Math 752 Topology Lecture Notes

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Contents

Chapter 1

Selected topics in Homology

Note: Knowledge of simplicial and singular homology will be assumed.

1.1 Cellular Homology

1.1.1 Degrees

Definition 1.1.1. *The degree of continuous map* $f : S^n \to S^n$ *is defined as:*

$$\deg f = f_*(1) \tag{1.1.1}$$

where $1 \in \mathbb{Z}$ *denotes the generator, and* $f_* : \widetilde{H}_n(S^n) = \mathbb{Z} \to \widetilde{H}_n(S^n) = \mathbb{Z}$ *is the homomorphism induced by* f *in homology.*

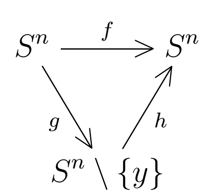
The degree has the following **properties**:

1. $\deg id_{S^n} = 1$.

   Proof. This is because $(id_{S^n})_* = id$ which is multiplication by the integer 1. □

2. If f is not surjective, then $\deg f = 0$.

   Proof. Indeed, suppose f is not surjective, then there is a $y \notin \text{Image} f$. Then we can factor f in the following way:



   Since $S^n - \{y\} \cong \mathbb{R}^n$ which is contractible, $H_n(S^n \setminus \{y\}) = 0$. Therefore $f_* = h_* g_* = 0$, so $\deg f = 0$. □

3. If $f \cong g$, then $\deg f = \deg g$.

   Proof. This is because $f_* = g_*$. Note that the converse is also true (by a theorem of Hopf). □

4. $\deg(g \circ f) = \deg g \cdot \deg f$.

   Proof. Indeed, we have that $(g \circ f)_* = g_* \circ f_*$. □

5. If f is a homotopy equivalence (so there exists a g so that $g \circ f \simeq id_{S^n}$), then $\deg f = \pm 1$.

   Proof. This follows directly from 1, 3, and 4 above, since $f \circ g \cong id_{S^n}$ implies that $\deg f \cdot \deg g = \deg id_{S^n} = 1$. □

6. If $r : S^n \to S^n$ is a reflection across some n-dimensional subspace of $\mathbb{R}^{n+1}$, that is, $r(x_0, \ldots x_n) \mapsto (-x_0, x_1, \ldots, x_n)$, then $\deg r = -1$.

   Proof. Without loss of generality we can assume the subspace is $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n-1}$. Choose a CW structure for S^n whose n-cells are given by Δ_1^n and Δ_2^n, the upper and lower hemispheres of S^n, attached by identifying their boundaries together in the standard way. Then consider the generator of $H_n(S^n)$: $[\Delta_1^n - \Delta_2^n]$. The reflection map r maps the cycle $\Delta_1^n - \Delta_2^n$ to $\Delta_2^n - \Delta_1^n = -(\Delta_1^n - \Delta_2^n)$. So

   $$r_*([\Delta_1^n - \Delta_2^n]) = [\Delta_2^n - \Delta_1^n] = [-(\Delta_1^n - \Delta_2^n)] = -1 \cdot [\Delta_1^n - \Delta_2^n]$$

   so $\deg r = -1$. □

7. If $a : S^n \to S^n$ is the antipodal map ($\underline{x} \mapsto -\underline{x}$), then $\deg a = (-1)^{n+1}$

   Proof. Note that a is a composition of $n+1$ reflections, since there are $n+1$ coordinates in $\underline{x}$, each getting mapped by an individual reflection. From 4 above we know that composition of maps leads to multiplication of degrees. □

8. If $f : S^n \to S^n$ and $Sf : S^{n+1} \to S^{n+1}$ is the suspension of f then $\deg Sf = \deg f$.

   Proof. Recall that if $f : X \to X$ is a continuos map and

   $$\Sigma X = X \times [-1, 1]/(X \times \{-1\}, X \times \{1\})$$

   denotes the suspension of X, then $Sf := f \times id_{[-1,1]}/\sim$, with the same equivalence as in ΣX. Note that $\Sigma S^n = S^{n+1}$.
   The Suspension Theorem states that

   $$\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X).$$

This can be proved by using the Mayer-Vietoris sequence for the decomposition

$$\Sigma X = C_+X \cup_X C_-X,$$

where C_+X and C_-X are the upper and lower cones of the suspension joined along their bases:

$$\to \widetilde{H}_{n+1}(C_+X) \oplus \widetilde{H}_{n+1}(C_-X) \to \widetilde{H}_{n+1}(\Sigma X) \to \widetilde{H}_n(X) \to \widetilde{H}_n(C_+X) \oplus \widetilde{H}_n(C_-X) \to$$

Since C_+X and C_-X are both contractible, the end groups in the above sequence are both zero. Thus, by exactness, we get $\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X)$, as desired.
Let C_+S^n denote the upper cone of ΣS^n. Note that the base of C_+S^n is $S^n \times \{0\} \subset \Sigma S^n$. Our map f induces a map $C_+f : (C_+S^n, S^n) \to (C_+S^n, S^n)$ whose quotient is Sf. The long exact sequence of the pair (C_+S^n, S^n) in homology gives the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow \widetilde{H}_{i+1}(C_+S^n, S^n) & \simeq & \widetilde{H}_{i+1}(C_+S^n/S^n) & \xrightarrow[\sim]{\partial} & \widetilde{H}_i(S^n) \longrightarrow 0 \\ & & \downarrow (Sf)_* & & \downarrow f_* \\ & & \widetilde{H}_{i+1}(S^{n+1}) & \xrightarrow[\sim]{\partial} & \widetilde{H}_i(S^n) \end{array}$$

Note that $C_+S^n/S^n \cong S^{n+1}$ so the boundary map ∂ at the top and bottom of the diagram are the same map. So by the commutativity of the diagram, since f_* is defined by multiplication by some integer m, then $(Sf)_*$ is multiplication by the same integer m. □

Example 1.1.2. Consider the reflection map: $r_n : S^n \to S^n$ defined by $(x_0, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n)$. Since r_n leaves $x_1, x_2, \dots, x_n$ unchanged we can unsuspend one at a time to get

$$\deg r_n = \deg r_{n-1} = \cdots = \deg r_0,$$

where $r_i : S^i \to S^i$ by $(x_0, x_1, \dots, x_i) \mapsto (-x_0, x_1, \dots, x_i)$. So $r_0 : S^0 \to S^0$ by $x_0 \mapsto -x_0$. Note that S^0 is two points but in reduced homology we are only looking at one integer. Consider

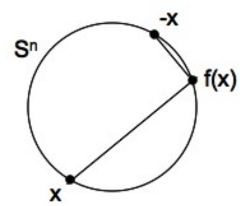
$$0 \to \widetilde{H}_0(S^0) \to H_0(S^0) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\widetilde{H}_0(S^0) = \{(a, -a) \mid a \in \mathbb{Z}\}$, $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$, and $\epsilon : (a, b) \mapsto a + b$. Then $(r_0)_* : \widetilde{H}_0(S^0) \to \widetilde{H}_0(S^0)$ is given by $(a, -a) \mapsto (-a, a) = (-1)(a, -a)$. So $\deg r_n = -1$.

9. If $f : S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$.

Proof. Consider the above figure. Since $f(x) \neq x$, the segment $(1-t)f(x)+t(-x)$ from $-x$ to $f(x)$ does not pass through the origin in $\mathbb{R}^{n+1}$ so we can normalize to obtain a homotopy:

$$g_t(x) := \frac{(1-t)f(x) + t(-x)}{|(1-t)f(x) + t(-x)|} : S^n \to S^n.$$



Note that this homotopy is well defined since $(1-t)f(x) - tx \neq 0$ for any $x \in S^n$ and $t \in [0,1]$, because $f(x) \neq x$ for all x. Then g_t is a homotopy from f to a, the antipodal map. □

Exercises

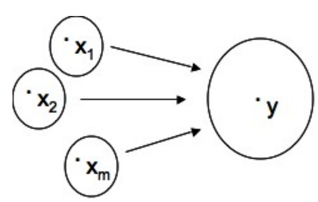
1. Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$.

2. Let $f : S^{2n} \to S^{2n}$ be a continuous map. Show that there is a point $x \in S^{2n}$ so that either $f(x) = x$ or $f(x) = -x$.

3. A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an *even map*. Show that an even map has even degree, and this degree is in fact zero when n is even. When n is odd, show there exist even maps of any given even degree.

1.1.2 How to Compute Degrees?

Assume $f : S^n \to S^n$ is surjective, and that f has the property that there exists some $y \in \text{Image}(S^n)$ so that $f^{-1}(y)$ is a finite number of points, so $f^{-1}(y) = \{x_1, x_2, \dots, x_m\}$. Let U_i be a neighborhood of x_i so that all U_i's get mapped to some neighborhood V of y. So $f(U_i - x_i) \subset V - y$. We can choose the U_i to be disjoint. We can do this because f is continuous.



Let $f|_{U_i} : U_i \to V$ be the restriction of f to U_i. Then:

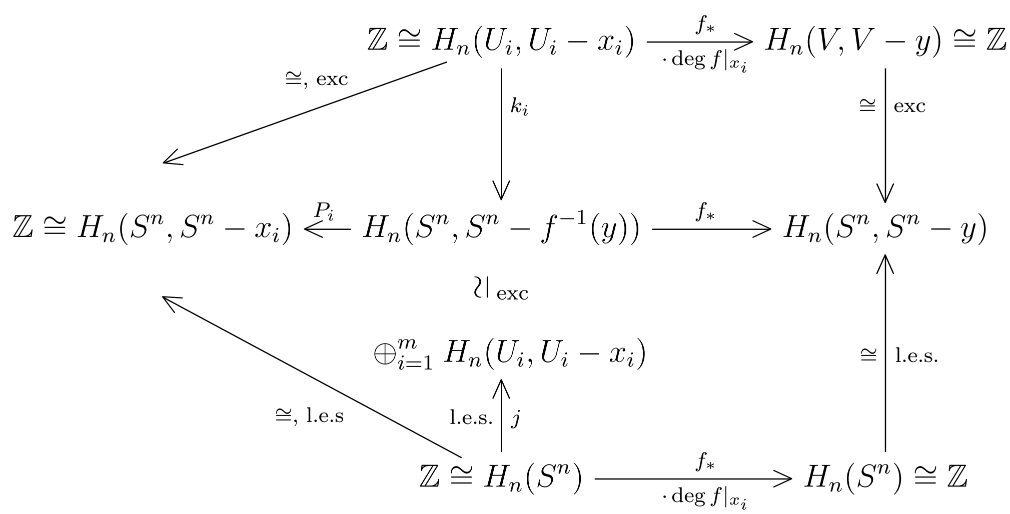
$$
\begin{array}{ccc}
H_n(U_i, U_i - x_i) & \xrightarrow{f_*} & H_n(V, V - y) \\
\wr| \text{ (excision)} & & \wr| \text{ (excision)} \\
H_n(S^n, S^n - x_i) & & H_n(S^n, S^n - y) \\
\wr| \text{ l.e.s.} & & \wr| \text{ l.e.s.} \\
\widetilde{H}_n(S^n) & & \widetilde{H}_n(S^n) \\
\wr| & & \wr| \\
\mathbb{Z} & & \mathbb{Z}
\end{array}
$$

Define the *local degree* of f at x_i, $\deg f_{x_i}$, to be the effect of $f_* : H_n(U_i, U_i - x_i) \to H_n(V, V - y)$. We then have the following result:

Theorem 1.1.3. *The degree of f equals the sum of local degrees at points in a generic fiber, that is,*

$$\deg f = \sum_{i=1}^{m} \deg f|_{x_i}.$$

Proof. Consider the commutative diagram, where the isomorphisms labelled by "exc" follow from excision, and "l.e.s" stands for a long exact sequence.



By examining the diagram above we have:

$$k_i(1) = (0, \ldots, 0, 1, 0, \ldots, 0)$$

where the entry 1 is in the ith place. Also, $P_i \circ j(1) = 1$, for all i, so

$$j(1) = (1, 1, \ldots, 1) = \sum_{i=1}^{m} k_i(1).$$

The commutativity of the lower rectangle gives:

$$\deg f = f_* j(1) = f_* \left(\sum_{i=1}^{m} k_i(1) \right) = \sum_{i=1}^{m} f_*(0, \ldots, 0, 1, 0, \ldots, 0) = \sum_{i=1}^{m} \deg f|_{x_i}$$

Thus we have shown that the degree of a map f is the sum of its local degrees. $\square$

Example 1.1.4. Let us consider the power map $f : S^1 \to S^1$, $f(x) = x^k$, $k \in \mathbb{Z}$. We claim that $\deg f = k$. We distinguish the following cases:

- If $k = 0$ then f is the constant map which has degree 0.
- If $k < 0$ we can compose f with a reflection $r : S^1 \to S^1$ by $(x, y) \to (x, -y)$. This reflection has degree -1. So since composition leads to multiplication of degrees, we can assume that $k > 0$.
- If $k > 0$, then for all $y \in S^1$, $f^{-1}(y)$ has k points (the k roots), call them $x_1, x_2, \dots, x_k$, and f has local degree 1 at each of these points. Indeed, for the above $y \in S^n$ we can find a small open neighborhood centered at y, call this neighborhood V, so that he pre-images of V are open neighborhoods U_i centered at each x_i, with $f|_{U_i} : U_i \to V$ a homeomorphism (which has possible degree ± 1). In this case, these homeomorphisms are a restriction of a rotation, which is homotopic to the identity, and thus the degree of $f|_{U_i}$ is 1 for each i.

So the degree of f is indeed k. Note that this implies that we can construct maps $S^n \to S^n$ of arbitrary degrees for any n, simply by suspending the power map f.

1.1.3 CW Complexes

Let us recall some notation from the theory of CW complexes. A CW complex X can be written as

$$X = \cup_n X_n,$$

where X_n is the n-skeleton, which contains all cells up to and including dimension n. Then

$$X_n = X_{n-1} \amalg_\lambda D^n_\lambda / \sim$$

with the identification $x \in \partial D^n_\lambda \sim \varphi^n_\lambda(x)$, for $\partial D^n_\lambda = S^{n-1} \xrightarrow{\varphi^n_\lambda} X_{n-1}$ the attaching map of the n-cell. So we are gluing the boundary of n-cells to X_{n-1} according to the attaching map φ_λ. A CW complex is endowed the weak topology: $A \subset X$ is open $\iff$ $A \cap X_n$ is open for any n. An n-cell will be denoted by $e^n_\lambda = \text{Int}(D^n_\lambda)$. One can think of X as a disjoint union of cells of various dimensions, or as $\amalg_{n,\lambda} D^n_\lambda / \sim$, where $\sim$ means that we are attaching the cells via their respective attaching maps.

A CW complex X is *finite* if it has finitely many cells, so there is a cell of maximum dimension and the dimension of X can be defined. A CW complex is of *finite type* if it has finitely many cells in each dimension. Note that a CW complex of finite type may have cells in infinitely many dimensions. If $X = \cup_n X_n$ and $X_m = X_n$ for all $m > n$ for some n, then $X = X_n$ and we say that the skeleton stabilizes.

Example 1.1.5. On the n-sphere S^n we have a CW structure with one 0-cell (e^0) and one n-cell (e^n). The attaching map for the n-cell is $\phi : S^{n-1} = \partial D^n \to$ point. There is only one such map, the collapsing map. Think of taking the disk D^n and collapsing the entire boundary to a single point, giving S^n.

Example 1.1.6. A different CW structure on S^n can be constructed so that there are two cells in each dimension from 0 to n. Start with $X_0 = S^0 = \{e_1^0, e_2^0\}$. Then $X_1 = S^1$ where the two 1-cells D_1^1, D_2^1 are attached to the 0-cells by homeomorphisms on the boundary. Similarly, two 2-cells can be attached to $X_1 = S^1$ by homeomorphism on the boundary giving $X_2 = S^2$. Keep working in this manner adding two cells in each new dimension. Note that if we identify each pair of cells in the same dimension by the antipodal map, we get a CW structure on $\mathbb{R}P^n$ with one cell in each dimension from 0 to n.

Example 1.1.7. The *complex projective space* $\mathbb{C}P^n = \mathbb{C}^{n+1}/\mathbb{C}^*$ is identified with the collection of complex lines through the origin. So we write $\mathbb{C}P^n = \{[z_0, \dots, z_n]\}$ where $[z_0, \dots, z_n] = (z_0 : \dots : z_n) \sim (\lambda z_0 : \dots : \lambda z_n)$. We have that

$$\mathbb{C}P^n \cong \mathbb{C}P^{n-1} \sqcup_\varphi D^{2n},$$

where $\psi : D^{2n} \to \mathbb{C}P^n$ is given by

$$(z_1 : \dots : z_n) \mapsto \left(z_1, : \dots : z_n : \sqrt{1 - \sum_{i=1}^n |z_i|^2} \right).$$

The attaching map of the $2n$-cell is $\varphi = \psi|_{S^{2n-1}} : S^{2n-1} \to \mathbb{C}P^{n-1}$. It follows that $\mathbb{C}P^n$ has a CW structure with one cell in each even dimension $0, 2, \dots, 2n$.

1.1.4 Cellular Homology

Let us start with the following preliminary result:

Lemma 1.1.8. *If X is a CW complex, then:*

(a) $H_k(X_n, X_{n-1}) = \begin{cases} 0 & \text{if } k \neq n \\ \mathbb{Z}^{\# \text{ of } n\text{-cells}} & \text{if } k = n. \end{cases}$

(b) $H_k(X_n) = 0$ *if* $k > n$.

(c) The inclusion $i : X_n \hookrightarrow X$ *induces an isomorphism* $H_k(X_n) \to H_k(X)$ *if* $k < n$.

Proof. (a) We know that X_n is obtained from X_{n-1} by attaching the n-cells $(e_\lambda^n)_\lambda$. Pick a point x_λ at the center of each n-cell e_λ^n. Let $A := X_n - \{x_\lambda\}_\lambda$. Then A deformation retracts to X_{n-1}, so we have that

$$H_k(X_n, X_{n-1}) \cong H_k(X_n, A).$$

By excising X_{n-1}, the latter group is isomorphic to $\oplus_\lambda H_k(D^n_\lambda, D^n_\lambda - \{x_\lambda\})$. Moreover, the homology long exact sequence of the pair $(D^n_\lambda, D^n_\lambda - \{x_\lambda\})$ yields that

$$H_k(D^n_\lambda, D^n_\lambda - \{x_\lambda\}) \cong \widetilde{H}_{k-1}(S^{n-1}_\lambda) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

So the claim follows.

(b) Consider the following portion of the long exact sequence of the pair for (X_n, X_{n-1}):

$$\to H_{k+1}(X_n, X_{n-1}) \to H_k(X_{n-1}) \to H_k(X_n) \to H_k(X_n, X_{n-1}) \to$$

If $k+1 \neq n$ and $k \neq n$, we have from part (a) that $H_{k+1}(X_n, X_{n-1}) = 0$ and $H_k(X_n, X_{n-1}) = 0$. Thus $H_k(X_{n-1}) \cong H_k(X_n)$. Hence if $k > n$ (so in particular, $n \neq k+1$ and $n \neq k$), we get by iteration that

$$H_k(X_n) \cong H_k(X_{n-1}) \cong \cdots \cong H_k(X_0).$$

Note that X_0 is a collection of points, so $H_k(X_0) = 0$. Thus when $k > n$ we have $H_k(X_n) = 0$ as desired.

(c) We only prove the statement for finite dimensional CW complexes. Let $k < n$ and consider the long exact sequence for the pair (X_{n+1}, X_n):

$$\to H_{k+1}(X_{n+1}, X_n) \to H_k(X_n) \to H_k(X_{n+1}) \to H_k(X_{n+1}, X_n) \to$$

Since $k < n$ we have $k+1 \neq n+1$ and $k \neq n+1$, so by part (a) we get that $H_{k+1}(X_{n+1}, X_n) = 0$ and $H_k(X_{n+1}, X_n) = 0$. Thus $H_k(X_n) \cong H_k(X_{n+1})$. By repeated iterations, we obtain:

$$H_k(X_n) \cong H_k(X_{n+1}) \cong H_k(X_{n+2}) \cong \cdots \cong H_k(X_{n+l}) = H_k(X).$$

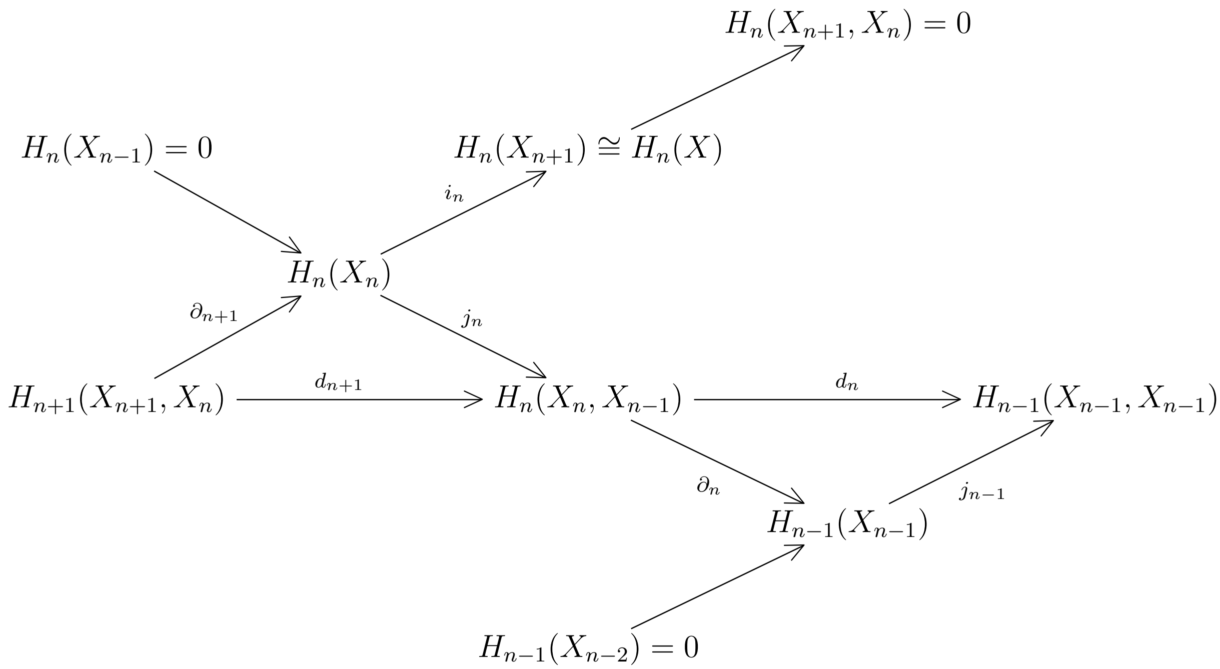
Since X is finite dimensional we know that $X = X_{n+l}$ for some l. This proves the claim. □

In what follows we defined the cellular homology of a CW complex X in terms of a given cell structure, then we show that it coincides with the singular homology, so it is in fact independent on the cell structure. Cellular homology is very useful for computations.

Definition 1.1.9. *The cellular homology $H^{CW}_*(X)$ of a CW complex X is the homology of the cellular chain complex $(\mathcal{C}_*(X), d_*)$ indexed by the cells of X, i.e.,*

$$\mathcal{C}_n(X) := H_n(X_n, X_{n-1}) = \mathbb{Z}^{\#n-cells}, \tag{1.1.2}$$

and with differentials $d_n : \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X)$ defined by the following diagram:



The diagonal arrows are induced from long exact sequences of pairs, and we use Lemma 1.1.8 for the identifications $H_n(X_{n-1}) = 0$, $H_{n-1}(X_{n-2}) = 0$ and $H_n(X_{n+1}) \cong H_n(X)$ in the diagram. In the notations of the above diagram, we now set:

$$d_n = j_{n-1} \circ \partial_n : \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X), \tag{1.1.3}$$

and note that we have

$$d_n \circ d_{n+1} = 0. \tag{1.1.4}$$

Indeed,

$$d_n \circ d_{n+1} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1} = 0$$

since $\partial_n \circ j_n = 0$ as the composition of two consecutive maps in a long exact sequence. So $(\mathcal{C}_*(X), d_*)$ is a chain complex.

The following result asserts that cellular homology is independent on the cell structure used for its definition:

Theorem 1.1.10. *There are isomorphisms*

$$H_n^{CW}(X) \cong H_n(X)$$

for all n, where $H_n(X)$ is the singular homology of X.

Proof. Since $H_n(X_{n+1}, X_n) = 0$ and $H_n(X) \cong H_n(X_{n+1})$, we get from the diagram above that

$$H_n(X) \cong H_n(X_n)/\ker i_n \cong H_n(X_n)/\text{Image } \partial_{n+1}.$$

Now, $H_n(X_n) \cong \text{Image } j_n \cong \ker \partial_n \cong \ker d_n$. The first isomorphism comes from j_n being injective, while the second follows by exactness. Finally, $\ker \partial_n = \ker d_n$ since $d_n = j_{n-1} \circ \partial_n$ and j_{n-1} is injective. Also, we have $\text{Image } \partial_{n+1} = \text{Image } d_{n+1}$. Indeed, $d_{n+1} = j_n \circ \partial_{n+1}$ and j_n is injective.

Altogether, we have

$$H_n(X) \cong H_n(X_n)/\text{Image } \partial_{n+1} = \ker d_n/\text{Image } d_{n+1} = H_n^{CW}(X)$$

So we have proved the theorem. □

Let us now discuss some **immediate consequences** of the above theorem.

(a) *If X has no n-cells, then $H_n(X) = 0$.*
Indeed, in this case we have $\mathcal{C}_n = H_n(X_n, X_{n-1}) = 0$, so $H_n^{CW}(X) = 0$.

(b) *If X is connected and has a single 0-cell then $d_1 : \mathcal{C}_1 \to \mathcal{C}_0$ is the zero map.*
Indeed, since X contains only a single 0-cell, $\mathcal{C}_0 = \mathbb{Z}$. Also, since X is connected, $H_0(X) = \mathbb{Z}$. So by the theorem above $\mathbb{Z} = H_0(X) = \ker d_0/\text{Image } d_1 = \mathbb{Z}/\text{Image } d_1$. This implies that $\text{Image } d_1 = 0$, so d_1 is the zero map as desired.

(c) *If X has no two cells in adjacent dimensions then $d_n = 0$ for all n and $H_n(X) \cong \mathbb{Z}^{\#n-cells}$ for all n.*
Indeed, in this case all maps d_n vanish. So for any n, $H_n^{CW}(X) \cong \mathcal{C}_n \cong \mathbb{Z}^{\#n-cells}$.

Example 1.1.11. Recall that $\mathbb{C}P^n$ has one cell in each dimension $0, 2, 4, \dots, 2n$. So $\mathbb{C}P^n$ has no two cells in adjacent dimensions, meaning we can apply Consequence (c) above to say:

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, 4, \dots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.1.12. When $n > 1$, $S^n \times S^n$ has one 0-cell, two n-cells, and one $2n$-cell. Since $n > 1$, these cells are not in adjacent dimensions so again Consequence (c) above applies to give:

$$H_i(S^n \times S^n) = \begin{cases} \mathbb{Z} & i = 0, 2n \\ \mathbb{Z}^2 & i = n \\ 0 & \text{otherwise.} \end{cases}$$

In the remaining of this section, we discuss how to compute in general the maps

$$d_n : \mathcal{C}_n(X) = \mathbb{Z}^{\#n-cells} \to \mathcal{C}_{n-1}(X) = \mathbb{Z}^{\#(n-1)-cells}$$

of the cellular chain complex. Let us consider the n-cells $\{e^n_\alpha\}_\alpha$ as the basis for $\mathcal{C}_n(X)$ and the $(n-1)$-cells $\{e^{n-1}_\beta\}_\beta$ as the basis for $\mathcal{C}_{n-1}(X)$. In particular, we can write:

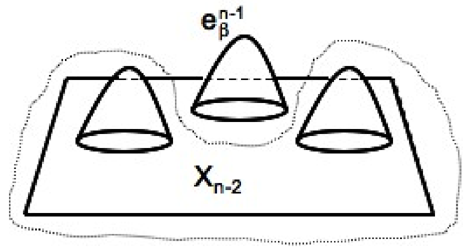
$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} \cdot e^{n-1}_\beta,$$

with $d_{\alpha\beta} \in \mathbb{Z}$. The following result provides a way of computing the coefficients $d_{\alpha\beta}$:

Theorem 1.1.13. *The coefficient* $d_{\alpha\beta}$ *is equal to the degree of the map* $\Delta_{\alpha,\beta} : S^{n-1}_\alpha \to S^{n-1}_\beta$ *defined by the composition:*

$$S^{n-1}_\alpha = \partial e^n_\alpha \xrightarrow{\varphi^n_\alpha} X_{n-1} = X_{n-2} \amalg_\gamma e^{n-1}_\gamma \xrightarrow{collapse} X_{n-1}/(X_{n-2} \sqcup_{\gamma\neq\beta} e^{n-1}_\gamma) = S^{n-1}_\beta,$$

where φ^n_α *is the attaching map of* e^n_α*, and the collapsing map sends* $X_{n-2} \coprod_{\gamma\neq\beta} e^{n-1}_\gamma$ *to a point.*



Proof. We will proceed with the proof by chasing the following diagram, and we note that the map $\Delta_{\alpha\beta*}$ is defined so that the top right square commutes.

$$\begin{array}{ccccccc}
H_n(D^n_\alpha, S^{n-1}_\alpha) & \xrightarrow[\simeq]{\partial} & H_{n-1}(S^{n-1}_\alpha) & \xrightarrow{\Delta_{\alpha\beta*}} & H_{n-1}(S^{n-1}_\beta) & & \\
\downarrow \Phi^n_* & & \downarrow \varphi^{n-1}_{\alpha*} & & \uparrow q_{\beta*} & & \\
\mathcal{C}_n(X) & \xrightarrow{\partial_n} & H_{n-1}(X_{n-1}) & \xrightarrow{q_*} & \tilde{H}_{n-1}(X_{n-1}/X_{n-2}) & & = \oplus_\beta \tilde{H}_{n-1}(e^{n-1}_\beta/\partial e^{n-1}_\beta) \\
& \searrow d_n & \downarrow j^{n-1}_* & & \downarrow \simeq & & \\
& & \mathcal{C}_{n-1}(X) & \xrightarrow{\simeq} & H_n(\frac{X_{n-1}}{X_{n-2}}, \frac{X_{n-2}}{X_{n-2}}) & &
\end{array}$$

Recall that our goal is to compute $d_n(e^n_\alpha)$. The upper left square is natural and therefore commutes (it is induced by the characteristic map $\Phi : (D^*, S^{*-1}) \to (X_*, X_{*-1})$ of a cell), while the lower left triangle is part of the exact diagram defining the chain complex $\mathcal{C}_*(X)$ and is defined to commute as well. Appealing to naturality, the map Φ gives a unique D^n_α so that $\Phi^n(D^n_\alpha) = e^n_\alpha$. Since the top left square and the bottom left triangle both commute, this gives that

$$d_n(e^n_\alpha) = j^{n-1}_* \circ \varphi^{n-1}_{\alpha*} \circ \partial(D^n_\alpha).$$

Looking to the bottom right square, recall that since X is a CW complex, (X_n, X_{n-1}) is a good pair. This gives the isomorphism $\mathcal{C}_{n-1}(X) = H_{n-1}(X_{n-1}, X_{n-2}) \simeq \tilde{H}_{n-1}(X_{n-1}/X_{n-2})$ But, we similarly have $\tilde{H}_{n-1}(X_{n-1}/X_{n-2}) \simeq H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})$, where the isomorphism is induced by the quotient map q collapsing X_{n-2}.
The bottom right square commutes by the definition of j^{n-1}_* and q_*, from which it follows that

$$d_n(e^\alpha_n) = q_* \circ \varphi^{n-1}_{\alpha*} \circ \partial(D^\alpha_n),$$

where formally we should precompose in the left hand side with the isomorphism between $\mathcal{C}_{n-1}(X)$ and $\tilde{H}_{n-1}(X_{n-1}/X_{n-2})$ so that everything is in the same space. This last map takes the generator D_n^α to some linear combination of generators in $\oplus_\beta \tilde{H}_{n-1}(e_\beta^{n-1}/\partial e_\beta^{n-1})$. To see which generators it maps to, we project down to the β summands to obtain

$$d_n(e_n^\alpha) = \sum_\beta q_{\beta *} q_* \varphi_{\alpha *}^{n-1} \partial D_n^\alpha.$$

As noted before, we have defined $\Delta_{\alpha\beta *} = q_{\beta *} q_* \varphi_{\alpha *}^{n-1}$. So writing

$$d_n(e_n^\alpha) = \sum_\beta \Delta_{\alpha\beta *} \partial D_n^\beta,$$

we see from the definition of the above maps that $\Delta_{\alpha\beta *}$ is multiplication by $d_{\alpha\beta}$. □

Example 1.1.14. Let M_g be the close oriented surface of genus g, with its usual CW structure: one 0-cell, $2g$ 1-cells $\{a_1, b_1, \cdots a_g, b_g\}$, and one 2-cell attached by product of commutators $[a_1, b_1] \cdots [a_g, b_g]$. The associated cellular chain complex of M_g is:

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Since M_g is connected and has only one 0-cell, we get that $d_1 = 0$. We claim that d_2 is also the zero map. This amounts to showing that $d_2(e) = 0$, where e denotes the 2-cell. Indeed, let us compute the coefficients d_{ea_i} and d_{eb_i} in our degree formula. As the attaching map sends the generator to $a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}$, when we collapse all 1-cells (except a_i, resp. b_i) to a point, the word defining the attaching map $a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}$ reduces to $a_i a_i^{-1}$ and resp. $b_i b_i^{-1}$. Hence $d_{ea_i} = 1 - 1 = 0$. Similarly, $d_{eb_i} = 1 - 1 = 0$, for each i. Altogether,

$$d_2(e) = a_1 + b_1 - a_1 - b_1 + \cdots a_g + b_g - a_g - b_g = 0.$$

So the homology groups of M_g are given by

$$H_n(M_g) = \begin{cases} \mathbb{Z} & \text{i=0,2} \\ \mathbb{Z}^{2g} & \text{i=1} \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.1.15. Let N_g be the closed nonorientable surface of genus g, with its cell structure consisting of one 0-cell, g 1-cells $\{a_1, \cdots, a_g\}$, and one 2-cell e attached by the word $a_1^2 \cdots a_g^2$. The cellular chain complex of N_g is given by

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

As before, $d_1 = 0$ since N_g is connected and there is only one cell in dimension zero. To compute $d_2 : \mathbb{Z} \to \mathbb{Z}^g$ we again apply the cellular boundary formula, and obtain

$$d_2(1) = (2, 2, \cdots, 2)$$

since each a_1 appears in the attaching word with total exponent 2, which means that each map $\Delta_{\alpha\beta}$ is homotopic to the map $z \mapsto z^2$ of degree 2. In particular, d_2 is injective, hence $H_2(N_g) = 0$. If we change the standard basis for $\mathbb{Z}^g$ by replacing the last standard basis element $e_n = (0, \cdots, 0, 1)$ by $e'_n(1, \cdots, 1)$, then $d_2(1) = 2 \cdot e'_n$, so

$$H_1(N_g) \cong \mathbb{Z}^g/\text{Image } d_2 \cong \mathbb{Z}^g/2\mathbb{Z} \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2.$$

Altogether,

$$H_n(N_g) = \begin{cases} \mathbb{Z} & \text{i=0} \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & \text{i=1} \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.1.16. Recall that $\mathbb{RP}^n$ has a CW structure with one cell e^k in each dimension $0 \leq k \leq n$. Moreover, the attaching map of e^k in $\mathbb{RP}^n$ is the two-fold cover projection $\varphi : S^{k-1} \to \mathbb{RP}^{k-1}$. The cellular chain complex for $\mathbb{RP}^n$ looks like:

$$0 \xrightarrow{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} ...\mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

To compute the differential d_k, we need to compute the degree of the composite map

$$\Delta : S^{k-1} \xrightarrow{\varphi} \mathbb{RP}^{k-1} \xrightarrow{q} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} = S^{k-1}.$$

The map Δ is a homeomorphism when restricted to each component of $S^{k-1} \setminus S^{k-2}$, and these homeomorphisms are obtained from each other by precomposing with the antipodal map a of S^{k-1}, which has degree $(-1)^k$. Hence, by our local degree formula, we get that:

$$\deg \Delta = \deg id + \deg a = 1 + (-1)^k.$$

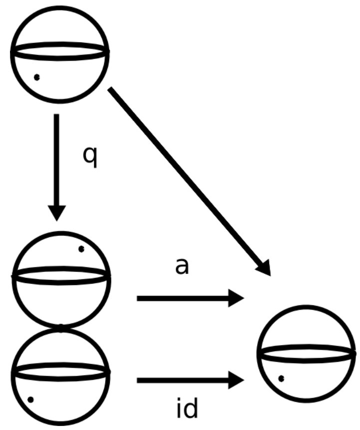
In particular,

$$d_k = \begin{cases} 0 & \text{if k is odd} \\ 2 & \text{if k is even,} \end{cases}$$

and therefore we obtain that

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}_2 & \text{if k is odd },\ 0 < k < n \\ \mathbb{Z} & k = 0, \text{ and } k = n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, note that an equivalent definition of the above map Δ is obtained by first collapsing the equatorial S^{k-2} to a point to get $S^{k-1} \vee S^{k-1}$, and then mapping the two copies of S^{k-1} onto S^{k-1}, the first one by the identity map, and the second by the antipodal map.



Exercises

1. Describe a cell structure on $S^n \vee S^n \vee \cdots \vee S^n$ and calculate $H_*(S^n \vee S^n \vee \cdots \vee S^n)$.

2. Let $f : S^n \to S^n$ be a map of degree m. Let $X = S^n \cup_f D^{n+1}$ be a space obtained from S^n by attaching a $(n+1)$-cell via f. Compute the homology of X.

3. Let G be a finitely generated abelian group, and fix $n \geq 1$. Construct a CW-complex X such that $H_n(X) \cong G$ and $\tilde{H}_i(X) = 0$ for all $i \neq n$. (Hint: Use the calculation of the previous exercise, together with know facts from Algebra about the structure of finitely generated abelian groups.) More generally, given finitely generated abelian groups $G_1, G_2, \cdots, G_k$, construct a CW-complex X whose homology groups are $H_i(X) = G_i$, $i = 1, \cdots, k$, and $\tilde{H}_i(X) = 0$ for all $i \notin \{1, 2, \cdots, k\}$.

4. Show that $\mathbb{RP}^5$ and $\mathbb{RP}^4 \vee S^5$ have the same homology and fundamental group. Are these spaces homotopy equivalent?

5. Let $0 \leq m < n$. Compute the homology of $\mathbb{RP}^n/\mathbb{RP}^m$.

6. The *mapping torus* T_f of a map $f : X \to X$ is the quotient of $X \times I$

$$T_f = \frac{X \times I}{(x, 0) \sim (f(x), 1)}.$$

Let A and B be copies of S^1, let $X = A \vee B$, and let p be the wedge point of X. Let $f : X \to X$ be a map that satisfies $f(p) = p$, carries A into A by a degree–3 map, and carries B into B by a degree–5 map.

(a) Equip T_f with a CW structure by attaching cells to $X \vee S^1$.

(b) Compute a presentation of $\pi_1(T_f)$.

(c) Compute $H_1(T_f; \mathbb{Z})$.

7. The closed oriented surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a space X. Compute the homology groups of X and the relative homology groups of (R, M_g).

8. Let X be the space obtained by attaching two 2-cells to S^1, one via the map $z \mapsto z^3$ and the other via $z \mapsto z^5$, where z denotes the complex coordinate on $S^1 \subset \mathbb{C}$.

(a) Compute the homology of X with coefficients in $\mathbb{Z}$.

(b) Is X homeomorphic to the 2-sphere S^2? Justify your answer!

1.2 Euler Characteristic

Definition 1.2.1. *Let X be a finite CW complex of dimension n and denote by c_i the number of i cells of X. The Euler characteristic of X is defined as:*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot c_i. \tag{1.2.1}$$

It is natural to question whether or not the Euler characteristic depends on the cell structure chosen for the space X. As we will see below, this is not the case. It suffices to show that the Euler characteristic depends only on the cellular homology of the space X. Indeed, cellular homology is isomorphic to singular homology, and the latter is independent of the cell structure on X.

Recall that if G is a finitely generated abelian group, then G decomposes into a free part and a torsion part, i.e.,

$$G \simeq \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \cdots \mathbb{Z}_{n_k}.$$

The integer $r := \mathrm{rk}(G)$ is the rank of G. The rank is additive in short exact sequences of finitely generated abelian groups.

Theorem 1.2.2. *The Euler characteristic can be computed as:*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot b_i(X) \tag{1.2.2}$$

with $b_i(X) := \mathrm{rk} H_i(X)$ the i-th Betti number of X. In particular, $\chi(X)$ is independent of the chosen cell structure on X.

Proof. We will follow the following notation: $B_i = \mathrm{Image}(d_{i+1})$, $Z_i = \ker(d_i)$, and $H_i = Z_i/B_i$. Consider a chain complex of finitely generated abelian groups and the short exact sequences defining homology:

$$0 \xrightarrow{d_{n+1}} \mathcal{C}_n \xrightarrow{d_n} \ldots \xrightarrow{d_2} \mathcal{C}_1 \xrightarrow{d_1} \mathcal{C}_0 \xrightarrow{d_0} 0$$

$$0 \longrightarrow Z_i \xhookrightarrow{\iota} \mathcal{C}_i \xrightarrow{d_i} B_{i-1} \longrightarrow 0$$

$$0 \longrightarrow B_i \xrightarrow{d_{i+1}} Z_i \xrightarrow{q} H_i \longrightarrow 0$$

The additivity of rank yields that

$$c_i := \mathrm{rk}\mathcal{C}_i = \mathrm{rk}Z_i + \mathrm{rk}B_{i-1}$$

and

$$\mathrm{rk}Z_i = \mathrm{rk}B_i + \mathrm{rk}H_i.$$

Substitute the second equality into the first, multiply the resulting equality by $(-1)^i$, and sum over i to get that $\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \mathrm{rk}H_i$.

Finally, apply this result to the cellular chain complex $\mathcal{C}_i = H_i(X_i, X_{i-1})$ and use the identification between cellular and singular homology. ☐

Example 1.2.3. If M_g and N_g denote the orientable and resp. nonorientable closed surfaces of genus g, then $\chi(M_g) = 1 - 2g + 1 = 2(1-g)$ and $\chi(N_g) = 1 - g + 1 = 2 - g$. So all the orientable and resp. non-orientable surfaces are distinguished from each other by their Euler characteristic, and there are only the relations $\chi(M_g) = \chi(N_{2g})$.

Exercises

1. A graded abelian group is a sequence of abelian groups $A_\bullet := (A_n)_{n\geq 0}$. We say that $A_\bullet$ is of *finite type* if

$$\sum_{n\geq 0} \mathrm{rank} A_n < \infty.$$

The *Euler characteristic* of a finite type graded abelian group $A_\bullet$ is the integer

$$\chi(A_\bullet) := \sum_{n\geq 0} (-1)^n \cdot \mathrm{rank} A_n.$$

A short exact sequence of graded groups $A_\bullet$, $B_\bullet$, $C_\bullet$, is a sequence of short exact sequences

$$0 \to A_n \to B_n \to C_n \to 0, \quad n \geq 0.$$

Prove that if $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a short exact sequence of graded abelian groups of finite type, then

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

2. Suppose we are given three finite type graded abelian groups $A_\bullet$, $B_\bullet$, $C_\bullet$, which are part of a long exact sequence

$$\cdots \to A_k \stackrel{i_k}{\to} B_k \stackrel{j_k}{\to} C_k \stackrel{\partial_k}{\to} A_{k-1} \to \cdots \to A_0 \to B_0 \to C_0 \to 0.$$

Show that

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

3. For finite CW complexes X and Y, show that

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

4. If a finite CW complex X is a union of subcomplexes A and B, show that

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

5. For a finite CW complex and $p : Y \to X$ an n-sheeted covering space, show that

$$\chi(Y) = n \cdot \chi(X).$$

6. Show that if $f : \mathbb{RP}^{2n} \to Y$ is a covering map of a CW-complex Y, then f is a homeomorphism.

1.3 Lefschetz Fixed Point Theorem

Recall that if G is a finitely generated abelian group, then G decomposes into a free part and a torsion part, i.e.,

$$G \simeq \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \cdots \mathbb{Z}_{n_k}.$$

Here $r = \mathrm{rk}(G)$ and $\mathrm{Torsion}(G) := \times_{i=1}^{k} \mathbb{Z}_{n_i}$. Given an endomorphism $\varphi : G \to G$, define its trace by

$$\mathrm{Tr}(\varphi) = \mathrm{Tr}\ \left(\bar{\varphi} : G/\mathrm{Torsion}(G) \to G/\mathrm{Torsion}(G)\right) \tag{1.3.1}$$

where the latter trace is the linear algebraic trace of the map $\bar{\varphi} : \mathbb{Z}^r \to \mathbb{Z}^r$.

Definition 1.3.1. *If X has the homotopy type of a finite simplicial or cellular complex and $f : X \to X$, then the Lefschetz number of f is defined to be*

$$\tau(f) = \sum_i (-1)^i \cdot \mathrm{Tr}(f_* : H_i(X) \to H_i(X)). \tag{1.3.2}$$

Remark 1.3.2. Notice that homotopic maps have the same Lefschetz number since they induce the same maps on homology.

Example 1.3.3. If $f \simeq id_X$, then $\tau(f) = \chi(X)$. This follows from the fact the map induced in homology by the identity map is the identity matrix and that the trace of the identity matrix in this case is the corresponding Betti number of X.

Theorem 1.3.4. *(Lefschetz)*
If X is a retract of a finite simplicial (or cellular) complex and if $f : X \to X$ satisfies $\tau(f) \neq 0$, then f has a fixed point.

Before proving this theorem, let us consider a few examples.

Example 1.3.5. Suppose that X has the homology of a point (up to torsion). Then

$$\tau(f) = \mathrm{Tr}\ (f_* : H_0(X) \to H_0(X)) = 1.$$

This follows from the fact that all the other homology groups are zero and that the map induced on H_0 is the identity.

This example leads immediately to two nontrivial results, the first of which is the Brouwer fixed point theorem.

Example 1.3.6. (Brouwer) If $f : D^n \to D^n$ is continuous then f has a fixed point.

Example 1.3.7. If $X = \mathbb{RP}^{2n}$ then modulo torsion X has the homology of a point. Therefore any continuos map $f : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point.

Finally we are led to an example which does not follow from the computation for a point.

Example 1.3.8. If $f : S^n \to S^n$ is a continuos map and $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point. To verify this, we compute

$$\begin{aligned}\tau(f) &= \mathrm{Tr}(f_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \cdot \mathrm{Tr}(f_* : H_n(S^n) \to H_n(S^n)) \\ &= 1 + (-1)^n \cdot \deg(f) \\ &\neq 0.\end{aligned}$$

Corollary 1.3.9. *If $a : S^n \to S^n$ is the antipodal map, then* $\deg(a) = (-1)^{n+1}$.

Now we return to outlining the proof:

Definition 1.3.10. *If K and L are simplicial complexes and $f : K \to L$ is a linear map which sends each simplex of K to a simplex in L so that vertices map to vertices, then f is said to be simplicial.*

Note that a simplicial map is uniquely determined by its values on vertices. The *simplicial approximation theorem* asserts that given any map f from a finite simplicial complex to an arbitrary simplicial complex, we can find a map g in the homotopy class of f so that g is simplicial in the above sense with respect to some finite iteration of barycentric subdivisions of the domain.

Theorem 1.3.11. *If K is a finite simplicial complex and L is an arbitrary simplicial complex, then for any map $f : K \to L$ there is a map in the homotopy class of f which is simplicial with respect to some interated barycentric subdivision of K.*

The proof of this result is omitted. We now proceed to the Lefschetz theorem.

Proof. (sketch)
Let us suppose that f has no fixed points. The general case reduces to the case when X is a finite simplicial complex. Indeed, if $r : K \to X$ is a retraction of a finite simplicial complex K onto X, the composition $f \circ r : K \to X \subset K$ has exactly the same fixed points as f and since $r_* : H_i(K) \to H_i(X)$ is projection onto a direct summand, we have that $\mathrm{Tr}(f_* \circ r_*) = \mathrm{Tr}(f_*)$, so $\tau(f \circ r) = \tau(f)$. We therefore take X to be a finite simplicial complex.

X is compact and there exists a metric d on X so that d restricts to the Euclidean metric on each simplex of X; choose such a metric. If f has no fixed points, we can find a uniform ϵ for which $d(x, f(x)) > \epsilon$ by the standard covering trick. Via repeated barycentric subdivision of X we can construct L so that for each vertex, the union of all simplicies containing that vertex has diameter less than $\frac{\epsilon}{2}$. Applying the simplicial approximation theorem we can find a subdivision K of L and a simplicial map $g : K \to L$ so that g lies in the homotopy class of f. Moreover, we may take g so that $f(\sigma)$ lies in the subcomplex of X consisting of all simplicies containing σ. Again, by repeated barycentric subdivision we may choose K so that each simplex in K has diameter less than $\frac{\epsilon}{2}$. In particular then $g(\sigma) \cap \sigma = \emptyset$ for each $\sigma \in K$. Notice $\tau(g) = \tau(f)$ since f and g are homotopic.

Since g is simplicial, K_n maps to L_n (that is, g sends n-skeletons to n-skeletons). We constructed K as a subdivision of L so that $g(K_n) \subset K_n$ for each n.

We will use the algebraic fact that trace is additive for short exact sequences to show that we can replace $H_i(X)$ with $H_i(K_i, K_{i-1})$ in our computation of the Lefschetz number. By essentially the same argument as was used above in the computation of the Euler characteristic and using this fact we obtain that

$$\tau(g) = \sum_i (-1)^i \cdot \mathrm{Tr}(g_* : H_i(K_i, K_{i-1}) \to H_i(K_i, K_{i-1}))$$

We have a natural basis for $H_i(K_i, K_{i-1})$ coming from the simplicies σ^i in K_i. But since $g(\sigma) \cap \sigma = \emptyset$ it follows that $\mathrm{Tr}(g_* : H_i(K_i, K_{i-1}) \to H_i(K_i, K_{i-1})) = 0$ for each i. So $\tau(f) = \tau(g) = 0$.

The cellular case is proved similarly, using instead a corresponding cellular approximation theorem.

□

Exercises

1. Is there a continuous map $f : \mathbb{RP}^{2k-1} \to \mathbb{RP}^{2k-1}$ with no fixed points? Explain.

1. Is there a continuous map $f : \mathbb{CP}^{2k-1} \to \mathbb{CP}^{2k-1}$ with no fixed points? Explain. We will see later that any map $f : \mathbb{CP}^{2k} \to \mathbb{CP}^{2k}$ has a fixed point.

1.4 Homology with General Coefficients

Let G be an abelian group and X a topological space. We define the homology of X with G coefficients, denoted $H_*(X;G)$, as the homology of the chain complex

$$C_i(X;G) = C_i(X) \otimes G \tag{1.4.1}$$

consisting of finite formal sums $\sum_i \eta_i \cdot \sigma_i$ ($\sigma : \Delta_i \to X$, $\eta_i \in G$), and with boundary maps given by

$$\partial_i^G := \partial_i \otimes id_G.$$

Since ∂_i satisfies $\partial_i \circ \partial_{i+1} = 0$ it follows that $\partial_i^G \circ \partial_{i+1}^G = 0$, so $(C_*(X;G), \partial_*^G)$ forms indeed a chain complex. We can construct versions of the usual modified homology groups (relative, reduced, etc.) in the natural way. Define relative chains by $C_i(X, A; G) := C_i(X;G)/C_i(A;G)$, and reduced homology via the augmented chain complex

$$\cdots \xrightarrow{\partial_{i+1}^G} C_i(X;G) \xrightarrow{\partial_i^G} \cdots \xrightarrow{\partial_2^G} C_1(X;G) \xrightarrow{\partial_1^G} C_0(X;G) \xrightarrow{\epsilon} G \to 0.$$

where $\epsilon(\sum_i \eta_i \sigma_i) = \sum_i \eta_i$. Notice that $H_i(X) = H_i(X, \mathbb{Z})$ by definition.

By looking directly at the chain maps, it follows that

$$H_i(pt;G) = \begin{cases} G & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Nothing (other than coefficients) needs to change in our previous proofs about the relationships between relative homology and reduced homology of quotient spaces so we can compute the homology of a sphere as before by induction and using the long exact sequence of the pair (D^n, S^n) to be

$$H_i(S^n;G) = \begin{cases} G & i = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we can build cellular homology in the same way, defining

$$\mathcal{C}_i^G(X) = H_i(X_i, X_{i-1}; G) = G^{\#\ \text{n-cells}}.$$

The cellular boundary maps are given by:

$$d_n(\sum_\alpha n_\alpha e_\alpha^n) = \sum_{\alpha,\beta} d_{\alpha\beta} n_\alpha e_\beta^{n-1},$$

where $d_{\alpha\beta}$ is as before the degree of a map $\Delta_{\alpha\beta} : S^{n-1} \to S^{n-1}$. This follows from the easy fact that if $f : S^k \to S^k$ has degree m, then $f_* : H_k(S^k;G) \simeq G \to H_k(S^k;G) \simeq G$ is the multiplication by m. As it is the case for integers, we get

$$H_i^{CW}(X;G) \simeq H_i(X;G)$$

for all i.

Example 1.4.1. We compute $H_i(\mathbb{RP}^n;\mathbb{Z}_2)$ using the calculation above. Notice that over $\mathbb{Z}$ the cellular boundary maps are $d_i = 0$ or $d_i = 2$ depending on the parity of i, and therefore with $\mathbb{Z}_2$-coefficients all of boundary maps vanish. Therefore,

$$H_i(\mathbb{RP}^n;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & 0 \le i \le n \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.4.2. Fix $n > 0$ and let $g : S^n \to S^n$ be a map of degree m. Define the CW complex

$$X = S^n \cup_g e^{n+1},$$

where the $(n+1)$-cell ∂e^{n+1} is attached to S^n via the map g. Let f be the quotient map $f : X \to X/S^n$. Define $Y = X/S^n = S^{n+1}$. The homology of X can be easily computed by using the cellular chain complex:

$$0 \xrightarrow{d_{n+2}} \mathbb{Z} \xrightarrow[m]{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} \dots \xrightarrow{d_1} 0 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Therefore,

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_m & i = n \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, as $Y = S^{n+1}$, we have

$$H_i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that f induces the trivial homomorphisms in homology with $\mathbb{Z}$-coefficients (except in degree zero, where f_* is the identity). So it is natural to ask if f is homotopic to the constant map. As we will see below, by considering $\mathbb{Z}_m$-coefficients we can show that this is not the case.

Let us now consider $H_*(X;\mathbb{Z}_m)$ where m is, as above, the degree of the map g. We return to the cellular chain complex level and observe that we have

$$0 \xrightarrow{d_{n+2}} \mathbb{Z}_m \xrightarrow[m]{d_{n+1}} \mathbb{Z}_m \xrightarrow{d_n} \dots \xrightarrow{d_1} 0 \xrightarrow{d_1} \mathbb{Z}_m \xrightarrow{d_0} 0$$

Multiplication by m is now the zero map, so we get

$$H_i(X;\mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & i = 0, n, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Also, as already discussesd,

$$H_i(Y;\mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & i = 0, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

We next consider the induced homomorphism $f_* : H_{n+1}(X;\mathbb{Z}_m) \to H_{n+1}(X;\mathbb{Z}_m)$. The claim is that this map is injective, thus non-trivial map, so f cannot be homotopic to the constant map. As noted before, we still have an isomorphism $\widetilde{H}_{n+1}(Y;\mathbb{Z}_m) \simeq H_{n+1}(X, S^n;\mathbb{Z}_m)$. This leads us to consider the long exact sequence of the pair (X, S^n) in dimension $n+1$. We have

$$\cdots \longrightarrow H_{n+1}(S^n;\mathbb{Z}_m) \longrightarrow H_{n+1}(X;\mathbb{Z}_m) \xrightarrow{f_*} H_{n+1}(X, S^n;\mathbb{Z}_m) \longrightarrow \cdots$$

But, $H_{n+1}(S^n;\mathbb{Z}_m) = 0$ and so f_* is injective on $H_{n+1}(X;\mathbb{Z}_m)$. Since $H_{n+1}(X;\mathbb{Z}_m) = \mathbb{Z}_m \neq 0$ and $H_{n+1}(X, S^n;\mathbb{Z}_m) \simeq \widetilde{H}_{n+1}(Y;\mathbb{Z}_m)$ it follows that f_* is not trivial on $H_{n+1}(X;\mathbb{Z}_m)$, which proves our claim.

Exercises

1. Calculate the homology of the 2-torus T^2 with coefficients in $\mathbb{Z}$, $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively. Do the same calculations for the Klein bottle.

1.5 Universal Coefficient Theorem for Homology

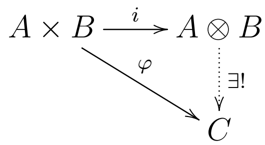
1.5.1 Tensor Products

Let A, B be abelian groups. Define the abelian group

$$A \otimes B = \{a \otimes b \mid a \in A,\ b \in B\}/\sim \tag{1.5.1}$$

where $\sim$ is generated by the relations $(a + a') \otimes b = a \otimes b + a' \otimes b$ and $a \otimes (b + b') = a \otimes b + a \otimes b'$. The zero element of $A \otimes B$ is $0 \otimes b = a \otimes 0 = 0 \otimes 0 = 0_{A \otimes B}$ since, e.g., $0 \otimes b = (0+0) \otimes b = 0 \otimes b + 0 \otimes b$ so $0 \otimes b = 0_{A\otimes B}$. Similarly, the inverse of an element $a \otimes b$ is $-(a \otimes b) = (-a) \otimes b = a \otimes (-b)$ since, e.g., $0_{A\otimes B} = 0 \otimes b = (a + (-a)) \otimes b = a \otimes b + (-a) \otimes b$.

Lemma 1.5.1. *The tensor product satisfies the following universal property which asserts that if $\varphi : A \times B \to C$ is any bilinear map, then there exists a unique map $\overline{\varphi} : A \otimes B \to C$ such that $\varphi = \overline{\varphi} \circ i$, where $i : A \times B \to A \otimes B$ is the natural map $(a, b) \mapsto a \otimes b$.*



Proof. Indeed, $\overline{\varphi} : A \otimes B \to C$ can be defined by $a \otimes b \mapsto \varphi(a, b)$. □

Proposition 1.5.2. *The tensor product satisfies the following properties:*

(1) $A \otimes B \cong B \otimes A$ *via the isomorphism* $a \otimes b \mapsto b \otimes a$.

(2) $(\bigoplus_i A_i) \otimes B \cong \bigoplus_i (A_i \otimes B)$ *via the isomorphism* $(a_i)_i \otimes b \mapsto (a_i \otimes b)_i$.

(3) $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ *via the isomorphism* $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$.

(4) $\mathbb{Z} \otimes A \cong A$ *via the isomorphism* $n \otimes a \mapsto na$.

(5) $\mathbb{Z}/n\mathbb{Z} \otimes A \cong A/nA$ *via the isomorphism* $l \otimes a \mapsto la$.

Proof. These are easy to prove by using the above universal property. We sketch a few.

(1) The map $\varphi : A \times B \to B \otimes A$ defined by $(a, b) \mapsto b \otimes a$ is clearly bilinear and therefore induces a homomorphism $\overline{\varphi} : A \otimes B \to B \otimes A$ with $a \otimes b \mapsto b \otimes a$. Similarly, there is the reverse map $\psi : B \times A \to A \otimes B$ defined by $(b, a) \mapsto a \otimes b$ which induces a homomorphism $\overline{\psi} : B \otimes A \to A \otimes B$ with $b \otimes a \mapsto a \otimes b$. Clearly, $\overline{\varphi} \circ \overline{\psi} = id_{B \otimes A}$ and $\overline{\psi} \circ \overline{\varphi} = id_{A \otimes B}$ and $A \otimes B \cong B \otimes A$.

(4) The map $\varphi : \mathbb{Z} \times A \to A$ defined by $(n, a) \mapsto na$ is a bilinear map and therefore induces a homomorphism $\overline{\varphi} : \mathbb{Z} \otimes A \to A$ with $n \otimes a \mapsto na$. Now suppose $\overline{\varphi}(n \otimes a) = 0$. Then $na = 0$ and $n \otimes a = 1 \otimes (na) = 1 \otimes 0 = 0_{\mathbb{Z} \otimes A}$. Thus $\overline{\varphi}$ is injective. Moreover, if $a \in A$, then $\overline{\varphi}(1 \otimes a) = a$ and $\overline{\varphi}$ is surjective as well.

(5) The map $\varphi : \mathbb{Z}/n\mathbb{Z} \times A \to A/nA$ defined by $(l, a) \mapsto la$ is a bilinear map and therefore induces a homomorphism $\overline{\varphi} : \mathbb{Z}/n\mathbb{Z} \otimes A \to A/nA$ with $l \otimes a \mapsto la$. Now suppose $\overline{\varphi}(l \otimes a) = la = 0$. Then $la = \sum_{i=1}^{k} na_i$ and $l \otimes a = 1 \otimes (la) = 1 \otimes (\sum_{i=1}^{k} na_i) = \sum_{i=1}^{k} (n \otimes a_i) = 0_{\mathbb{Z}/n\mathbb{Z} \otimes A}$, so $\overline{\varphi}$ is injective. Now let $a \in A/nA$. Then $\overline{\varphi}(1 \otimes a) = a$ and $\overline{\varphi}$ is surjective as well.

□

More generally, if R is a ring and A and B are R-modules, a tensor product $A \otimes_R B$ can be defined as follows:

(1) if R is commutative, define the R-module $A \otimes_R B := A \otimes B / \sim$, where $\sim$ is the relation generated by $ra \otimes b = a \otimes rb = r(a \otimes b)$.

(2) if R is not commutative, we need A a right R-module and B a left R-module and the relation is $ar \otimes b = a \otimes rb$. In this case $A \otimes_R B$ is only an abelian group.

In both cases, $A \otimes_R B$ is not necessarily isomorphic to $A \otimes B$.

Example 1.5.3. Let $R = \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Now $R \otimes_R R \cong R$ which is a 2-dimensional $\mathbb{Q}$-vector space. However, $R \otimes R$ as a $\mathbb{Z}$-module is a 4-dimesnional $\mathbb{Q}$-vector space.

Lemma 1.5.4. *If G is an abelian group, then the functor $- \otimes G$ is right exact, that is, if $A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is exact, then $A \otimes G \xrightarrow{i \otimes 1_G} B \otimes G \xrightarrow{j \otimes 1_G} C \otimes G \to 0$ is exact.*

Proof. Let $c \otimes g \in C \otimes G$. Since j is onto, there exists, $b \in B$ such that $j(b) = c$. Then $(j \otimes 1_G)(b \otimes g) = c \otimes g$ and $j \otimes 1_G$ is onto.

Since $j \circ i = 0$, we have $(j \otimes 1_G) \circ (i \otimes 1_G) = (j \circ i) \otimes 1_G = 0$ and thus, $\text{Image}(i \otimes 1_G) \subseteq \ker(j \otimes 1_G)$.

It remains to show that $\ker(j \otimes 1_G) \subseteq \text{Image}(i \otimes 1_G)$. It is enough to show that

$$\psi : B \otimes G / \text{Image}(i \otimes 1_G) \xrightarrow{\cong} C \otimes G,$$

where ψ is the map induced by $j \otimes 1_G$. Construct an inverse of ψ, induced from the homomorphism

$$\varphi : C \times G \to B \otimes G / \text{Image}(i \otimes 1_G)$$

defined by $(c, g) \mapsto b \otimes g$, where $j(b) = c$. We must show that φ is a well-defined bilinear map and that the induced map $\overline{\varphi}$ satisfies $\overline{\varphi} \circ \psi = id$ and $\psi \circ \overline{\varphi} = id$.

If $j(b) = j(b') = c$, then $b - b' \in \ker\ j = \text{Image}\ i$, so $b - b' = i(a)$ for some $a \in A$. Thus, $b \otimes g - b' \otimes g = (b - b') \otimes g = i(a) \otimes g \in \text{Image}(i \otimes 1_G)$. So φ is well defined.

Now $\varphi((c + c', g)) = d \otimes g$ where $j(d) = c + c'$. Since j is surjective, choose $b, b' \in B$ such that $j(b) = c$ and $j(b') = c'$. Then $d - (b + b') \in \ker\ j = \text{Image}\ i$ and so there exists $a \in A$ such that $i(a) = d - (b + b')$. Thus, $\varphi((c + c', g)) = d \otimes g = (b + b') \otimes g =$

$b \otimes g + b' \otimes g = \varphi(c,g) + \varphi(c',g)$ and φ is linear in the first component. For the second component, $\varphi(c, g+g') = b \otimes (g+g') = b \otimes g + b \otimes g' = \varphi(c,g) + \varphi(c,g')$. Thus, φ is bilinear.

Now by the universal property of the tensor product, the bilinear map φ induces a homomorphism

$$\overline{\varphi} : C \otimes G \to B \otimes G/\mathrm{Image}(i \otimes 1_G)$$

defined by $c \otimes g \mapsto \varphi(c,g) = b \otimes g$, where $j(b) = c$. For $c \otimes g \in C \otimes G$,

$$\psi \circ \overline{\varphi}(c \otimes g) = \psi(b \otimes g) = j(b) \otimes g = c \otimes g,$$

so $\psi \circ \overline{\varphi} = id_{C \otimes G}$. Similarly, for $b \otimes g \in B \otimes G/\mathrm{Image}(i \otimes 1_G)$, $\overline{\varphi} \circ \psi(b \otimes g) = \overline{\varphi}(j(b) \otimes g) = \varphi(j(b), g) = b \otimes g$. Thus $\overline{\varphi} \circ \psi = id$. □

1.5.2 The Tor functor and the Universal Coefficient Theorem

In this section we explain how to compute $H_*(X;G)$ in terms of $H_*(X;\mathbb{Z})$ and G. More generally, given a chain complex

$$C_\bullet : \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_0 \to 0$$

of free abelian groups and G an abelian group, we aim to compute $H_*(C_\bullet; G) = H_*(C_\bullet \otimes G)$ in terms of $H_*(C_\bullet;\mathbb{Z})$ and G. The answer is provided by the following result:

Theorem 1.5.5. *(Universal Coefficient Theorem)*
There are natural short exact sequences:

$$0 \to H_n(C_\bullet) \otimes G \to H_n(C_\bullet; G) \to \mathrm{Tor}(H_{n-1}(C_\bullet), G) \to 0 \text{ for all } n. \tag{1.5.2}$$

Naturality here means that if $C_\bullet \to C'_\bullet$ is a chain map, then there is an induced map of short exact sequences with commuting squares. Moreover, these short exact sequences split, but not naturally.

In particular, if $C_\bullet = C_*(X,A)$ is the relative singular chain complex, then there are natural short exact sequences

$$0 \to H_n(X,A) \otimes G \to H_n(X,A;G) \to \mathrm{Tor}(H_{n-1}(X,A), G) \to 0. \tag{1.5.3}$$

Naturality is with respect to maps of pairs $(X,A) \xrightarrow{f} (Y,B)$. The exact sequence (1.5.3) splits, but not naturally. Indeed, if we assume that $A = B = \emptyset$, then we have splittings $H_n(X;G) = \big(H_n(X) \otimes G\big) \oplus \mathrm{Tor}(H_{n-1}(X), G)$, $H_n(Y;G) = \big(H_n(Y) \otimes G\big) \oplus \mathrm{Tor}(H_{n-1}(Y), G)$. If these splittings were natural, and f induces the trivial map $f_* = 0$ on $H_*(-;\mathbb{Z})$ then f induces the trivial map on $H_*(-;G)$, for any coefficient group G. But this is in contradiction with Example 1.4.2.

Let us next explain the Tor functor appearing in the statement of the universal coefficient theorem.

Definition 1.5.6. *A free resolution of an abelian group H is an exact sequence:*

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0,$$

with each F_n free abelian.

Given an abelian group G, from a free resolution $F_\bullet$ of H, we obtain a modified chain complex:

$$F_\bullet \otimes G : \cdots \to F_2 \otimes G \to F_1 \otimes G \to F_0 \otimes G \to 0.$$

We define

$$\mathrm{Tor}_n(H, G) := H_n(F_\bullet \otimes G). \tag{1.5.4}$$

Note here that we have removed the final term of the complex to account for the fact that $- \otimes G$ is right exact.

Moreover, the following holds:

Lemma 1.5.7. *For any two free resolutions $F_\bullet$ and $F'_\bullet$ of H there are canonical isomorphisms $H_n(F_\bullet \otimes G) \cong H_n(F'_\bullet \otimes G)$ for all n. Thus, $\mathrm{Tor}_n(H, G)$ is independent of the free resolution $F_\bullet$.*

Proposition 1.5.8. *For any abelian group H, we have that*

$$\mathrm{Tor}_n(H, G) = 0 \text{ if } n > 1, \tag{1.5.5}$$

and

$$Tor_0(H, G) \cong H \otimes G. \tag{1.5.6}$$

Proof. Indeed, given an abelian group H, take F_0 to be the free abelian group on a set of generators of H to get $F_0 \overset{f_0}{\twoheadrightarrow} H \to 0$. Let $F_1 := \ker(f_0)$, and note that F_1 is a free group, as it is a subgroup of a free abelian group F_0. Let $F_1 \hookrightarrow F_0$ be the inclusion map. Then

$$0 \to F_1 \hookrightarrow F_0 \twoheadrightarrow H \to 0$$

is a free resolution of H. Thus, $\mathrm{Tor}_n(H, G) = 0$ if $n > 1$. Moreover, it follows readily that $\mathrm{Tor}_0(H, G) \cong H \otimes G$. □

Definition 1.5.9. *In what follows, we adopt the notation:*

$$Tor(H, G) := Tor_1(H, G).$$

Proposition 1.5.10. *The* Tor *functor satisfies the following properties:*

(1) $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$.

(2) $\mathrm{Tor}(\bigoplus_i A_i, B) \cong \bigoplus_i \mathrm{Tor}(A_i, B)$.

(3) $\mathrm{Tor}(A, B) = 0$ *if A or B is free or torsion-free.*

(4) $\operatorname{Tor}(A,B) \cong \operatorname{Tor}(\operatorname{Torsion}(A), B)$*, where* $\operatorname{Torsion}(A)$ *is the torsion subgroup of* A*.*

(5) $\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, A) \cong \ker(A \xrightarrow{n} A)$*.*

(6) For a short exact sequence: $0 \to B \to C \to D \to 0$ *of abelian groups, there is a natural exact sequence:*

$$0 \to \operatorname{Tor}(A,B) \to \operatorname{Tor}(A,C) \to \operatorname{Tor}(A,D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0.$$

Proof. (2) Choose a free resolution for $\bigoplus_i A_i$ as the direct sum of free resolutions for the A_i's.

(5) The exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}/n\mathbb{Z}$. Now $- \otimes A$ gives $\mathbb{Z} \otimes A \xrightarrow{n \otimes 1_A} \mathbb{Z} \otimes A \to 0$ which by property (4) of the tensor product is $A \xrightarrow{n} A \to 0$. Thus, $\operatorname{Tor}(\mathbb{Z}/n\mathbb{Z}, A) = \ker(A \xrightarrow{n} A)$.

(3) If A is free, we can choose the free resolution:

$$F_1 = 0 \to F_0 = A \to A \to 0$$

which implies that $\operatorname{Tor}(A,B) = 0$. On the other hand, if B is free, tensoring the exact sequence $0 \to F_1 \to F_0 \to A \to 0$ with $B = \mathbb{Z}^s$ gives a direct sum of copies of $0 \to F_1 \to F_0 \to A \to 0$. Hence, it is an exact sequence and so H_1 of this complex is 0. For the torsion free case, see below.

(6) Let $0 \to F_1 \to F_0 \to A \to 0$ be a free resolution of A, and tensor it with the short exact sequence $0 \to B \to C \to D \to 0$ to get a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_1 \otimes B & \to & F_1 \otimes C & \to & F_1 \otimes D & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_0 \otimes B & \to & F_0 \otimes C & \to & F_0 \otimes D & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Rows are exact since tensoring with a free group preserves exactness. Thus we get a short exact sequence of chain complexes. Recall now that for any short exact sequence of chain complexes $0 \to \mathcal{B}_\bullet \to \mathcal{C}_\bullet \to \mathcal{D}_\bullet \to 0$ (which means exactness for each level n: $0 \to B_n \to C_n \to D_n \to 0$, commuting with differential ∂), there is an associated long exact sequence of homology groups

$$\cdots \to H_n(\mathcal{B}_\bullet) \to H_n(\mathcal{C}_\bullet) \to H_n(\mathcal{D}_\bullet) \to H_{n-1}(\mathcal{B}_\bullet) \to \dots$$

So in our situation we obtain the homology long exact sequence:

$$0 \to H_1(F_\bullet \otimes B) \to H_1(F_\bullet \otimes C) \to H_1(F_\bullet \otimes D) \to H_0(F_\bullet \otimes B) \to H_0(F_\bullet \otimes C) \to H_0(F_\bullet \otimes D) \to 0$$

Since $H_1(F_\bullet \otimes B) = \mathrm{Tor}(A, B)$ and $H_0(F_\bullet \otimes B) = A \otimes B$, the above long exact sequence reduces to:

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0.$$

(1) Apply (6) to a free resolution $0 \to F_1 \to F_0 \to B \to 0$ of B, and get a long exact sequence:

$$0 \to \mathrm{Tor}(A, F_1) \to \mathrm{Tor}(A, F_0) \to \mathrm{Tor}(A, B) \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0.$$

Because F_1, F_0 are free, by (3) we have that $\mathrm{Tor}(A, F_1) = \mathrm{Tor}(A, F_0) = 0$, so the long exact sequence becomes:

$$0 \to \mathrm{Tor}(A, B) \to A \otimes F_1 \to A \otimes F_0 \to A \otimes B \to 0.$$

Also, by definition of Tor, we have a long exact sequence:

$$0 \to \mathrm{Tor}(B, A) \to F_1 \otimes A \to F_0 \otimes A \to B \otimes A \to 0.$$

So we get a diagram:

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Tor}(A, B) & \to & A \otimes F_1 & \to & A \otimes F_0 & \to & A \otimes B & \to 0 \\ & \downarrow \phi & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & \\ 0 \to & \mathrm{Tor}(B, A) & \to & F_1 \otimes A & \to & F_0 \otimes A & \to & B \otimes A & \to 0 \end{array}$$

with the arrow labeled ϕ defined as follows. The two squares on the right commute since $\otimes$ is naturally commutative. Hence, there exists $\phi : \mathrm{Tor}(A, B) \to \mathrm{Tor}(B, A)$ which makes the left square commutative. Moreover, by the 5-lemma, we get that ϕ is an isomorphism.

We can now prove the torsion free case of (3). Let $0 \to F_1 \xrightarrow{f} F_0 \to A \to 0$ be a free resolution of A. The claim about the vanishing of $\mathrm{Tor}(A, B)$ is equivalent to the injectivity of the map $f \otimes id_B : F_1 \otimes B \to F_0 \otimes B$. Assume $\sum_i x_i \otimes b_i \in \ker(f \otimes id_B)$. So $\sum_i f(x_i) \otimes b_i = 0 \in F_1 \otimes B$. In other words, $\sum_i f(x_i) \otimes b_i$ can be reduced to zero by a finite number of applications of the defining relations for tensor products. Only a finite number of elemnts of B, generating a finitely generated subgroup B_0 of B, are involved in this process, so in fact $\sum_i x_i \otimes b_i \in \ker(f \otimes id_{B_0})$. But B_0 is finitely generated and torsion free, hence free, so $\mathrm{Tor}(A, B_0) = 0$. Thus $\sum_i x_i \otimes b_i = 0$, which proves the claim. The case when A is torsion free follows now by using (1) to reduce to the previous case.

(4) Apply (6) to the short exact sequence: $0 \to \mathrm{Torsion}(A) \to A \to A/\mathrm{Torsion}(A) \to 0$ to get :

$$0 \to \mathrm{Tor}(G, \mathrm{Torsion}(A)) \to \mathrm{Tor}(G, A) \to \mathrm{Tor}(G, A/\mathrm{Torsion}(A)) \to \cdots$$

Because $A/\mathrm{Torsion}(A)$ is torsion free, $\mathrm{Tor}(G, A/\mathrm{Torsion}(A)) = 0$ by (3), so:

$$\mathrm{Tor}(G, \mathrm{Torsion}(A)) \simeq \mathrm{Tor}(G, A)$$

Now by (1), we get that $\mathrm{Tor}(A, G) \simeq \mathrm{Tor}(\mathrm{Torsion}(A), G)$.

□

Remark 1.5.11. It follows from (5) that

$$\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \frac{\mathbb{Z}}{(n,m)\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z},$$

where (n, m) is the greatest common divisor of n and m. More generally, if A and B are finitely generated abelian groups, then

$$\mathrm{Tor}(A, B) = \mathrm{Torsion}(A) \otimes \mathrm{Torsion}(B) \tag{1.5.7}$$

where $\mathrm{Torsion}(A)$ and $\mathrm{Torsion}(B)$ are the torsion subgroups of A and B respectively.

Let us conclude with some examples:

Example 1.5.12. Suppose $G = \mathbb{Q}$, then $\mathrm{Tor}(H_{n-1}(X), \mathbb{Q}) = 0$, so

$$H_n(X; \mathbb{Q}) \simeq H_n(X) \otimes \mathbb{Q}.$$

It follows that the n-th Betti number of X is given by

$$b_n(X) := \mathrm{rk} H_n(X) = \dim_{\mathbb{Q}} H_n(X; \mathbb{Q}).$$

Example 1.5.13. Suppose $X = T^2$, and $G = \mathbb{Z}/4$. Recall that $H_1(T^2) = \mathbb{Z}^2$. So:

$$H_0(T^2; \mathbb{Z}/4) = H_0(T^2) \otimes \mathbb{Z}/4 = \mathbb{Z}/4$$

$$H_1(T^2; \mathbb{Z}/4) = \big(H_1(T^2) \otimes \mathbb{Z}/4\big) \oplus \mathrm{Tor}(H_0(T^2), \mathbb{Z}/4) = \mathbb{Z}^2 \otimes \mathbb{Z}/4 = (\mathbb{Z}/4)^2$$

$$H_2(T^2; \mathbb{Z}/4) = \big(H_2(T^2) \otimes \mathbb{Z}/4\big) \oplus \mathrm{Tor}(H_1(T^2), \mathbb{Z}/4) = \mathbb{Z}/4.$$

Example 1.5.14. Suppose $X = K$ is the Klein bottle, and $G = \mathbb{Z}/4$. Recall that $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}/2$, and $H_2(K) = 0$, so:

$$H_2(K; \mathbb{Z}/4) = \big(H_2(K) \otimes \mathbb{Z}/4\big) \oplus \mathrm{Tor}(H_1(K), \mathbb{Z}/4) = \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/4) \oplus \mathrm{Tor}(\mathbb{Z}/2, \mathbb{Z}/4) = 0 \oplus \mathbb{Z}/2 = \mathbb{Z}/2.$$

Exercises

1. Prove Lemma 1.5.7.

2. Show that $\widetilde{H}_n(X; \mathbb{Z}) = 0$ for all n if, and only if, $\widetilde{H}_n(X; \mathbb{Q}) = 0$ and $\widetilde{H}_n(X; \mathbb{Z}/p) = 0$ for all n and for all primes p.

Chapter 2

Basics of Cohomology

Given a space X and an abelian group G, we will first define cohomology groups $H^i(X;G)$. In the next chapter we will show that, via the cup product operation, the graded group $\bigoplus_i H^i(X;G)$ becomes a ring. The ring structure will help us distinguish spaces X and Y which have isomorphic homology and cohomology groups but non-isomorphic cohomology rings, for example $X = \mathbb{CP}^2$ and $Y = S^2 \vee S^4$.

2.1 Cohomology of a chain complex: definition

Let G be an abelian group, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free abelian groups:

$$\cdots \longrightarrow C_{n+1} \overset{\partial_{n+1}}{\longrightarrow} C_n \overset{\partial_n}{\longrightarrow} C_{n-1} \overset{\partial_{n-1}}{\longrightarrow} \cdots \tag{2.1.1}$$

Dualize the chain complex (2.1.1), i.e., apply $\mathrm{Hom}(-;G)$ to it, to get the *cochain complex*:

$$\cdots \overset{\delta^{n+1}}{\longleftarrow} C^{n+1} \overset{\delta^n}{\longleftarrow} C^n \overset{\delta^{n-1}}{\longleftarrow} C^{n-1} \longleftarrow \cdots \tag{2.1.2}$$

with

$$C^n := \mathrm{Hom}(C_n, G), \tag{2.1.3}$$

and where the *coboundary map*

$$\delta^n : C^n \to C^{n+1} \tag{2.1.4}$$

is defined by

$$(\delta^n \psi)(\alpha) = \psi(\partial_{n+1}\alpha), \text{ for } \psi \in C^n \text{ and } \alpha \in C_{n+1}. \tag{2.1.5}$$

It follows that

$$(\delta^{n+1} \circ \delta^n)(\psi) = \psi \partial_{n+1} \partial_{n+2} = 0, \forall \psi \tag{2.1.6}$$

since $\partial_{n+1} \circ \partial_{n+2} = 0$ in the chain complex (2.1.1).

Definition 2.1.1. *The n-th cohomology group $H^n(C_\bullet;G)$ with G-coefficients of the chain complex $C_\bullet$ is defined by:*

$$H^n(C_\bullet;G) := H_n(C^\bullet;\delta^\bullet) := \ker(\delta : C^n \to C^{n+1})/\mathrm{Image}(\delta : C^{n-1} \to C^n). \tag{2.1.7}$$

2.2 Relation between cohomology and homology

In this section, we explain how each cohomology group $H^n(C_\bullet; G)$ can be computed only in terms of the coefficients G and the integral homology groups $H_*(C_\bullet)$ of $(C_\bullet, \partial_\bullet)$.

2.2.1 Ext groups

Let H and G be given abelian groups. Consider a free resolution of H:

$$F_\bullet : \ \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0$$

Dualize it with respect to G, i.e., apply $\mathrm{Hom}(-, G)$ to it, to get the cochain complex

$$\cdots \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \longleftarrow 0$$

where we set $H^* = \mathrm{Hom}(H, G)$ and similarly for F_i^*. After discarding H^*, we get the cochain complex involving only the F_i^*'s, and we consider its cohomology groups.

$$H^n(F_\bullet; G) = \ker f_{n+1}^* / \mathrm{Image} f_n^*$$

The *Ext groups* are defined as:

$$\mathrm{Ext}^n(H, G) := H^n(F_\bullet; G). \qquad (2.2.1)$$

The following result holds:

Lemma 2.2.1. *The* Ext *groups are well-defined, i.e., independent of the choice of resolution* $F_\bullet$ *of* H.

As in the case of the Tor functor, one can thus work with the free resolution of H given by

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H \longrightarrow 0,$$

where F_0 is the free abelian group on the generators of H, while F_1 is the free abelian group on the relations of H. In particular, it follows that

$$\mathrm{Ext}^n(H, G) = 0 \ , \ \forall n \geq 1.$$

We also get that

$$\mathrm{Ext}^0(H, G) = \mathrm{Hom}(H, G).$$

For simplicity, we set:

$$\mathrm{Ext}(H, G) := \mathrm{Ext}^1(H, G). \qquad (2.2.2)$$

Proposition 2.2.2. *The* Ext *group* $\mathrm{Ext}(H, G)$ *satisfies the following properties:*

(a) $\mathrm{Ext}(H \oplus H', G) = \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$.

(b) If H is free, then $\mathrm{Ext}(H, G) = 0$.

(c) $\mathrm{Ext}(\mathbb{Z}/n, G) = G/nG$.

Proof. For (a) use the fact that a free resolution of $H \oplus H'$ is a direct sum of free resolutions of H and resp. H'. For (b), if H is free, then $0 \longrightarrow H \longrightarrow H \longrightarrow 0$ is a free resolution of H, so $\mathrm{Ext}(H, G) = 0$. For part (c), start with the free resolution of $\mathbb{Z}/n$ given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0,$$

dualize it and use the fact that $\mathrm{Hom}(\mathbb{Z}, G) = G$ to conclude that $\mathrm{Ext}(\mathbb{Z}/n, G) = G/nG$. □

As an immediate consequence of these properties, we get the following:

Corollary 2.2.3. *If H is a finitely generated abelian group, then :*

$$\mathrm{Ext}(H, G) = \mathrm{Ext}(\mathrm{Torsion}(H), G) = \mathrm{Torsion}(H) \otimes_{\mathbb{Z}} G. \tag{2.2.3}$$

Proof. Indeed, H decomposes into a free part and a torsion part, and the claim follows by Proposition 2.2.2. □

2.2.2 Universal Coefficient Theorem

The following result shows that cohomology is entirely determined by its coefficients and the integral homology:

Theorem 2.2.4. *Given an abelian group G and a chain complex $(C_\bullet, \partial_\bullet)$ of free abelian groups with homology $H_*(C_\bullet)$, the cohomology group $H^n(C_\bullet; G)$ fits into a natural short exact sequence:*

$$0 \to \mathrm{Ext}(H_{n-1}(C_\bullet), G) \longrightarrow H^n(C_\bullet; G) \xrightarrow{h} \mathrm{Hom}(H_n(C_\bullet), G) \longrightarrow 0 \tag{2.2.4}$$

In addition, this sequence is split, that is,

$$H^n(C_\bullet; G) \cong \mathrm{Ext}(H_{n-1}(C_\bullet), G) \oplus \mathrm{Hom}(H_n(C_\bullet), G). \tag{2.2.5}$$

Proof. (Sketch)
The homomorphism $h : H^n(C_\bullet; G) \to \mathrm{Hom}(H_n(C_\bullet), G)$ is defined as follows. Let $Z_n = \ker \partial_n$, $B_n = \mathrm{Image}\ \partial_{n+1}$, $i_n : B_n \hookrightarrow Z_n$ the inclusion map, and $H_n = Z_n/B_n$. Let $[\phi] \in H^n(C_\bullet; G)$. Then ϕ is represented by a homomorphism $\phi : C_n \to G$, so that $\delta^n \phi := \phi \partial_{n+1} = 0$, which implies that $\phi|_{B_n} = 0$. Let $\phi_0 := \phi|_{Z_n}$, then ϕ_0 vanishes on B_n, so it induces a quotient homomorphism $\bar{\phi}_0 : Z_n/B_n \to G$, i.e., $\bar{\phi}_0 \in \mathrm{Hom}(H_n(C_\bullet), G)$. We define h by

$$h([\phi]) = \bar{\phi}_0.$$

Notice that if $\phi \in \mathrm{Image}\ \delta^{n-1}$, i.e., $\phi = \delta^{n-1}\psi = \psi \partial_n$, then $\phi|_{Z_n} = 0$, so $\bar{\phi}_0 = 0$, which shows that h is well-defined. It is not hard to show that h is an epimorphism, and

$$\ker h = \mathrm{Coker}(i^*_{n-1} : Z^*_{n-1} \to B^*_{n-1}) = \mathrm{Ext}(H_{n-1}(C_\bullet), G), \tag{2.2.6}$$

where the Ext group is defined with respect to the free resolution of $H_{n-1}(C_\bullet)$ given by

$$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C_\bullet) \longrightarrow 0.$$

□

Remark 2.2.5. The splitting in the above universal coefficient theorem is not natural; see Exercise 8 at the end of this chapter for an example.

The following special case of Theorem 2.2.4 is very useful in calculations:

Corollary 2.2.6. *Let $(C_\bullet, \partial_\bullet)$ be a chain complex so that its (integral) homology groups H_* are finitely generated, and let $T_n = \mathrm{Torsion}(H_n)$. Then we have natural short exact sequences:*

$$0 \to T_{n-1} \longrightarrow H^n(C_\bullet; \mathbb{Z}) \longrightarrow H_n/T_n \to 0 \tag{2.2.7}$$

This sequence splits, so:

$$H^n(C_\bullet; \mathbb{Z}) \cong T_{n-1} \oplus H_n/T_n. \tag{2.2.8}$$

Finally, we have the following easy application of Theorem 2.2.4:

Proposition 2.2.7. *If a chain map $\alpha : C_\bullet \to C'_\bullet$ between chain complexes $C_\bullet$ and $C'_\bullet$ induces isomorphisms α_* on integral homology groups, then α induces isomorphisms α^* on the cohomology groups $H^*(-; G)$ for any abelian group G.*

Proof. By the naturality part of Theorem 2.2.4, we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(C_\bullet), G) & \longrightarrow & H^n(C_\bullet; G) & \longrightarrow & \mathrm{Hom}(H_n(C_\bullet), G) & \longrightarrow & 0 \\ & & \uparrow (\alpha_*)^* & & \uparrow \alpha^* & & \uparrow (\alpha_*)^* & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(C'_\bullet), G) & \longrightarrow & H^n(C'_\bullet; G) & \longrightarrow & \mathrm{Hom}(H_n(C'_\bullet), G) & \longrightarrow & 0 \end{array}$$

The claim follows by the five-lemma, since α_* and its dual are isomorphisms. □

2.3 Cohomology of spaces

2.3.1 Definition and immediate consequences

Suppose X is a topological space with singular chain complex $(C_\bullet(X), \partial_\bullet)$. The group of *singular n-cochains* of X is defined as:

$$C^n(X; G) := \mathrm{Hom}(C_n(X), G). \tag{2.3.1}$$

So n-cochains are functions from singular n-simplices to G.

The *coboundary map*

$$\delta^n : C^n(X; G) \to C^{n+1}(X; G)$$

is defined as the dual of the corresponding boundary map $\partial_{n+1} : C_{n+1} \to C_n$, i.e., for $\psi \in C^n(X;G)$, we let

$$\delta^n \psi := \psi \partial_{n+1} : C_{n+1}(X) \overset{\partial_{n+1}}{\to} C_n(X) \overset{\psi}{\to} G. \tag{2.3.2}$$

It follows that

$$\delta^{n+1} \circ \delta^n = 0, \tag{2.3.3}$$

and for a singular $(n+1)$-simplex $\sigma : \Delta_{n+1} \to X$ we have:

$$\delta^n \psi(\sigma) = \sum_{i=0}^{n+1} (-1)^i \cdot \psi(\sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_{n+1}]}). \tag{2.3.4}$$

Definition 2.3.1. *The cohomology groups of X with G-coefficients are defined as:*

$$H^n(X;G) := \ker(\delta^n : C^n(X;G) \to C^{n+1}(X;G))/\mathrm{Image}(\delta^{n-1} : C^{n-1}(X;G) \to C^n(X;G)). \tag{2.3.5}$$

Elements of $\ker\ \delta^n$ *are called n-cocycles, and elements of* $\mathrm{Image}\ \delta^{n-1}$ *are called n-coboundaries.*

Remark 2.3.2. Note that ψ is an n-cocycle if, by definition, it vanishes on n-boundaries.

Since the groups $C_n(X)$ of singular chains are free, we can employ Theorem 2.2.4 to compute the cohomology groups $H^n(X;G)$ in terms of the coefficients G and the integral homology of X. More precisely, we have natural short exact sequences:

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X;G) \longrightarrow \mathrm{Hom}(H_n(X), G). \longrightarrow 0 \tag{2.3.6}$$

Moreover, these sequences split, though not naturally.

Let us now derive some immediate consequences from (2.3.6):

(a) If $n = 0$, (2.3.6) yields that

$$H^0(X;G) = \mathrm{Hom}(H_0(X), G), \tag{2.3.7}$$

or equivalently, $H^0(X;G)$ consists of all functions from the set of path-connected components of X to the group G.

(b) If $n = 1$, the Ext-term in (2.3.6) vanishes since $H_0(X)$ is free, so we get:

$$H^1(X;G) = \mathrm{Hom}(H_1(X), G). \tag{2.3.8}$$

Remark 2.3.3. Theorem 2.2.4 also works for modules over a PID. In particular, if $G = F$ is a field, then

$$H^n(X;F) \simeq \mathrm{Hom}(H_n(X), F) \simeq Hom_F(H_n(X;F), F) = H_n(X, F)^\vee$$

Thus, with field coefficients, cohomology is the dual of homology.

Example 2.3.4. Let X be a point space. From (2.3.6), we have:

$$H^i(X;G) = \mathrm{Hom}(H_i(X), G) \oplus \mathrm{Ext}(H_{i-1}(X), G).$$

And since

$$H_i(X) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & \text{otherwise,} \end{cases}$$

we get

$$\mathrm{Hom}(H_i(X), G) = \begin{cases} G, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, since $H_i(X)$ is free for all i, we also have that $\mathrm{Ext}(H_{i-1}(X), G) = 0$, for all i. Altogether,

$$H^i(X;G) = \begin{cases} G, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.3.5. Let $X = S^n$. Then we have

$$H_i(X) = \begin{cases} \mathbb{Z}, & i = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

Thus the Ext-term in the universal coefficient theorem vanishes and we get:

$$H^i(X;G) = \mathrm{Hom}(H_i(X), G) = \begin{cases} G, & i = 0 \text{ or } n \\ 0, & \text{otherwise.} \end{cases}$$

2.3.2 Reduced cohomology groups

We start with the augmented singular chain complex for X:

$$\cdots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

with $\epsilon(\sum_i n_i x_i) = \sum_i n_i$. After dualizing it (i.e., applying $\mathrm{Hom}(-;G)$), we get the augmented cochain complex

$$\cdots \xleftarrow{\delta} C^1(X;G) \xleftarrow{\delta} C^0(X;G) \xleftarrow{\epsilon^*} G \longleftarrow 0.$$

Note that since $\epsilon\partial = 0$, we get by dualizing that $\delta\epsilon^* = 0$. The homology of this augmented cochain complex is the *reduced cohomology* of X with G-coefficients, denoted by $\widetilde{H}^i(X;G)$.

It follows by definition that $\widetilde{H}^i(X;G) = H^i(X;G)$, if $i > 0$, and by the universal coefficient theorem (applied to the augmented chain complex), we get $\widetilde{H}^0(X;G) = \mathrm{Hom}(\widetilde{H}_0(X), G)$.

2.3.3 Relative cohomology groups

To define relative cohomology groups $H^n(X, A; G)$ for a pair (X, A), we dualize the relative chain complex by setting

$$C^n(X, A; G) := \mathrm{Hom}(C_n(X, A), G). \tag{2.3.9}$$

The group $C^n(X, A; G)$ can be identified with functions from n-simplices in X to G that vanish on simplices in A, so we have a natural inclusion

$$C^n(X, A; G) \hookrightarrow C^n(X; G). \tag{2.3.10}$$

The relative coboundary maps

$$\delta : C^n(X, A; G) \to C^{n+1}(X, A; G) \tag{2.3.11}$$

are obtained by restrcting the absolute ones, so they satisfy $\delta^2 = 0$. So the *relative cohomology groups* $H^n(X, A; G)$ are defined.

We next dualize the short exact sequence

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \longrightarrow 0$$

to get another short exact sequence

$$0 \longleftarrow C^n(A; G) \xleftarrow{i^*} C^n(X; G) \xleftarrow{j^*} C^n(X, A; G) \longleftarrow 0, \tag{2.3.12}$$

where the exactness at $C^n(A; G)$ follows by extending a cochain in A “by zero". More precisely, for $\psi \in C^n(A; G)$, we define a function $\widehat{\psi} : C_n(X) \to G$ by

$$\widehat{\psi}(\sigma) = \begin{cases} \psi(\sigma), & \text{if } \sigma \in C_n(A) \\ 0, & \text{if } \mathrm{Image}(\sigma) \cap A = \emptyset \end{cases}$$

$\widehat{\psi}$ is a well-defined element of $C^n(X; G)$ since $C_n(X)$ has a basis made of simplices contained in A and those contained in $X \setminus A$. It is clear that $i^*(\widehat{\psi}) = \psi$.

Since i and j commute with ∂, it follows that i^* and j^* commute with δ. So we obtain a short exact sequence of cochain complexes:

$$0 \longleftarrow C^*(A; G) \xleftarrow{i^*} C^*(X; G) \xleftarrow{j^*} C^*(X, A; G) \longleftarrow 0. \tag{2.3.13}$$

By taking the associated long exact sequence of homology groups, we get the long exact sequence for the cohomology groups of the pair (X, A):

$$\cdots \longrightarrow H^n(X, A; G) \xrightarrow{j^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \longrightarrow \cdots \tag{2.3.14}$$

We can also consider above the augmented chain complexes on X and A, and get a long exact sequence for the reduced cohomology groups, with $\widetilde{H}^n(X, A; G) = H^n(X, A; G)$:

$$\cdots \longrightarrow H^n(X, A; G) \longrightarrow \widetilde{H}^n(X; G) \longrightarrow \widetilde{H}^n(A; G) \longrightarrow H^{n+1}(X, A; G) \longrightarrow \cdots \tag{2.3.15}$$

In particular, if $A = x_0$ is a point in X, we get by (2.3.15) that

$$\widetilde{H}^n(X; G) \cong H^n(X, x_0; G). \tag{2.3.16}$$

2.3.4 Induced homomorphisms

If $f : X \to Y$ is a continuous map, we have induced chain maps

$$\begin{aligned} f_\# : \quad & C_n(X) \longrightarrow C_n(Y) \\ & (\sigma : \Delta_n \to X) \longmapsto (f \circ \sigma : \Delta_n \xrightarrow{\sigma} X \xrightarrow{f} Y) \end{aligned}$$

satisfying $f_\# \partial = \partial f_\#$.
Dualizing $f_\#$ with respect to G, we get maps

$$f^\# : C^n(Y; G) \to C^n(X; G),$$

with $f^\#(\psi) = \psi(f_\#)$ and $\delta f^\# = f^\# \delta$ (which is obtained by dualizing $f_\# \partial = \partial f_\#$). Thus, we get induced homomorphisms on cohomology groups:

$$f^* : H^n(Y, G) \to H^n(X, G).$$

In fact, we can repeat the above for maps of pairs, say $f : (X, A) \to (Y, B)$. And note that the universal coefficient theorem also works for pairs because $C_n(X, A) = C_n(X)/C_n(A)$ is free abelian. So, by naturality, we get a commutative diagram for a map of pairs $f : (X, A) \to (Y, B)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X, A), G) & \longrightarrow & H^n(X, A; G) & \longrightarrow & \mathrm{Hom}(H_n(X, A), G) & \longrightarrow & 0 \\ & & \uparrow (f_*)^* & & \uparrow f^* & & \uparrow (f_*)^* & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(Y, B), G) & \longrightarrow & H^n(Y, B; G) & \longrightarrow & \mathrm{Hom}(H_n(Y, B), G) & \longrightarrow & 0 \end{array}$$

2.3.5 Homotopy invariance

Theorem 2.3.6. *If $f \simeq g : (X, A) \to (Y, B)$ and G is an abelian group, then*

$$f^* = g^* : H^n(Y, B; G) \to H^n(X, A; G).$$

Proof. Recall from the proof of the similar statement for homology that there is a *prism operator*

$$P : C_n(X, A) \to C_{n+1}(Y, B) \tag{2.3.17}$$

satisfying

$$f_\# - g_\# = P\partial + \partial P \tag{2.3.18}$$

with $f_\#$ and $g_\#$ the induced maps on singular chain complexes. (In fact, if $F : X \times I \to Y$ denotes the homotopy, with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, then the prism operator is defined on generators $(\sigma : \Delta_n \to X) \in C_n(X)$ by pre-composing $F \circ (\sigma \times id) : \Delta^n \times I \to Y$ with an appropriate decomposition of $\Delta^n \times I$ into $(n+1)$-dimensional simplices. Then one notes that such a P takes $C_n(A)$ to $C_{n+1}(B)$, hence it induces the relative prism operator of (2.3.17).)

So the difference of the middle maps in the following diagram equals to the sum of the two side “paths":

$$\begin{array}{ccccc} & & C_n(X,A) & \xrightarrow{\partial} & C_{n-1}(X,A) \\ & \swarrow^{P} & f_\# \downarrow\downarrow g_\# & \swarrow^{P} & \\ C_{n+1}(Y,B) & \xrightarrow{\partial} & C_n(Y,B) & & \end{array}$$

Then it follows from (2.3.18) that $f_* = g_*$ on relative homology groups.

The claim about cohomology follows by dualizing the prism operator (2.3.17) to get

$$P^* : C^{n+1}(Y, B; G) \to C^n(X, A; G) \tag{2.3.19}$$

which satisfies an identity dual to (2.3.18), that is,

$$f^\# - g^\# = \delta P^* + P^*\delta. \tag{2.3.20}$$

This implies readily that $f^* = g^*$ on relative cohomology groups. □

The following is an immediate consequence of Theorem 2.3.6:

Corollary 2.3.7. *If $f : X \to Y$ is a homotopy equivalence, then $f^* : H^n(Y;G) \to H^n(X;G)$ is an isomorphism, for any coefficient group G.*

Example 2.3.8. We have:

$$H^i(\mathbb{R}^n; G) = \begin{cases} G, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

This follows immediately by the homotopy invariance of cohomology groups, since $\mathbb{R}^n$ is contractible.

2.3.6 Excision

Theorem 2.3.9. *Given a topological space X, suppose that $Z \subset A \subset X$, with $cl(Z) \subseteq int(A)$. Then the inclusion of pairs $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms*

$$i^* : H^n(X, A; G) \to H^n(X \setminus Z, A \setminus Z; G) \tag{2.3.21}$$

for all n. Equivalently, if A and B are subsets of X with $X = int(A) \cup int(B)$, then the inclusion map $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms in cohomology.

Proof. By the naturality of universal coefficient theorem, we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}(H_{n-1}(X,A),G) & \to & H^n(X,A;G) & \to & \mathrm{Hom}(H_n(X,A),G) & \to & 0 \\ & & \uparrow (i_*)^* & & \uparrow i^* & & \uparrow (i_*)^* & & \\ 0 & \to & \mathrm{Ext}(H_{n-1}(X\setminus Z,A\setminus Z),G) & \to & H^n(X\setminus Z,A\setminus Z;G) & \to & \mathrm{Hom}(H_n(X\setminus Z,A\setminus Z),G) & \to & 0 \end{array}$$

By excision for homology, the maps i_*, hence $(i_*)^*$, are isomorphisms. So by the five-lemma, it follows that i^* is also an isomorphism. □

2.3.7 Mayer-Vietoris sequence

Theorem 2.3.10. *Let X be a topological space, and A and B be subsets of X so that*

$$X = int(A) \cup int(B).$$

Then there is a long exact sequence of cohomology groups:

$$\cdots \longrightarrow H^n(X;G) \xrightarrow{\psi} H^n(A;G) \oplus H^n(B;G) \xrightarrow{\phi} H^n(A \cap B;G) \longrightarrow H^{n+1}(X;G) \longrightarrow \cdots \tag{2.3.22}$$

Proof. There is a short exact sequence of cochain complexes, which at level n is given by:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^n(A+B;G) & \xrightarrow{\psi} & C^n(A;G) \oplus C^n(B;G) & \xrightarrow{\phi} & C^n(A \cap B;G) & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & \mathrm{Hom}(C_n(A+B),G) & & & & & & \end{array}$$

where $C_n(A+B)$ is the set of simplices in X which are sums of simplices in either A or B, and the maps are defined by

$$\psi(\eta) = (\eta|_{C_n(A)}, \eta|_{C_n(B)})$$

and

$$\phi(\alpha, \beta) = \alpha|_{C_n(A \cap B)} - \beta|_{C_n(A \cap B)}.$$

Moreover, since $C_*(A+B) \hookrightarrow C_*(X)$ is a chain homotopy, it follows by dualizing that $C^*(A+B;G)$ and $C^*(X;G)$ are chain homotopic, and thus $H^*(A+B;G) \cong H^*(X;G)$. The cohomology Mayer-Vietoris sequence (2.3.22) is the long exact cohomology sequence of the above short exact sequence of cochain complexes. □

Remark 2.3.11. A similar Mayer-Vietoris sequence holds can be obtained for the reduced cohomology groups.

Example 2.3.12. Let us compute the cohomology groups of S^n by using the above Mayer-Vietoris sequence. Cover S^n by two open sets $A = S^n \setminus N$ and $B = S^n \setminus S$, where N and S are the North and resp. South pole of S^n. Then we have $A \cap B \simeq S^{n-1}$ and $A \simeq B \simeq \mathbb{R}^n$. Thus by the Mayer-Vietoris sequence for reduced cohomology, together with Example 2.3.8, homotopy invariance and induction, we get:

$$\widetilde{H}^i(S^n;G) \cong \widetilde{H}^{i-1}(S^{n-1};G) \cong \cdots \cong \widetilde{H}^{i-n}(S^0;G) \cong \begin{cases} G, & i = n \\ 0, & \text{otherwise.} \end{cases}$$

2.3.8 Cellular cohomology

Definition 2.3.13. *Let X be a CW complex. The* cellular cochain complex *of X, $(\mathcal{C}^\bullet(X;G), d^\bullet)$, is defined by setting:*

$$\mathcal{C}^n(X;G) := H^n(X_n, X_{n-1}; G),$$

for X_n the n-skeleton of X, and with coboundary maps

$$d^n = \delta^n \circ j^n$$

fitting in the following diagram (where the coefficient group for cohomology is by default G):

$$\begin{array}{ccccccc}
 & & H^{n-1}(X_{n-1}) & & & & \\
 & \overset{j^{n-1}}{\nearrow} & & \overset{\delta^{n-1}}{\searrow} & & & \\
\cdots \to H^{n-1}(X_{n-1}, X_{n-2}) & & \xrightarrow{\quad d^{n-1} \quad} & & H^n(X_n, X_{n-1}) & \xrightarrow{\quad d^n \quad} & H^{n+1}(X_{n+1}, X_n) \to \cdots \\
 & & & & \underset{j^n}{\searrow} \qquad \underset{\delta^n}{\nearrow} & & \\
 & & & & H^n(X_n) & &
\end{array}$$

Here, the diagonal arrows are part of cohomology long exact sequences for the relevant pairs. For this reason, it follows that $j^n\delta^{n-1} = 0$, and therefore

$$d^n d^{n-1} = \delta^n j^n \delta^{n-1} j^{n-1} = 0.$$

So $(\mathcal{C}^\bullet(X;G), d^\bullet)$ is indeed a cochain complex.
The cellular cohomology of X with G-coefficients is by definition the cohomology of the cellular cochain complex $(\mathcal{C}^\bullet(X;G), d^\bullet)$

Just like in the case of cellular homology, we have the following identification:

Theorem 2.3.14. *The singular and cellular cohomology of X are isomorphic, i.e.,*

$$H^n(X;G) \cong H^n(\mathcal{C}^\bullet(X;G)) \tag{2.3.23}$$

for all n and any coefficient group G. Moreover, the cellular cochain complex $(\mathcal{C}^\bullet(X;G), d^\bullet)$ is isomorphic to the dual of the cellular chain complex $(\mathcal{C}_\bullet(X), d_\bullet)$, obtained by applying $\mathrm{Hom}(-;G)$.

Proof. Recall from Section 1.1.4 that for the cellular chain complex of X we have that

$$\mathcal{C}_n(X) := H_n(X_n, X_{n-1}) \cong \mathbb{Z}^{\#\text{ of n-cells}}$$

and $H_i(X_n, X_{n-1}) = 0$ whenever $i \neq n$. So by the universal coefficient theorem, we obtain:

$$\mathcal{C}^n(X;G) := H^n(X_n, X_{n-1}; G) \cong \mathrm{Hom}(\mathcal{C}_n(X), G) \tag{2.3.24}$$

since the Ext term vanishes. The universal coefficient theorem also yields that

$$H^i(X_n, X_{n-1}; G) = 0 \text{ if } i \neq n, \tag{2.3.25}$$

since the groups $H_i(X_n, X_{n-1})$ are either free or trivial.
From the long exact sequence of the pair (X_n, X_{n-1}), that is,

$$\cdots \longrightarrow H^k(X_n, X_{n-1}; G) \longrightarrow H^k(X_n; G) \longrightarrow H^k(X_{n-1}; G) \longrightarrow H^{k+1}(X_n, X_{n-1}; G) \longrightarrow \cdots,$$

we thus get for $k \neq n, n-1$ the isomorphisms

$$H^k(X_n; G) \cong H^k(X_{n-1}; G). \tag{2.3.26}$$

Therefore, if $k > n$, we obtain by induction:

$$H^k(X_n; G) \cong H^k(X_{n-1}; G) \cong H^k(X_{n-2}; G) \cong \cdots \cong H^k(X_0; G) = 0 \tag{2.3.27}$$

since X_0 is just a set of points.

We next claim that there is an isomorphism

$$H^n(X_{n+1}; G) \cong H^n(X; G). \tag{2.3.28}$$

First recall from Lemma 1.1.8(c) that the inclusion $X_{n+1} \hookrightarrow X$ induces isomorphisms on homology groups H_k, for $k < n+1$. So by the naturality of the universal coefficient theorem, we get the following diagram with commutative squares:

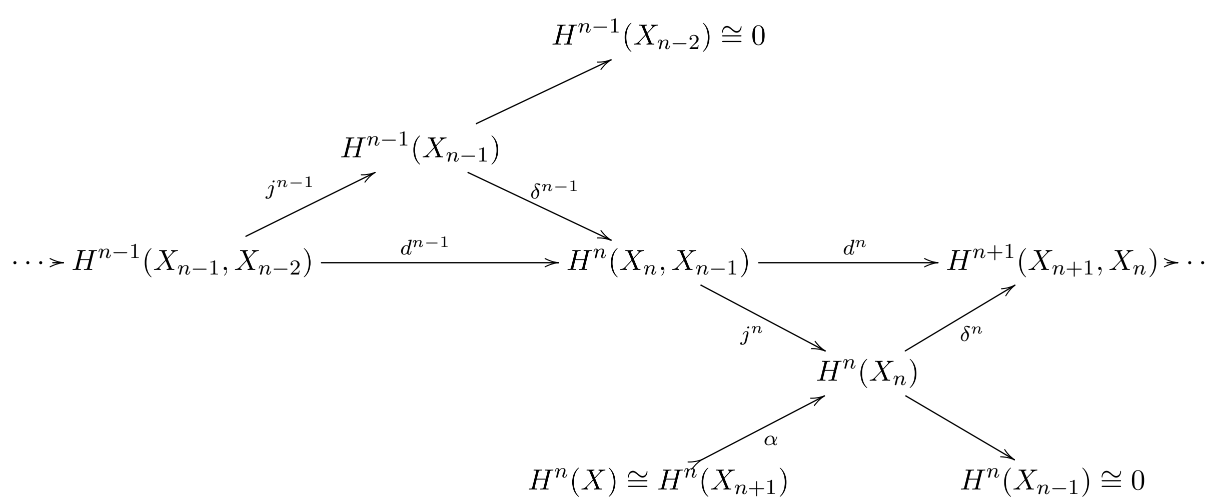
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X), G) & \longrightarrow & H^n(X; G) & \xrightarrow{h} & \mathrm{Hom}(H_n(X), G) & \longrightarrow & 0 \\
 & & \cong \big\downarrow (i_*)^* & & \big\downarrow i^* & & \cong \big\downarrow (i_*)^* & & \\
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X_{n+1}), G) & \longrightarrow & H^n(X_{n+1}; G) & \xrightarrow{h} & \mathrm{Hom}(H_n(X_{n+1}), G) & \longrightarrow & 0
\end{array}$$

Then, by using the five-lemma, it follows that the middle map

$$i^* : H^n(X; G) \to H^n(X_{n+1}; G)$$

is also an isomorphism.

Altogether, by using (2.3.27) and (2.3.28), we get the following diagram (where the diagonal arrows are part of long exact sequences of pairs):



Thus, by using the definition $d^n = \delta^n j^n$ of the cellular coboundary maps, and after noting that j^{n-1} and j^n are onto and α is injective, we obtain the following sequence of isomorphisms:

$$\begin{aligned} H^n(X;G) &\cong H^n(X_{n+1};G) \\ &\cong \text{Image}(\alpha) \\ &\cong \ker(\delta^n) \\ &\cong \ker(d^n)/\ker(j^n) \\ &\cong \ker(d^n)/\text{Image}(\delta^{n-1}) \\ &\cong \ker(d^n)/\text{Image}(\delta^{n-1}j^{n-1}) \\ &\cong \ker(d^n)/\text{Image}(d^{n-1}). \end{aligned} \tag{2.3.29}$$

The only claim left to prove is that

$$d^n = (d_{n+1})^*. \tag{2.3.30}$$

By definition, the cellular coboundary map d^n is the composition:

$$d^n : H^n(X_n, X_{n-1};G) \xrightarrow{j^n} H^n(X_n : G) \xrightarrow{\delta^n} H^{n+1}(X_{n+1}, X_n;G),$$

and, similarly, the boundary map d_{n+1} of the cellular chain complex is given by:

$$d_{n+1} : H_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial_{n+1}} H_n(X_n) \xrightarrow{j_n} H_n(X_n, X_{n-1}).$$

Let us now consider the following diagram:

$$\begin{array}{llcccc} d^n : & H^n(X_n, X_{n-1};G) & \xrightarrow{j^n} & H^n(X_n;G) & \xrightarrow{\delta^n} & H^{n+1}(X_{n+1}, X_n;G) \\ & \cong\downarrow h & & \downarrow h & & \cong\downarrow h \\ (d_{n+1})^* : & \text{Hom}(H_n(X_n, X_{n-1}), G) & \xrightarrow{(j_n)^*} & \text{Hom}(H_n(X_n), G) & \xrightarrow{(\partial_{n+1})^*} & \text{Hom}(H_{n+1}(X_{n+1}, X_n), G) \end{array}$$

The composition across the top is the cellular coboundary map d^n, and we want to conclude that it is the same as the composition $(d_{n+1})^*$ across the bottom row. The extreme vertical arrows labelled h are isomorphisms by the universal coefficient theorem, since the relevant Ext terms vanish (by using (2.3.25)). So it suffices to show that the diagram commutes. The left square commutes by the naturality of universal coefficient theorem for the inclusion map $(X_n, \emptyset) \hookrightarrow (X_n, X_{n-1})$, and the right square commutes by a simple diagram chase. □

Example 2.3.15. Let $X = \mathbb{RP}^2$. Then X has one cell in each dimension 0, 1, and 2, and the cellular chain complex of X is:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

To compute the (cellular) cohomology $H^*(X;\mathbb{Z})$, we dualize (i.e., apply $\mathrm{Hom}(-,\mathbb{Z})$) the above cellular chain complex, and get:

$$0 \longleftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow 0$$

Thus, we have

$$H^i(\mathbb{RP}^2;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i=0 \\ \mathbb{Z}/2, & i=2 \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, in order to calculate $H^*(X;\mathbb{Z}/2)$, we dualize the cellular chain complex of X with respect to $\mathbb{Z}/2$ (i.e., by applying $\mathrm{Hom}(-,\mathbb{Z}/2)$) to get:

$$0 \longleftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \longleftarrow 0$$

We then have:

$$H^i(\mathbb{RP}^2;\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2, & i=0,1,\text{or } 2 \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.3.16. Let K be the Klein bottle and let us compute $H_*(K;\mathbb{Z}/3)$ and $H^*(K;\mathbb{Z}/3)$. The cellular chain complex of K is given by:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(2,0)} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So the cellular chain complex of K with $\mathbb{Z}/3$-coefficients is given by:

$$0 \longrightarrow \mathbb{Z}/3 \xrightarrow{(2,0)} \mathbb{Z}/3\oplus\mathbb{Z}/3 \xrightarrow{0} \mathbb{Z}/3 \longrightarrow 0$$

Note that the map $(2,0) : \mathbb{Z}/3 \to \mathbb{Z}/3 \oplus \mathbb{Z}/3$ is an isomorphism on the first component, so we get:

$$H_i(K;\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & i=0 \text{ or } 1 \\ 0, & \text{otherwise.} \end{cases}$$

In order to compute the cohomology with $\mathbb{Z}/3$-coefficients, we dualize the cellular chain complex of K with respect to $\mathbb{Z}/3$ to get:

$$0 \longleftarrow \mathbb{Z}/3 \xleftarrow{(2,0)} \mathbb{Z}/3\oplus\mathbb{Z}/3 \xleftarrow{0} \mathbb{Z}/3 \longleftarrow 0$$

Therefore, we have

$$H^i(K;\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & i=0 \text{ or } 1 \\ 0, & \text{otherwise.} \end{cases}$$

Exercises

1. Prove Lemma 2.2.1.

2. Show that the functor $\mathrm{Ext}(-,-)$ is contravariant in the first variable, that is, if H, H' and G are abelian groups, a homomorphism $\alpha : H \to H'$ induces a homomorphism $\alpha^* : \mathrm{Ext}(H', G) \to \mathrm{Ext}(H, G)$.

3. For a topological space X, let

$$\langle\ ,\ \rangle : C^n(X) \otimes C_n(X) \to \mathbb{Z}$$

be the Kronecker pairing given by $\langle \phi, \sigma \rangle := \phi(\sigma)$. In terms of this pairing, the coboundary map $\delta : C^n(X) \to C^{n+1}(X)$ is defined by $\langle \delta(\phi), \sigma \rangle = \langle \phi, \partial\sigma \rangle$ for all $\sigma \in C_{n+1}(X)$. Show that this pairing induces a pairing between cohomology and homology:

$$\langle\ ,\ \rangle : H^n(X;\mathbb{Z}) \otimes H_n(X;\mathbb{Z}) \to \mathbb{Z}.$$

4. Compute $H^*(S^n; G)$ by using the long exact sequence of a pair, coupled with excision.

5. Compute the cohomology of the spaces $S^1 \times S^1$, $\mathbb{RP}^2$ and the Klein bottle first with $\mathbb{Z}$ coefficients, then with $\mathbb{Z}/2$ coefficients.

6. Show that if $f : S^n \to S^n$ has degree d, then $f^* : H^n(S^n; G) \to H^n(S^n; G)$ is multiplication by d.

7. Show that if A is a closed subspace of X that is a deformation retract of some neighborhood, then the quotient map $X \to X/A$ induces isomorphisms

$$H^n(X, A; G) \cong \widetilde{H}^n(X/A; G)$$

for all n.

8. Let X be a space obtained from S^n by attaching a cell e^{n+1} by a degree m map.

- Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $\widetilde{H}_i(-;\mathbb{Z})$ for all i, but not on $H^{n+1}(-;\mathbb{Z})$. Conclude that the splitting in the universal coefficient theorem for cohomology cannot be natural.
- Show that the inclusion $S^n \hookrightarrow X$ induces the trivial map on $\widetilde{H}^i(-;\mathbb{Z})$ for all i, but not on $H_n(-;\mathbb{Z})$.

9. Let X and Y be path-connected and locally contractible spaces such that $H^1(X;\mathbb{Q}) \neq 0$ and $H^1(Y;\mathbb{Q}) \neq 0$. Show that $X \vee Y$ is not a retract of $X \times Y$.

10. Let X be the space obtained by attaching two 2-cells to S^1, one via the map $z \mapsto z^3$ and the other via $z \mapsto z^5$, where z denotes the complex coordinate on $S^1 \subset \mathbb{C}$. Compute the cohomology groups $H^*(X; G)$ of X with coefficients:

(a) $G = \mathbb{Z}$.

(b) $G = \mathbb{Z}/2$.

(c) $G = \mathbb{Z}/3$.

Chapter 3

Cup Product in Cohomology

Let us motivate this chapter with the following simple, but hopefully convincing example. Consider the spaces $X = \mathbb{C}P^2$ and $Y = S^2 \vee S^4$. As CW complexes, both X and Y have one 0-cell, one 2-cell and one 4-cell. Hence the cellular chain complex for both X and Y is:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So X and Y have the same homology and cohomology groups. Note that X and Y also have the same fundamental groups: $\pi_1(X) = \pi_1(Y) = 0$. A natural question is then whether X and Y are homotopy equivalent. Similarly, one can ask if there is a map $f : X \to Y$ inducing isomorphisms on (co)homology groups. We will see below that by using cup products in cohomology, we can show that the answer to both questions is negative.

3.1 Cup Products: definition, properties, examples

Definition 3.1.1. *Let X be a topological space, and fix a coefficient ring R (e.g., $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Q}$). Let $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$. The cup product $\phi \smile \psi \in C^{k+l}(X;R)$ is defined by:*

$$(\phi \smile \psi)(\sigma : \Delta_{k+l} \to X) = \phi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,v_{k+l}]}), \tag{3.1.1}$$

where "$\cdot$" denotes the multiplication in ring R.

The aim is to show that this cup product of cochains induces a cup product of cohomology classes. We need the following result which relates the cup product to coboundary maps.

Lemma 3.1.2.

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi \tag{3.1.2}$$

for $\phi \in C^k(X;R)$, and $\psi \in C^l(X;R)$.

Proof. For $\sigma : \Delta^{k+l+1} \to X$ we have

$$(\delta\phi \smile \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma|_{[v_0,\cdots,\widehat{v}_i,\cdots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\cdots,v_{k+l+1}]})$$

and

$$(-1)^k(\phi \smile \delta\psi)(\sigma) = \sum_{i=k}^{k+l+1} (-1)^i \phi(\sigma|_{[v_0,\cdots,v_k]}) \cdot \psi(\sigma|_{[v_k,\cdots,\widehat{v}_i,\cdots,v_{k+l+1}]}).$$

When we add these two expressions, the last term of the first sum cancels with the first term of the second sum, and the remaining terms are exactly $\delta(\phi \smile \psi)(\sigma) = (\phi \smile \psi)(\partial\sigma)$ since $\partial\sigma = \sum_{i=0}^{k+l+1}(-1)^i \sigma\ |_{[v_0,\cdots,\widehat{v}_i,\cdots,v_{k+l+1}]}$. □

As immediate consequences of the above Lemma, we have:

Corollary 3.1.3. *The cup product of two cocycles is again a cocycle. That is, if ϕ, ψ are cocycles, then $\delta(\phi \smile \psi) = 0$.*

Proof. This is true, since $\delta\phi = 0$ and $\delta\psi = 0$ imply by (3.1.2) that $\delta(\phi \smile \psi) = 0$. □

Moreover,

Corollary 3.1.4. *If either one of ϕ or ψ is a cocycle and the other a coboundary, then $\phi \smile \psi$ is a coboundary.*

Proof. Say $\delta\phi = 0$ and $\psi = \delta\eta$. Then $\phi \smile \psi = \phi \smile \delta\eta = \pm\delta(\phi \smile \eta)$. Similarly, if $\delta\psi = 0, \phi = \delta\eta$ then $\phi \smile \psi = \delta\eta \smile \psi = \delta(\eta \smile \psi)$. □

It follows from Corollary 3.1.3 and Corollary 3.1.4 that we get an induced cup product on cohomology:

$$H^k(X;R) \times H^l(X;R) \overset{\smile}{\longrightarrow} H^{k+l}(X;R). \tag{3.1.3}$$

It is distributive and associative since it is so on the cochain level. If R has an identity element, then there is an identity element for the cup product, namely the class $1 \in H^0(X;R)$ defined by the 0-cocycle taking the value 1 on each singular 0-simplex.

Considering the cup product as an operation on the the direct sum of all cohomology groups, we get a (graded) ring structure on the cohomology $\oplus_i H^i(X;R)$. We will elaborate on the ring structure on cohomology groups induced by the cup product after looking at a few examples and properties of the cup product.

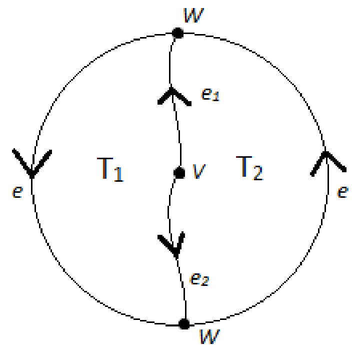
Example 3.1.5. Let us consider the real projective plane $\mathbb{RP}^2$. Its $\mathbb{Z}/2\mathbb{Z}$-cohomology is computed by:

$$H^i(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } i = 0,1,2 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha \in H^1(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ be the generator, and consider

$$\alpha^2 := \alpha \smile \alpha \in H^2(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}).$$

We claim that $\alpha^2 \neq 0$, so α^2 is in fact the generator of $H^2(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z})$.



Consider the cell structure on $\mathbb{RP}^2$ with two 0-cells v and w, three 1-cells e, e_1 and e_2, and two 2-cells T_1 and T_2. The 2-cell T_1 is attached by the word $e_1 e e_2^{-1}$, and the 2-cell T_2 is attached by the word $e_2 e e_1^{-1}$ (see the figure below).

Since α is a generator of $H^1(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(H_1(\mathbb{RP}^2), \mathbb{Z}/2\mathbb{Z})$, it is represented by a cocycle

$$\phi : C_1(\mathbb{RP}^2) \to \mathbb{Z}/2\mathbb{Z}$$

with $\phi(e) = 1$, where we use the fact that e represents the generator of $H_1(\mathbb{RP}^2)$. The cocycle condition for ϕ translates into the identities:

$$0 = (\delta\phi)(T_1) = \phi(\partial T_1) = \phi(e_1) + \phi(e) - \phi(e_2).$$

$$0 = (\delta\phi)(T_2) = \phi(\partial T_2) = \phi(e_2) + \phi(e) - \phi(e_1).$$

As $\phi(e) = 1$, without loss of generality we may take $\phi(e_1) = 1$ and $\phi(e_2) = 0$.

Next, note that $\alpha^2 = \alpha \smile \alpha$ is represented by $\phi \smile \phi$, and we have:

$$(\phi \smile \phi)(T_1) = \phi(e_1) \cdot \phi(e) = 1$$

since $T_1 : [vww] \to \mathbb{RP}^2$. Similarly,

$$(\phi \smile \phi)(T_2) = \phi(e_2) \cdot \phi(e) = 0.$$

Since the generator of $H_2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$ is $T_1 + T_2$, and we have

$$(\phi \smile \phi)(T_1 + T_2) = (\phi \smile \phi)(T_1) + (\phi \smile \phi)(T_2) = 1 + 0 = 1,$$

it follows that α^2 (which is represented by $\phi \smile \phi$) is the generator of $H^2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$. $\square$

The cup product on cochains

$$C^k(X; R) \times C^l(X; R) \longrightarrow C^{k+l}(X; R)$$

restricts to cup products:

$$C^k(X,A;R) \times C^l(X;R) \longrightarrow C^{k+l}(X,A;R),$$

$$C^k(X,A;R) \times C^l(X,A;R) \longrightarrow C^{k+l}(X,A;R),$$

and

$$C^k(X;R) \times C^l(X,A;R) \longrightarrow C^{k+l}(X,A;R)$$

since $C^i(X,A;R)$ can be regarded as the set of cochains vanishing on chains in A, and if ϕ or ψ vanishes on chains in A, then so does $\phi \smile \psi$. So there exist relative cup products:

$$H^k(X,A;R) \times H^l(X;R) \overset{\smile}{\longrightarrow} H^{k+l}(X,A;R),$$

$$H^k(X,A;R) \times H^l(X,A;R) \overset{\smile}{\longrightarrow} H^{k+l}(X,A;R),$$

and

$$H^k(X;R) \times H^l(X,A;R) \overset{\smile}{\longrightarrow} H^{k+l}(X,A;R).$$

In particular, if A is a point, we get a cup product on the reduced cohomology $\widetilde{H}^*(X;R)$.

More generally, there is a cup product

$$H^k(X,A;R) \times H^l(X,B;R) \overset{\smile}{\longrightarrow} H^{k+l}(X,A \cup B;R)$$

when A and B are open subsets of X or subcomplexes of the CW complex X. Indeed, the absolute cup product restricts first to a cup product

$$C^k(X,A;R) \times C^l(X,B;R) \longrightarrow C^{k+l}(X,A+B;R),$$

where $C^{k+l}(X,A+B;R)$ is the subgroup of $C^{k+l}(X;R)$ consisting of cochains vanishing on sums of chains in A and chains in B. If A and B are opens in X, then $C^{k+l}(X,A \cup B;R) \hookrightarrow C^{k+l}(X,A+B;R)$ induces an isomorphism in cohomology, via the five-lemma and the fact that the restriction maps $C^i(A \cup B;R) \to C^i(A+B;R)$ induce cohomology isomorphisms.

Let us now prove the following simple but important fact:

Lemma 3.1.6. *The cup product is functorial, i.e., for a map $f : X \to Y$ the induced maps $f^* : H^i(Y;R) \to H^i(X;R)$ satisfy*

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta), \tag{3.1.4}$$

and similarly in the relative case.

Proof. It sufficies to show the following cochain formula

$$f^\#(\phi \smile \psi) = f^\#(\phi) \smile f^\#(\psi).$$

For $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ we have:

$$\begin{aligned} f^{\#}(\phi) \smile f^{\#}(\psi)(\sigma : \Delta_{k+l} \to X) &= (f^{\#}\phi)(\sigma|_{[v_0,\cdots,v_k]}) \cdot (f^{\#}\psi)(\sigma|_{[v_k,\cdots,v_{k+l}]}) \\ &= \phi((f_{\#}\sigma)|_{[v_0,\cdots,v_k]}) \cdot \psi((f_{\#}\sigma)|_{[v_k,\cdots,v_{k+l}]}) \\ &= (\phi \smile \psi)(f_{\#}\sigma) \\ &= (f^{\#}(\phi \smile \psi))(\sigma). \end{aligned}$$

□

Definition 3.1.7. *A graded ring is a ring A with a sum decomposition $A = \oplus_k A_k$ where the A_k are additive subgroups so that the multiplication of A takes $A_k \times A_l$ to A_{k+l}. Elements of A_k are called elements of degree k.*

Definition 3.1.8. *The cohomology ring of a topological space X is the graded ring*

$$H^*(X;R) := (\bigoplus_{k\geq 0} H^k(X;R), \smile),$$

with respect to the cup product operation. If R has an identity, then so does $H^*(X;R)$. Similarly, we define the cohomology ring of a pair $H^*(X,A;R)$ by using the relative cup product.

Remark 3.1.9. By scalar multiplication with elements of R, we can regard these cohomology rings as R-algebras.

The following is an immediate consequence of Lemma 3.1.6:

Corollary 3.1.10. *If $f : X \to Y$ is a continuous map then we get an induced ring homomorphism*

$$f^* : H^*(Y;R) \to H^*(X;R).$$

Example 3.1.11. The isomorphisms

$$H^*(\bigsqcup_\alpha X_\alpha; R) \xrightarrow{\cong} \prod_\alpha H^*(X_\alpha; R) \tag{3.1.5}$$

whose coordinates are induced by the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigsqcup_\alpha X_\alpha$ is a ring isomorphism with respect to the coordinatewise multiplication in a ring product, since each coordinate function i_α^* is a ring homomorphism. Similarly, the group isomorphism

$$\widetilde{H}^*(\bigvee_\alpha X_\alpha; R) \cong \prod_\alpha \widetilde{H}^*(X_\alpha; R) \tag{3.1.6}$$

is a ring isomorphism. Here the reduced cohomology is identified to cohomology relative to a basepoint, and we use relative cup products. (We also assume the basepoints $x_\alpha \in X_\alpha$ are deformation retracts of neighborhoods.)

Example 3.1.12. From our calculations in Example 3.1.5 we have that:

$$\begin{aligned} H^*(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) &= \{a_0 + a_1\alpha + a_2\alpha^2 \mid a_i \in \mathbb{Z}/2\mathbb{Z}\} \\ &= (\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^3), \end{aligned}$$

where α is a generator of $H^1(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z})$.

Example 3.1.13.

$$H^*(S^n,\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2)$$

where α is a generator of $H^n(S^n;\mathbb{Z})$. Indeed, we have

$$H^i(S^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

So if α is a generator of $H^n(S^n;\mathbb{Z})$, then the only possible cup products are $\alpha \smile 1$ and $\alpha \smile \alpha$. However, $\alpha \smile \alpha \in H^{2n}(S^n;\mathbb{Z}) = 0$. Hence $\alpha^2 = 0$.

Let us now recall that the cell structure on

$$\mathbb{RP}^\infty = \cup_{n\geq 0}\mathbb{RP}^n$$

consists of one cell in each non-negative dimension. The following result will be proved later on in this section:

Theorem 3.1.14. *The cohomology rings of the real (resp. complex) projective spaces are given by:*

(a)

$$H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2[\alpha]/(\alpha^{n+1})$$

where α is the generator of $H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$.

(b)

$$H^*(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2[\alpha]$$

where α is the generator of $H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$.

(c)

$$H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[\beta]/(\beta^{n+1})$$

where β is the generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$.

(d)

$$H^*(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}[\beta]$$

where β is the generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$.

Before discussing the proof of the above theorem, let us get back to the following motivating example:

Example 3.1.15. We saw at the beginning of this chapter that the spaces $X = \mathbb{CP}^2$ and $Y = S^2 \vee S^4$ have the same homology and cohomology groups, and even the same CW structure. The cup products can be used to decide whether these spaces are homotopy equivalent. Indeed, let us consider the cohomology rings $H^*(X;\mathbb{Z})$ and $H^*(Y;\mathbb{Z})$. From the above theorem, we have that:

$$H^*(\mathbb{CP}^2;\mathbb{Z}) = \mathbb{Z}[\beta]/(\beta^3),$$

where β is the generator of $H^2(\mathbb{CP}^2;\mathbb{Z})$. We also have a ring isomorphism

$$\widetilde{H}^*(S^2 \vee S^4;\mathbb{Z}) \cong \widetilde{H}^*(S^2;\mathbb{Z}) \oplus \widetilde{H}^*(S^4;\mathbb{Z}),$$

where $H^*(S^2;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2)$ and $H^*(S^4;\mathbb{Z}) = \mathbb{Z}[\gamma]/(\gamma^2)$, with degree of α equal to 2 and degree of γ equal to 4. Moreover, $\alpha^2 = 0 = \gamma^2$ and $\alpha \smile \gamma = 0$.
Consider the cohomology generators in degree 2 and square them. In the case of $H^*(\mathbb{CP}^2;\mathbb{Z})$, β^2 is a generator of $H^4(\mathbb{CP}^2;\mathbb{Z})$, hence $\beta^2 \neq 0$. However, in the case of $H^*(S^2 \vee S^4;\mathbb{Z})$, $\alpha^2 \in H^4(S^2;\mathbb{Z}) = 0$. Hence the two cohomology rings of the two spaces are not isomorphic, hence the two spaces are not homotopy equivalent.

Let us now get back to the proof of Theorem 3.1.14. We will discuss below the proof in the case of $\mathbb{RP}^n$. The result in the case of $\mathbb{RP}^\infty$ follows from the finite-dimensional case since the inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^\infty$ induces isomorphisms on $H^i(-;\mathbb{Z}/2)$ for $i \leq n$ by cellular cohomology. The complex projective spaces are handled in precisely the same manner, using $\mathbb{Z}$-coefficients and replacing H^k by H^{2k} and $\mathbb{R}$ by $\mathbb{C}$.

We next prove the following result:

Theorem 3.1.16.

$$H^*(\mathbb{RP}^n;\mathbb{Z}/2) = \mathbb{Z}/2[\alpha]/(\alpha^{n+1}), \tag{3.1.7}$$

where α is the generator of $H^1(\mathbb{RP}^n;\mathbb{Z}/2)$.

Proof. For simplicity, let us use the notation $\mathbb{P}^n := \mathbb{RP}^n$ and all coefficients for the cohomology groups are understood to be $\mathbb{Z}/2$-coefficients.

We prove (3.1.7) by induction on n. Let α_i be a generator for $H^i(\mathbb{P}^n)$ and α_j be a generator for $H^j(\mathbb{P}^n)$, with $i + j = n$. Since for any $k < n$ the inclusion map $u : \mathbb{P}^k \hookrightarrow \mathbb{P}^n$ induces isomorphisms on cohomology groups H^l, for $l \leq k$, it suffices by induction on n to show that $\alpha_i \smile \alpha_j \neq 0$.

Recall now that $\mathbb{P}^n = S^n/(\mathbb{Z}/2)$, with

$$S^n = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} | \sum_{l=0}^{n} x_l^2 = 1\}.$$

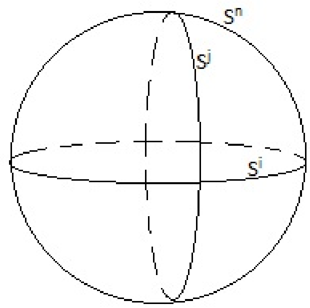
Let

$$S^i = \{(x_0, \cdots, x_i, 0, \cdots, 0) \mid \sum_{l=0}^{i} x_l^2 = 1\}$$

and

$$S^j = \{(0, \cdots, 0, x_{n-j}, \cdots x_n) \mid \sum_{l=n-j}^{n} x_l^2 = 1\}$$

be the i-th and j-th (sub)sphere respectively. Note that since $i + j = n$, we have that $x_{n-j} = x_i$. Hence $S^i \cap S^j = \{(0, \cdots, 0, \pm 1, 0, \cdots, 0)\}$ with ± 1 is in the i-th position, i.e., the intersection consists of the two antipodal points with i-th coordinate ± 1 and all other coordinates zero.



Hence, $\mathbb{P}^i = S^i/(\mathbb{Z}/2)$ and $\mathbb{P}^j = S^j/(\mathbb{Z}/2)$ are subsets of $\mathbb{P}^n = S^n/(\mathbb{Z}/2)$ so that

$$\mathbb{P}^i \cap \mathbb{P}^j = \{p\} = (0 : \cdots : 0 : 1 : 0 : \cdots : 0)$$

with 1 is in the i-th place.

Let $U \subset \mathbb{P}^n$ be the open subset consisting of points $(x_0 : \cdots : x_n)$ with $x_i \neq 0$, i.e.,

$$U = \{(x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n)\},$$

and notice that the map

$$\phi\big((x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n)\big) = (x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

is a homeomorphism $U \cong \mathbb{R}^n$ which takes p to $0 \in \mathbb{R}^n$.

We clearly have that $\mathbb{P}^n = \mathbb{P}^{n-1} \cup U$, where $\mathbb{P}^{n-1}$ is identified to the set of points in $\mathbb{P}^n$ with the i-th coordinate equal to zero. Regarding U as the interior of the n-cell of $\mathbb{P}^n$ (attached to $\mathbb{P}^{n-1}$), it follows that $\mathbb{P}^n - \{p\}$ deformation retracts to $\mathbb{P}^{n-1}$. Similarly, as $\{p\} = \mathbb{P}^i \cap \mathbb{P}^j$, we have that $\mathbb{P}^i - \{p\} \simeq \mathbb{P}^{i-1}$ and $\mathbb{P}^j - \{p\} \simeq \mathbb{P}^{j-1}$. All of this is represented schematically in the figure below, where $\mathbb{P}^n$ is represented by a disc with its antipodal boundary points identified.

Let us now write $\mathbb{R}^n = \mathbb{R}^i \times \mathbb{R}^j$, with coordinates of factors denoted by $(x_0, \cdots, x_{i-1})$ and $(x_{i+1}, \cdots, x_n)$, respectively. Consider the following commutative diagram with horizontal

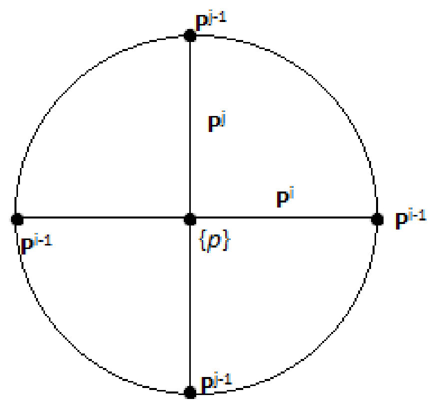


Figure 3.1:

arrows given by the (relative) cup product:

$$\begin{array}{ccc}
H^i(\mathbb{P}^n) \times H^j(\mathbb{P}^n) & \longrightarrow & H^n(\mathbb{P}^n) \\
\uparrow & & \uparrow \\
H^i(\mathbb{P}^n, \mathbb{P}^n - \mathbb{P}^j) \times H^j(\mathbb{P}^n, \mathbb{P}^n - \mathbb{P}^i) & \longrightarrow & H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \\
\downarrow & & \downarrow \\
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \times H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) & \longrightarrow & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})
\end{array}$$

The diagram commutes by naturality of the cup product. Let us examine the bottom row in the above diagram. Let D^i denote a small closed i-disc in $\mathbb{R}^i$ with boundary S^{i-1}. Then by homotopy equivalence and excision we have:

$$\begin{aligned}
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) &\cong H^i(\mathbb{R}^n, \mathbb{R}^n - int(D^i) \times \mathbb{R}^j) \\
&\cong H^i(D^i \times \mathbb{R}^j, S^{i-1} \times \mathbb{R}^j) \\
&\cong H^i(D^i \times D^j, S^{i-1} \times D^j) \\
&\cong H^i((D^i, S^{i-1}) \times D^j) \\
&\cong H^i(D^i, S^{i-1}).
\end{aligned}$$

Similarly,

$$H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) \cong H^j((D^j, S^{j-1}) \times D^i) \cong H^j(D^j, S^{j-1})$$

and

$$H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong H^n(D^n, S^{n-1}) \cong H^n(D^i \times D^j, S^{i-1} \times D^j \cup S^{j-1} \times D^i).$$

Since D^n is an n-cell, its class $[D^n]$ (in the $\mathbb{Z}/2$-cellular cohomology) generates $H^n(D^n, S^{n-1})$, and similar considerations apply to $[D^i \in H^i(D^i, S^{i-1})$ and $[D^j] \in H^j(D^j, S^{j-1})$. So the above isomorphisms and cellular cohomology show that the cup product of the bottom arrow in the above commutative diagram takes the product of generators to a generator, i.e., it is given by

$$[D^i] \times [D^j] \mapsto [D^n].$$

The same will be true for the top row, provided we show that the four vertical maps in the above diagram are isomorphisms.
For the bottom right vertical arrow, we have by excision that

$$H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \cong H^n(U, U - \{p\}) \cong H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}), \tag{3.1.8}$$

where the last isomorphism follows by using the homeomorphism $\phi : U \to \mathbb{R}^n$.
For the top right vertical arrow, we already noted that $\mathbb{P}^n - \{p\}$ deformation retracts to $\mathbb{P}^{n-1}$, so we have

$$H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \cong H^n(\mathbb{P}^n, \mathbb{P}^{n-1}) \cong \mathbb{Z}/2, \tag{3.1.9}$$

where the second isomorphism follows by cellular cohomology. Moreover, by using the long exact sequence for the cohomology of the pair $(\mathbb{P}^n, \mathbb{P}^{n-1})$ and the fact that $H^n(\mathbb{P}^{n-1}) = 0$, we get that the map $\mathbb{Z}/2 = H^n(\mathbb{P}^n, \mathbb{P}^{n-1}) \to H^n(\mathbb{P}^n) \cong \mathbb{Z}/2$ is onto, hence an isomorphism. Thus we get:

$$H^n(\mathbb{P}^n, \mathbb{P}^n - \{p\}) \cong H^n(\mathbb{P}^n) \tag{3.1.10}$$

To show that the two left vertical arrows are isomorphisms, consider the following commutative diagram.

$$\begin{array}{ccccccc}
H^i(\mathbb{P}^n) & \xleftarrow[(2)]{} & H^i(\mathbb{P}^n, \mathbb{P}^{i-1}) & \xleftarrow[(4)]{} & H^i(\mathbb{P}^n, \mathbb{P}^n - \mathbb{P}^j) & \xrightarrow{(5)} & H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (3)} & & \downarrow{\scriptstyle (6)} & & \downarrow{\scriptstyle (7)} \\
H^i(\mathbb{P}^i) & \xleftarrow[(8)]{} & H^i(\mathbb{P}^i, \mathbb{P}^{i-1}) & \xleftarrow[(9)]{} & H^i(\mathbb{P}^i, \mathbb{P}^i - \{p\}) & \xrightarrow{(10)} & H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\})
\end{array}$$

It suffices to show that all these maps are isomorphisms. (Then to finish the proof of the theorem, just interchange i and j.) First note that $(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) = (\mathbb{R}^i, \mathbb{R}^i - \{0\}) \times \mathbb{R}^j$ deformation retract to $(\mathbb{R}^i, \mathbb{R}^i - \{0\})$, so the arrow (7) is an isomorphism. As already pointed out, (10) is an isomorphism by (3.1.8). Moreover, (9) is an isomorphism as in (3.1.9), and (8) is an isomorphism as in (3.1.10). The arrow (1) is an isomorphism by cellular homology, and the arrow (3) is an isomorphism by cellular homology and the naturality of the cohomology long exact sequence. By commutativity of the left square, it then follows that (2) is an isomorphism. In order to show that (4) is an isomorphism, we note that $\mathbb{P}^n - \mathbb{P}^j$ deformation retracts onto $\mathbb{P}^{i-1}$. Indeed, a point $v = (x_0 : \cdots : x_n) \in \mathbb{P}^n - \mathbb{P}^j$ has at least one of the first i coordinates non-zero, so the function

$$f_t(v) := (x_0 : \cdots : x_{i-1} : tx_i : \cdots : tx_n)$$

gives, as t decreases from 1 to 0, a deformation retract from $\mathbb{P}^n - \mathbb{P}^j$ onto $\mathbb{P}^{i-1}$.
Since (3), (4) and (9) are isomorphisms, the commutativity of the middle square yields that (6) is an isomorphism. Finally, since (6), (7) and (10) are isomorphisms, the commutativity of the right square yields that (5) is an is0morphism, which completes the proof of the theorem. $\square$

Example 3.1.17. Let us consider the spaces $\mathbb{RP}^{2n+1}$ and $\mathbb{RP}^{2n} \vee S^{2n+1}$. First note that these spaces have the same CW structure and the same cellular chain complex, so they have the same homology and cohomology groups. However, we claim that $\mathbb{RP}^{2n+1}$ and $\mathbb{RP}^{2n} \vee S^{2n+1}$ are not homotopy equivalent. In order to justify the claim, we first compute their $\mathbb{Z}/2\mathbb{Z}$-cohomology rings.
From the above theorem, the cohomology ring of $\mathbb{RP}^{2n+1}$ is:

$$H^*(\mathbb{RP}^{2n+1}; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{2n+2}),$$

where α is a degree one element, namely the generator of $H^1(\mathbb{RP}^{2n+1}; \mathbb{Z}/2\mathbb{Z})$.
We also have a ring isomorphism

$$\widetilde{H}^*(\mathbb{RP}^{2n} \vee S^{2n+1}; \mathbb{Z}/2\mathbb{Z}) \cong \widetilde{H}^*(\mathbb{RP}^{2n}; \mathbb{Z}/2\mathbb{Z}) \oplus \widetilde{H}^*(S^{2n+1}; \mathbb{Z}/2\mathbb{Z})$$

with $H^*(\mathbb{RP}^{2n}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\beta]/(\beta^{2n+1})$ for β the degree 1 generator of $H^1(\mathbb{RP}^{2n}; \mathbb{Z}/2\mathbb{Z})$, and $H^*(S^{2n+1}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\gamma]/(\gamma^2)$ for γ the generator of $H^{2n+1}(S^{2n+1}; \mathbb{Z}/2\mathbb{Z})$ of degree $2n+1$.
If there was a homotopy equivalence $f : \mathbb{RP}^{2n+1} \to \mathbb{RP}^{2n} \vee S^{2n+1}$, then the generators of degree one would correspond isomorphically to each other, i.e., we would get $f^*(\beta) = \alpha$. But as f^* is a ring isomorphism, this would then imply that: $f^*(\beta^{2n+1}) = (f^*(\beta))^{2n+1} = \alpha^{2n+1}$. However, this yields a contradiction, since $\beta^{2n+1} = 0$, thus $f^*(\beta^{2n+1}) = 0$, while $\alpha^{2n+1} \neq 0$ since α^{2n+1} generates $H^{2n+1}(\mathbb{RP}^{2n+1}; \mathbb{Z}/2\mathbb{Z})$.

3.2 Application: Borsuk-Ulam Theorem

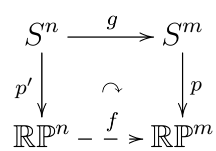
In this section we use cup products in order to prove the following result:

Theorem 3.2.1 (Borsuk-Ulam)**.** *If $n > m \geq 1$, there are no maps $g : S^n \to S^m$ commuting with the antipodal maps, i.e., for which $g(-x) = -g(x)$, for all $x \in S^n$.*

Proof. We prove the theorem by contradiction. Assume that there is a map $g : S^n \to S^m$ commuting with the antipodal maps. Then g carries pairs of antipodal points $(x, -x)$ in S^n to pairs of antipodal points $\big(g(x), g(-x) = -g(x)\big)$ in S^m. So, by passage to the quotient, g induces a map

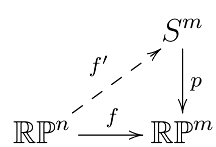
$$f : \mathbb{RP}^n \to \mathbb{RP}^m$$
$$[x] \mapsto [g(x)]$$

which makes the following diagram commutative:

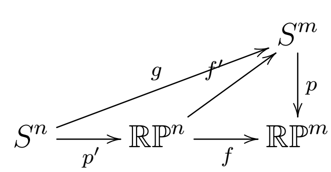


Here p and p' are the two-sheeted covering maps.

We claim that there exists a lift f' of f, i.e., $f = pf'$ in the following diagram:



Let us for now assume the claim and complete the proof of the theorem. Consider the following diagram:



We have $pg = fp' = pf'p'$, the second equality following from the above claim. This implies that both g and $f'p'$ are lifts of fp'. Under the two-sheeted covering map p, antipodal points in S^m are mapped to the same point in $\mathbb{RP}^m$. Therefore, $pg = pf'p'$ implies that at a point $x \in S^n$, we have $g(x) = f'p'(x)$ or $ag(x) = f'p'(x)$, where $a : S^m \to S^m$ is the antipodal map. But $ag(x) = -g(x) = g(-x)$ and $f'p'(x) = f'p'(-x)$. Thus at $x \in S^n$, one of following equalities holds: $g(x) = f'p'(x)$ or $g(-x) = f'p'(-x)$. Since g and $f'p'$ are lifts of fp' and they coincide at a point, it follows by the uniqueness of the lift that $g = f'p'$. But this is a contradiction since $p'(x) = p'(-x)$, hence $f'p'(x) = f'p'(-x)$, while $g(x) \neq g(-x) = -g(x)$.

It remains to prove the claim. A lift for f exists iff

$$f_*(\pi_1(\mathbb{RP}^n)) \subseteq p_*(\pi_1(S^m)). \tag{3.2.1}$$

If $m = 1$, the only homomorphism

$$f_* : \pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2 \to \pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$$

is the trivial one, so (3.2.1) is satisfied.
If $m > 1$, both groups $\pi_1(\mathbb{RP}^n)$ and $\pi_1(\mathbb{RP}^m)$ are $\mathbb{Z}/2$. We will use cup products to show that the induced map $f_* : \mathbb{Z}/2 \to \mathbb{Z}/2$ on fundamental groups is the trivial map. Let $\alpha_m \in H^*(\mathbb{RP}^m; \mathbb{Z}/2)$ and $\alpha_n \in H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ be the generators of degree 1, and consider the induced ring homomorphism

$$f^* : H^*(\mathbb{RP}^m; \mathbb{Z}/2) \to H^*(\mathbb{RP}^n; \mathbb{Z}/2).$$

We have:

$$0 = f^*(\alpha_m^{m+1}) = f^*(\alpha_m)^{m+1},$$

so $f^*(\alpha_m) \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$ has order $m + 1 < n + 1$. Therefore,

$$f^*(\alpha_m) \neq \alpha_n.$$

Since $H^1(\mathbb{RP}^n; \mathbb{Z}/2) = \mathbb{Z}/2 = \langle \alpha_n \rangle$, this implies that

$$f^*(\alpha_m) = 0.$$

Let $i : \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^n$ and $j : \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^m$ be the inclusions obtained by setting all but the first two homogeneous coordinates equal to zero. By cellular cohomology, the map $j^* : H^1(\mathbb{RP}^m) \to H^1(\mathbb{RP}^1)$ is an isomorphism, so $j^*(\alpha_m)$ is the generator of $H^1(\mathbb{RP}^1)$, and in particular,

$$j^*(\alpha_m) \neq 0.$$

On the other hand,

$$(f \circ i)^*(\alpha_m) = i^*(f^*(\alpha_m)) = 0.$$

So $(f \circ i)^* \neq j^*$, hence the maps $f \circ i$ and j are not homotopic.

But the homotopy classes of i and j generate $\pi_1(\mathbb{RP}^n)$ and $\pi_1(\mathbb{RP}^m)$, respectively. So the homomorphisms

$$\begin{array}{rcl} f_* : \pi_1(\mathbb{RP}^n) \simeq \mathbb{Z}/2 & \longrightarrow & \pi_1(\mathbb{RP}^m) \simeq \mathbb{Z}/2 \\ {[i]} & \mapsto & [f \circ i] \neq [j] \end{array}$$

maps the generator $[i]$ to an element of $\mathbb{Z}/2$ other than the generator $[j]$, i.e., $f_* = 0$. This proves the claim, and completes the theorem. □

Exercises

1. Show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X)$ are zero. In particular, this is the case if X is a suspension. Conclude that spaces such as $\mathbb{RP}^2$ and T^2 cannot be written as unions of two open contractible subsets.

2. Is the Hopf map

$$f : S^3 \subset \mathbb{C}^2 \to S^2 = \mathbb{C} \cup \{\infty\}, \;\; (z, w) \mapsto \tfrac{z}{w}$$

nullhomotopic? Explain.

3. Is there a continuous map $f : X \to Y$ inducing isomorphisms on all of the cohomology *groups* (i.e., $f^* : H^i(Y;\mathbb{Z}) \overset{\cong}{\to} H^i(X;\mathbb{Z})$, for all i) but X and Y do not have isomorphic cohomology *rings* (with $\mathbb{Z}$ coefficients)? Explain your answer.

4. Show that $\mathbb{RP}^3$ and $\mathbb{RP}^2 \vee S^3$ have the same cohomology rings with integer coefficients.

5.

(a) Show that $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, with x the generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$.

(a) Show that the Lefschetz number τ_f of a map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ is given by

$$\tau_f = 1 + d + d^2 + \cdots + d^n,$$

where $f^*(x) = dx$ for some $d \in \mathbb{Z}$, and with x as in part (a).

(c) Show that for n even, any map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ has a fixed point.

(d) When n is odd, show that there is a fixed point unless $f^*(x) = -x$, where x denotes as before a generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$.

6. Use cup products to compute the map $H^*(\mathbb{CP}^n;\mathbb{Z}) \to H^*(\mathbb{CP}^n;\mathbb{Z})$ induced by the map $\mathbb{CP}^n \to \mathbb{CP}^n$ that is a quotient of the map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ raising each coordinate to the d-th power, $(z_0, \cdots, z_n) \mapsto (z_0^d, \cdots, z_n^d)$, for a fixed integer $d > 0$. (*Hint*: First do the case $n = 1$.)

7. Describe the cohomology ring $H^*(X \vee Y)$ of a join of two spaces.

8. Let $\mathbb{H} = \mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot i \oplus \mathbb{R}\cdot j \oplus \mathbb{R}\cdot k$ be the skew-field of quaternions, where $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. For a quaternion $q = a + bi + cj + dk$, $a, b, c, d \in \mathbb{R}$, its conjugate is defined by $\bar{q} = a - bi - cj - dk$. Let $|q| := \sqrt{a^2 + b^2 + c^2 + d^2}$.

(a) Verify the following formulae in $\mathbb{H}$: $q \cdot \bar{q} = |q|^2$, $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$, $|q_1 q_2| = |q_1| \cdot |q_2|$.

(b) Let $S^7 \subset \mathbb{H} \oplus \mathbb{H}$ be the unit sphere, and let $f : S^7 \to S^4 = \mathbb{HP}^1 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1 {q_2}^{-1}$. Show that for any $p \in S^4$, the fiber $f^{-1}(p)$ is homeomorphic to S^3.

(c) Let $\mathbb{HP}^n$ be the quaternionic projective space defined exactly as in the complex case as the quotient of $\mathbb{H}^{n+1} \setminus \{0\}$ by the equivalence relation $v \sim \lambda v$, for $\lambda \in \mathbb{H} \setminus \{0\}$. Show that the CW structure of $\mathbb{HP}^n$ consists of only one cell in each dimension $0, 4, 8, \cdots, 4n$, and calculate the homology of $\mathbb{HP}^n$.

(d) Show that $H^*(\mathbb{HP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, with x the generator of $H^4(\mathbb{HP}^n; \mathbb{Z})$.

(e) Show that $S^4 \vee S^8$ and $\mathbb{HP}^2$ are not homotopy equivalent.

9. For a map $f : S^{2n-1} \to S^n$ with $n \geq 2$, let $X_f = S^n \cup_f D^{2n}$ be the CW complex obtained by attaching a $2n$-cell to S^n by the map f. Let $a \in H^n(X_f; \mathbb{Z})$ and $b \in H^{2n}(X_f; \mathbb{Z})$ be the generators of respective groups. The *Hopf invariant* $H(f) \in \mathbb{Z}$ of the map f is defined by the identity $a^2 = H(f)b$.

(a) Let $f : S^3 \to S^2 = \mathbb{C} \cup \{\infty\}$ be given by $f(z_1, z_2) = z_1/z_2$, for $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$. Show that $X_f = \mathbb{CP}^2$ and $H(f) = \pm 1$.

(b) Let $f : S^7 \to S^4 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1 {q_2}^{-1}$ in terms of quaternions $(q_1, q_2) \in S^7$, the unit sphere in $\mathbb{H}^2$. Show that $X_f = \mathbb{HP}^2$ and $H(f) = \pm 1$.

3.3 Künneth Formula

3.3.1 Cross product

Let us motivate this section by consider the spaces $S^2 \times S^3$ and $S^2 \vee S^3 \vee S^5$. Both spaces are CW complexes with cells $\{e^0, e^2, e^3, e^5\}$ in degrees, $0, 2, 3$ and 5, respectively. So the cellular chain complex for both spaces is:

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$$

Hence both spaces have the same homology and cohomology groups. It is then natural to ask the following:

Question 3.3.1. *Are the spaces $S^2 \times S^3$ and $S^2 \vee S^3 \vee S^5$ homotopy equivalent?*

The aim of this section is to convince the reader that the answer is *No*. More precisely, we will show that the two spaces have different cohomology rings.
The cohomology ring $H^*(S^2 \vee S^3 \vee S^5; \mathbb{Z})$ can be computed from the ring isomorphism

$$\widetilde{H}^*(S^2 \vee S^3 \vee S^5; \mathbb{Z}) \cong \widetilde{H}^*(S^2; \mathbb{Z}) \oplus \widetilde{H}^*(S^3; \mathbb{Z}) \oplus \widetilde{H}^*(S^5; \mathbb{Z}),$$

with $H^*(S^2; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2)$, $H^*(S^3; \mathbb{Z}) \cong \mathbb{Z}[\beta]/(\beta^2)$ and $H^*(S^5; \mathbb{Z}) \cong \mathbb{Z}[\gamma]/(\gamma^2)$, where α is the generator of $H^2(S^2; \mathbb{Z})$, β is the generator of $H^3(S^3; \mathbb{Z})$ and γ is the generator of $H^5(S^5; \mathbb{Z})$. Moreover, we have that $\alpha \smile \beta = 0$. Indeed, let

$$p : S^2 \vee S^3 \vee S^5 \to S^2 \vee S^3$$

be the natural retraction map. Then p^* induces isomorphisms on H^2 and H^3. So if $\bar{\alpha}$ and $\bar{\beta}$ are the generators of $H^2(S^2 \vee S^3)$ and $H^3(S^2 \vee S^3)$, then $\alpha = p^*\bar{\alpha}$ and $\beta = p^*\bar{\beta}$. So

$$\alpha \smile \beta = p^*\bar{\alpha} \smile p^*\bar{\beta} = p^*(\bar{\alpha} \smile \bar{\beta}) = 0$$

since $\bar{\alpha} \smile \bar{\beta} = 0$.
By the end of this section, we will show that the product of the generators of degree 2 and degree 3 in the cohomology ring of $S^2 \times S^3$ is the generator in degree 5, so it is non-zero. This will then completely answer the above question.

The following result is proved in [Hatcher, Theorem 3.14]:

Theorem 3.3.2. *Let R be a commutative ring, and $\alpha \in H^k(X, A; R)$ and $\beta \in H^l(X, A; R)$. Then the following holds:*

$$\alpha \smile \beta = (-1)^{kl} \cdot \beta \smile \alpha. \tag{3.3.1}$$

Definition 3.3.3. *A graded ring which satisfies a condition as in the above theorem is called graded commutative. Hence the cohomology ring $H^*(X, A; R)$ is a graded commutative ring.*

Corollary 3.3.4. *If $\alpha \in H^*(X;R)$ is of odd degree and if $H^*(X;R)$ has no elements of order two, then $\alpha \smile \alpha = 0$.*

Definition 3.3.5. *Cross product or External cup product*
Let X and Y be topological spaces, and denote by p and q the projections $p : X \times Y \longrightarrow X$ and $q : X \times Y \longrightarrow Y$. By using the cohomology maps defined by these projections, we have an induced map denoted by $\times$:

$$\begin{array}{ccccc} H^*(X;R) & \times & H^*(Y;R) & \stackrel{\times}{\longrightarrow} & H^*(X \times Y;R) \\ a & & b & \mapsto & a \times b := p^*(a) \smile q^*(b) \end{array}$$

All cohomology groups $H^i(X;R)$ and $H^i(Y;R)$ have an R-module structure, hence so do the corresponding cohomology rings $H^*(X;R)$ and $H^*(Y;R)$. Since the map $\times$ is bilinear, the universal property for tensor products yields a group homomorphism called the cross product, which we again denote by $\times$:

$$H^*(X;R) \otimes_R H^*(Y;R) \stackrel{\times}{\longrightarrow} H^*(X \times Y;R) \tag{3.3.2}$$

So, by definition, we have that:

$$\times(a \otimes b) := a \times b.$$

The cross-product becomes a ring isomorphism if we put a ring structure on $H^*(X;R) \otimes_R H^*(Y;R)$ by the following multiplication operation:

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg(b) \cdot \deg(c)}(ac \otimes bd) \tag{3.3.3}$$

Indeed, we have:

$$\begin{aligned} \times((a \otimes b) \cdot (c \otimes d)) &= (-1)^{\deg(b) \cdot \deg(c)} \times (ac \otimes bd) \\ &= (-1)^{\deg(b) \cdot \deg(c)} \ (ac \times bd) \\ &= (-1)^{(\deg b) \cdot \deg(c)} \ p^*(a \smile c) \smile q^*(b \smile d) \\ &= (-1)^{\deg(b) \cdot \deg(c)} \ p^*(a) \smile p^*(c) \smile q^*(b) \smile q^*(d) \\ &\stackrel{(3.3.1)}{=} p^*(a) \smile q^*(b) \smile p^*(c) \smile q^*(d) \\ &= \times(a \otimes b) \smile \times(c \otimes d). \end{aligned}$$

3.3.2 Künneth theorem in cohomology. Examples

The following result is very helpful in finding the cohomology ring of a product of CW complexes:

Theorem 3.3.6. *Künneth Formula*
If X and Y are CW complexes, and $H^k(Y;R)$ is a finitely generated free R-module for all k, then the cross product

$$H^*(X;R) \otimes_R H^*(Y;R) \stackrel{\times}{\longrightarrow} H^*(X \times Y;R)$$

is a ring isomorphism. Moreover, we have the following isomorphism of groups:

$$H^n(X \times Y; R) \cong \bigoplus_{i+j=n} H^i(X; R) \otimes_R H^j(Y; R) \tag{3.3.4}$$

In the next section, we will explain the content of Theorem 3.3.6 in a more general context. Let us now work out some examples.

Example 3.3.7. Let us find the cohomology ring of $S^2 \times S^3$, which appeared at the beginning of this section. According to the Künneth formula, we have the following ring isomorphism:

$$H^*(S^2 \times S^3; \mathbb{Z}) \cong H^*(S^2; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^3; \mathbb{Z})$$

If we let $a \in H^*(S^2; \mathbb{Z})$ denote the degree 2 element which generates $H^2(S^2; \mathbb{Z})$ and $b \in H^*(S^3; \mathbb{Z})$ the degree 3 element which generates $H^3(S^3; \mathbb{Z})$, then $\times(a \otimes 1)$ and $\times(1 \otimes b)$ (where 1 denotes the identity in the respective cohomology rings) will be the generators in $H^*(S^2 \times S^3; \mathbb{Z})$ of degree 2 and 3, respectively. Moreover, $\times(a \otimes 1) \smile \times(1 \otimes b) = \times(a \otimes b)$ will be a generator of degree 5 in $H^*(S^2 \times S^3; \mathbb{Z})$.

In order to simplify the notations, we make the following definition.

Definition 3.3.8. *Exterior Algebra*
Let R be a commutative ring with identity. The exterior algebra over R, denoted

$$\Lambda_R[\alpha_1, \alpha_2, \ldots],$$

is the free R-module generated by products of the form:

$$\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_k}, \text{ with } i_1 < i_2 < \cdots < i_k,$$

and with associative and distributive multiplication defined by the rules:

$$\begin{aligned} \alpha_i\alpha_j &= -\alpha_j\alpha_i, \text{ if } i \neq j \\ \alpha_i^2 &= 0. \end{aligned}$$

The empty product of α_i's is allowed and it gives the identity element $1 \in \Lambda_R[\alpha_1, \alpha_2, \ldots]$.

Example 3.3.9. Let us now show that

$$H^*(S^3 \times S^5 \times S^7; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[a_3, a_5, a_7], \tag{3.3.5}$$

where a_i is the generator of degree i in $H^*(S^3 \times S^5 \times S^7; \mathbb{Z})$, for $i = 3, 5, 7$.
By the Künneth formula applied to the product of CW complexes $S^3 \times S^5 \times S^7$, we have the following ring isomorphism:

$$H^*(S^3 \times S^5 \times S^7; \mathbb{Z}) \cong H^*(S^3; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^5; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^7; \mathbb{Z}).$$

Let α_i be the generator of degree i in $H^*(S^i; \mathbb{Z})$ for $i = 3, 5, 7$. Then the generators of degree 3, 5 and 7 in $H^*(S^3 \times S^5 \times S^7; \mathbb{Z})$ are given respectively by:

- $a_3 = \times(\alpha_3 \otimes 1 \otimes 1)$
- $a_5 = \times(1 \otimes \alpha_5 \otimes 1)$
- $a_7 = \times(1 \otimes 1 \otimes \alpha_7)$

The product of these generators produce generators of higher degrees, i.e., 8, 10, 12 and 15, in the cohomology ring $H^*(S^3 \times S^5 \times S^7; \mathbb{Z})$. Let us compute some products of the elements:

$$\begin{aligned} a_3^2 &= \times(\alpha_3 \otimes 1 \otimes 1) \smile \times(\alpha_3 \otimes 1 \otimes 1) \\ &= \times\big[(\alpha_3 \otimes 1 \otimes 1) \cdot \times(\alpha_3 \otimes 1 \otimes 1)\big] \\ &= \times(\alpha_3^2 \otimes 1 \otimes 1) \\ &= 0 \end{aligned}$$

and a similar result for a_5^2 and a_7^2.

$$\begin{aligned} a_3 a_5 &= \times(\alpha_3 \otimes 1 \otimes 1) \smile \times(1 \otimes \alpha_5 \otimes 1) \\ &= \times\big[(\alpha_3 \otimes 1 \otimes 1) \cdot (1 \otimes \alpha_5 \otimes 1)\big] \\ &= (-1)^{0 \cdot 0} \times (\alpha_3 \otimes \alpha_5 \otimes 1) \\ &= \times(\alpha_3 \otimes \alpha_5 \otimes 1) \end{aligned}$$

$$\begin{aligned} a_5 a_3 &= \times(1 \otimes \alpha_5 \otimes 1) \smile \times(\alpha_3 \otimes 1 \otimes 1) \\ &= \times\big[(1 \otimes \alpha_5 \otimes 1) \cdot (\alpha_3 \otimes 1 \otimes 1)\big] \\ &= (-1)^{3 \cdot 5} \times (\alpha_3 \otimes \alpha_5 \otimes 1) \\ &= -a_3 a_5 \end{aligned}$$

We have similar results for the other products too. The above calculations show that we have an isomorphism $H^*(S^3 \times S^5 \times S^7; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[a_3, a_5, a_7]$.

Remark 3.3.10. It is easy to see that a similar result holds for the cohomology ring of any (finite) product of odd dimensional spheres.

Example 3.3.11. By the Künneth formula we have the following ring isomorphism:

$$\begin{aligned} H^*(\mathbb{RP}^\infty \times \mathbb{RP}^\infty; \mathbb{Z}/2) &= H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) \otimes_{\mathbb{Z}_2} H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) \\ &= \mathbb{Z}/2[\alpha] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\beta] \\ &= \mathbb{Z}/2[\alpha, \beta] \end{aligned}$$

where α and β are generators of degree 1, and they commute since we work with $\mathbb{Z}/2$-coefficients.

Example 3.3.12. Let us now investigate if the spaces $\mathbb{C}P^6$ and $S^2 \times S^4 \times S^6$ are homotopy equivalent. Fortunately, there is an easy answer to this question. Consider the usual CW structure for $\mathbb{C}P^6$ and the product CW structure for $S^2 \times S^4 \times S^6$. Both spaces have cells only in even dimensions, but $\mathbb{C}P^6$ has one cell in dimension 6, whereas $S^2 \times S^4 \times S^6$ has two cells in dimension 6. It follows that $H_6(\mathbb{C}P^6) = \mathbb{Z}$, whereas $H_6(S^2 \times S^4 \times S^6) = \mathbb{Z} \oplus \mathbb{Z}$. So $\mathbb{C}P^6$ and $S^2 \times S^4 \times S^6$ are not homotopy equivalent. A more difficult approach to answer the question would be to show that the cohomolgy rings for these spaces are not isomorphic. We will do this in the following example.

Example 3.3.13. Let us show that if $n > 1$, the spaces $\mathbb{C}P^{\frac{n(n+1)}{2}}$ and $S^2 \times S^4 \times \cdots \times S^{2n}$ are not homotopy equivalent. Consider the following cases:

- If $n = 1$, then $\mathbb{C}P^1$ is homeomorphic to S^2.
- If $n = 2$, then both the spaces $\mathbb{C}P^3$ and $S^2 \times S^4$ have one cell in each of the dimensions $\{0, 2, 4, 6\}$. Thus they also have the same cellular chain/cochain complex and, in particular, their homology/cohomology groups are isomorphic. We will, however, distinguish these spaces by their cohomology rings.
- If $n \geq 3$, then $\mathbb{C}P^n$ has one cell in each of the dimensions $\{0, 2, 4, \ldots, 2n\}$, but the cell structure of $S^2 \times S^4 \times \cdots \times S^{2n}$ is different from that of $\mathbb{C}P^n$ since, for example, $S^2 \times S^4 \times \cdots \times S^{2n}$ has two 6-cells. As both spaces have cells only in even dimensions, we can already conclude that they have different homology and cohomology groups since they have different cell structures.

We will now show that for $n > 1$ the two spaces have non-isomorphic cohomology rings. First, the Künneth formula yields that:

$$H^*(S^2 \times S^4 \times \cdots \times S^{2n}; \mathbb{Z}) \cong H^*(S^2; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^4; \mathbb{Z}) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} H^*(S^{2n}; \mathbb{Z})$$

So a degree 2 element in this ring looks like $\times(a \otimes 1 \otimes 1 \otimes \cdots \otimes 1)$, where $a \in H^2(S^2)$. The square of this element is:

$$\begin{aligned} [\times(a \otimes 1 \otimes 1 \otimes \cdots \otimes 1)]^2 &= \times[(a \otimes 1 \otimes 1 \otimes \cdots \otimes 1)^2] \\ &= \times(a^2 \otimes 1 \otimes 1 \otimes \cdots \otimes 1) \\ &= 0 \end{aligned}$$

since $a^2 \in H^4(S^2) = 0$. However, in the case of $\mathbb{C}P^{\frac{n(n+1)}{2}}$, we know that square of a non-zero degree 2 element is a non-zero degree 4 element. Hence the cohomology rings of the two spaces are not isomorphic.

Example 3.3.14. Let us use cup products and the Künneth formula in order to show that $S^n \vee S^m$ is not a retract of $S^n \times S^m$, for $n, m \geq 1$. First, consider the product CW structure on $S^n \times S^m$: it consists of cells $\{e^0, e^m, e^n, e^{m+n}\}$ with attaching maps $\phi : \partial e^m \to e^0$ and

$\phi' : \partial e^n \to e^0$ coming from the factors. Hence $S^n \vee S^m$ is a subset of $S^n \times S^m$. (Note that we also allow the case $n = m$.) Next, suppose by contradiction that there is a retract

$$r : S^n \times S^m \to S^n \vee S^m.$$

So, if $i : S^n \vee S^m \hookrightarrow S^n \times S^m$ denotes the inclusion, then the composition $r \circ i$ is the identity map on $S^n \vee S^m$. It follows that the cohomology map $(r \circ i)^* = i^* \circ r^*$ is the identity, so

$$r^* : H^*(S^n \vee S^m) \longrightarrow H^*(S^n \times S^m)$$

is a monomorphism. By the Künneth formula, we have a ring isomorphism

$$H^*(S^n) \otimes H^*(S^m) \overset{\times}{\cong} H^*(S^n \times S^m).$$

Hence, a non-zero element in $H^n(S^n \times S^m)$ is of the form $a \times 1 := \times(a \otimes 1)$, with $a \in H^n(S^n)$ a non-zero class. Similarly, a non-zero element in $H^m(S^n \times S^m)$ is of the form $1 \times b := \times(1 \otimes b)$, for some non-zero class $b \in H^m(S^m)$. Let us now consider the product of non-zero elements $a \times 1 \in H^n(S^n \times S^m)$ and $1 \times b \in H^m(S^n \times S^m)$ in the ring $H^*(S^n \times S^m)$. We get:

$$\begin{aligned} (a \times 1) \smile (1 \times b) &= \times(a \otimes 1) \smile \times(1 \otimes b) \\ &= \times[(a \otimes 1) \cdot (1 \otimes b)] \\ &= \times(a \otimes b) \\ &= a \times b \\ &\neq 0, \end{aligned} \tag{3.3.6}$$

since $a \otimes b \neq 0$ in $H^*(S^n) \otimes H^*(S^m)$. We also have a ring isomorphism

$$\widetilde{H}^*(S^n \vee S^m) \cong \widetilde{H}^*(S^n) \oplus \widetilde{H}^*(S^m).$$

Let $\alpha, \beta \in H^*(S^n \vee S^m)$ be the generators of degree n and m, respectively. Then

$$\alpha \smile \beta \in H^{n+m}(S^n \vee S^m) = 0.$$

On the other hand, since r^* is a monomorphism, the classes $r^*(\alpha)$ and $r^*(\beta)$ are non-zero elements of degree n and resp. m in the cohomology ring $H^*(S^n \times S^m)$, so by the above calculation, their product is non zero. But

$$r^*(\alpha) \smile r^*(\beta) = r^*(\alpha \smile \beta) = r^*(0) = 0,$$

which gives us a contradiction.

3.3.3 Künneth exact sequence and applications

In this section, we aim to provide the necessary background for Künneth-type theorems.

Let us fix coefficients in a PID ring R.

Given two chain complexes $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$, we define $(C \otimes C')_\bullet$ to be the complex with:

$$(C \otimes C')_n = \bigoplus_{p=0}^{n} (C_p \otimes C'_{n-p}) \tag{3.3.7}$$

and boundary map $d_n : (C \otimes C')_n \to (C \otimes C')_{n-1}$ which on $C_p \otimes C'_{n-p}$ is given by:

$$d_n(a \otimes b) = (\partial_p a) \otimes b + (-1)^p (a \otimes \partial'_{n-p} b). \tag{3.3.8}$$

Then we have:

$$\begin{aligned} (d \circ d)(a \otimes b) &= d\big((\partial a) \otimes b + (-1)^p (a \otimes \partial' b)\big) \\ &= (\partial^2 a) \otimes b + (-1)^{p-1} (\partial a) \otimes (\partial' b) + (-1)^p (\partial a) \otimes (\partial' b) + (-1)^p a \otimes (\partial'^2 b) \\ &= 0. \end{aligned}$$

So $\big((C \otimes C')_\bullet, d_\bullet\big)$ is a chain complex. It is therefore natural to ask the following question:

Question 3.3.15. *How is the homology $H_*((C \otimes C')_\bullet)$ related to $H_*(C_\bullet)$ and $H_*(C'_\bullet)$?*

The answer is provided by the following result from homological algebra:

Theorem 3.3.16. *Künneth exact sequence*
Let R be a PID, and assume that for each i, C_i is a free R-module. Then for all n, there is a split short exact sequence:

$$0 \longrightarrow \bigoplus_p \big(H_p(C_\bullet) \otimes_R H_{n-p}(C'_\bullet)\big) \longrightarrow H_n((C \otimes C)_\bullet) \longrightarrow \bigoplus_p \operatorname{Tor}_R\big(H_p(C_\bullet), H_{n-p-1}(C'_\bullet)\big) \longrightarrow 0 \tag{3.3.9}$$

In what follows we discuss several applications of Theorem 3.3.16.

Künneth Formula for homology.

Let X and Y be two spaces, and let $C_\bullet$ and $C'_\bullet$ denote the singular chain complexes of X and Y, respectively. Then it is not hard to see that the singular chain complex $C_\bullet(X \times Y)$ of $X \times Y$ is chain homotopy equivalent to $(C \otimes C')_\bullet$, so they have the same homology groups. We thus have the following important consequence of Theorem 3.3.16:

Corollary 3.3.17. *Künneth Formula for homology*
If X and Y are topological spaces, then the following holds:

$$H_n(X \times Y) \cong \bigoplus_{p=0}^{n} \big(H_p(X) \otimes H_{n-p}(Y)\big) \oplus \bigoplus_{p=0}^{n-1} \operatorname{Tor}\big(H_p(X), H_{n-p-1}(Y)\big). \tag{3.3.10}$$

In particular, if all homology groups of X or Y are free R-modules, then:

$$H_n(X \times Y) \cong \bigoplus_{p=0}^{n} H_p(X) \otimes H_{n-p}(Y). \tag{3.3.11}$$

As a consequence of Corollary 3.3.17, we have:

Corollary 3.3.18. *If the Euler characteristics $\chi(X)$ and $\chi(Y)$ are defined, then $\chi(X \times Y)$ is defined, and:*

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y). \tag{3.3.12}$$

Universal Coefficient Theorem for homology

The *Universal Coefficient Theorem* for homology can be seen as a consequence of Theorem 3.3.16 as follows: take $C_\bullet$ to be the singular chain complex of X and let $C'_\bullet$ to be the chain complex defined by: $C'_n = 0$ if $n \neq 0$, $C'_0 = R$, and $\partial'_n = 0$ for all $n \geq 0$. We then get by Theorem 3.3.16 that:

$$H_n(X; R) \cong \big(H_n(X) \otimes R\big) \oplus \operatorname{Tor}(H_{n-1}(X), R). \tag{3.3.13}$$

Remark 3.3.19. Note that (3.3.13) can also be obtained from (3.3.10) by taking Y to be a point.

Künneth formula for cohomology

Finally, we also have the following cohomology Künneth formula:

Corollary 3.3.20. *Künneth formula for cohomology*
If R is a PID, and all homology groups $H_i(X; R)$ are finitely generated, then there is a split exact sequence (with R-coefficients):

$$0 \longrightarrow \bigoplus_{p=0}^{n} \big(H^p(X) \otimes H^{n-p}(Y)\big) \longrightarrow H^n(X \times Y) \longrightarrow \bigoplus_{p=0}^{n+1} \operatorname{Tor}\big(H^p(X), H^{n-p+1}(Y)\big) \longrightarrow 0. \tag{3.3.14}$$

Moreover, if all cohomology groups $H^i(X)$ of X (or Y) are free over R, we get the following isomorphism:

$$H^n(X \times Y) \cong \bigoplus_{p=0}^{n} H^p(X) \otimes H^{n-p}(Y). \tag{3.3.15}$$

Proof. (Sketch.) Let us indicate how this result is obtained from Theorem 3.3.16. We would like to apply the Künneth exact sequence to the chain complexes defined by:

$$C_{-n} := C^n(X; R), \quad \partial_{-n} := \delta^n_X$$

and

$$C'_{-n} := C^n(Y;R), \quad \partial'_{-n} := \delta^n_Y.$$

However, note that C_i and C'_i are not necessarily R-free. Indeed,

$$C^n(X;R) = \mathrm{Hom}_R(C_n(X;R), R),$$

but $C_n(X;R)$ is not necessarily a finitely generated R-module. In order to get around this problem, the idea is to replace the chain complex $C_\bullet(X;R)$ by a chain homotopic one, which has finitely generated components. Here is where the assumption that $H_i(X;R)$ are finitely generated is used. □

Exercises

1. Are the spaces $S^2 \times \mathbb{RP}^4$ and $S^4 \times \mathbb{RP}^2$ homotopy equivalent? Justify your answer!

2. Using cup products, show that every map $S^{k+l} \to S^k \times S^l$ induces the trivial homomorphism $H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$, assuming $k > 0$ and $l > 0$.

3. Describe $H^*(\mathbb{CP}^\infty/\mathbb{CP}^1;\mathbb{Z})$ as a ring with finitely many multiplicative generators. How does this ring compare with $H^*(S^6 \times \mathbb{HP}^\infty;\mathbb{Z})$?

4. Show that if $H_n(X;\mathbb{Z})$ is finitely generated and free for each n, then $H^*(X;\mathbb{Z}_p)$ and $H^*(X;\mathbb{Z}) \otimes \mathbb{Z}_p$ are isomorphic as rings, so in particular the ring structure with $\mathbb{Z}$-coefficients determines the ring structure with $\mathbb{Z}_p$-coefficients.

5. Show that the cross product map $H^*(X;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \to H^*(X \times Y;\mathbb{Z})$ is not an isomorphism if X and Y are infinite discrete sets.

6. Show that for n even S^n is not an *H-space*, i.e., there is no map $\mu : S^n \times S^n \to S^n$ so that $\mu \circ i_1 = id_{S^n}$ and $\mu \circ i_2 = id_{S^n}$, where i_1, i_2 are the inclusions on factors.

7. Let A be the union of two once linked circles in S^3, and B be the union of two unlinked circles. Show that the cohomology groups of $S^3 \setminus A$ and $S^3 \setminus B$ are isomorphic, but their cohomology rings are not.

8. Compute the ring structure of $H^*(T^n;\mathbb{Z})$, where T^n is the n-dimensional torus (a product of n circles). Do the same for $H^*(T^n \setminus \{x\};\mathbb{Z})$, where $x \in T^n$ is any point.

Chapter 4

Poincaré Duality

4.1 Introduction

In this chapter, we show that oriented n-manifolds enjoy a very special symmetry on their (co)homology groups:

Theorem 4.1.1. *Let M be a closed (i.e., compact without boundary), oriented and connected manifold of dimension n. Then for all $i \geq 0$ we have isomorphisms:*

$$H_i(M;\mathbb{Z}) \cong H^{n-i}(M;\mathbb{Z}). \tag{4.1.1}$$

In particular, we get:

Corollary 4.1.2. *For all $i \geq 0$, the isomorphisms*

$$H_i(M;\mathbb{Q}) \overset{(4.1.1)}{\cong} H^{n-i}(M;\mathbb{Q}) \overset{(UCT)}{\cong} \mathrm{Hom}(H_{n-i}(M;\mathbb{Q}),\mathbb{Q}) \tag{4.1.2}$$

yield a non-degenerate bilinear pairing

$$H_i(M;\mathbb{Q}) \times H_{n-i}(M;\mathbb{Q}) \to \mathbb{Q}.$$

Moreover, the complementary Betti numbers are equal, i.e.,

$$\beta_i(M) = \beta_{n-i}(M).$$

In the next section we will explain in more detail the notion of orientability of manifolds. Later on, we will describe explicitly the nature of the isomorphism (4.1.1) by using the *cap product* operation $\frown$, i.e., we will show that it is realized by

$$\frown [M] : H^{n-i}(M;\mathbb{Z}) \longrightarrow H_i(M;\mathbb{Z}), \tag{4.1.3}$$

where $[M] \in H_n(M)$ is the “fundamental (orientation) class" of the manifold M.

4.2 Manifolds. Orientation of manifolds

Definition 4.2.1. *A Hausdorff space M is a (topological) manifold if any point $x \in M$ has a neighborhood U_x homeomorphic to $\mathbb{R}^n$ (where such a homeomorphism takes x to 0).*

Let us now compute the *local homology groups* of a manifold M at some point $x \in M$:

$$\begin{aligned} H_i(M, M \setminus \{x\}; \mathbb{Z}) &\overset{(1)}{\cong} H_i(U_x, U_x \setminus \{x\}; \mathbb{Z}) \\ &\overset{(2)}{\cong} H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \\ &\overset{(3)}{\cong} \widetilde{H}_{i-1}(\mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \\ &\overset{(4)}{\cong} \widetilde{H}_{i-1}(S^{n-1}; \mathbb{Z}) \\ &= \begin{cases} \mathbb{Z}, & \text{if } \ i = n \\ 0\ , & \text{otherwise,} \end{cases} \end{aligned} \tag{4.2.1}$$

where (1) follows by excision, (2) by using the homeomorphism $U_x \cong \mathbb{R}^n$, (3) by the homology long exact sequence of a pair, and (4) by using a deformation retract.

Definition 4.2.2. *The dimension of a manifold M, denoted* $\dim(M)$*, is the only non-vanishing degree of the local homology groups of M.*

Definition 4.2.3. *A local orientation of an n-manifold M at $x \in M$ is a choice μ_x of one of the two generators of the local homology group $H_n(M, M \setminus \{x\}; \mathbb{Z}) = \mathbb{Z}$.*

Remark 4.2.4. A local orientation μ_x at $x \in M$ induces local orientations at all nearby points y, i.e., if x and y are contained in a small ball B, then we have induced isomorphisms:

$$\mu_x \in \mathbb{Z} = H_n(M, M \setminus \{x\}) \overset{\cong}{\longleftarrow} H_n(M, M \setminus B) = \mathbb{Z} \overset{\cong}{\longrightarrow} H_n(M, M \setminus \{y\}) = \mathbb{Z} \in \mu_y, \tag{4.2.2}$$

where the above isomorphisms are induced by deformation retracts.

Definition 4.2.5. *A (global) orientation on an n-manifold M is a continuous choice of local orientations, i.e., for every $x \in M$ there exists a closed ball of finite positive dimension $B \subset U_x \cong \mathbb{R}^n$ and a (generating) class $\mu_B \in H_n(M, M \setminus B)$ such that $\rho_y : H_n(M, M \setminus B) \to H_n(M, M \setminus \{y\})$ takes μ_B to μ_y for all $y \in B$.*

Definition 4.2.6. *The pair consisting of manifold and orientation is called an oriented manifold.*

Notation: Let M be an n-manifold and $K \subset L \subset M$ be compact subsets. Consider the map induced by inclusion of pairs:

$$\rho_K : H_i(M, M \setminus L) \to H_i(M, M \setminus K).$$

Then for $a \in H_i(M, M \setminus L)$, $\rho_K(a)$ is called the *restriction* of a to K.

In the above notations, we have the following important result:

Theorem 4.2.7. *For any oriented manifold M of dimension n and any compact $K \subset M$, there is a unique $\mu_K \in H_n(M, M \setminus K; \mathbb{Z})$ such that $\rho_x(\mu_K) = \mu_x$ for all $x \in K$.*

An immediate corollary of the above theorem is the existence of the fundamental class of compact oriented manifolds. More precisely, by taking $K = M$ in Theorem 4.2.7, we get the following:

Corollary 4.2.8. *If M is a compact oriented n-manifold, there exists a unique $\mu_M \in H_n(M; \mathbb{Z})$ so that $\rho_x(\mu_M) = \mu_x$ for all $x \in M$.*

Definition 4.2.9. *The homology class $[M] := \mu_M$ of Corollary 4.2.8 is called the fundamental class of M.*

The proof of Theorem 4.2.7 uses the following:

Lemma 4.2.10. *If K is a compact subset of an n-manifold M, we have:*

(i) $H_i(M, M \setminus K) = 0$ if $i > n$.

(ii) $a \in H_n(M, M \setminus K)$ is equal to 0 if and only if $\rho_x(a) = 0$ for all $x \in K$.

Before proving the above lemma, let us finish the proof of Theorem 4.2.7.

Proof. (of Theorem 4.2.7)
For the *uniqueness* part, if μ_K^1 and μ_K^2 are as in the statement of the theorem, then for all $x \in K$ we have $\rho_x(\mu_K^1 - \mu_K^2) = \mu_x - \mu_x = 0$. Then by using Lemma 4.2.10(ii), we get that $\mu_K^1 - \mu_K^2 = 0$, or $\mu_K^1 = \mu_K^2$.

We prove the *existence* part in several steps:

Step I: If K is contained in a sufficiently small euclidean closed ball (of finite positive radius) B centered at a point $y \in M$, as in the definition of orientability, then for all $x \in K$, the composition

$$H_n(M, M \setminus B) \xrightarrow{\rho_K} H_n(M, M \setminus K) \xrightarrow{\rho_x} H_n(M, M \setminus \{x\}) \tag{4.2.3}$$

is an isomorphism. Then set $\mu_K := \rho_K(\mu_B)$, with $\mu_B \in H_n(M, M \setminus B)$ as in the definition of orientability.

Step II: If the theorem holds for compact subsets K_1 and K_2 and for their intersection $K_1 \cap K_2$, we show that it holds for their union $K = K_1 \cup K_2$. Indeed, the Mayer-Vietoris sequence for the open cover

$$M \setminus (K_1 \cap K_2) = (M \setminus K_1) \cup (M \setminus K_2),$$

with intersection

$$M \setminus K = (M \setminus K_1) \cap (M \setminus K_2)$$

gives the long exact sequence:

$$0 \to H_n(M, M \setminus K) \xrightarrow{\varphi} H_n(M, M \setminus K_1) \oplus H_n(M, M \setminus K_2) \xrightarrow{\psi} H_n(M, M \setminus (K_1 \cap K_2)) \to \dots \tag{4.2.4}$$

where $\varphi(a) = \rho_{K_1}(a) \oplus \rho_{K_2}(a)$ and $\psi(b \oplus c) = \rho_{K_1 \cap K_2}(b) - \rho_{K_1 \cap K_2}(c)$. By our assumption, there exist unique $\mu_{K_1} \in H_n(M, M \setminus K_1)$ and $\mu_{K_2} \in H_n(M, M \setminus K_2)$ restricting to local orientations at points $x \in K_1$ and resp. $x \in K_2$, hence

$$\rho_x \circ \rho_{K_1 \cap K_2}(\mu_{K_i}) = \rho_x(\mu_{K_i}) = \mu_x \tag{4.2.5}$$

for all $x \in K_1 \cap K_2$ and $i = 1, 2$. Then we have

$$\rho_x(\rho_{K_1 \cap K_2}(\mu_{K_1}) - \rho_{K_1 \cap K_2}(\mu_{K_2})) = \mu_x - \mu_x = 0 \tag{4.2.6}$$

for all $x \in K_1 \cap K_2$. So by Lemma 4.2.10 we get that

$$\psi(\mu_{K_1} \oplus \mu_{K_2}) = \rho_{K_1 \cap K_2}(\mu_{K_1}) - \rho_{K_1 \cap K_2}(\mu_{K_2}) = 0, \tag{4.2.7}$$

i.e., $\mu_{K_1} \oplus \mu_{K_2} \in \ker \psi = \text{Image } \varphi$. Since φ is injective, there exists a unique

$$\mu_K \in H_n(M, M \setminus K)$$

such that $\varphi(\mu_K) = \mu_{K_1} \oplus \mu_{K_2}$. By the uniqueness part, we also have that μ_K restricts to local orientations at points $x \in K$.

Step III: For an arbitrary compact K, we write K as a finite union $K = K_1 \cup K_2 \cup \ldots \cup K_r$ with each K_i as in Step I. Then the claim follows by induction on r by using Step II. □

Let us now get back to proving Lemma 4.2.10:

Proof. (of Lemma 4.2.10)
The proof is done in several steps, as indicated below.

Step I: Assume that $M = \mathbb{R}^n$ and K is a convex compact subset. Let B be a large ball in $\mathbb{R}^n$ with $K \subset B$, and let $S = \partial B$ be the bounding sphere. Then for all $x \in K$, both $M \setminus K$ and $M \setminus \{x\}$ deformation retract to S. So we have:

$$\begin{aligned} H_i(M, M \setminus K) &\cong H_i(M, M \setminus \{x\}) \\ &\cong H_i(\mathbb{R}^n, S^{n-1}) \\ &\cong \widetilde{H}_{i-1}(S^{n-1}) \\ &= \begin{cases} \mathbb{Z} & \text{for } i = n \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.2.8}$$

Step II: We next show that if the Lemma holds for compact sets K_1, K_2 and for their intersection $K_1 \cap K_2$, then it holds for $K := K_1 \cup K_2$. Indeed, we have the Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to H_{i+1}(M, M \setminus (K_1 \cap K_2)) \to H_i(M, M \setminus K) \xrightarrow{\varphi} H_i(M, M \setminus K_1) \oplus H_i(M, M \setminus K_2) \\ \xrightarrow{\psi} H_i(M, M \setminus (K_1 \cap K_2)) \to \cdots \end{aligned} \tag{4.2.9}$$

If $i > n$, we have by our assumption that $H_{i+1}(M, M \setminus (K_1 \cap K_2)) = 0$, $H_i(M, M \setminus K_1) = 0$ and $H_i(M, M \setminus K_2) = 0$. Therefore, $H_i(M, M \setminus K) = 0$.

If $i = n$, the Mayer-Vietoris sequence takes the form

$$0 \to H_n(M, M \setminus K) \xrightarrow{\varphi} H_n(M, M \setminus K_1) \oplus H_n(M, M \setminus K_2) \xrightarrow{\psi} H_n(M, M \setminus (K_1 \cap K_2)) \to \dots \tag{4.2.10}$$

with φ injective. So for $a \in H_n(M, M \setminus K)$, we have the following sequence of equivalences:

$$\begin{aligned} a = 0 &\iff 0 = \varphi(a) = \rho_{K_1}(a) \oplus \rho_{K_2}(a) \\ &\iff \rho_{K_1}(a) = 0 \text{ and } \rho_{K_2}(a) = 0 \\ &\iff \rho_x \rho_{K_1}(a) = 0 \ \forall x \in K_1, \text{ and } \rho_y \rho_{K_2}(a) = 0 \ \forall y \in K_2 \\ &\text{(since, by assumption, the lemma holds for } K_1 \text{ and } K_2) \\ &\iff \rho_x(a) = 0, \ \forall x \in K_1 \cup K_2. \end{aligned} \tag{4.2.11}$$

Step III: If $M = \mathbb{R}^n$ and $K = K_1 \cup K_2 \cup \cdots \cup K_r$ with each K_i convex and compact (which also implies that $K_1 \cap K_2$ is convex and compact), then the lemma holds for K by Step I and Step II.

Step IV: Assume that $M = \mathbb{R}^n$ and K is an arbitrary compact subset in $\mathbb{R}^n$. Choose a compact neighborhood N of K in $\mathbb{R}^n$. Then for any $a \in H_i(M, M \setminus K)$ there exists $a' \in H_i(M, M \setminus N)$ such that $\rho_K(a') = a$. Indeed, if γ is a cycle representative of a, we have that $\gamma \in C_i(\mathbb{R}^n)$ and $\partial\gamma \in C_{i-1}(\mathbb{R}^n \setminus K)$. So $\partial\gamma \cap K = \emptyset$. Choose N small enough so that $\partial\gamma \cap N = \emptyset$. Next, we cover K by a union of closed balls B_i such that $B_i \subset N$ and $B_i \cap K \neq \emptyset$. Then ρ_K factors as

$$\begin{array}{ccc} H_i(\mathbb{R}^n, \mathbb{R}^n \setminus N) & \xrightarrow{\quad \rho_K \quad} & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K) \\ & \searrow^{\rho_{\cup_i B_i}} \qquad \nearrow^{\rho_K} & \\ & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \cup_i B_i) & \end{array}$$

If $i > n$, then $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \cup_i B_i) = 0$ by Step III. So for any $a \in H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K)$, we have that

$$a = \rho_K(a') = \rho_K(\rho_{\cup_i B_i}(a')) = 0.$$

If $i = n$, then $\rho_x(a) = 0$ for all $x \in K$ implies by a deformation retract argument that $\rho_x(a) = 0$ for all $x \in \cup_i B_i$. By using Step III, we then get that $\rho_{\cup_i B_i}(a') = 0$. Hence we have $a = \rho_K(\rho_{\cup_i B_i}(a')) = 0$.

Step V: If K is contained in some euclidean neighborhood in (arbitrary) M, we have by excision

$$H_i(M, M \setminus K) \cong H_i(\mathbb{R}^n, \mathbb{R}^n \setminus K). \tag{4.2.12}$$

So the Lemma holds for K by Step IV.

Step VI: Finally, note that any compact subset K of M can be written as a union $K = K_1 \cup K_2 \cup \ldots \cup K_r$ with each K_i as in Step V. Then the Lemma follows by using Step V, Step II and induction. □

Exercises

1. Show that every covering space of an orientable manifold is an orientable manifold.

2. Given a covering space action of a group G on an orientable manifold M by orientation-preserving homeomorphisms, show that M/G is also orientable.

3. For a map $f : M \to N$ between connected closed orientable n-manifolds with fundamental classes $[M]$ and $[N]$, the degree of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes. Show that for any connected closed orientable n-manifold M there is a degree 1 map $M \to S^n$.

4. Show that a p-sheeted covering space projection $M \to N$ has degree p, when M and N are connected closed orientable manifolds.

5. Given two disjoint connected n-manifolds M_1 and M_2, a connected n-manifold $M_1 \# M_2$, their *connected sum*, can be constructed by deleting the interiors of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. (Assume that each B_i embeds nicely in a larger ball in M_i.)

(a) Show that if M_1 and M_2 are closed then there are isomorphisms

$$H_i(M_1 \# M_2; \mathbb{Z}) \simeq H_i(M_1; \mathbb{Z}) \oplus H_i(M_2; \mathbb{Z}), \quad \text{for } 0 < i < n,$$

with one exception: If both M_1 and M_2 are non-orientable, then $H_{n-1}(M_1 \# M_2; \mathbb{Z})$ is obtained from $H_{n-1}(M_1; \mathbb{Z}) \oplus H_{n-1}(M_2; \mathbb{Z})$ by replacing one of the two $\mathbb{Z}_2$-summands by a $\mathbb{Z}$-summand.

(b) Show that $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n)$ if M_1 and M_2 are closed.

4.3 Cohomolgy with Compact Support

Let X be a topological space and define the *compactly supported i-cochains* on X by:

$$C_c^i(X) := \bigcup_{K \text{ compact in } X} C^i(X, X \setminus K) \subset C^i(X). \tag{4.3.1}$$

Equivalently,

$$C_c^i(X) = \{\varphi : C_i(X) \to \mathbb{Z} \mid \exists \text{ compact } K_\varphi \subset X \text{ s.t. } \varphi = 0 \text{ on chains in } X \setminus K_\varphi\}. \tag{4.3.2}$$

Define a coboundary operator by

$$\delta\varphi(\sigma) := \varphi(\partial\sigma),$$

and note that if $\varphi \in C_c^i(X)$ vanishes on chains in $X \setminus K_\varphi$ then $\delta\varphi$ is also zero on all chains in $X \setminus K_\varphi$, and so $\delta\varphi \in C_c^{i+1}(X)$. Therefore we get a cochain (sub)complex $(C_c^\bullet(X), \delta^\bullet)$.

Definition 4.3.1. *The i-th cohomology of X with compact support is defined by*

$$H_c^i(X) := H^i(C_c^\bullet(X)).$$

In what follows, we give an alternative characterization of the cohomology with compact support, which is more useful for calculations. We begin by recalling the notion of *direct limit of groups.*

Definition 4.3.2. *Let G_α be abelian groups indexed by some directed set I, i.e., I has a partial order $\leq$ and for any $\alpha, \beta \in I$, there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Suppose also that for each pair $\alpha \leq \beta$ there is a homomorphism $f_{\alpha\beta} : G_\alpha \to G_\beta$ such that $f_{\alpha\alpha} = id_{G_\alpha}$ and $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$. Consider the set*

$$\amalg_\alpha G_\alpha / \sim$$

where the equivalence relation $\sim$ is defined as: if $x \in G_\alpha, x' \in G_{\alpha'}$, then $x \sim x'$ if $f_{\alpha\gamma}(x) = f_{\alpha'\gamma}(x')$ with $\alpha, \alpha' \leq \gamma$. Any two equivalence classes $[x]$ and $[x']$ have representatives lying in the same G_γ, with $\alpha, \alpha' \leq \gamma$, so we can define

$$[x] + [x'] = [f_{\alpha\gamma}(x) + f_{\alpha'\gamma}(x')].$$

This is a well-defined binary operation, and it gives an abelian group structure on the set $\amalg_\alpha G_\alpha / \sim$. The direct limit of the groups G_α is then the group defined as:

$$\varinjlim_{\alpha \in I} G_\alpha := \amalg_\alpha G_\alpha / \sim . \tag{4.3.3}$$

Remark 4.3.3. If $J \subset I$ so that $\forall \alpha \in I, \exists \beta \in J$ with $\alpha \leq \beta$, then $\varinjlim_{\alpha \in I} G_\alpha = \varinjlim_{\beta \in J} G_\beta$. In particular, if $J = \{\beta\}$ (i.e, I contains a maximal element), then $\varinjlim_{\alpha \in I} G_\alpha = G_\beta$.

We can now prove the following result:

Proposition 4.3.4. *There is an isomorphism*

$$H^i_c(X) \cong \varinjlim_{K\in I} H^i(X, X\setminus K) \tag{4.3.4}$$

where $I := \{K \subset X|\ K \text{ compact}\}$.

Proof. First note that I is a directed set since it is partially ordered by inclusion, and the union of two compact sets is also compact. Moreover, if $K \subseteq L$ are compact subsets of X, then there is a homomorphism $f_{KL} : H^i(X, X \setminus K) \to H^i(X, X \setminus L)$ induced by inclusion. Hence the direct limit group $\varinjlim_{K\in I} H^i(X, X \setminus K)$ is well-defined.

Each element of $\varinjlim_{K\in I} H^i(X, X \setminus K)$ is represented by some cocycle $\varphi \in C^i(X, X \setminus K)$ for some compact subset K of X. Regarding φ as an i-cochain with compact support, its cohomology class yields an element $[\varphi] \in H^i_c(X)$. Moreover, such a cocycle $\varphi \in C^i(X, X\setminus K)$ is the zero element in $\varinjlim H^i(X, X \setminus K)$ iff $\varphi = \delta\psi$ for some $\psi \in C^i(X, X \setminus L)$ with $L \supset K$, and so $[\varphi] = 0$ in $H^i_c(X)$. □

Remark 4.3.5. If X is compact, then $H^i_c(X) = H^i(X)$, for all $i \geq 0$, since in this case there is a unique maximal compact set $K \subset X$, namely X itself.

Example 4.3.6. Let us compute the cohomology with compact support of $\mathbb{R}^n$. By the above proposition,

$$H^i_c(\mathbb{R}^n) = \varinjlim_{K} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K),$$

where the direct limit is over the directed set of compact subsets of $\mathbb{R}^n$. Note that it suffices to let K range over closed balls B_k of integer radius k centered at the origin since each compact $K \subset \mathbb{R}^n$ is contained in such a ball. So we have that

$$\varinjlim_{K} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K) = \varinjlim_{k\in\mathbb{Z}_{\geq 0}} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k)$$

Moreover, we have isomorphisms

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_{k+1})$$

induced by inclusion, since for all k:

$$H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \cong H^i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Altogether,

$$H^i_c(\mathbb{R}^n) \cong \varinjlim H^i(\mathbb{R}^n, \mathbb{R}^n \setminus K) = \varinjlim_{k\in\mathbb{Z}_{\geq 0}} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.3.7. It follows from the previous example that the cohomology with compact support $H_c^*(-)$ is *not* a homotopy invariant.

Remark 4.3.8. Let $\widehat{X} = X \cup \hat{x}$ be the one point compactification of X. Then

$$H_c^i(X) \cong H^i(\widehat{X}, \hat{x}) \cong \widetilde{H}^i(\widehat{X}). \tag{4.3.5}$$

For example, $H_c^i(\mathbb{R}^n) \cong \widetilde{H}^i(S^n)$. This follows from the following general fact. If U is an open subset of a topological space V, with closed complement $Z := V \setminus U$, then there exists a long exact sequence for the cohomology with compact support

$$\cdots \to H_c^i(U) \to H_c^i(V) \to H_c^i(Z) \to H_c^{i+1}(U) \to \cdots$$

If we apply this fact to the case $\widehat{X} = X \cup \hat{x}$, we get a long exact sequence

$$\cdots \to H_c^i(X) \to H_c^i(\widehat{X}) \to H_c^i(\hat{x}) \to \cdots$$

Since $\widehat{X}$ and $\hat{x}$ are compact, this yields that $H_c^i(X) \cong H^i(\widehat{X}, \hat{x}) \cong \widetilde{H}^i(\widehat{X})$, as claimed.

4.4 Cap Product and the Poincaré Duality Map

Definition 4.4.1. *We define the cap product operation*

$$C^i(X) \otimes C_n(X) \xrightarrow{\frown} C_{n-i}(X) \tag{4.4.1}$$

as follows: for $b \in C^i(X)$ and $\xi \in C_n(X)$, $b \frown \xi \in C_{n-i}(X)$ is defined by

$$a(b \frown \xi) := (a \smile b)\xi \tag{4.4.2}$$

where $a \in C^{n-i}(X)$.

Remark 4.4.2. It is not hard to see that if $\sigma : \Delta_n \to X$ is an n-simplex and $b \in C^i(X)$, then

$$b \frown \sigma = \underbrace{b(\sigma|_{[v_{n-i},\cdots,v_n]})}_{\in\mathbb{Z}} \cdot \underbrace{\sigma|_{[v_0,\cdots,v_{n-i}]}}_{\in C_{n-i}(X)}. \tag{4.4.3}$$

The reader is encouraged to show that these two notions of cap product are equivalent.

The following result is a direct consequence of the definition:

Lemma 4.4.3. *For any $b \in C^i(X)$ and $\xi \in C_n(X)$, we have:*

$$\partial(b \frown \xi) = \delta b \frown \xi + (-1)^i\, b \frown \partial\xi. \tag{4.4.4}$$

As a consequence, the cap product descends to (co)homology:

Corollary 4.4.4. *There is an induced cap product operation*

$$H^i(X) \otimes H_n(X) \xrightarrow{\frown} H_{n-i}(X). \tag{4.4.5}$$

Remark 4.4.5. A relative cap product

$$H^i(X, A) \otimes H_n(X, A) \xrightarrow{\frown} H_{n-i}(X) \tag{4.4.6}$$

can be defined as follows. First note that the restriction

$$C^i(X, A) \otimes C_n(X) \xrightarrow{\frown} C_{n-i}(X)$$

of absolute cap product (4.4.1) vanishes on $C^i(X, A) \otimes C_n(A)$, so it induces:

$$C^i(X, A) \otimes C_n(X, A) \xrightarrow{\frown} C_{n-i}(X).$$

Since (4.4.4) still holds in this relative setting, we get a relative cap product operation:

$$H^i(X, A) \otimes H_n(X, A) \xrightarrow{\frown} H_{n-i}(X).$$

The following result states that the cap product $\frown$ is functorial. Its proof is a direct consequence of the definition of cap products and is left as an exercise:

Lemma 4.4.6. *If $f : X \to Y$ is a continuous map, then*

$$\varphi \frown f_*\xi = f_*((f^*\varphi) \frown \xi) \tag{4.4.7}$$

for all $\varphi \in H^i(Y)$ and $\xi \in H_n(X)$. This fact is illustrated in the following diagram:

$$\begin{array}{ccccc} H^i(X) & \otimes & H_n(X) & \xrightarrow{\frown} & H_{n-i}(X) \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ H^i(Y) & \otimes & H_n(Y) & \xrightarrow{\frown} & H_{n-i}(Y) \end{array}$$

Let us next move towards the definition of the Poincaré duality map. Let M be a n-dimensional orientable connected manifold (not necessarily compact), and let $K \subset L \subset M$ where K, L are compact subsets. Consider the diagram:

$$\begin{array}{ccccc} H^i(M, M \setminus L) & \otimes & H_n(M, M \setminus L) & \xrightarrow{\frown} & H_{n-i}(M) \\ {\scriptstyle i^*}\uparrow & & {\scriptstyle i_*}\downarrow & & \Vert \\ H^i(M, M \setminus K) & \otimes & H_n(M, M \setminus K) & \xrightarrow{\frown} & H_{n-i}(M) \end{array}$$

By the functoriality of the cap product, we have for any $\varphi \in H^i(M, M \setminus K)$ that:

$$(i^*\varphi) \frown \mu_L = \varphi \frown i_*(\mu_L), \tag{4.4.8}$$

where μ_K and μ_L denote the orientation classes of Theorem 4.2.7. Moreover, the following identification holds:

Lemma 4.4.7. *For compact subsets $K \subset L$ of M, we have:*

$$i_*(\mu_L) = \mu_K. \tag{4.4.9}$$

Proof. The claim follows from the commutativity of the following diagram and the uniqueness of μ_K in $H_n(M, M \setminus K)$ which restricts to local orientations μ_x, $\forall x \in K$.

$$\begin{array}{ccc} \mu_K \in H_n(M, M \setminus K) & \longrightarrow & H_n(M, M \setminus x) \\ {\scriptstyle i_*}\uparrow & \nearrow & \\ \mu_L \in H_n(M, M \setminus L) & & \end{array}$$

□

Therefore, we have from (4.4.8) and (4.4.9) that:

$$(i^*\varphi) \frown \mu_L = \varphi \frown i_*(\mu_L) = \varphi \frown \mu_K, \tag{4.4.10}$$

for all $\varphi \in H^i(M, M \setminus K)$. Let us now recall from Proposition 4.3.4 that we have an isomorphism:

$$H^i_c(M) \cong \varinjlim_K H^i(M, M \setminus K), \tag{4.4.11}$$

where the direct limit on the right-hand side is taken over all compact subsets K of M. We can now define the *Poincaré duality map*

$$H_c^i(M) \xrightarrow{\frown} H_{n-i}(M) \tag{4.4.12}$$

as follows: its value on $\varphi \in H_c^i(M)$ is defined as $\varphi_K \frown \mu_K$, where $\varphi_K \in H^i(M, M \setminus K)$ is a representative of φ and $\mu_K \in H_n(M, M \setminus K)$ is the orientation class defined by K (cf. Theorem 4.2.7). Note that the Poincaré duality map (4.4.12) is well-defined (i.e., independent of the choice of the representative φ_K) by the commutativity of the following diagram (which follows from the identity (4.4.10)):

$$\begin{array}{ccc} H^i(M, M \setminus K) & \xrightarrow{\quad i^* \quad} & H^i(M, M \setminus L) \\ & {\scriptstyle \frown \mu_K} \searrow \quad \swarrow {\scriptstyle \frown \mu_L} & \\ & H_{n-i}(M) & \end{array}$$

We have now all the necessary ingredients to formulate the main theorem of this chapter:

Theorem 4.4.8. *(Poincaré Duality)*
If M is an n-dimensional oriented connected manifold, then the Poincaré duality map:

$$H_c^i(M) \xrightarrow{\frown} H_{n-i}(M)$$

is an isomorphism for all i.

An an immediate corollary, we get the following:

Corollary 4.4.9. *If M is an n-dimensional closed oriented connected manifold, then the map*

$$H^i(M) \xrightarrow{\frown} H_{n-i}(M)$$

defined by the cap product with the fundamental class of M, that is, $\varphi \mapsto \varphi \cap [M]$, is an isomorphism for all i.

4.5 The Poincaré Duality Theorem

This section is devoted to proving The Poincaré Duality Theorem, which we recall below for the convenience of the reader.

Theorem 4.5.1. *(Poincaré Duality)*
If M is an n-dimensional oriented connected manifold, then the Poincaré duality map:

$$H_c^i(M) \xrightarrow{\frown} H_{n-i}(M)$$

is an isomorphism for all i.

Proof. Recall that on an element $\varphi \in H_c^i(M) \cong \varinjlim_{\substack{K\subset X \\ K compact}} H^i(M, M \setminus K)$, the Poincaré duality map takes the value $\varphi_K \frown \mu_K$, where $\varphi_K \in H^i(M, M \setminus K)$ is a representative of φ, and μ_K is the orientation class of $H_n(M, M \setminus K)$.

The proof of the theorem will be divided into several steps. We first show that the statement holds locally, then we glue the local isomorphisms by a Mayer-Vietoris argument.

Step I: We first show that the theorem holds for $M = \mathbb{R}^n$.
Let B_k denote the closed ball of integer radius k in $\mathbb{R}^n$. Then

$$H_c^i(\mathbb{R}^n) \cong \varinjlim_{B_k} H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_{n-i}(\mathbb{R}^n) \simeq \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

The Universal Coefficient Theorem yields that

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \simeq \mathrm{Hom}(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k); \mathbb{Z}).$$

So $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k)$ is generated by some class a_k so that $a_k(\mu_{B_k}) = 1 \in \mathbb{Z}$. Let $1 \in H^0(\mathbb{R}^n) = \mathbb{Z}$ be the generator. Then:

$$1 = a_k(\mu_{B_k}) = (1 \smile a_k)(\mu_{B_k}) = 1(a_k \frown \mu_{B_k})$$

Hence $a_k \frown \mu_{B_k}$ is a generator of $H_0(R^n)$. In particular, the map

$$\frown \mu_{B_k} : H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) \to H_0(\mathbb{R}^n)$$

is an isomorphism. Taking the direct limit over the B_k's, we get an isomorphism

$$H_c^n(\mathbb{R}^n) \xrightarrow{\cong} H_0(\mathbb{R}^n),$$

which by the above considerations coincides with the Poincaré duality map. Also, both groups are trivial for $i \neq n$, so the claim follows.

Step II: Assuming the theorem holds for opens $U, V \subset M$ and for their intersection $U \cap V$, we show that it holds for the union $U \cup V$.
For this purpose, we construct a commutative diagram

$$\begin{array}{ccccccccc} \cdots \to & H^i_c(U \cap V) & \to & H^i_c(U) \oplus H^i_c(V) & \to & H^i_c(U \cup V) & \to & H^{i+1}_c(U \cap V) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H_{n-i}(U \cap V) & \to & H_{n-i}(U) \oplus H_{n-i}(V) & \to & H_{n-i}(U \cup V) & \to & H_{n-i-1}(U \cap V) & \to \cdots \end{array} \tag{4.5.1}$$

Once the diagram is constructed, the claim follows by the 5-lemma.
The bottom row in (4.5.1) is just the Mayer-Vietoris homology sequence. The top row of the above diagram can be constructed as follows. For compact subsets $K \subset U$ and $L \subset V$, consider the cohomology Mayer-Vietoris sequence for the pairs $(M, M \setminus K)$ and $(M, M \setminus L)$:

$$\cdots \to H^i(M, M \setminus (K \cap L)) \to H^i(M, M \setminus K) \oplus H^i(M, M \setminus L) \to H^i(M, M \setminus (K \cup L)) \to \cdots$$

By excision, we get a long exact sequence:

$$\cdots \to H^i(U \cap V, U \cap V \setminus K \cap L) \to H^i(U, U \setminus K) \oplus H^i(V, V \setminus L) \to H^i(U \cup V, U \cup V \setminus K \cup L) \to \cdots$$

Taking direct limits over $K \subset U$ and $L \subset V$, we get the top long exact sequence in (4.5.1):

$$\cdots \to H^i_c(U \cap V) \to H^i_c(U) \oplus H^i_c(V) \to H^i_c(U \cup V) \to \cdots$$

The commutativity follows by using the definition of the Poincaré duality map.

Step III: Assume M is a union of nested open subsets U_α so that the theorem holds for each U_α. We show that the theorem holds for M.
First note that any compact subset in M (in particular, the support of a singular (co)chain) is contained in some U_α. Then we claim that the following identifications hold:

$$H_i(M) = \varinjlim_{\alpha} H_i(U_\alpha) \tag{4.5.2}$$

and

$$H^i_c(M) = \varinjlim_{\alpha} H^i_c(U_\alpha). \tag{4.5.3}$$

This claim and Poincaré duality for each U_α imply the Poincaré duality isomorphism for M, since the direct limit of isomorphisms is an isomorphism. In order to prove the claim, we note that the inclusions $i_\alpha : U_\alpha \hookrightarrow M$ induce homomorphisms $i_{\alpha*} : H_i(U_\alpha) \to H_i(M)$ so that for $U_\alpha \hookrightarrow U_\beta$ the following diagram commutes:

$$\begin{array}{ccc} H_i(U_\alpha) & \longrightarrow & H_i(U_\beta) \\ & \searrow \quad \swarrow & \\ & H_i(M) & \end{array}$$

We therefore get a well-defined map

$$f : \varinjlim_{\alpha} H_i(U_\alpha) \to H_i(M).$$

We next show that f is an isomorphism.

- f is onto: any $[\xi] \in H_i(M)$ is represented by a cycle whose support is contained in a compact subset of M, thus in some U_α. The corresponding homology class in $H_i(U_\alpha)$ maps onto $[\xi]$.

- f is one-to-one: if $\xi = \partial\eta$, for $\eta \in C_{i+1}(M)$, then ξ is a cycle in some U_α, but not necessarily a boundary in U_α. On the other hand, η is contained in some larger U_β, so ξ can be regarded as a boundary in U_β. Therefore, $[\xi] = 0 \in H_i(U_\beta)$, hence it represents the zero class in $\varinjlim_{\alpha} H_i(U_\alpha)$.

So (4.5.2) follows. The identification in (4.5.3) is obtained similarly.

Step IV: We next show that the theorem holds when M is an open subset of $\mathbb{R}^n$.
If M is convex, then M is homeomorphic to $\mathbb{R}^n$, so the theorem holds by Step I. If M is not convex, then $M = \bigcup_{k \in \mathbb{Z}_{>0}} V_k$, with each V_k open and convex in $\mathbb{R}^n$. By induction and Step II, the theorem holds for the sets $U_k = V_1 \cup \cdots \cup V_k$. Note that $\{U_k\}_k$ forms a nested cover of opens for M, hence the theorem follows by Step III.

Step V: Finally, we show that the Poincaré duality isomorphism holds for an arbitrary M.
We first cover M by open sets V_α, each of which is homeomorphic to $\mathbb{R}^n$. We next choose a well ordering $<$ of the index set, which exists by Zorn's lemma (if M has a countable basis, the we can choose the positive integer as index set). Then the sets

$$U_\alpha := \cup_{\beta < \alpha} V_\beta.$$

form a nested open cover of M. So by Step III, it suffices to show that the theorem holds for each U_α. But $U_\alpha = \cup_{\beta<\alpha} V_\beta$, with $V_\beta \cong \mathbb{R}^n$ for each β, and the theorem holds for each V_β. By Step II and transfinite induction, the theorem holds for each U_α, and the claim follows. □

Remark 4.5.2. By taking coefficients in any commutative ring R, we can prove the Poincaré duality isomorphism over R via the coefficient map $\mathbb{Z} \to R$. Moreover, for $R = \mathbb{Z}/2$, Poincaré duality holds even without the orientability assumption.

Exercises

1. Show that if M^n is connected, non-compact manifold, then $H_i(M;\mathbb{Z}) = 0$ for $i \geq n$.

2. Show that the Euler characteristic of a closed, oriented, $(4n+2)$-dimensional manifold is even.

3. Let M be a closed oriented manifold with fundamental class $[M]$. Consider the following *cup product pairing* between cohomology groups of complementary dimensions (after moding out by the corresponding torsion subgroups):

$$(\ ,\ ) : H^i(M;\mathbb{Z})/\text{Torsion} \otimes H^{n-i}(M;\mathbb{Z})/\text{Torsion} \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$. Here $\langle\ ,\ \rangle : H^n(X;\mathbb{Z}) \otimes H_n(X;\mathbb{Z}) \to \mathbb{Z}$ is the Kronecker pairing defined in Homework #1.

(i) Show that the cup product pairing is *nonsingular* in the following sense: for each choice of a $\mathbb{Z}$-basis $\{\beta_1, \cdots, \beta_r\}$ of $H^{n-i}(M;\mathbb{Z})/\text{Torsion}$, there exists a $\mathbb{Z}$-basis $\{\alpha_1, \cdots, \alpha_r\}$ of $H^i(M;\mathbb{Z})/\text{Torsion}$ such that $(\alpha_i, \beta_j) = \delta_{ij}$. (Hint: Use the Universal Coefficient Theorem and Poincaré Duality.)

(ii) As an application, re-prove the following facts about the ring structures on the cohomology of projective spaces:

(a) $H^*(\mathbb{RP}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^{n+1}), \quad |x| = 1,$

(b) $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[y]/(y^{n+1}), \quad |y| = 2,$

(c) $H^*(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}[w]/(w^{n+1}), \quad |w| = 4.$

4. Let M be a closed, oriented $4n$-dimensional manifold, with fundamental class $[M]$. The middle *intersection pairing*

$$(\ ,\ ) : H^{2n}(M;\mathbb{Z})/\text{Torsion} \otimes H^{2n}(M;\mathbb{Z})/\text{Torsion} \to \mathbb{Z}$$

given by $(\alpha, \beta) = \langle \alpha \cup \beta, [M] \rangle$ is symmetric and nondegenerate. Let $\{\alpha_1, \cdots, \alpha_r\}$ be a $\mathbb{Z}$-basis of $H^{2n}(M;\mathbb{Z})/\text{Torsion}$, and let $A = (a_{ij})$ for $a_{ij} := (\alpha_i, \alpha_j) \in \mathbb{Z}$. Then A is a symmetric matrix with $\det(A) = \pm 1$, so it is diagonalizable over $\mathbb{R}$. Define the *signature* of M to be

$$\sigma(M) := (\text{the number of positive eigenvalues}) - (\text{the number of negative eigenvalues})$$

(a) Compute $\sigma(\mathbb{CP}^n)$, $\sigma(S^2 \times S^2)$.

(b) Show that the signature $\sigma(M)$ is congruent mod 2 to the Euler characteristic $\chi(M)$.

5. Show that if a connected manifold M is the boundary of a compact manifold, then the Euler characteristic of M is even. Conclude that $\mathbb{RP}^{2n}$, $\mathbb{CP}^{2n}$, $\mathbb{HP}^{2n}$ cannot be boundaries.

6. Show that if M^{4n} is a connected manifold which is the boundary of a compact oriented $(4n+1)$-dimensional manifold V, then the signature of M is zero.

7. Show that if M is a compact contractible n-manifold then ∂M is a homology $(n-1)$-sphere, that is, $H_i(\partial M;\mathbb{Z}) \simeq H_i(S^{n-1};\mathbb{Z})$ for all i.

8. Let M be a closed, connected, orientable 4-manifold with fundamental group $\pi_1(M) \cong \mathbb{Z}/3 * \mathbb{Z}/3$ and Euler characteristic $\chi(M) = 5$.

(a) Compute $H_i(M, \mathbb{Z})$ for all i.

(b) Prove that M is not homotopy equivalent to any CW complex with no 3-cells.

9. Let M be a closed, connected, oriented n-manifold and let $f : S^n \to M$ be a continuous map of non-zero degree, i.e., the morphism

$$f_* : H_n(S^n;\mathbb{Z}) \to H_n(M;\mathbb{Z})$$

is non-trivial. Show that M and S^n have the same $\mathbb{Q}$-homology.

10. Show that there is no orientation-reversing self-homotopy equivalence $\mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$.

4.6 Immediate applications of Poincaré Duality

In this section we derive several applications of the Poincaré duality isomorphism of Theorem 4.5.1. (In particular, we provide answers to some of the exercises listed in the previous section.)

Proposition 4.6.1. *If M^n is a closed odd dimensional manifold, then $\chi(M) = 0$.*

Proof. Let $n = 2k+1$.

If M is oriented, then (with $\mathbb{Z}$-coefficients):

$$\text{rk}H_i(M) \overset{(P.D.)}{=} \text{rk}H^{n-i}(M) \overset{(UCT)}{=} \text{rk}H_{n-i}(M).$$

So:

$$\chi(M) = \sum_{i=0}^{2k+1}(-1)^i \cdot \text{rk}H_i(M) = \sum_{i=0}^{k}\left((-1)^i + (-1)^{n-i}\right) \cdot \text{rk}H_i(M) = 0.$$

If M is non orientable, the Poincaré duality isomorphism holds with $\mathbb{Z}/2$-coefficients, and we get:

$$\chi(M) := \sum_{n=0}^{2k+1}(-1)^n \cdot \text{rk}H_i(M;\mathbb{Z}) \overset{(*)}{=} \sum_{n=0}^{2k+1}(-1)^i \cdot \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2) = 0,$$

where the vanishing follows as before by Poincaré duality (over $\mathbb{Z}/2$). The equality $(*)$ follows from the Universal Coefficient Theorem as follows:

$$H^i(M,\mathbb{Z}/2) = \text{Hom}(H_i(M),\mathbb{Z}/2) \oplus \text{Ext}(H_{i-1}(M),\mathbb{Z}/2).$$

- a $\mathbb{Z}$-summand of $H_i(M;\mathbb{Z})$ contributes
  - $\text{Hom}(\mathbb{Z},\mathbb{Z}/2) = \mathbb{Z}/2$ to $H^i(M;\mathbb{Z}/2)$, and
  - $\text{Ext}(\mathbb{Z},\mathbb{Z}/2) = 0$ to $H^{i+1}(M;\mathbb{Z}/2)$.
- a $\mathbb{Z}/m$ summand of $H_i(M;\mathbb{Z})$, with m odd, contributes:
  - $\text{Hom}(\mathbb{Z}/m,\mathbb{Z}/2) = 0$ to $H^i(M;\mathbb{Z}/2)$, and
  - $\text{Ext}(\mathbb{Z}/m,\mathbb{Z}/2) = 0$ to $H^{i+1}(M;\mathbb{Z}/2)$.
- a $\mathbb{Z}/m$ summand of $H_i(M;\mathbb{Z})$, with m even, contributes:
  - $\text{Hom}(\mathbb{Z}/m,\mathbb{Z}/2) = \mathbb{Z}/2$ to $H^i(M;\mathbb{Z}/2)$, and
  - $\text{Ext}(\mathbb{Z}/m,\mathbb{Z}/2) = \mathbb{Z}/2$ to $H^{i+1}(M;\mathbb{Z}/2)$, so these $\mathbb{Z}/2$ contributions cancel out in $\sum_i(-1)^i \cdot \dim_{\mathbb{Z}/2} H^i(M;\mathbb{Z}/2)$.

Finally, note that $\dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^i(M;\mathbb{Z}/2)$, so the claim follows. ∎

Proposition 4.6.2. *If M^n is a closed, oriented, connected manifold, then*

$$\text{Torsion}(H_{n-1}(M)) = 0.$$

Proof. Indeed,

$$\text{Torsion}(H_i(M)) \overset{(P.D.)}{=} \text{Torsion}(H^{n-i}(M)) \overset{(UCT)}{=} \text{Ext}(H_{n-1-i}(M), \mathbb{Z}) = \text{Torsion}(H_{n-1-i}(M))$$

Since M is connected, $H_0(M)$ is free, so the claim follows. □

We will show later the following:

Proposition 4.6.3. *If M^n is a closed, connected, non-orientable manifold, then*

$$\text{Torsion}(H_{n-1}(M)) = \mathbb{Z}/2$$

and

$$H^n(M) = \mathbb{Z}/2.$$

The second part of Proposition 4.6.3 follows from the Universal Coefficient Theorem and the following consequence of Poincaré duality (to be proved in the next section):

Lemma 4.6.4. *If M^n is an n-dimensional closed, connected manifold, then*

$$H_n(M) = \begin{cases} \mathbb{Z} & , \text{ if } M\text{-oriented} \\ 0 & , \text{ if } M\text{-non-oriented.} \end{cases}$$

4.7 Addendum to orientations of manifolds

Before we explain the proof of Proposition 4.6.3, we need to elaborate on orientations of manifolds.

Recall that if M^n is a n-manifold, a local orientation at $x \in M$ is a generator $\mu_x \in H_n(M, M \setminus x) \cong \mathbb{Z}$. We say that M is *oriented* if there exists a global orientation, i.e., a continuous choice $x \to \mu_x$ of local orientations. This means that for all $x \in M$, there is a closed euclidian ball B (of finite positive radius) around x so that

$$\mathbb{Z} \cong H_n(M, M \setminus B) \overset{\rho_x}{\to} H_n(M, M \setminus y)$$

sends the generator μ_{B_x} to the local orientation class μ_y, for all $y \in B_x$.

Proposition 4.7.1. *Any manifold M (oriented or not) has an oriented double cover $\widetilde{M}$.*

Proof. (Sketch)
Define

$$\widetilde{M} := \{\mu_x | \ x \in M, \mu_x \text{ a local orientation of } M \text{ at } x\}$$

and $\pi : \widetilde{M} \to M$ by $\mu_x \to x$. Clearly, π is a $2:1$ map.

We need to put a topology on $\widetilde{M}$ so that it becomes a manifold and π is a covering map. For an open ball $B \subset \mathbb{R}^n \subset M$ of finite radius, with a generator $\mu_B \in H_n(M, M \setminus B)$, define

$$U(\mu_B) = \{\mu_x \in \widetilde{M} | \ x \in B, \ \mu_x = \rho_x(\mu_B)\},$$

where ρ_x denotes the natural map $H_n(M, M \setminus B) \to H_n(M, M \setminus x)$. Then

$$\pi^{-1}(B) = U(\mu_B) \sqcup U(-\mu_B)$$

and both $U(\mu_B)$ and $U(-\mu_B)$ are in bijection to B. Moreover, it can be shown that the sets $\{U(\mu_B)\}_B$ form basis of opens for the topology of $\widetilde{M}$ so that π is continuous. So π is 2-fold covering and $\widetilde{M}$ is manifold.

Moreover, $\widetilde{M}$ is orientable. Indeed, we have,

$$H_n(\widetilde{M}, \widetilde{M} \setminus \mu_x) \cong H_n(U(\mu_B), U(\mu_B) \setminus \mu_x) \cong H_n(B, B \setminus x) \cong H_n(M, M \setminus x). \tag{4.7.1}$$

So at the point $\mu_x \in \widetilde{M}$ there exists a canonical local orientation

$$\widetilde{\mu}_x \in H_n(\widetilde{M}, \widetilde{M} \setminus \mu_x) \cong \mathbb{Z}$$

corresponding to μ_x under the above isomorphism (4.7.1). The consistency of such local orientations follows by construction. $\square$

Example 4.7.2. (a) The oriented double cover of $M = \mathbb{RP}^2$ is $\widetilde{M} = S^2$.

(b) The oriented double cover of the Klein bottle K is the 2-torus T^2.

Proposition 4.7.3. *If M is a connected manifold, then M is orientable if, and only if, $\widetilde{M}$ has two components. In particular, if $\pi_1(M) = 0$ or has no index 2 subgroup, then M is orientable.*

Proof. The oriented double cover $\widetilde{M}$ can have one or two components. If $\widetilde{M}$ has two components, each is oriented and homeomorphic to M, so M is orientable. Conversely, if M is orientable, it can have exactly two orientations at each point, each defining a sheet of $\widetilde{M}$. □

Example 4.7.4. $\mathbb{CP}^n$ is orientable.

The oriented double cover $\widetilde{M}$ can be embedded in a larger covering space $M_{\mathbb{Z}}$ of M as follows. Let

$$M_{\mathbb{Z}} = \{\alpha_x \mid \ x \in M,\ \alpha_x \in H_n(M, M \setminus x) = \mathbb{Z}\}.$$

We then have the $\mathbb{Z}$-fold projection map

$$\pi_{\mathbb{Z}} : M_{\mathbb{Z}} \to M$$

defined by $\alpha_x \to x$. A basis of opens $\{U(B)\}$ for $M_{\mathbb{Z}}$ can be defined by the following recipe: for an open ball $B \subset \mathbb{R}^n \subset M$, set

$$U(B) = \{\alpha_x \mid x \in B, \alpha_x = \rho_x(\alpha_B) \text{ for } \alpha_B \in H_n(M, M \setminus B) \cong \mathbb{Z})\}$$

with $\rho_x : H_n(M, M \setminus B) \xrightarrow{\cong} H_n(M, M \setminus x)$ induced by inclusion as before. For any $k \in \mathbb{Z}$, we then get a subcover $M_k \subset M_{\mathbb{Z}}$ by selecting $\pm k\mu_x$ in the fibre above x. So

$$M_{\mathbb{Z}} = \bigcup_{k \geq 0} M_k$$

with $M_0 \cong M, M_k \cong M_{-k}$, and $M_k \cong \widetilde{M}$, for any integer k.

Definition 4.7.5. *A section of $\pi_{\mathbb{Z}} : M_{\mathbb{Z}} \to M$ is a continuous map $\alpha : M \to M_{\mathbb{Z}}$ defined by $x \mapsto \alpha_x \in H_n(M, M \setminus x) = \mathbb{Z}$. An orientation of M is a section of $\pi_{\mathbb{Z}}$ assigning μ_x to each $x \in M$.*

One can generalize the definition of orientability by replacing $\mathbb{Z}$ any commutative ring R with unit. Note that by the universal coefficient theorem for homology, we have:

$$H_n(M, M \setminus x; R) \cong H_n(M, M \setminus x) \otimes R \cong \mathbb{Z} \otimes R \cong R.$$

The covering $M_{\mathbb{Z}}$ can be generalized to:

$$M_R = \{\alpha_x \mid \ x \in M, \alpha_x \in H_n(M, M \setminus x; R) \cong R\}.$$

The corresponding covering map $\pi_R : M_R \to M$ is defined by $\alpha_x \mapsto x$ (so the fibre over $x \in M$ is R). Each $r \in R$ determines a subcovering M_r by selecting the points $\pm\mu_x \otimes r \in$

$H_n(M, M \setminus x; R)$ in each fibre. If r is an element of order 2 in R, then M_r is a copy of M. (Indeed, $\pm\mu_x \otimes r = \mu_x \otimes \pm r = \mu_x \otimes r$.) Otherwise, M_r is homeomorphic to the oriented double cover $\widetilde{M}$. We have

$$M_R = \bigcup_{r \in R} M_r,$$

with all M_r being disjoint except for $M_r = M_{-r}$, and $M_r = M$ if $2r = 0$.

Definition 4.7.6. *An R-orientation of an n-dimensional manifold M is a section of M_R assigning to each $x \in M$ a generator u of $H_n(M, M \setminus x; R) \cong R$.*

Remark 4.7.7. Note that a generator of R is an element u so that $Ru = R$. Since R has a unit, this is equivalent to saying that u is invertible in R.

Remark 4.7.8. An orientable manifold is R-orientable, for all commutative rings R with unit. A non-orientable manifold is R-orientable iff R contains a unit of order 2. Thus every manifold is $\mathbb{Z}/2$-orientable.

We are now ready to prove the following result, which shows that orientability of a closed manifold is reflected in the structure of its homology:

Theorem 4.7.9. *Let M be a closed connected n-manifold. Then:*

(a) if M is (R-)orientable, then $H_n(M; R) \to H_n(M, M \setminus x; R) \cong R$ is an isomorphism for any $x \in M$.

(b) if M is not orientable, then $H_n(M; R) \to H_n(M, M \setminus x; R) \cong R$ is one-to-one, with image the group generated by the set of elements of order 2 in R.

(c) $H_i(M; R) = 0$, for all $i > n$.

The proof of Theorem 4.7.9 is based on the Theorem 4.2.7 and Lemma 4.2.10 (which we formulate here with R-coefficients in parts (a) and (b) below), together with a slight generalization of Theorem 4.2.7 (see part (c) below) which holds without the orientability assumption:

Lemma 4.7.10. *Let M be a connected n-manifold and K a compact subset of M. Then:*

(a) if M is R-oriented, then there exists a unique $\mu_K \in H_n(M, M \setminus K; R)$ such that $\rho_x(\mu_K) = \mu_x \in H_n(M, M \setminus x; R)$, for all $x \in K$.

(b) $H_i(M, M \setminus K; R) = 0$ for $i > n$, and a class $\alpha_K \in H_n(M, M \setminus K; R)$ is zero iff $\rho_x(\alpha_K) = 0$ for any $x \in K$.

(c) if $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique class $\alpha_K \in H_n(M, M \setminus K; R)$ so that $\rho_x(\alpha_K) = \alpha_x \in H_n(M, M \setminus x; R)$, for all $x \in K$.

Note that the proof of part (c) of the above lemma is almost identical to that of Theorem 4.2.7 (with the uniqueness following from part (b)), with the only easy modification appearing in Step I of loc.cit. (where the orientation assumption used in the proof of Theorem 4.2.7 is replaced by the continuity of the section). We leave the details to the reader.

To deduce parts (a) and (b) of Theorem 4.7.9, choose $K = M$ in the above lemma, and let $\Gamma_R(M)$ be the set of sections of the covering map $M_R \to M$. With respect to the addition of functions and multiplication by scalars in R, $\Gamma_R(M)$ becomes an R-module. Moreover, there exists a homomorphism

$$H_n(M; R) \longrightarrow \Gamma_R(M)$$

defined by

$$\alpha \to (x \mapsto \alpha_x),$$

where α_x is the image of α under the map $\rho_x : H_n(M; R) \to H_n(M, M \setminus \{x\}; R)$. The above lemma asserts that this is in fact an isomorphism.
Let us now translate the statements about $H_n(M; R)$ in Theorem 4.7.9 into statements about the R-module $\Gamma_R(M)$:

1. For the oriented case: $H_n(M; R) \cong \Gamma_R(M) \to H_n(M, M \setminus x; R)$ is an isomorphism, defined by $\alpha \mapsto (x \mapsto \alpha_x) \mapsto \alpha_x$ for a given x.

2. For the non-oriented case: $H_n(M; R) \cong \Gamma_R(M) \to H_n(M, M \setminus x; R)$ is a monomorphism, with image the group generated by the elements of order 2 in R.

Note that since M is connected, each section in $\Gamma_R(M)$ is determined by its value at one point $x \in M$. The injectivity statements in part (a) and (b) of Theorem 4.7.9 follow from Lemma 4.7.10(b). Also, the surjectivity in part (a), as reformulated in part 1 above, follows from Lemma 4.7.10(a). The remaining statement in part 2 above can be seen as follows. Since π_R is a covering map, the section group $\Gamma_R(M)$ can be identified with the connected components of M_R which map homeomorphically via π_R to M. Since M is non-orientable, the oriented double cover $\pi : \widetilde{M} \to M$ is non-trivial (i.e., connected), thus the components of M_R are of the form $r(\widetilde{M})$, with $r : \widetilde{M} \to M_R$ the continuous map defined by $\mu \mapsto \mu \otimes r$. The only points in $r(\widetilde{M})$ which under π_R map to $x \in M$ are $\mu_x \otimes r$ and $-\mu_x \otimes r = \mu_x \otimes (-r)$. Thus, $\pi_R|_{r\widetilde{M}}$ is a homeomorphism iff $r = -r$, or $2r = 0$. □

Corollary 4.7.11. *If M is orientable, then $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$. If M is non-orientable, then $H_n(M; \mathbb{Z}) = 0$. In either case, $H_n(M; \mathbb{Z}/2) = \mathbb{Z}/2$.*

We can now prove the following:

Corollary 4.7.12. *Let M be a closed and connected n-manifold. If M is oriented, then*

$$\text{Torsion}(H_{n-1}(M)) = 0.$$

Otherwise,

$$\text{Torsion}(H_{n-1}(M)) = \mathbb{Z}/2.$$

Proof. By the universal coefficient theorem for homology, and using the fact that the homology groups of a closed manifold are finitely generated (e.g., see Cor.A.8 and A.9 in Hatcher's book), we have:

$$\begin{aligned} H_n(M;\mathbb{Z}/p) &= H_n(M;\mathbb{Z}) \otimes \mathbb{Z}/p \oplus \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{Z}/p) \\ &= H_n(M;\mathbb{Z}) \otimes \mathbb{Z}/p \oplus \mathrm{Torsion}(H_{n-1}(M;\mathbb{Z})) \otimes \mathbb{Z}/p. \end{aligned}$$

In the orientable case, if $H_{n-1}(M)$ contained torsion, then for some prime p, the group $H_n(M;\mathbb{Z}/p) = \mathbb{Z}/p$ would be larger than the $\mathbb{Z}/p$ coming from the first summand (here we use that $H_n(M) = \mathbb{Z}$), which is impossible. This means $\mathrm{Torsion}(H_{n-1}(M)) = 0$.
In the non-orientable case, we have by Theorem 4.7.9 that $H_n(M;\mathbb{Z}/m)$ is either $\mathbb{Z}/2$ or 0, depending on whether m is even or odd. (Indeed, in this case the map $H_n(M;\mathbb{Z}/m) \to \mathbb{Z}/m$ is injective with image the elements of order 2 in $\mathbb{Z}/m$. So, if m is odd, there are no elements of order 2 in $\mathbb{Z}/m$, while if $m = 2k$ is even, then k is the only element of order 2 in $\mathbb{Z}/m$.) Since in this case we have $H_n(M;\mathbb{Z}) = 0$, this forces the torsion subgroup of $H_{n-1}(M)$ to be $\mathbb{Z}/2$. □

Remark 4.7.13. By using the universal coefficient theorem for the cohomology of a closed n-manifold, we have:

$$H^n(M) = \mathrm{Free}(H_n(M)) \oplus \mathrm{Torsion}(H_{n-1}(M)).$$

So by using the result of and the previous corollary, we get that if M is oriented then $H^n(M) = \mathbb{Z}$. Otherwise, $H^n(M) = \mathbb{Z}/2$.

4.8 Cup product and Poincaré Duality

Let R be a fixed commutative coefficient ring, and fix $\varphi \in C^l(M;R)$, $\psi \in C^k(M;R)$ and $\sigma \in C_{k+l}(M;R)$. Then $\psi \frown \sigma \in C_l(M;R)$ is defined by

$$\varphi(\psi \frown \sigma) = (\varphi \smile \psi)(\sigma) \in R. \tag{4.8.1}$$

Alternatively, if σ is a $(k+l)$-simplex, then

$$\psi \frown \sigma = \psi(\sigma|_{[v_l, v_{l+1}, \ldots, v_{k+l}]}) \cdot \sigma|_{[v_0, v_1, \ldots, v_l]}. \tag{4.8.2}$$

Indeed,

$$\varphi(\psi \frown \sigma) = \psi(\sigma|_{[v_l, v_{l+1}, \ldots, v_{k+l}]}) \cdot \varphi(\sigma|_{[v_0, v_1, \ldots, v_l]}) = (\varphi \smile \psi)(\sigma). \tag{4.8.3}$$

This means that $- \smile \psi : C^l(M;R) \to C^{k+l}(M;R)$ is dual to $\psi \frown - : C_{k+l}(M;R) \to C_l(M;R)$. Passing to (co)homology, we get the following commutative diagram:

$$\begin{array}{ccc} H^l(M;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_l(M;R), R) \\ {\scriptstyle \smile\psi}\big\downarrow & & \big\downarrow{\scriptstyle (\psi\frown)^*} \\ H^{k+l}(M;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_{k+l}(M;R), R) \end{array}$$

In particular, if h is an isomorphism (e.g., R is a field, or we work over $\mathbb{Z}$ but H_* is torsion-free), then $- \smile \psi$ and $\psi \frown -$ determine each other.

Definition 4.8.1. *Let M be a closed connected R-oriented n-manifold. Then the cup product pairing*

$$H^k(M;R) \times H^{n-k}(M;R) \longrightarrow H^n(M;R) \xrightarrow{\frown [M]} H_0(M;R) = R \tag{4.8.4}$$

is defined by

$$(\varphi, \psi) \mapsto (\varphi \smile \psi) \mapsto (\varphi \smile \psi) \frown [M].$$

Definition 4.8.2. *Let A and B be R-modules. A pairing $\alpha : A \times B \to R$ is non-singular if $f : A \to \mathrm{Hom}_R(B,R)$ is an isomorphism, with f defined by $f(a)(b) = \alpha(a,b)$, and $g : B \to \mathrm{Hom}_R(A,R)$ is an isomorphism, with $g(b)(a) = \alpha(a,b)$.*

We then have the following:

Proposition 4.8.3. *The cup product pairing is non-singular if R is a field, or if $R = \mathbb{Z}$ and torsion is factored out.*

Proof. Consider the composition

$$f : H^k(M;R) \xrightarrow{h} \mathrm{Hom}_R(H_k(M;R), R) \xrightarrow{(P.D.)^*} \mathrm{Hom}_R(H^{n-k}(M;R), R),$$

where $(P.D.)^*$ denotes the dual of the Poincaré duality isomorphism. Under our assumptions on R, h is isomorphism. Moreover, by Poincaré Duality, $(PD)^*$ is also an isomorphism, hence f is an isomorphism. For $\varphi \in H^k(M;R)$ and $\psi \in H^{n-k}(M;R)$, we have:

$$\begin{aligned} f(\varphi)(\psi) &= ((P.D.)^* \circ h(\varphi))(\psi) \\ &= h(\varphi)(P.D.(\psi)) \\ &= h(\varphi)(\psi \frown [M]) \\ &= \varphi(\psi \frown [M]) \\ &= (\varphi \smile \psi)[M]. \end{aligned}$$

We obtain a similar isomorphism by interchanging k with $n-k$, so the claim follows. □

Corollary 4.8.4. *Let M be a closed connected $\mathbb{Z}$-oriented n-manifold. Then for any $\alpha \in H^k(M)$ a generator of a $\mathbb{Z}$-summand, there exists $\beta \in H^{n-k}(M)$ such that $\alpha \smile \beta$ generates $H^n(M) \cong \mathbb{Z}$.*

Proof. By hypothesis, there exists a homomorphism (i.e., the projection to some $\mathbb{Z}$-summand)

$$\varphi : H^k(M) \to \mathbb{Z}$$

such that $\varphi(\alpha) = 1$. By the non-singularity of the cup product pairing, φ is realized by taking the cup product with some $\beta \in H^{n-k}(M)$ and evaluating on the fundamental class $[M]$. We therefore get

$$1 = \varphi(\alpha) = (\alpha \smile \beta)[M].$$

This means $\alpha \smile \beta$ is the generator of $H^n(M)$. □

Corollary 4.8.5. $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$*, with* $\deg(\alpha) = 2$.

Proof. Let α be the generator of $H^2(\mathbb{CP}^n) = \mathbb{Z}$. By induction, we can assume that α^{n-1} generates $H^{2n-2}(\mathbb{CP}^n) = \mathbb{Z}$. Using the previous corollary, there exists $\beta \in H^2(\mathbb{CP}^n)$ so that $\alpha^{n-1} \smile \beta$ generates $H^{2n}(\mathbb{CP}^n) = \mathbb{Z}$. Note that since α is the generator of $H^2(\mathbb{CP}^n) = \mathbb{Z}$, it follows that $\beta = m\alpha$, for some $m \in \mathbb{Z}$. This means that $\alpha^{n-1} \smile \beta = m\alpha^n$ generates $\mathbb{Z}$. Thus $m = \pm 1$, whence α^n generates $H^{2n}(\mathbb{CP}^n)$. □

We can now ask the following:

Question 4.8.6. *Does there exist a $2n$-dimensional closed manifold whose cohomology is additively isomorphic to that of $\mathbb{CP}^n$, but with a different cup product structure?*

If $n = 2$, the answer is *No*. Indeed, $H^*(\mathbb{CP}^2;\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^3)$, with $\deg(\alpha) = 2$. If there is such manifold M, then α also generates $H^2(M) = H^2(\mathbb{CP}^2) = \mathbb{Z}$, so there exists $\beta \in H^2(M)$ such that $\alpha \smile \beta$ generates $H^4(M) = \mathbb{Z}$. So, $\beta = m\alpha$, for some $m \in \mathbb{Z}$. Hence $\alpha \smile \beta = m\alpha^2$ generates $H^4(M)$, which yields $m = \pm 1$. This means that M has the same cup product structure as $\mathbb{CP}^2$.
If $n \geq 3$, the answer is *Yes*. Indeed, $S^2 \times S^4$ and $\mathbb{CP}^3$ have isomorphic cohomology groups, but different cup product structures on their cohomology rings.

Another application of Poincaré duality is the following:

Corollary 4.8.7. *If M is a closed oriented manifold of dimension $m = 4n+2$, then $\chi(M)$ is even.*

Proof. By the definition of the Euler characteristic, $\chi(M)$, we have

$$\chi(M) = \sum_{i=0}^{4n+2} (-1)^i \cdot \mathrm{rk}(H_i(M)).$$

By Poincaré duality, we obtain

$$\mathrm{rk}(H_i(M)) = \mathrm{rk}(H_{m-i}(M)).$$

Therefore,

$$\chi(M) \equiv \mathrm{rk}(H_{2n+1}(M)) \pmod 2.$$

Let us now consider the following cup product pairing

$$H^{2n+1}(M) \times H^{2n+1}(M) \overset{\smile}{\to} H^{4n+2}(M) \overset{\frown [M]}{\longrightarrow} \mathbb{Z}$$

defined by

$$(\alpha, \beta) \mapsto (\alpha \smile \beta) \mapsto (\alpha \smile \beta) \frown [M].$$

By Poincaré Duality, after moding out by torsion, this pairing is non-singular. As a result, the matrix A of the cup product pairing is non-singular and anti-symmetric. By linear algebra, A is similar to a matrix with diagonal blocks

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Therefore,

$$\mathrm{rk}(H^{2n+1}(M)) = \mathrm{rk}(A),$$

which is clearly even. □

Remark 4.8.8. Dualizing the cup product pairing of Proposition 4.8.3, we get the non-singular *intersection pairing*

$$H_k(M) \times H_{n-k}(M) \to \mathbb{Z}$$

defined by

$$([\sigma], [\eta]) \to \sharp(\sigma \cap \eta'),$$

where η' is chosen so that it is homologous to η but transversal to σ (so $\sigma \cap \eta'$ is a finite number of points).

Example 4.8.9. Let T be the 2-dimensional torus and S be a meridian of T. Let M be the *pinched torus* T/S. Then Poincaré duality fails for M. If not, let α be the longitude of M (and T) and β be the a meridian of M. Then Poincaré duality for M would yield $([\alpha], [\beta]) \to \sharp(\alpha \cap \beta) = 1$. However, $[\beta] = 0$. This is impossible since the intersection pairing is non-singular. The reason for the failure of Poincaré duality is that the pinched torus $M := T/S$ is not a manifold. Indeed, a neighborhood of the pinch point is a join of two 2-disks, thus it is not homeomorphic to $\mathbb{R}^2$.

4.9 Manifolds with boundary: Poincaré duality and applications

In this section, we discuss the Poincaré duality theorem for manifolds with boundary. The proofs are routine adaptation of those for closed manifolds.

Definition 4.9.1. *A Hausdorff topological space M is an n-manifold with boundary if any point $x \in M$ has a neighborhood U_x homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. In particular,*

(a) if $U_x \cong \mathbb{R}^n$, then $H_n(M, M \setminus x) \cong H_n(U_x, U_x \setminus x) \cong \mathbb{Z}$.

(b) if $U_x \cong \mathbb{R}^n_+$, then

$$H_n(M, M \setminus x) \cong H_n(U_x, U_x \setminus x) \cong H_n(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{0\}) \cong 0.$$

The boundary of M is defined to be

$$\partial M := \{x \in M \mid H_n(M, M \setminus x) = 0\}.$$

Example 4.9.2. $\partial(D^n) = S^{n-1}$, $\partial(\mathbb{R}^n_+) = \mathbb{R}^{n-1}$.

Remark 4.9.3. If M is an n-manifold with boundary, then the boundary set ∂M is a manifold of dimension $n-1$.

Definition 4.9.4. *We say that a manifold with boundary $(M, \partial M)$ is orientable if $M \setminus \partial M$ is orientable as a manifold with no boundary.*

We have the following:

Proposition 4.9.5. *If $(M, \partial M)$ is a compact, orientable n-manifold with oriented boundary, then there exists a unique class $\mu_M \in H_n(M, \partial M)$ inducing local orientations $\mu_x \in H_n(M, M \setminus x)$ at all points $x \in M \setminus \partial M$.*

Remark 4.9.6. If $(M, \partial M)$ is a compact, orientable n-manifold with boundary, then in the long exact sequence for the pair $(M, \partial M)$ we have:

$$\begin{array}{ccc} H_n(M, \partial M) & \stackrel{\partial}{\longrightarrow} & H_{n-1}(\partial M) \\ [M] = \mu_M & \longmapsto & [\partial M] \end{array}$$

Theorem 4.9.7 (Poincaré Duality)**.** *If $(M, \partial M)$ is a connected, oriented n-manifold with boundary, then there are isomorphisms*

$$H^i_c(M) \xrightarrow[\cong]{\frown \mu_M} H_{n-i}(M, \partial M) \tag{4.9.1}$$

and

$$H^i_c(M, \partial M) \xrightarrow[\cong]{\frown \mu_M} H_{n-i}(M) \tag{4.9.2}$$

where $H^i_c(M, \partial M) := \varinjlim_{\substack{K compact \\ K \subset M \setminus \partial M}} H^i(M, (M \setminus K) \cup \partial M)$ is the cohomology with compact support for the pair $(M, \partial M)$.

Let us now describe some applications of Poincaré duality for manifolds with boundary.

Proposition 4.9.8. *If $M^n = \partial V^{n+1}$ is a connected manifold with V a compact $(n+1)$-dimensional manifold with boundary, then the Euler characteristic $\chi(M)$ is even.*

An immediate consequence of Proposition 4.9.8 is the following:

Corollary 4.9.9. $\mathbb{RP}^{2n}, \mathbb{CP}^{2n}, \mathbb{HP}^{2n}$ *cannot be boundaries of compact manifolds.*

In order to prove Proposition 4.9.8, we need the following result:

Proposition 4.9.10. *Assume V^{2n+1} is an oriented, $(2n+1)$-dimensional compact manifold with connected boundary $\partial V = M^{2n}$. If R is a field (e.g., $\mathbb{Z}/2\mathbb{Z}$ if M is non-orientable), then* $\dim_R H^n(M;R) = \dim_R H_n(M;R)$ *is even.*

Proof of Proposition 4.9.10. Consider the long exact sequence for the pair (V, M):

$$\begin{array}{ccccc} H^n(V;R) & \xrightarrow{i^*} & H^n(M;R) & \xrightarrow{\delta} & H^{n+1}(V,M;R) \\ & & \cong \big\downarrow \frown[M] & & \cong \big\downarrow \frown[V] \\ & & H_n(M;R) & \xrightarrow{i_*} & H_n(V;R) \end{array}$$

where i^*, i_* are induced by the inclusion $i : M = \partial V \hookrightarrow V$. By exactness, we have that Image $i^* \cong \ker \delta \overset{\text{P.D.}}{\cong} \ker i_*$, so

$$\dim(\text{Image } i^*) = \dim(\ker i_*) = \dim H_n(M;R) - \dim(\text{Image } i_*).$$

Since i^*, i_* are Hom-dual, we have that $\dim(\text{Image } i^*) = \dim(\text{Image } i_*)$. Altogether,

$$\dim H^n(M;R) = \dim H_n(M;R) = 2\dim(\text{Image } i_*)$$

is even. □

Proof of Proposition 4.9.8. If $n = \dim M$ is odd, then Proposition 4.6.1 yields that $\chi(M) = 0$, thus even. If $n = 2m$ is even, then we work with $\mathbb{Z}/2\mathbb{Z}$-coefficients and get:

$$\begin{aligned} \chi(M) &= \sum_{i=0}^{2m} (-1)^i \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2) \\ &\overset{(1)}{=} 2\sum_{i=0}^{m-1} (-1)^i \dim_{\mathbb{Z}/2} H_i(M;\mathbb{Z}/2) + (-1)^m \dim_{\mathbb{Z}/2} H_m(M;\mathbb{Z}/2) \\ &\equiv \dim_{\mathbb{Z}/2} H_m(M;\mathbb{Z}/2) \pmod 2 \\ &\overset{(2)}{\equiv} 0 \pmod 2, \end{aligned}$$

where equation (1) follows by Poincaré Duality, and congruence (2) is by Proposition 4.9.10. □

The proof of Proposition 4.9.10 also yields the following:

Corollary 4.9.11. Under the assumptions of Proposition 4.9.10, we have the following:

(a) Image $i^* \subset H^n(M^{2n}; R)$ is self-annihilating with respect to cup product $\smile$, i.e., if $\alpha, \beta \in$ Image i^*, then $\alpha \smile \beta = 0$.

(b) $\dim(\text{Image } i^*) = \frac{1}{2} \dim H^n(M^{2n}; R)$.

Proof. For any $\alpha = i^*(\overline{\alpha}), \beta = i^*(\overline{\beta})$ with $\overline{\alpha}, \overline{\beta} \in H^n(V; R)$, we have

$$\delta(\alpha \smile \beta) = \delta(i^*(\overline{\alpha}) \smile i^*(\overline{\beta})) = \delta i^*(\overline{\alpha} \smile \overline{\beta}) = 0$$

Hence, $\alpha \smile \beta \in \ker\big(\delta : H^{2n}(M; R) \to H^{2n+1}(V, M; R)\big) \cong 0$, where the last isomorphism follows by the following commutative diagram

$$\begin{array}{ccc} H^{2n}(M; R) & \xrightarrow{\delta} & H^{2n+1}(V, M; R) \\ \cong \downarrow \text{P.D.} & & \cong \downarrow \text{P.D.} \\ H_0(M; R) & \longrightarrow & H_0(V; R) \end{array}$$

with the bottom arrow an injection. $\square$

4.9.1 Signature

Definition 4.9.12. *Let M be a closed oriented manifold. If $\dim M = 4k$, the signature $\sigma(M)$ of M is defined to be the signature of the symmetric non-singular cup product pairing*

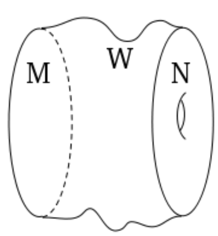
$$\begin{array}{rcl} H^{2k}(M; \mathbb{R}) \times H^{2k}(M; \mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\alpha, \beta) & \mapsto & (\alpha \smile \beta)[M] \end{array}$$

Otherwise, if $\dim M$ is not divisible by 4, we let $\sigma(M) = 0$.

Remark 4.9.13. Recall that a symmetric non-singular bilinear pairing has only real (non-zero) eigenvalues, and its signature is defined by subtracting the number of negative eigenvalues from the number of positive eigenvalues.

Example 4.9.14. $\sigma(S^2 \times S^2) = \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$, $\sigma(\mathbb{CP}^{2n}) = 1$, $\sigma(\mathbb{CP}^2 \# \mathbb{CP}^2) = 2$.

The signature σ is a *cobordism invariant*, i.e. if $\partial W = M \sqcup -N$, then $\sigma(M) = \sigma(N)$. Here $-N$ denotes the manifold N but with the opposite orientation.



Here we prove the following special case of this fact:

Theorem 4.9.15. *If, in the above notations, $M^{4k} = \partial V^{4k+1}$ is connected with V compact and orientable, then $\sigma(M) = 0$.*

Proof. Let $A = H^{2k}(M; \mathbb{R})$. The cup product yields a non-singular and symmetric pairing

$$\varphi : A \times A \to \mathbb{R}.$$

Let A_+ be the subspace on which the pairing is positive-definite, and A_- the subspace on which the pairing is negative-definite. Let $r = \dim A_+$, $2l = \dim A$ (which is even by Proposition 4.9.10). Then, $\dim A_- = 2l - r$ since the pairing is non-singular, and

$$\sigma(M) = r - (2l - r) = 2r - 2l.$$

In order to prove that $\sigma(M) = 0$, it suffices to show that $r = l$.

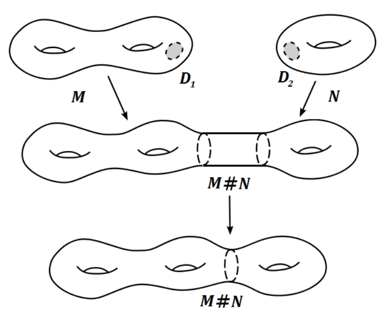
Let $B \subset A$ be the self-annihilating l-dimensional subspace given by Proposition 4.9.8. Then $A_+ \cap B = \{0\}$ and $A_- \cap B = \{0\}$. Hence,

$$\begin{aligned} \dim A_+ + \dim B \leq \dim A = 2l, \quad &\text{i.e.,} \quad r + l \leq 2l \quad \text{i.e.,} \quad r \leq l \\ \dim A_- + \dim B \leq \dim A = 2l, \quad &\text{i.e.,} \quad 2l - r + l \leq 2l \quad \text{i.e.,} \quad r \geq l \end{aligned}$$

In conclusion, $r = l$ and $\sigma(M) = 0$. □

4.9.2 Connected Sums

Definition 4.9.16. *Let M^n, N^n be closed, connected, oriented n-manifolds. Their connected sum*



is defined to be

$$M \# N := (M \setminus D_1^n) \cup_f (N \setminus D_2^n)$$

where $f : \partial D_1^n = S^{n-1} \to \partial D_2^n = S^{n-1}$ is an orientation-reversing homeomorphism.

Remark 4.9.17. The connected sum $M\#N$ of closed, connected, oriented n-manifolds is itself a closed, connected, oriented n-manifold. The cohomology ring $H^*(M\#N)$ is isomorphic to the ring resulting from the direct product of $H^*(M)$ and $H^*(N)$, with the unity elements identified, and the orientation classes identified. In particular, $H^0(M\#N) = \mathbb{Z}$, $H^n(M\#N) = \mathbb{Z}$ and $H^k(M\#N) \cong H^k(M) \oplus H^k(N)$, $0 < k < n$. Moreover, cup products of positive dimensional classes, one from each of the two original manifolds, are zero, i.e., $\alpha \smile \beta = 0$ for any $\alpha \in H^k(M)$ and $\beta \in H^l(N)$ with $k, l > 0$.

Example 4.9.18. By the above description of cup products of a connected sum, we get:

$$\sigma(\mathbb{CP}^2 \# -\mathbb{CP}^2) = 0.$$

In fact, it can be shown that $\mathbb{CP}^2\#-\mathbb{CP}^2$ is the boundary of a connected, oriented 5-manifold;

Example 4.9.19. The spaces $S^2 \times S^2$ and $\mathbb{CP}^2\#\mathbb{CP}^2$ have the same cohomology groups,

$$H^0 = \mathbb{Z},\ H^2 = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta,\ H^4 = \mathbb{Z},$$

but different cohomogy rings, since $\alpha \smile \beta \neq 0$ in $H^*(S^2 \times S^2)$, but $\alpha \smile \beta = 0$ in $H^*(\mathbb{CP}^2\#\mathbb{CP}^2)$.

Example 4.9.20. We have

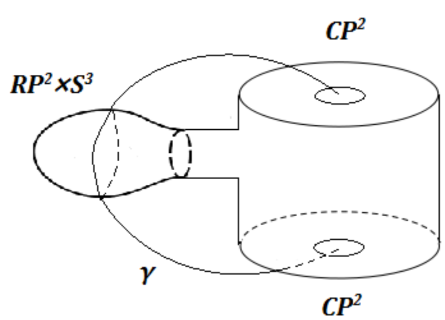
$$\sigma(\mathbb{CP}^2\#\mathbb{CP}^2) = 2 \neq 0,$$

so in view of Theorem 4.9.15, $\mathbb{CP}^2\#\mathbb{CP}^2$ cannot be the boundary of a compact, oriented 5-manifold. However, $\mathbb{CP}^2\#\mathbb{CP}^2 = \partial W^5$, where W^5 is a compact non-orientable 5-manifold. The compact manifold W can be constructed as follows:

(a) Start with $(\mathbb{CP}^2 \times I)\#(\mathbb{RP}^2 \times S^3)$.

(b) Run an orientation reversing path γ from one $\mathbb{CP}^2$ to the other, by traveling along an orientation reversing path in $\mathbb{RP}^2$.

(c) Enlarge the path to a tube and remove its interior. What is left is a 5-dimensional non-orientable manifold with $\partial W = \mathbb{CP}^2\#\mathbb{CP}^2$.



Chapter 5

Basics of Homotopy Theory

5.1 Homotopy Groups

Definition 5.1.1. *For each $n \geq 0$ and X a topological space with $x_0 \in X$, the n-th homotopy group of X is defined as*

$$\pi_n(X, x_0) = \big\{ f : (I^n, \partial I^n) \to (X, x_0) \big\} / \sim$$

where $\sim$ is the usual homotopy of maps.

Remark 5.1.2. Note that we have the following diagram of sets:

$$\begin{array}{ccc} (I^n, \partial I^n) & \xrightarrow{\ f\ } & (X, x_0) \\ \downarrow & \nearrow_{g} & \\ (I^n/\partial I^n, \partial I^n/\partial I^n) & & \end{array}$$

with $(I^n/\partial I^n, \partial I^n/\partial I^n) \simeq (S^n, s_0)$. So we can also define

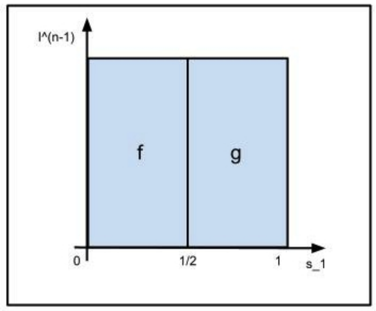
$$\pi_n(X, x_0) = \big\{ g : (S^n, s_0) \to (X, x_0) \big\} / \sim .$$

Remark 5.1.3. If $n = 0$, then $\pi_0(X)$ is the set of connected components of X. Indeed, we have $I^0 = \text{pt}$ and $\partial I^0 = \varnothing$, so $\pi_0(X)$ consists of homotopy classes of maps from a point into the space X.

Now we will prove several results analogous to the case $n = 1$, which corresponds to the fundamental group.

Proposition 5.1.4. *If $n \geq 1$, then $\pi_n(X, x_0)$ is a group with respect to the operation $+$ defined as:*

$$(f + g)(s_1, s_2, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, \ldots, s_n) & \frac{1}{2} \leq s_1 \leq 1 \end{cases}$$



(Note that if $n = 1$, this is the usual concatenation of paths/loops.)

Proof. First note that since only the first coordinate is involved in this operation, the same argument used to prove that π_1 is a group is valid here as well. Then the identity element is the constant map taking all of I^n to x_0 and the inverse element is given by

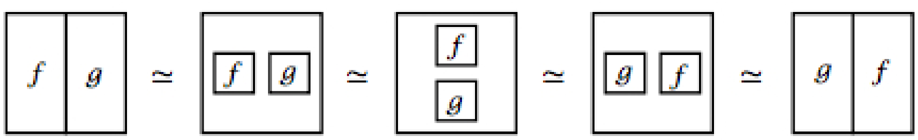
$$-f(s_1, s_2, \ldots, s_n) = f(1 - s_1, s_2, \ldots, s_n).$$

□

Proposition 5.1.5. *If $n \geq 2$, then $\pi_n(X, x_0)$ is abelian.*

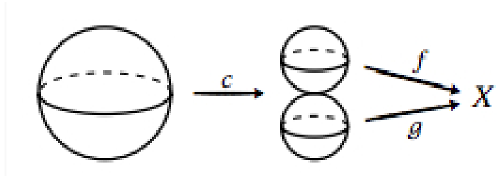
Intuitively, since the + operation only involves the first coordinate, if $n \geq 2$, there is enough space to "slide f past g ".

Proof. Let $n \geq 2$ and let $f, g \in \pi_n(X, x_0)$. We wish to show $f + g \simeq g + f$. Consider the following figures:



We first shrink the domains of f and g to smaller cubes inside I^n and map the remaining region to the base point x_0. Note that this is possible since both f and g map to x_0 on the boundaries, so the resulting map is continuous. Then there is enough room to slide f past g inside I^n. We then enlarge the domains of f and g back to their original size and get $g + f$. So we have constructed a homotopy between $f + g$ and $g + f$ and hence $\pi_n(X, x_0)$ is abelian. □

Remark 5.1.6. If we view $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, s_0) \to (X, x_0)$, then we have the following visual representation of $f + g$ (one can see this by collapsing boundaries in the above cube interpretation).



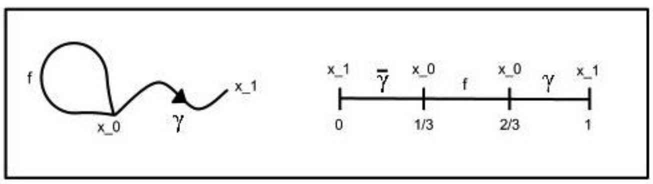
Next recall that if X is path-connected and $x_0, x_1 \in X$, then there is an isomorphism

$$\beta_\gamma : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

where γ is a path from x_0 to x_1, i.e., $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. The isomorphism β_γ is given by

$$\beta_\gamma([f]) = [\gamma^{-1} \cdot f \cdot \gamma]$$

for any $[f] \in \pi_1(X, x_0)$.

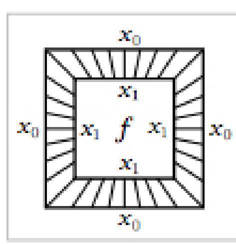


We next show a similar fact holds for all $n \geq 1$.

Proposition 5.1.7. *If $n \geq 1$ and X is path-connected, then there is an isomorphism β_γ : $\pi_n(X, x_1) \to \pi_n(X, x_0)$ given by*

$$\beta_\gamma([f]) = [\gamma \cdot f],$$

where γ is a path in X from x_1 to x_0, and $\gamma \cdot f$ is constructed by first shrinking the domain of f to a smaller cube inside I^n, and then inserting the path γ radially from x_1 to x_0 on the boundaries of these cubes.



Proof. It is easy to check that the following properties hold:

1. $\gamma \cdot (f + g) \simeq \gamma \cdot f + \gamma \cdot g$
2. $(\gamma \cdot \eta) \cdot f \simeq \gamma \cdot (\eta \cdot f)$, for η a path from x_0 to x_1

3. $c_{x_0} \cdot f \simeq f$, where c_{x_0} denotes the constant path based at x_0.

4. β_γ is well-defined with respect to homotopies of γ or f.

Note that (1) implies that β_γ is a group homomorphism, while (2) and (3) show that β_γ is invertible. Indeed, if $\overline{\gamma}(t) = \gamma(1-t)$, then $\beta_\gamma^{-1} = \beta_{\overline{\gamma}}$. □

So, as in the case $n = 1$, if the space X is path-connected, then π_n is independent of the choice of base point. Further, if $x_0 = x_1$, then (2) and (3) also imply that $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$:

$$\pi_1 \times \pi_n \to \pi_n$$
$$(\gamma, [f]) \mapsto [\gamma \cdot f]$$

Definition 5.1.8. *We say X is an* abelian space *if π_1 acts trivially on π_n for all $n \geq 1$.*

In particular, this means π_1 is abelian, since the action of π_1 on π_1 is by inner-automorphisms, which must all be trivial.

We next show that π_n is a functor.

Proposition 5.1.9. *A map $\phi : X \to Y$ induces group homomorphisms $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, \phi(x_0))$ given by $[f] \mapsto [\phi \circ f]$, for all $n \geq 1$.*

Proof. First note that, if $f \simeq g$, then $\phi \circ f \simeq \phi \circ g$. Indeed, if ψ_t is a homotopy between f and g, then $\phi \circ \psi_t$ is a homotopy between $\phi \circ f$ and $\phi \circ g$. So ϕ_* is well-defined. Moreover, from the definition of the group operation on π_n, it is clear that we have $\phi \circ (f + g) = (\phi \circ f) + (\phi \circ g)$. So $\phi_*([f + g]) = \phi_*([f]) + \phi_*([g])$. Hence ϕ_* is a group homomorphism. □

The following is a consequence of the definition of the above induced homomorphisms:

Proposition 5.1.10. *The homomorphisms induced by $\phi : X \to Y$ on higher homotopy groups satisfy the following two properties:*

1. $(\phi \circ \psi)_* = \phi_* \circ \psi_*$.

2. $(id_X)_* = id_{\pi_n(X, x_0)}$.

We thus have the following important consequence:

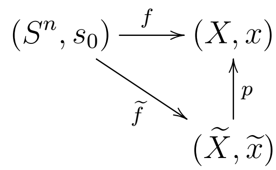
Corollary 5.1.11. *If $\phi : (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, \phi(x_0))$ is an isomorphism, for all $n \geq 1$.*

Example 5.1.12. Consider $\mathbb{R}^n$ (or any contractible space). We have $\pi_i(\mathbb{R}^n) = 0$ for all $i \geq 1$, since $\mathbb{R}^n$ is homotopy equivalent to a point.

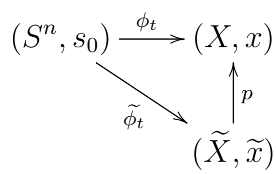
The following result is very useful for computations:

Proposition 5.1.13. *If $p : \widetilde{X} \to X$ is a covering map, then $p_* : \pi_n(\widetilde{X}, \widetilde{x}) \to \pi_n(X, p(\widetilde{x}))$ is an isomorphism for all $n \geq 2$.*

Proof. First we claim p_* is surjective. Let $x = p(\widetilde{x})$ and consider $f : (S^n, s_0) \to (X, x)$. Since $n \geq 2$, we have that $\pi_1(S^n) = 0$, so $f_*(\pi_1(S^n, s_0)) = 0 \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}))$. So f admits a lift, i.e., there is $\widetilde{f} : (S^n, s_0) \to (\widetilde{X}, \widetilde{x})$ such that $p \circ \widetilde{f} = f$. Then $[f] = [p \circ \widetilde{f}] = p_*([\widetilde{f}])$. So p_* is surjective.



Next, we show that p_* is injective. Suppose $[\widetilde{f}] \in \ker p_*$. So $p_*([\widetilde{f}]) = [p \circ \widetilde{f}] = 0$. Let $p \circ \widetilde{f} = f$. Then $f \simeq c_x$ via some homotopy $\phi_t : (S^n, s_0) \to (X, x_0)$ with $\phi_1 = f$ and $\phi_0 = c_x$. Again, by the lifting criterion, there is a unique $\widetilde{\phi}_t : (S^n, s_0) \to (\widetilde{X}, \widetilde{x})$ with $p \circ \widetilde{\phi}_t = \phi_t$.



Then we have $p \circ \widetilde{\phi}_1 = \phi_1 = f$ and $p \circ \widetilde{\phi}_0 = \phi_0 = c_x$, so by the uniqueness of lifts, we must have $\widetilde{\phi}_1 = \widetilde{f}$ and $\widetilde{\phi}_0 = c_{\widetilde{x}}$. Then $\widetilde{\phi}_t$ is a homotopy between $\widetilde{f}$ and $c_{\widetilde{x}}$. So $[\widetilde{f}] = 0$. Thus p_* is injective. □

Example 5.1.14. Consider S^1 with its universal covering map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi i t}$. We already know $\pi_1(S^1) = \mathbb{Z}$. If $n \geq 2$, Proposition 5.1.13 yields that $\pi_n(S^1) = \pi_n(\mathbb{R}) = 0$.

Example 5.1.15. Consider $T^n = S^1 \times S^1 \times \cdots \times S^1$, the n-torus. We have $\pi_1(T^n) = \mathbb{Z}^n$. By using the universal covering map $p : \mathbb{R}^n \to T^n$, we have by Proposition 5.1.13 that $\pi_i(T^n) = \pi_i(\mathbb{R}^n) = 0$ for $i \geq 2$.

Definition 5.1.16. *If $\pi_n(X) = 0$ for all $n \geq 2$, the space X is called* aspherical.

Proposition 5.1.17. *Let $\{X_\alpha\}_\alpha$ be a collection of path-connected spaces. Then*

$$\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n(X_\alpha)$$

for all n.

Proof. First note that a map $f : Y \to \prod_\alpha X_\alpha$ is a collection of maps $f_\alpha : Y \to X_\alpha$. For elements of π_n, take $Y = S^n$ (note that since all spaces are path-connected, we may drop the reference to base points). For homotopies, take $Y = S^n \times I$. □

Example 5.1.18. It is a natural question to find two spaces X and Y such that $\pi_n(X) \cong \pi_n(Y)$ for all n, but with X and Y not homotopy equivalent. Whitehead's Theorem (to be discussed later on) states that if a map of CW complexes $f : X \to Y$ induces isomorphisms on all π_n, then f is a homotopy equivalence. So we must find X and Y so that there is no continuous map $f : X \to Y$ inducing the isomorphisms on π_n's. Consider $X = S^2 \times \mathbb{R}P^3$ and $Y = \mathbb{R}P^2 \times S^3$. Then $\pi_n(X) = \pi_n(S^2 \times \mathbb{R}P^3) = \pi_n(S^2) \times \pi_n(\mathbb{R}P^3)$. Since S^3 is a covering of $\mathbb{R}P^3$, for all $n \geq 2$ we have that $\pi_n(X) = \pi_n(S^2) \times \pi_n(S^3)$. We also have $\pi_1(X) = \pi_1(S^2) \times \pi_1(\mathbb{R}P^3) = \mathbb{Z}/2$. Similarly, we have $\pi_n(Y) = \pi_n(\mathbb{R}P^2 \times S^3) = \pi_n(\mathbb{R}P^2) \times \pi_n(S^3)$. And since S^2 is a covering of $\mathbb{R}P^2$, for $n \geq 2$ we have that $\pi_n(Y) = \pi_n(S^2) \times \pi_n(S^3)$. Finally, $\pi_1(Y) = \pi_1(\mathbb{R}P^2) \times \pi_1(S^3) = \mathbb{Z}/2$. So $\pi_n(X) = \pi_n(Y)$ for all n. By considering homology groups, however, we see that X and Y are not homotopy equivalent. Indeed, by the Künneth formula, we get that $H_5(X) = \mathbb{Z}$ while $H_5(Y) = 0$ (since $\mathbb{R}P^3$ is oriented while $\mathbb{R}P^2$ is not).

Just like there is a homomorphism $\pi_1(X) \to H_1(X)$, we can also construct homomorphisms

$$\pi_n(X) \to H_n(X)$$

defined by $[f : S^n \to X] \mapsto f_*[S^n]$, where $[S^n]$ is the fundamental class of S^n. A very important result in homotopy theory is the following:

Theorem 5.1.19. *(Hurewicz)*
If $n \geq 2$ and $\pi_i(X) = 0$ for all $i < n$, then $H_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$.

Moreover, there is also a relative version of the Hurewicz theorem (see the next section for a definition of the relative homotopy groups), which can be used to prove the following:

Corollary 5.1.20. *If X and Y are CW complexes with $\pi_1(X) = \pi_1(Y) = 0$, and a map $f : X \to Y$ induces isomorphisms on all integral homology groups H_n, then f is a homotopy equivalence.*

5.2 Relative Homotopy Groups

Given a triple (X, A, x_0) where $x_0 \in A \subset X$, we define relative homotopy groups as follows:

Definition 5.2.1. *Let X be a space and let $A \subseteq X$ and $x_0 \in A$. Let*

$$I^{n-1} = \{(s_1, \ldots, s_n) \in I^n |\ s_n = 0\}$$

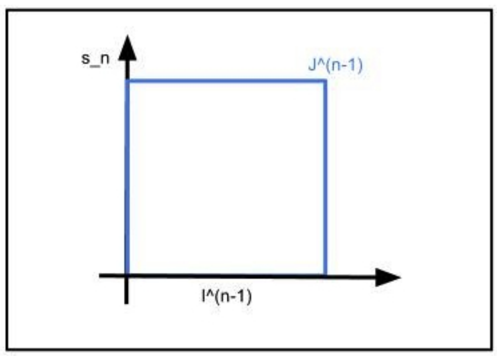
and set

$$J^{n-1} = \overline{\partial I^n \backslash I^{n-1}}.$$

Then define the n-th homotopy group of the pair (X, A) as:

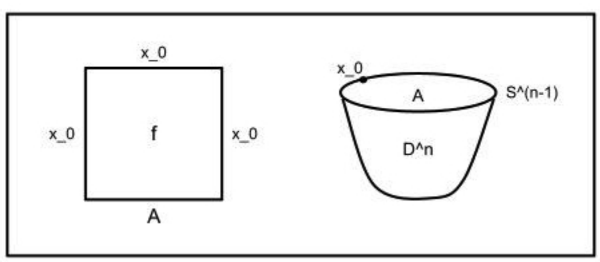
$$\pi_n(X, A, x_0) = \big\{f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)\big\}/\sim$$

where, as before, $\sim$ is the homotopy equivalence relation.



Alternatively, by collapsing J^{n-1} to a point, we can take

$$\pi_n(X, A, x_0) = \big\{g : (D^n, S^{n-1}, s_0) \to (X, A, x_0)\big\}/\sim .$$



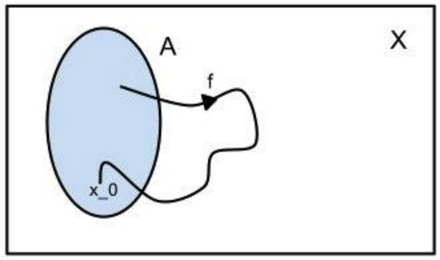
A sum operation is defined in $\pi_n(X, A, x_0)$ by the same formulas as for $\pi_n(X, x_0)$, except that the coordinate s_n now plays a special role and is no longer available for the sum operation. Thus, we have:

Proposition 5.2.2. *If $n \geq 2$, then $\pi_n(X, A, x_0)$ forms a group under the usual sum operation. Further, if $n \geq 3$, then $\pi_n(X, A, x_0)$ is abelian.*

Remark 5.2.3. Note that the proposition fails in the case $n = 1$. Indeed, we have that

$$\pi_1(X, A, x_0) = \big\{ f : (I, \{0,1\}, \{1\}) \to (X, A, x_0) \big\} / \sim .$$

Then $\pi_1(X, A, x_0)$ consists of homotopy classes paths starting anywhere A and ending at x_0, so we cannot always concatenate two paths.



Just as in the absolute case, a map of pairs $\phi : (X, A, x_0) \to (Y, B, y_0)$ induces homomorphisms $\phi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ for all n.

A very important feature of the relative homotopy groups is the following:

Proposition 5.2.4. *The relative homotopy groups of (X, A, x_0) fit into a long exact sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0) \to 0,$$

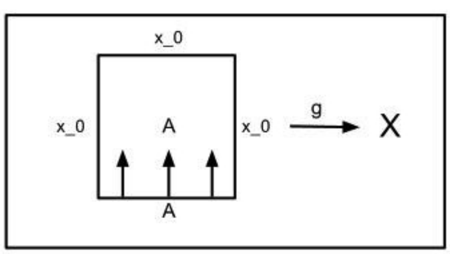
where the map ∂ is defined by $\partial[f] = [f|_{I^{n-1}}]$ and all others are induced by inclusions.

Remark 5.2.5. Near the end of the above sequence, where group structures are not defined, exactness still makes sense: the image of one map is the kernel of the next, those elements mapping to the homotopy class of the constant map.

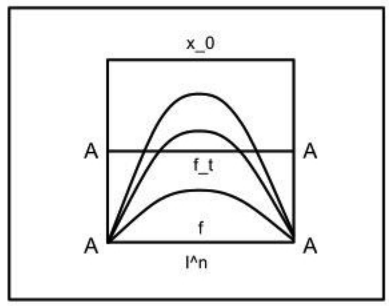
For what follows, it will be important to have a good description of the zero element $0 \in \pi_n(X, A, x_0)$.

Lemma 5.2.6. *Let $[f] \in \pi_n(X, A, x_0)$. Then $[f] = 0$ iff $f \simeq g$ for some map g with image contained in A.*

Proof. ($\Leftarrow$) Suppose $f \simeq g$ for some g with Image $g \subset A$.



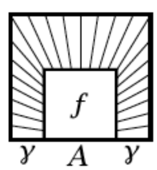
Then we can deform I^n to J^{n-1} as indicated in the above picture, and so $g \simeq c_{x_0}$. Since homotopy is a transitive relation, we then get that $f \simeq c_{x_0}$.
($\Rightarrow$) Suppose $[f] = 0$ in $\pi_n(X, A, x_0)$. So $f \simeq c_{x_0}$ via some homotopy $F : I^{n+1} \to X$. Then we may deform I^n inside I^{n+1} (while fixing the boundary) to J^n. Composing with F, we get a homotopy from f to a map g with Image $g \subset A$.



□

Recall that if X is path-connected, then $\pi_n(X)$ is independent of our choice of base point and $\pi_1(X)$ acts on $\pi_n(X)$ for all n. In the relative case, we have:

Lemma 5.2.7. *If A is path-connected, then $\beta_\gamma : \pi_n(X, A, x_0) \to \pi_n(X, A, x_1)$ is an isomorphism, where γ is a path in A from x_0 to x_1.*



Remark 5.2.8. In particular, if $x_0 = x_1$, we get an action of $\pi_1(A)$ on $\pi_n(X, A)$.

Definition 5.2.9. *We say that the pair (X, A) is n-*connected *if $\pi_i(X, A) = 0$ for $i \leq n$ and X is n-*connected *if $\pi_i(X) = 0$ for $i \leq n$.*

5.3 Homotopy Groups of Spheres

We now turn our attention to computing (some of) the homotopy groups $\pi_i(S^n)$. For $i \leq n, i = n+1, n+2, n+3$ and a few more cases, this is known. In general, however, this is a very difficult problem. For $i = n$, we would expect to have $\pi_n(S^n) = \mathbb{Z}$ by associating to each (homotopy class of a) map $f : S^n \to S^n$ its degree. For $i < n$, we will show that $\pi_i(S^n) = 0$. Note that if $f : S^i \to S^n$ is not surjective, i.e., there is $y \in S^n \setminus f(S^i)$, then f factors through $\mathbb{R}^n$, which is contractible. By composing f with the retraction $\mathbb{R}^n \to x_0$ we get that $f \simeq c_{x_0}$. However, there are surjective maps $S^i \to S^n$ for $i < n$, in which case the above proof fails. To make things work, we "alter" f to make it *cellular*.

Definition 5.3.1. *Let X and Y be CW-complexes. A map $f : X \to Y$ is called* cellular *if $f(X_n) \subset Y_n$ for all n, where X_n denotes the n-skeleton of X and similarly for Y.*

Theorem 5.3.2. *Any map between CW-complexes is homotopic to a cellular map. A similar statement holds for maps of pairs.*

Corollary 5.3.3. *For $i < n$, we have $\pi_i(S^n) = 0$.*

Proof. Choose the standard CW-structure on S^i and S^n. For $[f] \in \pi_i(S^n)$, we may assume by the above theorem that $f : S^i \to S^n$ is cellular. Then $f(S^i) \subset (S^n)_i$. But $(S^n)_i$ is a point, so f is a constant map. □

Corollary 5.3.4. *Let $A \subset X$ and suppose that all cells of $X \setminus A$ have dimension $> n$. Then $\pi_i(X, A) = 0$ for $i \leq n$.*

Proof. Let $[f] \in \pi_i(X, A)$. By the relative version of the cellular approximation, the map of pairs $f : (D^i, S^{i-1}) \to (X, A)$ is homotopic to a map g with $g(D^i) \subset X_i$. But for $i \leq n$, we have that $X_i \subset A$, so Image $g \subset A$. Therefore, $[f] = [g] = 0$. □

Corollary 5.3.5. *$\pi_i(X, X_n) = 0$ for all $i \leq n$.*

Therefore, the long exact sequence for the homotopy groups of the pair (X, X_n) yields the following:

Corollary 5.3.6. *For $i < n$, we have $\pi_i(X) \cong \pi_i(X_n)$.*

Theorem 5.3.7. *(Suspension Theorem)*
Let $f : S^i \to S^n$ be a map, and consider its suspension,

$$\Sigma f : \Sigma S^i = S^{i+1} \to \Sigma S^n = S^{n+1}.$$

The assignment

$$[f] \in \pi_i(S^n) \ \mapsto [\Sigma f] \in \pi_{i+1}(S^{n+1})$$

defines a homomorphism $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$. Moreover, this is an isomorphism $\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$ for $i < 2n - 1$ and a surjection for $i = 2n - 1$.

Corollary 5.3.8. *$\pi_n(S^n)$ is either $\mathbb{Z}$ or a finite quotient of $\mathbb{Z}$ (for $n \geq 2$), generated by the degree map.*

Proof. By the Suspension Theorem, we have the following:

$$\mathbb{Z} \cong \pi_1(S^1) \twoheadrightarrow \pi_2(S^2) \cong \pi_3(S^3) \cong \pi_4(S^4) \cong \cdots$$

□

To show that $\pi_1(S^1) \cong \pi_2(S^2)$, we can use *the long exact sequence for the homotopy groups of a fibration.* (Note: Covering maps are a good example of a fibration with F discrete).

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

$$\cdots \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \cdots \tag{5.3.1}$$

Applying the above long exact sequence to the Hopf fibration $S^1 \hookrightarrow S^3 \xrightarrow{f} S^2$, we obtain:

$$\cdots \to \pi_2(S^1) \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3) \to \cdots$$

Using the fact that $\pi_2(S^3) = 0$ and $\pi_1(S^3) = 0$, we therefore have an isomorphism:

$$\pi_2(S^2) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Note that by using the vanishing of the higher homotopy groups of S^1, the above long exact sequence also yields that

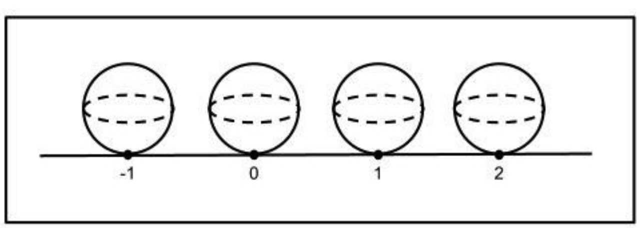
$$\pi_3(S^2) \cong \pi_2(S^2) \cong \mathbb{Z}.$$

Remark 5.3.9. Unlike the homology and cohomology groups, the homotopy groups of a finite CW-complex can be infinitely generated. This fact is discussed in the next example.

Example 5.3.10. For $n \geq 2$, consider the finite CW complex $S^1 \vee S^n$. We then have that

$$\pi_n(S^1 \vee S^n) = \pi_n(\widetilde{S^1 \vee S^n}),$$

where $\widetilde{S^1 \vee S^n}$ is the universal cover of $S^1 \vee S^n$, depicted below:



By contracting the segments between integers, we have that

$$\widetilde{S^1 \vee S^n} \simeq \bigvee_{k\in\mathbb{Z}} S^n_k.$$

So for any $n \geq 2$, we have:

$$\pi_n(S^1 \vee S^n) = \pi_n(\bigvee_{k\in\mathbb{Z}} S^n_k),$$

which is the free abelian group generated by the inclusions $S^n_k \hookrightarrow \bigvee_{k\in\mathbb{Z}} S^n_k$. Indeed, we have the following:

Lemma 5.3.11. $\pi_n(\bigvee_\alpha S^n_\alpha)$ *is free abelian and generated by the inclusions of factors.*

Proof. First note that, since the image of any $f : S^n \to \bigvee_\alpha S^n_\alpha$ is compact hence contained in the wedge of finitely many S^n_α's, we can assume that there are only finitely many S^n_α's in the wedge $\bigvee_\alpha S^n_\alpha$. Then we can regard $\bigvee_\alpha S^n_\alpha$ as the n-skeleton of $\prod_\alpha S^n_\alpha$. The cell structure of a particular S^n_α consists of a single 0-cell e^0_α and a single n-cell, e^n_α. Thus, in the product $\prod_\alpha S^n_\alpha$ there is one 0-cell $e^0 = \prod_\alpha e^0_\alpha$, which, together with the n-cells

$$\bigcup_\alpha (\prod_{\beta\neq\alpha} e^0_\beta) \times e^n_\alpha,$$

form the n-skeleton of $\bigvee_\alpha S^n_\alpha$. Hence $\prod_\alpha S^n_\alpha \setminus \bigvee_\alpha S^n_\alpha$ has only cells of dimension at least $2n$, which by Corollary 5.3.5 yields that the pair $(\prod_\alpha S^n_\alpha, \bigvee_\alpha S^n_\alpha)$ is $(2n-1)$-connected. In particular, as $n \geq 2$, we get:

$$\pi_n(\bigvee_\alpha S^n_\alpha) \cong \pi_n\Big(\prod_\alpha S^n_\alpha\Big) \cong \prod_\alpha \pi_n(S^n_\alpha) = \bigoplus_\alpha \pi_n(S^n_\alpha) = \bigoplus_\alpha \mathbb{Z}.$$

□

To conclude our example, we showed that $\pi_n(S^1 \vee S^n) \cong \pi_n(\bigvee_{k\in\mathbb{Z}} S^n_k)$, and $\pi_n(\bigvee_{k\in\mathbb{Z}} S^n_k)$ is free abelian generated by the inclusion of each of the infinite number of n-spheres. Therefore, $\pi_n(S^1 \vee S^n)$ is infinitely generated.

Remark 5.3.12. Under the action of π_1 on π_n, we can regard π_n as a $\mathbb{Z}[\pi_1]$-module, with

$$\mathbb{Z}[\pi_1] = \{\sum_\alpha n_\alpha \gamma_\alpha |\ n_\alpha \in \mathbb{Z}, \gamma_\alpha \in \pi_1\}.$$

Since all S^n_k in the universal cover $\bigvee_{k\in\mathbb{Z}} S^n_k$ are identified under the π_1-action, π_n is a free $\mathbb{Z}[\pi_1]$-module of rank 1, i.e.,

$$\begin{aligned} \pi_n \cong \mathbb{Z}[\pi_1] \cong \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}&[t, t^{-1}], \\ 1 &\mapsto t \\ -1 &\mapsto t^{-1} \\ n &\mapsto t^n, \end{aligned}$$

which is infinitely generated (by the powers of t) over $\mathbb{Z}$ (i.e., as an abelian group).

5.4 Whitehead's Theorem

In this section, we discuss the following important result:

Theorem 5.4.1. *(Whitehead)*
If X and Y are CW complexes, and a map $f : X \to Y$ induces isomorphisms on the homotopy groups π_n for all n, then f is a homotopy equivalence. Moreover, if X is a subcomplex of Y, and f is the inclusion map, then X is a deformation retract of Y.

The following consequence is very useful in practice:

Corollary 5.4.2. *If X and Y are CW complexes with $\pi_1(X) = \pi_1(Y) = 0$, and $f : X \to Y$ induces isomorphisms on homology groups H_n for all n, then f is a homotopy equivalence.*

The above corollary follows from Whitehead's theorem and the following relative version of the Hurewicz theorem:

Theorem 5.4.3. *(Hurewicz)*
If $n \geq 2$, and $\pi_i(X, A) = 0$ for $i < n$, with A simply-connected and non-empty, then $H_i(X, A) = 0$ for $i < n$ and $\pi_n(X, A) \cong H_n(X, A)$.

Before discussing the proof of Whitehead's theorem, let us give an example that shows that having induced isomorphisms on all homology groups is not sufficient for having a homotopy equivalence (so the simply-connectedness assumption in Corollary 5.4.2 is really important):

Example 5.4.4. Let

$$X = S^1 \hookrightarrow (S^1 \vee S^n) \cup e^{n+1} = Y \qquad (n \geq 2),$$

where f is the inclusion of X in Y. We know from Example 5.3.10 that $\pi_n(S^1 \vee S^n) \cong \mathbb{Z}[t, t^{-1}]$. We define Y by attaching the $(n+1)$-cell e^{n+1} to $S^1 \vee S^n$ by a map $g : S^n = \partial e^{n+1} \to S^1 \vee S^n$ so that $[g] \in \pi_n(S^1 \vee S^n)$ corresponds to the element $2t - 1 \in \mathbb{Z}[t, t^{-1}]$. We then see that

$$\pi_n(Y) = \mathbb{Z}[t, t^{-1}]/(2t - 1) \neq 0 = \pi_n(X),$$

since by setting $t = \frac{1}{2}$ we get that $\mathbb{Z}[t, t^{-1}]/(2t - 1) \cong \mathbb{Z}[\frac{1}{2}] = \{\frac{a}{2^k} \mid k \in \mathbb{Z}_{\geq 0}\} \subset \mathbb{Q}$. Moreover, from the long exact sequence of homotopy groups for the $(n-1)$-connected pair (Y, X), the inclusion $X \hookrightarrow Y$ induces an isomorphism on homotopy groups π_i for $i < n$. Finally, this inclusion map also induces isomorphisms on all homology groups, $H_k(X) \cong H_k(Y)$ for all k, as can be seen from cellular homology. Indeed, the cellular boundary map

$$H_{n+1}(Y_{n+1}, Y_n) \to H_n(Y_n, Y_{n-1})$$

is an isomorphism since the degree of the composition of the attaching map $S^n \to S^1 \vee S^n$ of e^{n+1} with the collapse map $S^1 \vee S^n \to S^n$ is $2 - 1 = 1$.

Let us now get back to the proof of Whitehead's Theorem 5.4.1:

Proof. (of Whitehead's theorem) To prove Whitehead's theorem, we will use the following *compression lemma*:

Lemma 5.4.5. *(Compression Lemma)*
Let (X, A) be a CW pair, and (Y, B) be a pair with Y path-connected and $B \neq \emptyset$. Suppose that for each $n > 0$ for which $X \setminus A$ has cells of dimension n, $\pi_n(Y, B, b_0) = 0$ for all $b_0 \in B$. Then any map $f : (X, A) \to (Y, B)$ is homotopic to some map $f' : X \to B$ fixing A (i.e. $f'|_A = f|_A$).

We can then split the proof of Whitehead's theorem into two cases:
Case 1: If f is an inclusion $X \hookrightarrow Y$, since $\pi_n(X) = \pi_n(Y)$ for all n, we get by the long exact sequence for the homotopy groups of the pair (Y, X) that $\pi_n(Y, X) = 0$ for all n. Applying the above compression lemma to the identity map $id : (Y, X) \to (Y, X)$, we get that the identity map id_Y is homotopic to a deformation retract $r : Y \to X$.
Case 2: The general case of a map $f : X \to Y$ can be reduced to the above case of an inclusion by using the *mapping cylinder* of f, i.e.,

$$M_f := (X \times I) \sqcup Y/(x, 1) \sim f(x).$$

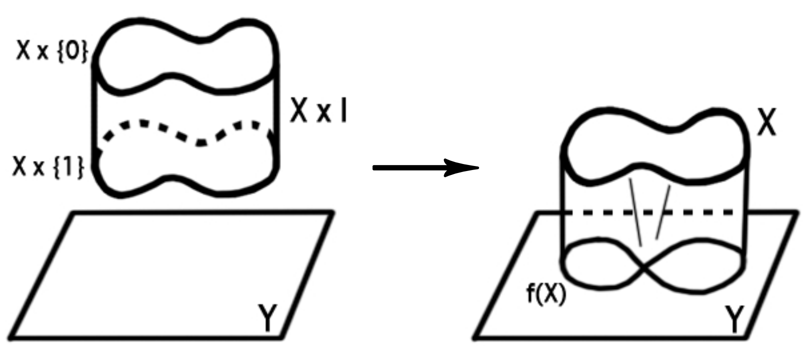


Figure 5.1: The mapping cylinder of f, M_f

Note that M_f contains both $X = X \times \{0\}$ and Y as subspaces, and M_f deformation retracts onto Y. Moreover, the map f can be written as the composition of the inclusion i of X into M_f, and the retraction r from M_f to Y:

$$X = X \times \{0\} \hookrightarrow M_f \xrightarrow{r} Y, \qquad (f = r \circ i, \text{ for } i : X \times \{0\} \hookrightarrow M_f).$$

Since M_f is homotopy equivalent to Y via r, it suffices to show that M_f deformation retracts onto X, so we can replace f with the inclusion map i. If f is a cellular map, then M_f is a CW complex having X as a subcomplex. So we can apply Case 1. If f is not cellular, than f is homotopic to some cellular map g, so we may work with g and the mapping cylinder M_g and again reduce to Case 1. □

We can now prove Corollary 5.4.2:

Proof. (of Corollary 5.4.2)
After replacing Y by the mapping cylinder M_f, we may take f to be an inclusion $X \hookrightarrow Y$. As $H_n(X) \cong H_n(Y)$ for all n, we have by the long exact sequence for the homology of the pair (Y, X) that $H_n(Y, X) = 0$ for all n.

Since X and Y are simply-connected, we have $\pi_1(Y, X) = 0$. So by the relative Hurewicz Theorem 5.4.3, the first non-zero $\pi_n(Y, X)$ is isomorphic to the first non-zero $H_n(Y, X)$. So $\pi_n(Y, X) = 0$ for all n. Then, by the homotopy long exact sequence for the pair (Y, X), we get that

$$\pi_n(X) \cong \pi_n(Y)$$

for all n, with isomorphisms induced by the inclusion map f. Finally, Whitehead's theorem yields that f is a homotopy equivalence. □

Example 5.4.6. Let $X = \mathbb{RP}^2$ and $Y = S^2 \times \mathbb{RP}^\infty$. First note that $\pi_1(X) = \pi_1(Y) \cong \mathbb{Z}/2$. Also, since S^2 is a covering of $\mathbb{RP}^2$, we have that

$$\pi_i(X) \cong \pi_i(S^2), \quad i \geq 2.$$

Moreover, $\pi_i(Y) \cong \pi_i(S^2) \times \pi_i(\mathbb{RP}^\infty)$, and as $\mathbb{RP}^\infty$ is covered by $S^\infty = \bigcup_{n \geq 0} S^n$, we get that

$$\pi_i(Y) \cong \pi_i(S^2) \times \pi_i(S^\infty), \quad i \geq 2.$$

To calculate $\pi_i(S^\infty)$, we use cellular approximation. More precisely, we can approximate any $f : S^i \to S^\infty$ by a cellular map g so that Image $g \subset S^n$ for $i \ll n$. Thus, $[f] = [g] \in \pi_i(S^n) = 0$, and we see that

$$\pi_i(X) \cong \pi_i(S^2) \cong \pi_i(Y), \quad i \geq 2.$$

Altogether, we have that X and Y have the same homotopy groups. However, as can be easily seen by considering homology groups, X and Y are not homotopy equivalent. In particular, by Whitehead's theorem, there cannot exist a map $f : \mathbb{RP}^2 \to S^2 \times \mathbb{RP}^\infty$ inducing isomorphisms on π_n for all n. Indeed, if such a map existed, it would have to be a homotopy equivalence.

Example 5.4.7. As we will see later on, the CW complexes S^2 and $S^3 \times \mathbb{CP}^\infty$ have isomorphic homotopy groups, but they are not homotopy equivalent.

Let us now prove another important result:

Theorem 5.4.8. *If $f : X \to Y$ induces isomorphisms on homotopy groups π_n for all n, then it induces isomorphisms on homology and cohomology groups with G coefficients, for any group G.*

Proof. By the universal coefficient theorems, it suffices to show that f induces isomorphisms on integral homology groups $H_*(-; \mathbb{Z})$.

We only prove the assertion here under the extra condition that X is simply connected (the general case follows easily from spectral sequence theory). As before, we can also assume that f by an inclusion (by replacing Y with the homotopy equivalent space defined by mapping cylinder M_f of f). Since by the hypothesis, $\pi_n(X) \cong \pi_n(Y)$ for all n, with isomorphisms induced by the inclusion f, the homotopy long exact sequence of the pair (Y, X) yields that $\pi_n(Y, X) = 0$ for all n. By the relative Hurewicz theorem (as $\pi_1(X) = 0$), this gives that $H_n(Y, X) = 0$ for all n. Hence, by the long exact sequence for homology, $H_n(X) \cong H_n(Y)$ for all n, and the proof is complete. □

Example 5.4.9. Take $X = \mathbb{RP}^2 \times S^3$ and $Y = S^2 \times \mathbb{RP}^3$. They have isomorphic homotopy groups π_n for all n, but $H_5(X) \ncong H_5(Y)$. So there cannot exist a map $f : X \to Y$ inducing the isomorphisms on the π_n.

Example 5.4.10. Any abelian group G can be realized as $\pi_n(X)$ with $n \geq 2$ for some space X. In fact, for a finitely generated group $G = \langle g_1, \ldots, g_s \mid r_1, \ldots, r_k \rangle$, we can can take

$$X = \left(\bigvee_{i=1}^{s} S_i^n \right) \cup \bigcup_{j=1}^{k} e_j^{n+1},$$

for e_j^{n+1} attached to $\bigvee_{i=1}^{s} S_i^n$ by the map $f : S_j^n \to \bigvee_{n=1}^{s} S^n$ with $[f] = r_j$.

Example 5.4.11. *Eilenberg-MacLane spaces*
For any group G and $n \in \mathbb{Z}$, one can define a space $K(G, n) = X$ with $\pi_n(X) = G$ and $\pi_i(X) = 0$ for all $i \neq n$. (These spaces unique up to homotopy!) Some familiar such spaces are:

- $K(\mathbb{Z}/2, 1) = \mathbb{RP}^\infty$
- $K(\mathbb{Z}, 1) = S^1$
- $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$

For the last example, we can see that $\pi_2(\mathbb{CP}^\infty) \cong \pi_1(S^1) = \mathbb{Z}$ by using the *fibration* (see next section for a definition)

$$S^1 \hookrightarrow S^\infty \longrightarrow \mathbb{CP}^\infty$$

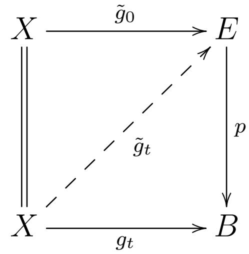
which gives the long exact sequence of homotopy groups

$$\cdots \to \pi_2(S^\infty) \to \pi_2(\mathbb{CP}^\infty) \to \pi_1(S^1) \to \pi_1(S^\infty) \to \cdots$$

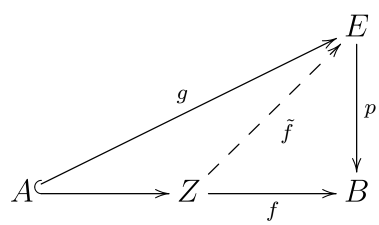
together with the fact that $\pi_2(S^\infty) = \pi_1(S^\infty) = 0$, giving $\pi_2(\mathbb{CP}^\infty) \cong \pi_1(S^1)$ by exactness.

5.5 Fibrations and Fiber Bundles

Definition 5.5.1. *Homotopy Lifting Property*
A map $p : E \to B$ has the homotopy lifting property with respect to X if, given a homotopy $g_t : X \to B$, and a lift $\tilde{g}_0 : X \to E$, of g_0, there exists a homotopy $\tilde{g}_t : X \to E$ lifting g_t and extending $\tilde{g}_0$.



Remark 5.5.2. This is a special case of the *lift extension property.* A map $p : E \to B$ has the lift extension property with respect to a pair (Z, A) if for all maps $f : Z \to B$ and $g : A \to E$, there exists a lift $\tilde{f} : Z \to E$ of f extending g. (Think of $Z = X \times [0, 1]$, and $A = X \times \{0\}$.)



Definition 5.5.3. *A fibration $p : E \to B$ is a map having the homotopy lifting property with respect to all spaces X.*

Definition 5.5.4. *Homotopy Lifting Property for a pair (X, A)*
A map $p : E \to B$ has the homotopy lifting property with respect to a pair (X, A) if each homotopy $g_t : X \to B$ lifts to a homotopy $\tilde{g}_t : X \to E$ starting with a given lift $\tilde{g}_0$ and extending a given lift $\tilde{g}_t : A \to E$.

Remark 5.5.5. The homotopy lifting property with respect to the pair (X, A) is the lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$.

Remark 5.5.6. The homotopy lifting property for a disk D^n is equivalent for the homotopy lifting property for $(D^n, \partial D^n)$ since the pairs $(D^n \times I, D^n \times \{0\})$ and $(D^n \times I, D^n \times \{0\} \cup \partial D^n \times I)$ are homeomorphic. This implies that *a fibration has has the homotopy lifting property with respect to all CW pairs* (X, A). Indeed, the homotopy lifting property for discs is in fact equivalent to the homotopy lifting property with respect to all CW pairs (X, A). This can be easily seen by induction over the skeleta of X, so it suffices to construct a lifting $\tilde{g}_t$ one cell of $X \setminus A$ at a time. Composing with the characteristic map $D^n \to X$ of a cell then gives the reduction to the case $(X, A) = (D^n, \partial D^n)$.

Theorem 5.5.7. *Given a fibration $p : E \to B$, points $b \in B$ and $e \in F := p^{-1}(b)$, there is an isomorphism $p_* : \pi_n(E, F, e) \xrightarrow{\cong} \pi_n(B, b)$ for all $n \geq 1$. Hence, if B is path connected there a long exact sequence of homotopy groups:*

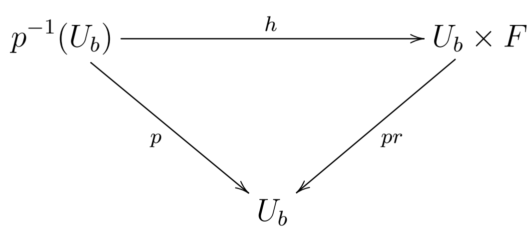
$$\cdots \to \pi_n(F, e) \to \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \to \pi_{n-1}(F, e) \to \cdots \pi_0(E, e) \to 0$$

Proof. To show that p_* is onto, represent an element of $\pi_n(B, b)$ by a map $f : (I^n, \partial I^n) \to (B, b)$, and note that the constant map to e is a lift of f to E over $J^{n-1} \subset I^n$. The homotopy lifting property for the pair $(I^{n-1}, \partial I^{n-1})$ extends this to a lift $\widetilde{f} : I^n \to E$. This lift satisfies $\widetilde{f}(\partial I^n) \subset F$ since $f(\partial I^n) = b$. So $\widetilde{f}$ represents an element of $\pi_n(E, F, e)$ with $p_*([\widetilde{f}]) = [f]$ since $p\widetilde{f} = f$.

To show injectivity of p_*, let $\widetilde{f_0}, \widetilde{f_1} : (I^n, \partial I^n, J^{n-1}) \to (E, F, e)$ be so that $p_*(\widetilde{f_0}) = p_*(\widetilde{f_1})$. Let $H : (I^n \times I, \partial I^n \times I) \to (B, b)$ be a homotopy from $p\widetilde{f_0}$ to $p\widetilde{f_1}$. We have a partial lift given by $\widetilde{f_0}$ on $I^n \times \{0\}$, $\widetilde{f_1}$ on $I^n \times \{1\}$ and the constant map to e on $J^{n-1} \times I$. The homotopy lifting property for CW pairs extends this to a lift $\widetilde{H} : I^n \times I \to E$ giving a homotopy $\widetilde{f_t} : (I^n, \partial I^n, J^{n-1}) \to (E, F, e)$ from $\widetilde{f_0}$ to $\widetilde{f_1}$.

Finally, the long exact sequence of fibration follows by plugging $\pi_n(B, b)$ in for $\pi_n(E, F, e)$ in the long exact sequence for the pair (E, F). The map $\pi_n(E, e) \to \pi_n(E, F, e)$ in the latter sequence becomes the composition $\pi_n(E, e) \to \pi_n(E, F, e) \xrightarrow{p_*} \pi_n(B, b)$, which is exactly $p_* : \pi_n(E, e) \to \pi_n(B, b)$. The surjectivity of $\pi_0(F, e) \to \pi_0(E, e)$ follows from the path-connectedness of B, since a path in E from an arbitrary point $x \in E$ to F can be obtained by lifting a path in B from $p(x)$ to b. □

Definition 5.5.8. *Fiber Bundle*
A map $p : E \to B$ is a fiber bundle with fiber F if, for all points $b \in B$, there exists neighborhood U_b of b with a homeomorphism $h : p^{-1}(U_b) \to U_b \times F$ so that the following diagram commutes:



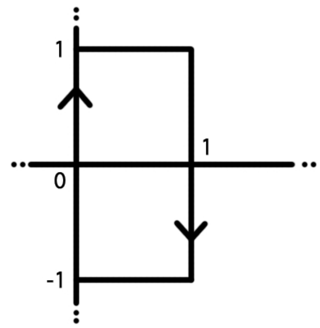
Remark 5.5.9. Fibers of fibrations are homotopy equivalent, while fibers of fiber bundles are homeomorphic.

Theorem 5.5.10. *(Hurewicz)*
Fiber bundles over paracompact spaces are fibrations.

Example 5.5.11. *Examples of fiber bundles*

1. If F is discrete, a fiber bundle with fiber F is a covering map.

2. The Möbius band $I \times [-1, 1]/(0, x) \sim (1, -x) \to S^1$ is a fiber bundle with fiber $[-1, 1]$, induced from the projection map $I \times [-1, 1] \to I$.



3. By glueing the unlabeled edges of a Möbius band, we get $K \to S^1$ (where K is the Klein bottle), a fiber bundle with fiber S^1.

4. The following is a fiber bundle with fiber S^1:

$$\begin{aligned} S^1 \hookrightarrow S^{2n+1}(\subset \mathbb{C}^{n+1}) &\to \mathbb{CP}^n \\ (z_0, \dots, z_n) &\mapsto [z_0 : \dots : z_n] \end{aligned}$$

   For $[\underline{z}] \in \mathbb{CP}^n$, there is an i such that $z_i \neq 0$. Then we have

$$U_{[\underline{z}]} = \{[z_0 : \dots : 1 : \dots : z_n]\} \cong \mathbb{C}^n$$

   (with the entry 1 in place of the ith coordinate), with a homeomorphism

$$\begin{aligned} p^{-1}(U_{[\underline{z}]}) &\to U_{[\underline{z}]} \times S^1 \\ (z_0, \dots, z_n) &\mapsto ([z_0 : \dots : z_n], z_i/|z_i|). \end{aligned}$$

   From this we get the fibration diagram from our discussion of Eilenberg-MacLane spaces,

$$\begin{array}{ccccccccc} S^1 & & & S^1 & & \cdots & S^1 & & \\ & \searrow & & & \searrow & & & \searrow & \\ & S^{2n+1} & & & S^{2n+3} & \cdots & & S^\infty & \simeq \{\text{pt}\} \\ & \downarrow & & & \downarrow & & & \downarrow & \\ & \mathbb{CP}^n & \subset & & \mathbb{CP}^{n+1} & \cdots & & \mathbb{CP}^\infty & \end{array}$$

   In particular, from the long exact sequence of the fibration

$$S^1 \hookrightarrow S^\infty \to \mathbb{CP}^\infty$$

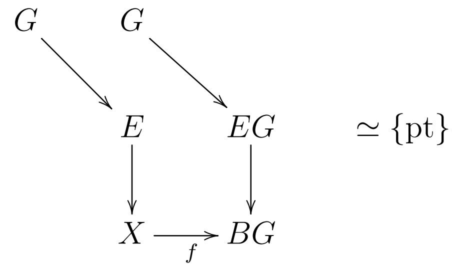
   with S^∞ contactible, we obtain that

$$\pi_i(\mathbb{CP}^\infty) \cong \pi_{i-1}(S^1) = \begin{cases} \mathbb{Z} & i = 2 \\ 0 & i \neq 2 \end{cases}$$

   i.e.,

$$\mathbb{CP}^\infty = K(\mathbb{Z}, 2).$$

Remark 5.5.12. For any topological group G, there exists a "universal fiber bundle" classifying the space of (principal) G-bundles. That is, any G-bundle over a space X is determined (by pull-back) by (the homotopy class of) a map $f : X \to BG$.



5. Other examples of fibrations (in fact fiber bundles) are provided by the orthogonal and unitary groups:

$$O(n-1) \hookrightarrow O(n) \to S^{n-1}$$
$$A \mapsto Ax,$$

where x is a fixed unit vector in $\mathbb{R}^n$. (If we assume n is large, the associated long exact sequence will give us that $\pi_i(O(n))$ is independent of n.) Similarly, there is a fibration

$$U(n-1) \hookrightarrow U(n) \to S^{2n-1}$$
$$A \mapsto Ax,$$

with x a fixed unit vector in $\mathbb{C}^n$.

In the remaining of this section, we show that *any map is homotopic to a fibration.* Given $f : A \to B$, define

$$E_f := \{(a, \gamma) \mid a \in A,\ \gamma : [0,1] \to B \text{ with } \gamma(0) = f(a)\}.$$

Then A can be regarded as a subset of E_f, by mapping $a \in A$ to $(a, c_{f(a)})$, where $c_{f(a)}$ is the constant path based at the image of a under f. Define

$$E_f \xrightarrow{p} B$$
$$(a, \gamma) \mapsto \gamma(1)$$

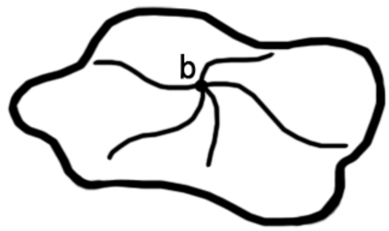
Then $p|_A = f$, and $f = p \circ i$ where i is the inclusion of A in E_f. Moreover, i is a homotopy equivalence, and p is a fibration with fiber A.

Remark 5.5.13. If $A = \{b\} \hookrightarrow B$, where f is the inclusion of b in B, then E_f is the contractible space of paths in B starting at b:
In this case, the above construction yields a fibration

$$\Omega B = p^{-1}(b) \hookrightarrow E_f \to B,$$

where ΩB is the space of all loops based at b. Since E_f is contractible, the associated long exact sequence of the fibration yields that

$$\pi_i(B) \cong \pi_{i-1}(\Omega B).$$



Exercises

1. Let $f : X \to Y$ be a homotopy equivalence. Let Z be any other space. Show that f induces bijections:

$$f_* : [Z, X] \to [Z, Y] \quad \text{and} \quad f^* : [Y, Z] \to [X, Z] \ ,$$

where $[A, B]$ denotes the set of homotopy classes of maps from the space A to B.

2. Find examples of spaces X and Y which have the same homology groups, cohomology groups, and cohomology rings, but with different homotopy groups.

3. Use homotopy groups in order to show that there is no retraction $\mathbb{RP}^n \to \mathbb{RP}^k$ if $n > k > 0$.

4. Show that an n-connected, n-dimensional CW complex is contractible.

5. (*Extension Lemma*)
Given a CW pair (X, A) and a map $f : A \to Y$ with Y path-connected, show that f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X \setminus A$ has cells of dimension n.

6. Show that a CW complex retracts onto any contractible subcomplex. (Hint: Use the above extension lemma.)

7. If $p : (\tilde{X}, \tilde{A}, \tilde{x}_0) \to (X, A, x_0)$ is a covering space with $\tilde{A} = p^{-1}(A)$, show that the map $p_* : \pi_n(\tilde{X}, \tilde{A}, \tilde{x}_0) \to \pi_n(X, A, x_0)$ is an isomorphism for all $n > 1$.

8. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic. Conclude that S^∞ is contractible, and more generally, this is true for the infinite suspension $\Sigma^\infty(X) := \bigcup_{n \geq 0} \Sigma^n(X)$ of any CW complex X.

9. Use cellular approximation to show that the n-skeletons of homotopy equivalent CW complexes without cells of dimension $n+1$ are also homotopy equivalent.

10. Show that a closed simply-connected 3-manifold is homotopy equivalent to S^3. (Hint: Use Poincaré Duality, and also the fact that closed manifolds are homotopy equivalent to CW complexes.)

11. Show that a map $f : X \to Y$ of connected CW complexes is a homotopy equivalence if it induces an isomorphism on π_1 and if a lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ to the universal covers induces an isomorphism on homology.

12. Show that $\pi_7(S^4)$ is non-trivial. [Hint: It contains a $\mathbb{Z}$-summand.]

13. Prove that the space $SO(3)$ of orthogonal 3×3 matrices with determinant 1 is homeomorphic to $\mathbb{RP}^3$.

14. Show that if $S^k \to S^m \to S^n$ is a fiber bundle, then $k = n - 1$ and $m = 2n - 1$.

15. Show that if there were fiber bundles $S^{n-1} \to S^{2n-1} \to S^n$ for *all* n, then the groups $\pi_i(S^n)$ would be finitely generated free abelian groups computable by induction, and non-zero if $i \geq n \geq 2$.