NAME: ______________________

FINAL EXAM

Due Friday 5/14 in Tommy Wong's mailbox (or you can email it to him if you type your exam in Tex). You may use your texts and notes, but may not work in groups on the exam.

1. Use homotopy groups to show that there is no retraction $\mathbb{RP}^n \to \mathbb{RP}^k$ if $n > k > 0$.

2. Show that a closed simply-connected 3-manifold is homotopy equivalent to S^3. (Hint: Use Poincaré duality and the fact that closed manifolds are homotopy equivalent to CW complexes.)

3. Let M be a closed, connected, orientable 4-manifold with fundamental group $\pi_1(M) \cong \mathbb{Z}_3 * \mathbb{Z}_3$ and Euler characteristic $\chi(M) = 5$.

(a) Compute $H_i(M, \mathbb{Z})$ for all i.
(b) Prove that M is not homotopy equivalent to any CW complex with no 3-cells.

4.

(a) For a finite CW complex X and an n-sheeted covering space $p : Y \to X$, show that the Euler characteristic satisfies:

$$\chi(X) = n \cdot \chi(X).$$

(b) Show that if $f : \mathbb{RP}^{2n} \to X$ is a covering map of a CW complex X, then f is a homeomorphism.

5. For each of the following statements, either give specific examples of (connected) topological spaces X and Y which satisfy the statement or explain (in one or two sentences) why no such examples exist. Homology and cohomology groups are considered with $\mathbb{Z}$-coefficients.

(a) X is homotopy equivalent to Y, but X is not homeomorphic to Y.
(b) $H_i(X) \cong H_i(Y)$ for all i, but $\pi_1(X) \ncong \pi_1(Y)$.
(c) X and Y have isomorphic homology groups (in all dimensions) but non-isomorphic cohomology groups (in some dimension).
(d) X and Y have isomorphic homology and cohomology groups (in all dimensions), and $\pi_1(X) \cong \pi_1(Y)$, but X is not homotopy equivalent to Y.

(e) X and Y have isomorphic homology and cohomology groups, but X and Y are not homotopy equivalent.
(f) X and Y have isomorphic homology and cohomology groups, but different homotopy groups.