map $i : \mathbb{R}P^n \to \mathbb{R}P^\infty$ classifies the bundle π, as the universal bundle $S^\infty \to \mathbb{R}P^\infty$ pulls back under the inclusion to $S^m \to \mathbb{R}P^m$. In particular,

$$w_1(\pi) = i^* w_1 = i^* x = x,$$

where x is the generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2) = H^1(\mathbb{R}P^m; \mathbb{Z}/2)$. Therefore,

$$w(\pi) = 1 + x.$$

We next show that

$$T\mathbb{R}P^m \oplus \mathcal{E}^1 \cong \underbrace{\pi \oplus \cdots \oplus \pi}_{m+1 \text{ times}}, \tag{4.3.4}$$

from which the computation of $w(\mathbb{R}P^m)$ follows by an application of the Whitney sum formula.

To prove (4.3.4), start with $S^m \hookrightarrow \mathbb{R}^{m+1}$ with (rank one) normal bundle denoted by $\mathcal{E}_\nu$. Note that $\mathcal{E}_\nu$ is a trivial line bundle on S^m, as it has a global non-zero section (mapping $y \in S^m$ to the normal vector ν_y at y). We then have

$$TS^m \oplus \mathcal{E}_\nu \cong T\mathbb{R}^{m+1}|_{S^m} = \mathcal{E}^{m+1} \cong \underbrace{\mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^1}_{m+1 \text{ times}},$$

with $\mathcal{E}^{m+1}$ the trivial bundle of rank $n+1$ on S^m, i.e., the Whitney sum of $m+1$ trivial line bundles $\mathcal{E}^1$ on S^m, each of which is generated by the global non-zero section $y \mapsto \frac{d}{dx_i}|_y$, $i = 1, \cdots, m+1$.

Let $a : S^m \to S^m$ be the antipodal map, with differential $da : TS^m \to TS^m$. Let $\gamma : (-\epsilon, \epsilon) \to S^m$, $\gamma(0) = y$, $v = \gamma'(0) \in T_y S^m$. Then $da(v) = \frac{d}{dt}(a \circ \gamma(t))|_{t=0} = -\gamma'(0) = -v \in T_{a(y)} S^m$. Therefore da is an involution on TS^m, commuting with a, and hence

$$TS^m / da = T\mathbb{R}P^m.$$

Next note that the normal bundle $\mathcal{E}_\nu$ on S^m is invariant under the antipodal action (as $da(\nu_y) = \nu_{a(y)}$), so it descends to the trivial line bundle on $\mathbb{R}P^m$, i.e.,

$$\mathcal{E}_\nu / da \cong \mathcal{E}^1.$$

Finally,

$$S^m \times \mathbb{R}/da \cong S^m \times \mathbb{R}/(y, t\frac{d}{dx_i}) \sim (-y, -t\frac{d}{dx_i}) \cong S^n \times_{\mathbb{Z}/2} \mathbb{R},$$

which is the associated bundle of π with fiber $\mathbb{R}$. So,

$$\mathcal{E}^1 / da \cong \pi.$$

This concludes the proof of (4.3.4) and of the theorem. $\square$

Remark 4.3.13. Note that $\mathbb{R}P^3 \cong SO(3)$ is a Lie group, so its tangent bundle is trivial. In this case, once can conclude directly that $w(\mathbb{R}P^3) = 1$, but this fact can also be seen from formula (4.3.3).

Boundary Problem.

For a closed manifold M^n, let $\mu_M \in H_n(M, \mathbb{Z}/2)$ be the fundamental class. We will associate to M certain $\mathbb{Z}/2$-invariants, called its *Stiefel-Whitney numbers*.

Definition 4.3.14. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a tuple of non-negative integers such that $\sum_{i=1}^n i\alpha_i = n$. Set*

$$w^{[\alpha]}(M) := w_1(M)^{\alpha_1} \cup \cdots \cup w_n(M)^{\alpha_n} \in H^n(M; \mathbb{Z}/2).$$

The Stiefel-Whitney number of M with index α is defined as

$$w_{(\alpha)}(M) := \langle w^{[\alpha]}(M), \mu_M \rangle \in \mathbb{Z}/2,$$

where $\langle -, - \rangle : H^n(M; \mathbb{Z}/2) \times H_n(M; \mathbb{Z}/2) \to \mathbb{Z}/2$ is the Kronecker evaluation pairing.

We have the following result:

Theorem 4.3.15 (Pontrjagin-Thom)**.** *A closed n-dimensional manifold M is the boundary of a smooth compact $(n+1)$-dimensional manifold W if and only if all Stiefel-Whitney numbers of M vanish.*

Proof. We only show here one implication (due to Pontrjagin), namely that if $M = \partial W$ then $w_{(\alpha)}(M) = 0$, for any $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\sum_{i=1}^n i\alpha_i = n$.

If $i : M \hookrightarrow W$ denotes the boundary embedding, then

$$i^*TW \cong TM \oplus \nu^1,$$

where ν^1 is the rank-one normal bundle of M in W.

Assume that TW has a Euclidean metric. Then the normal bundle ν^1 is trivialized by picking the inward unit normal vector at every point in M. Hence

$$i^*TW \cong TM \oplus \mathcal{E}^1,$$

where $\mathcal{E}^1$ is the trivial line bundle on M. In particular, the Whitney sum formula yields that

$$w_k(M) = i^* w_k(W),$$

for $k = 1, \cdots, n$, so $w^{[\alpha]}(M) = i^* w^{[\alpha]}(W)$ for any α as above.

Let μ_W be the fundamental class of (W, M) i.e., the generator of $H_{n+1}(W, M; \mathbb{Z}/2)$, and let μ_M be the fundamental class of M as above.

From the long exact homology sequence for the pair (W, M) and Poincaré duality, we have that

$$\partial(\mu_W) = \mu_M.$$

Let $\delta : H^n(M;\mathbb{Z}/2) \to H^{n+1}(W, M;\mathbb{Z}/2)$ be the map adjoint to ∂. The naturality of the cap product yields the identity:

$$\langle y, \mu_M\rangle = \langle y, \partial\mu_W\rangle = \langle \delta y, \mu_W\rangle$$

for any $y \in H^n(M;\mathbb{Z}/2)$. Putting it all together we have:

$$\begin{aligned} w_{(\alpha)}(M) &= \langle w^{[\alpha]}(M), \mu_M\rangle \\ &= \langle i^* w^{[\alpha]}(W), \partial\mu_W\rangle \\ &= \langle \delta(i^* w^{[\alpha]}(W)), \mu_W\rangle \\ &= \langle 0, \mu_W\rangle \\ &= 0, \end{aligned}$$

since $\delta \circ i^* = 0$, as can be seen from the long exact cohomology sequence for the pair (W, M). □

Example 4.3.16. Suppose $M = X \sqcup X$, i.e., M is the disjoint union of two copies of a closed n-dimensional manifold X. Then for any α, $w_{(\alpha)}(M) = 2w_{(\alpha)}(X) = 0$. This is consistent with the fact that $X \sqcup X$ is the boundary of the cylinder $X \times [0, 1]$.

Example 4.3.17. Every $\mathbb{R}P^{2k-1}$ is a boundary. Indeed, the total Stiefel-Whitney class of $\mathbb{R}P^{2k-1}$ is $(1 + x)^{2k} = (1 + x^2)^k$, with x the generator of $H^1(\mathbb{R}P^{2k-1};\mathbb{Z}/2)$. Thus, all the odd degree Stiefel-Whitney classes of $\mathbb{R}P^{2k-1}$ are 0. Since every monomial in the Stiefel-Whitney classes of $\mathbb{R}P^{2k-1}$ of total degree $2k - 1$ must contain a factor w_j with j odd, all Stiefel-Whitney numbers of $\mathbb{R}P^{2k-1}$ vanish. This implies the claim by the Pontrjagin-Thom Theorem 4.3.15.

Example 4.3.18. The real projective space $\mathbb{R}P^{2k}$ is not a boundary, for any integer $k \geq 0$. Indeed, the total Stiefel-Whitney class of $\mathbb{R}P^{2k}$ is

$$\begin{aligned} w(\mathbb{R}P^{2k}) = (1 + x)^{2k+1} &= 1 + \binom{2k+1}{1}x + \cdots + \binom{2k+1}{2k}x^{2k} \\ &= 1 + x + \cdots + x^{2k} \end{aligned}$$

In particular, $w_{2k}(\mathbb{R}P^{2k}) = x^{2k}$. It follow that for $\alpha = (0, 0, \ldots, 1)$ we have

$$w_{(\alpha)}(\mathbb{R}P^{2k}) = 1 \neq 0.$$

4.4 *Pontrjagin classes*

In this section, unless specified, we use the symbol π to denote real vector bundles (or $O(n)$-bundles), and use ω for complex vector bundles (or $U(n)$-bundles) on a topological space X.

Given a real vector bundle π, we can consider its *complexification* $\pi \otimes \mathbb{C}$, i.e., the complex vector bundle with same transition functions as π:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(n) \subset U(n),$$

and fiber $\mathbb{R}^n \otimes \mathbb{C} \cong \mathbb{C}^n$.

Given a complex vector bundle ω, we can consider its *realization* $\omega_{\mathbb{R}}$, obtained by forgeting the complex structure, i.e., with transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(n) \hookrightarrow O(2n).$$

Given a complex vector bundle ω, its *conjugation* $\overline{\omega}$ is defined by transition functions

$$\overline{g_{\alpha\beta}} : U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} U(n) \xrightarrow{\bar{\cdot}} U(n),$$

with the second homomorphism given by conjugation. $\overline{\omega}$ has the same underlying real vector bundle as ω, but the opposite complex structure on its fibers.

Lemma 4.4.1. *If ω is a complex vector bundle, then*

$$\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}.$$

Proof. Let $\jmath$ be the linear transformation on $F_{\mathbb{R}} \otimes \mathbb{C}$ given by multiplication by i. Here F is the fiber of complex vector bundle ω, and $F_{\mathbb{R}}$ is the fiber of its realization $\omega_{\mathbb{R}}$. Then $\jmath^2 = -id$, so we have

$$F_{\mathbb{R}} \otimes \mathbb{C} \cong Eigen(i) \oplus Eigen(-i),$$

where $\jmath$ acts as multiplication by i on $Eigen(i)$, and it acts as multiplication by $-i$ on $Eigen(-i)$. Moreover, we have $F \subseteq Eigen(i)$ and $\overline{F} \subseteq Eigen(-i)$. By a dimension count we then get that $F_{\mathbb{R}} \otimes \mathbb{C} \cong F \oplus \overline{F}$. □

Lemma 4.4.2. *Let π be a real vector bundle. Then*

$$\overline{\pi \otimes \mathbb{C}} \cong \pi \otimes \mathbb{C}.$$

Proof. Indeed, since the transition functions of $\pi \otimes \mathbb{C}$ are real-values (same as those of π), they are also the transition functions for $\overline{\pi \otimes \mathbb{C}}$. □

Lemma 4.4.3. *If ω is a rank n complex vector bundle, the Chern classes of its conjugate $\overline{\omega}$ are computed by*

$$c_k(\overline{\omega}) = (-1)^k \cdot c_k(\omega),$$

for any $k = 1, \cdots, n$.

Proof. Recall that one way to define (universal) Chern classes is by induction by using the fibration

$$S^{2k-1} \hookrightarrow BU(k-1) \to BU(k).$$

In fact,

$$c_k = d_{2k}(a),$$

where a is the generator of $H^{2k-1}(S^{2k-1};\mathbb{Z})$.

The complex conjugation on the fiber S^{2k-1} of the above fibration is a map of degree $(-1)^k$ (it keeps k out of $2k$ real basis vectors invariant, and it changes the sign of the other k; each sign change is a reflection and it has degree -1). In particular, the homomorphism $H^{2k-1}(S^{2k-1};\mathbb{Z}) \to H^{2k-1}(S^{2k-1};\mathbb{Z})$ induced by conjugation is defined by $a \mapsto (-1)^k \cdot a$. Altogether, this gives $c_k(\overline{\omega}) = (-1)^k \cdot c_k(\omega)$. □

Combining the results from Lemma 4.4.2 and Lemma 4.4.3, we have the following:

Corollary 4.4.4. *For any real vector bundle π,*

$$c_k(\pi \otimes \mathbb{C}) = c_k(\overline{\pi \otimes \mathbb{C}}) = (-1)^k c_k(\pi \otimes \mathbb{C}).$$

In particular, for any odd integer k, $c_k(\pi \otimes \mathbb{C})$ is an integral cohomology class of order 2.

Definition 4.4.5 (Pontryagin classes of real vector bundles)**.** *Let $\pi : E \to X$ be a real vector bundle of rank n. The i-th Pontrjagin class of π is defined as:*

$$p_i(\pi) := (-1)^i c_{2i}(\pi \otimes \mathbb{C}) \in H^{4i}(X;\mathbb{Z}).$$

If ω a complex vector bundle of rank n, we define its i-th Pontryagin class as

$$p_i(\omega) := p_i(\omega_{\mathbb{R}}) = (-1)^i c_{2i}(\omega \oplus \overline{\omega}).$$

Remark 4.4.6. Note that $p_i(\pi) = 0$ for all $i > \frac{n}{2}$.

Definition 4.4.7. *If π is a real vector bundle on X, its total Pontrjagin class is defined as*

$$p(\pi) = p_0 + p_1 + \cdots \in H^*(X;\mathbb{Z}).$$

Theorem 4.4.8 (Product formula)**.** *If π_1 and π_2 are real vector bundles on X, then*

$$p(\pi_1 \oplus \pi_2) = p(\pi_1) \cup p(\pi_2) \text{ mod 2-torsion.}$$

Proof. We have $(\pi_1 \oplus \pi_2) \otimes \mathbb{C} \cong (\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C})$. Therefore,

$$\begin{aligned} p_i(\pi_1 \oplus \pi_2) &= (-1)^i c_{2i}((\pi_1 \oplus \pi_2) \otimes \mathbb{C}) \\ &= (-1)^i \sum_{k+l=2i} c_k(\pi_1 \otimes \mathbb{C}) \cup c_l(\pi_2 \otimes \mathbb{C}) \end{aligned}$$

$$= (-1)^i \sum_{a+b=i} c_{2a}(\pi_1 \otimes \mathbb{C}) \cup c_{2b}(\pi_2 \otimes \mathbb{C}) + \{\text{elements of order 2}\}$$
$$= \sum_{a+b=i} p_a(\pi_1) \cup p_b(\pi_2) + \{\text{elements of order 2}\},$$

thus proving the claim. □

Definition 4.4.9. *If M is a real smooth manifold, we define*

$$p(M) := p(TM).$$

If M is a complex manifold, we define

$$p(M) := p((TM)_{\mathbb{R}}).$$

Here TM is the tangent bundle of the manifold M.

In order to give applications of Pontrjagin classes, we need the following computational result:

Theorem 4.4.10 (Chern and Pontrjagin classes of complex projective space)**.** *The total Chern and Pontrjagin classes of the complex projective space $\mathbb{C}P^n$ are computed by:*

$$c(\mathbb{C}P^n) = (1+c)^{n+1}, \tag{4.4.1}$$

$$p(\mathbb{C}P^n) = (1+c^2)^{n+1}, \tag{4.4.2}$$

where $c \in H^2(\mathbb{C}P^n;\mathbb{Z})$ is a generator.

Proof. The arguments involved in the computation of $c(\mathbb{C}P^n)$ are very similar to those of Theorem 4.3.5. Indeed, one first shows that there is a splitting

$$T\mathbb{C}P^n \oplus \mathcal{E}^1 = \underbrace{\gamma \oplus \cdots \oplus \gamma}_{n+1 \text{ times}},$$

were $\mathcal{E}^1$ is the trivial complex line bundle on $\mathbb{C}P^n$ and γ is the complex line bundle associated to the principle S^1-bundle $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$. Then γ is classified by the inclusion

$$\begin{tikzcd}
S^{2n+1} \arrow[r, hook] \arrow[d] & S^\infty \arrow[d] \\
\mathbb{C}P^n \arrow[r, hook] & \mathbb{C}P^\infty = BU(1)
\end{tikzcd}$$

and hence $c_1(\gamma) = c$, the generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z}) = H^2(\mathbb{C}P^n;\mathbb{Z})$. The Whitney sum formula for Chern classes then yields:

$$c(\mathbb{C}P^n) = c(T\mathbb{C}P^n) = c(\gamma)^{n+1} = (1+c)^{n+1}.$$

By conjugation, one gets

$$c(\overline{T\mathbb{C}P^n}) = (1-c)^{n+1}.$$

Therefore,

$$\begin{aligned} c((T\mathbb{C}P^n)_{\mathbb{R}} \otimes \mathbb{C}) &= c(T\mathbb{C}P^n \oplus \overline{T\mathbb{C}P^n}) \\ &= c(T\mathbb{C}P^n) \cup c(\overline{T\mathbb{C}P^n}) \\ &= (1-c^2)^{n+1}, \end{aligned}$$

from which one can readily deduce that $p(\mathbb{C}P^n) = (1+c^2)^{n+1}$. $\square$

Applications to the embedding problem

After forgetting the complex structure, $\mathbb{C}P^n$ is a $2n$-dimensional real smooth manifold. Suppose that there is an embedding

$$\mathbb{C}P^n \hookrightarrow \mathbb{R}^{2n+k},$$

and we would like to find contraints on the embedding codimension k by means of Pontrjagin classes.

Let $(T\mathbb{C}P^n)_{\mathbb{R}}$ be the realization of the tangent bundle for $\mathbb{C}P^n$. Then the existence of an embedding as above implies that there exists a normal (real) bundle ν^k of rank k such that

$$(T\mathbb{C}P^n)_{\mathbb{R}} \oplus \nu^k \cong T\mathbb{R}^{2n+k}|_{\mathbb{C}P^n} \cong \mathcal{E}^{2n+k}, \tag{4.4.3}$$

with $\mathcal{E}^{2n+k}$ denoting the trivial real vector bundle of rank $2n+k$.

By applying the Pontrjagin class p to (4.4.3) and using the product formula of Theorem 4.4.8 together with the fact that there are no elements of order 2 in $H^*(\mathbb{C}P^n;\mathbb{Z})$, we have

$$p(\mathbb{C}P^n) \cdot p(\nu^k) = 1.$$

Therefore, we get

$$p(\nu^k) = p(\mathbb{C}P^n)^{-1}. \tag{4.4.4}$$

And by the definition of the Pontryagin classes, we know that if $p_i(\nu^k) \neq 0$, then $i \leq \frac{k}{2}$.

Example 4.4.11. *In this example, we use Pontrjagin classes to show that* $\mathbb{C}P^2$ *does not embed in* $\mathbb{R}^5$. *First,*

$$p(\mathbb{C}P^2) = (1+c^2)^3 = 1 + 3c^2,$$

with $c \in H^2(\mathbb{C}P^2;\mathbb{Z})$ *a generator (hence* $c^3 = 0$*). If there is an embedding* $\mathbb{C}P^2 \hookrightarrow \mathbb{R}^{4+k}$ *with normal bundle* ν^k*, then*

$$p(\nu^k) = p(\mathbb{C}P^2)^{-1} = 1 - 3c^2.$$

Hence $p_1(\nu^k) \neq 0$*, which implies that* $k \geq 2$.

4.5 Oriented cobordism and Pontrjagin numbers

If M is a smooth oriented manifold, we denote by $-M$ the same manifold but with the opposite orientation.

Definition 4.5.1. *Let M^n and N^n be smooth, closed, oriented real manifolds of dimension n. We say M and N are oriented cobordant if there exists a smooth, compact, oriented $(n+1)$-dimensional manifold W^{n+1}, such that $\partial W = M \sqcup (-N)$.*

Remark 4.5.2. Let us say a word of convention about orienting a boundary. For any $x \in \partial W$, there exist an inward normal vector $\nu_+(x)$ and an outward normal vector $\nu_-(x)$ to the boundary at x. By using a partition of unity, one can construct an inward/outward normal vector field $\nu_\pm : \partial W \to TW|_{\partial W}$. By convention, a frame $\{e_1, \cdots, e_n\}$ on $T_x(\partial W)$ is positive if $\{e_1, \cdots, e_n, \nu_-(x)\}$ is a positive frame for $T_x W$.

Lemma 4.5.3. *Oriented cobordism is an equivalence relation.*

Proof. M and $-M$ are clearly oriented cobordant because $M \sqcup (-M)$ is diffeomorphic to the boundary of $M \times [0,1]$. Hence oriented cobordism is reflexive. The symmetry can be deduced from the fact that, if $M \sqcup (-N) \simeq \partial W$, then $N \sqcup (-M) \simeq \partial(-W)$. Finally, if $M_1 \sqcup (-M_2) \simeq \partial W$, and $M_2 \sqcup (-M_3) \simeq \partial W'$, then we can glue W and W' along the common boundary and get a new manifold with boundary $M_1 \sqcup (-M_3)$. Hence oriented cobordism is also transitive. □

Definition 4.5.4. *Let Ω_n be the set of cobordism classes of closed, oriented, smooth n-manifolds.*

Corollary 4.5.5. *The set Ω_n is an abelian group with the disjoint union operation.*

Proof. This is an immediate consequence of Lemma 4.5.3. The zero element in Ω_n is the class of $\emptyset$, or equivalently, $[M] = 0 \in \Omega_n$ if and only if $M = \partial W$, for some compact manifold W. The inverse of $[M]$ is $[-M]$, since $M \sqcup (-M)$ is a boundary. □

A natural problem to investigate is to describe the group Ω_n by generators and relations. For example, both $[\mathbb{C}P^4]$ and $[\mathbb{C}P^2 \times \mathbb{C}P^2]$ are elements of Ω_8. Do they represent the same element, i.e., are $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ oriented cobordant? A lot of insight is gained by using *Pontrjagin numbers*.

Definition 4.5.6. *Let M^n be a smooth, closed, oriented real n-manifold, with fundamental class $\mu_M \in H_n(M;\mathbb{Z})$. Let $k = [\frac{n}{4}]$ and choose a partition $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{Z}^k$ such that $\sum_{j=1}^k 4j\alpha_j = n$. The Pontrjagin number of M associated to the partition α is defined as:*

$$p_{(\alpha)}(M) = \langle p_1(M)^{\alpha_1} \cup \cdots \cup p_k(M)^{\alpha_k}, \mu_M \rangle \in \mathbb{Z}.$$

Remark 4.5.7. If n is not divisible by 4, then $p_{(\alpha)}(M) = 0$ by definition.

Theorem 4.5.8. *For $n = 4k$, each $p_{(\alpha)}$ defines a homomorphism*

$$\Omega_n \longrightarrow \mathbb{Z}, \ \ [M] \mapsto p_{(\alpha)}(M).$$

Hence oriented cobordant manifolds have the same Pontrjagin numbers. In particular, if $M^n = \partial W^{n+1}$, then $p_{(\alpha)}(M) = 0$ for any partition α.

Proof. If $M = M_1 \sqcup M_2$, then $[M] = [M_1] + [M_2] \in \Omega_n$ and $\mu_M = \mu_{M_1} + \mu_{M_2} \in H_n(M;\mathbb{Z})$. It follows readily that $p_{(\alpha)}(M) = p_{(\alpha)}(M_1) + p_{(\alpha)}(M_2)$.

If $M = \partial N$, then it can be shown as in the proof of Theorem 4.3.15 that $p_{(\alpha)}(M) = 0$ for any partition α. □

Example 4.5.9. By Theorem 4.4.10, we have that $p(\mathbb{C}P^{2n}) = (1 + c^2)^{2n+1}$, where c is a generator of $H^2(\mathbb{C}P^{2n};\mathbb{Z})$. Hence $p_i(\mathbb{C}P^{2n}) = \binom{2n+1}{i} c^{2i}$. For the partition $\alpha = (0,\ldots,0,1)$, we find that $p_{(\alpha)}(\mathbb{C}P^{2n}) = \left\langle \binom{2n+1}{n} c^{2n}, \mu_{\mathbb{C}P^{2n}} \right\rangle = \binom{2n+1}{n} \neq 0$. We conclude that $\mathbb{C}P^{2n}$ is not an oriented boundary.

Remark 4.5.10. If we reverse the orientation of a manifold M of real dimension $n = 4k$, the Pontrjagin classes remain unchanged, but the fundamental class μ_M changes sign. Therefore, all Pontrjagin numbers $p_{(\alpha)}(M)$ change sign. This shows that, if some Pontrjagin number $p_{(\alpha)}(M)$ is nonzero, then M cannot have any orientation-reversing diffeomorphism.

Example 4.5.11. The above remark and Example 4.5.9 show that $\mathbb{C}P^{2n}$ does not have any orientation-reversing diffeomorphism. However, $\mathbb{C}P^{2n+1}$ has an orientation-reversing diffeomorphism induced by complex conjugation.

Example 4.5.12. Let us consider Ω_4. As $\mathbb{C}P^2$ is not an oriented boundary by Example 4.5.9, we have $[\mathbb{C}P^2] \neq 0 \in \Omega_n$. Recall that $p(\mathbb{C}P^2) = 1 + 3c^2$, so $p_1(\mathbb{C}P^2) = 3c^2$. For the partition $\alpha = (1)$, we then get that $p_{(1)}(\mathbb{C}P^2) = 3$. So

$$\Omega_4 \xrightarrow{p_{(1)}} 3\mathbb{Z} \longrightarrow 0$$

is exact, thus $\operatorname{rank}(\Omega_4) \geq 1$.

Example 4.5.13. We next consider Ω_8. The partitions to work with in this case are $\alpha_1 = (2,0)$ and $\alpha_2 = (0,1)$, and Theorem 4.5.8 yields a homomorphism

$$\Omega_8 \xrightarrow{(p_{(\alpha_1)}, p_{(\alpha_2)})} \mathbb{Z} \oplus \mathbb{Z}.$$

We aim to show that

$$\operatorname{rank}(\Omega_8) = \dim_{\mathbb{Q}}(\Omega_8 \otimes \mathbb{Q}) \geq 2.$$

We start by noting that both $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ are compact oriented 8-manifolds which are not boundaries. We calculate the Pontrjagin numbers of these two spaces. First,

$$p(\mathbb{C}P^4) = (1+c^2)^5 = 1 + 5c^2 + 10c^4,$$

where c is a generator of $H^2(\mathbb{C}P^4;\mathbb{Z})$. Hence, $p_1(\mathbb{C}P^4) = 5c^2$ and $p_2(\mathbb{C}P^4) = 10c^4$. The Pontrjagin numbers of $\mathbb{C}P^4$ corresponding to the partitions $\alpha_1 = (2,0)$ and $\alpha_2 = (0,1)$ are given as:

$$p_{(\alpha_1)}(\mathbb{C}P^4) = \langle p_1(\mathbb{C}P^4)^2, \mu_{\mathbb{C}P^4} \rangle = 25,$$
$$p_{(\alpha_2)}(\mathbb{C}P^4) = \langle p_2(\mathbb{C}P^4), \mu_{\mathbb{C}P^4} \rangle = 10.$$

In order to compute the corresponding Pontrjagin numbers for $\mathbb{C}P^2 \times \mathbb{C}P^2$, let $pr_i : \mathbb{C}P^2 \times \mathbb{C}P^2 \to \mathbb{C}P^2$, $i = 1, 2$, be the projections on factors. Then

$$T(\mathbb{C}P^2 \times \mathbb{C}P^2) \cong pr_1^* T(\mathbb{C}P^2) \oplus pr_2^* T(\mathbb{C}P^2),$$

so Theorem 4.4.8 yields that

$$p(\mathbb{C}P^2 \times \mathbb{C}P^2) = pr_1^* p(\mathbb{C}P^2) \cup pr_2^* p(\mathbb{C}P^2) = p(\mathbb{C}P^2) \times p(\mathbb{C}P^2),$$

where $\times$ denotes the external product. Let c_1 and c_2 denote the generators of the second integral cohomology of the two $\mathbb{C}P^2$ factors. Then:

$$\begin{aligned} p(\mathbb{C}P^2 \times \mathbb{C}P^2) &= (1+c_1^2)^3 \cdot (1+c_2^2)^3 = (1+3c_1^2) \cdot (1+3c_2^2) \\ &= 1 + 3c_1^2 + 3c_2^2 + 9c_1^2c_2^2. \end{aligned}$$

Hence, $p_1(\mathbb{C}P^2 \times \mathbb{C}P^2) = 3(c_1^2 + c_2^2)$ and $p_2(\mathbb{C}P^2 \times \mathbb{C}P^2) = 9c_1^2c_2^2$. Therefore, the Pontrjagin numbers of $\mathbb{C}P^2 \times \mathbb{C}P^2$ corresponding to the partitions α_1 and α_2 are computed by (here we use the fact that $c_1^4 = 0 = c_2^4$):

$$p_{(\alpha_1)}(\mathbb{C}P^2 \times \mathbb{C}P^2) = 18, \quad p_{(\alpha_2)}(\mathbb{C}P^2 \times \mathbb{C}P^2) = 9.$$

The values of the homomorphism $(p_{(\alpha_1)}, p_{(\alpha_2)}) : \Omega_8 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ on $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ fit into the 2×2 matrix $\begin{bmatrix} 25 & 18 \\ 10 & 9 \end{bmatrix}$ with nonzero determinant. Hence $\operatorname{rank}(\Omega_8) \geq 2$.

More generally, we the following qualitative description of Ω_n, which we mention here without proof.

Theorem 4.5.14 (Thom)**.** *The oriented cobordism group Ω_n is finitely generated of rank $|I|$, where I is the collection of partitions α satisfying $\sum_j 4j\alpha_j = n$. In fact, modulo torsion, Ω_n is generated by products of even complex projective spaces. Moreover, $\bigoplus_{\alpha \in I} p_{(\alpha)} : \Omega_n \to \mathbb{Z}^{|I|}$ is an injective homomorphism onto a subgroup of the same rank.*

Example 4.5.15. Our computations from Examples 4.5.12 and 4.5.13 together with Theorem 4.5.14 yield that in fact we have: $\operatorname{rank}(\Omega_4) = 1$ and $\operatorname{rank}(\Omega_8) = 2$.

4.6 Signature as an oriented cobordism invariant

Recall that if M^{4k} is a closed, oriented manifold of real dimension $n = 4k$, then we can define its *signature* $\sigma(M)$ as the signature of the bilinear symmetric pairing

$$H^{2k}(M;\mathbb{Q}) \times H^{2k}(M;\mathbb{Q}) \to \mathbb{Q},$$

which is non-degenerate by Poincaré duality. Recall also that if M is an oriented boundary then $\sigma(M) = 0$. This suffices to deduce the folowing result:

Theorem 4.6.1 (Thom). *$\sigma : \Omega_{4k} \to \mathbb{Z}$ is a homomorphism.*

It follows from Theorems 4.5.14 and 4.6.1 that the *signature is a rational combination of Pontrjagin numbers*, i.e.,

$$\sigma = \sum_{\alpha \in I} a_\alpha p_{(\alpha)} \tag{4.6.1}$$

for some coefficients $a_\alpha \in \mathbb{Q}$. The *Hirzebruch signature theorem* provides an explicit formula for these coefficients a_α. In what follows we compute by hand the coefficients a_α in the cases of Ω_4 and Ω_8.

Example 4.6.2. On closed oriented 4-manifolds, the signature is computed by

$$\sigma = a p_{(1)}, \tag{4.6.2}$$

with $a \in \mathbb{Q}$ to be determined. Since a is the same for any $[M] \in \Omega_4$, we will determine it by performing our calculations on $M = \mathbb{C}P^2$. Recall that $\sigma(\mathbb{C}P^2) = 1$, and if $c \in H^2(\mathbb{C}P^2;\mathbb{Z})$ is a generator then $p_1(\mathbb{C}P^2) = 3c^2$. Hence $p_{(1)}(\mathbb{C}P^2) = 3$, and (4.6.2) implies that $1 = 3a$, or $a = \frac{1}{3}$. Therefore, for any closed oriented 4-manifold M^4 we have that

$$\sigma(M) = \langle \frac{1}{3} p_1(M), \mu_M \rangle = \frac{1}{3} p_{(1)}(M) \in \mathbb{Z}.$$

Example 4.6.3. On closed oriented 8-manifolds, the signature is computed by (4.6.1) as

$$\sigma = a_{(2,0)} p_{(2,0)} + a_{(0,1)} p_{(0,1)}, \tag{4.6.3}$$

with $a_{(2,0)}, a_{(0,1)} \in \mathbb{Q}$ to be determined. Since Ω_8 is generated rationally by $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$, we can calculate $a_{(2,0)}$ and $a_{(0,1)}$ by evaluating (4.6.3) on $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$. Using our computations from Example 4.5.13, we have:

$$1 = \sigma(\mathbb{C}P^4) = 25 a_{(2,0)} + 10 a_{(0,1)}, \tag{4.6.4}$$

and

$$1 = \sigma(\mathbb{C}P^2 \times \mathbb{C}P^2) = 18 a_{(2,0)} + 9 a_{(0,1)}. \tag{4.6.5}$$

Solving for $a_{(2,0)}$ and $a_{(0,1)}$ in (4.6.4) and (4.6.5), we get:

$$a_{(2,0)} = -\frac{1}{45}, \quad a_{(0,1)} = \frac{7}{45}.$$

Altogether, the signature of a closed oriented manifold M^8 is computed by the following formula:

$$\sigma(M^8) = \frac{1}{45}\langle 7p_2(M) - p_1(M)^2, \mu_M \rangle. \tag{4.6.6}$$

4.7 *Exotic 7-spheres*

Now we turn to the construction of exotic 7-spheres. Start with M a smooth, 3-connected orientable 8-manifold. Up to homotopy, $M \simeq (S^4 \vee \cdots \vee S^4) \cup_f e^8$. Assume further that $\beta_4(M) = 1$, i.e., $M \simeq S^4 \cup_f e^8$, for some map $f : S^7 \to S^4$. By Whitney's embedding theorem, there is a smooth embedding $S^4 \hookrightarrow M$. Let E be a tubular neighborhood of this embedded S^4 in M; in other words, E is a D^4-bundle on S^4 inside M. Such D^4-bundles on S^4 are classified by

$$\pi_3(SO(4)) \cong \pi_3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

(Here we use the fact that $S^3 \times S^3$ is a 2-fold covering of $SO(4)$.) That means that E corresponds to a pair of integers (i, j).

Let X^7 be the boundary of E, so X is a S^3-bundle over S^4. If X is diffeomorphic to a 7-sphere, one can recover M from E by attaching an 8-cell to $X = \partial E$. So the question to investigate is: *for which pairs of integers (i, j) is X diffeomorphic to S^7?*

One can show the following:

Lemma 4.7.1. *X is homotopy equivalent to S^7 if and only if $i + j = \pm 1$.*

Suppose $i + j = 1$. Then for each choice of i, we get an S^3-bundle over S^4, namely $X = \partial E$, which has the homotopy type of S^7. If X is in fact diffeomorphic to S^7, then we can recover M by attaching an 8-cell to X, and in this case the signature of M is computed by

$$\sigma(M) = \frac{1}{45}\left(7p_{(0,1)}(M) - p_{(2,0)}(M)\right).$$

Moreover, one can show that:

Lemma 4.7.2. $p_{(2,0)}(M) = 4(i-j)^2 = 4(2i-1)^2$.

Note that $\sigma(M) = \pm 1$ since $H^4(M; \mathbb{Z}) = \mathbb{Z}$, and let us fix the orientation according to which $\sigma(M) = 1$. Our assumption that X was diffeomorphic to S^7 leads now to a contradiction, since

$$p_{(0,1)}(M) = \frac{4(2i-1)^2 + 45}{7}$$

is by definition an integer for all i, which is contradicted e.g., for $i = 2$.

So far (for $i = 2$ and $j = -1$), we constructed a space X which is homotopy equivalent to S^7, but which is not diffeomorphic to S^7. In fact, one can further show the following:

Lemma 4.7.3. *X is homeomorphic to S^7, so in fact X is an exotic 7-sphere.*

This latest fact can be shown by constructing a Morse function $g : X \to \mathbb{R}$ with only two nondegenerate critical points (a maximum and a minimum). An application of Reeb's theorem then yields that X is homeomorphic to S^7.

4.8 Exercises

1. Construct explicitly the bounding manifold for $\mathbb{R}P^3$.

2. Let ω be a rank n complex vector bundle on a topological space X, and let $c_i(\omega) \in H^{2i}(X;\mathbb{Z})$ be its i-th Chern class. Via $\mathbb{Z} \to \mathbb{Z}/2$, $c_i(\omega)$ determines a cohomology class $\bar{c}_i(\omega) \in H^{2i}(X;\mathbb{Z}/2)$. By forgetting the complex structure on the fibers of ω, we obtain the realization $\omega_{\mathbb{R}}$ of ω, as a rank $2n$ real vector bundle on X.

Show that the Stiefel-Whitney classes of $\omega_{\mathbb{R}}$ are computed as follows:

(a) $w_{2i}(\omega_{\mathbb{R}}) = \bar{c}_i(\omega)$, for $0 \leq i \leq n$.

(b) $w_{2i+1}(\omega_{\mathbb{R}}) = 0$ for any integer i.

3. Let M be a $2n$-dimensional smooth manifold with tangent bundle TM. Show that, if M admits a complex structure, then $w_{2i}(M)$ is the mod 2 reduction of an integral class for any $0 \leq i \leq n$, and $w_{2i+1}(M) = 0$ for any integer i. In particular, Stiefel-Whitney classes give obstructions to the existence of a complex structure on an even-dimensional real smooth manifold.

4. Show that a real smooth manifold M is orientable if and only if $w_1(M) = 0$.

5. Show that $\mathbb{C}P^3$ does not embed in $\mathbb{R}^7$.

6. Show that $\mathbb{C}P^4$ does not embed in $\mathbb{R}^{11}$.

7. Example 4.5.9 shows that $\mathbb{C}P^2$ is not the boundary on an oriented compact 5-manifold. Can $\mathbb{C}P^2$ be the boundary on some non-orientable compact 5-manifold?

8. Show that $\mathbb{C}P^{2n+1}$ is the boundary of a compact manifold.